Second Order and Higher Order Equations

Introduction

Second order and higher order equations occur frequently in science and engineering (like pendulum problem etc.) and hence has its own importance. It has its own flavour also. We devote this section for an elementary introduction.

DEFINITION 8.1.1 (Second Order Linear Differential Equation) ***The equation***

$$p(x)y'' + q(x)y' + r(x)y = c(x), \quad x \in I \tag{8.1.1}$$

is called a SECOND ORDER LINEAR DIFFERENTIAL EQUATION.

Here I is an interval contained in $\mathbb{R}$; and the functions $p(\cdot), q(\cdot), r(\cdot),$ and $c(\cdot)$ are real valued continuous functions defined on $\mathbb{R}$. the functions $p(\cdot), q(\cdot),$ and $r(\cdot)$ are called the coefficients of Equation (8.1.1) and $c(x)$ is called the non-homogeneous term or the force function.

Equation (8.1.1) is called linear homogeneous if $c(x) \equiv 0$ and non-homogeneous if $c(x) \neq 0$.

Recall that a second order equation is called nonlinear if it is not linear.

EXAMPLE 8.1.2

1. The equation

   $$y'' + \sqrt{\frac{g}{\ell}} \sin y = 0$$

   is a second order equation which is nonlinear.

2. $y'' - y = 0$ is an example of a linear second order equation.
3. $y'' + y' + y = \sin x$ is a non-homogeneous linear second order equation.

4. $ax^2y'' + bxy' + cy = 0$ $c \neq 0$ is a homogeneous second order linear equation. This equation is called EULER EQUATION OF ORDER 2. Here $a, b,$ and c are real constants.

DEFINITION 8.1.3 ***A function y defined on I is called a solution of Equation (8.1.1) if y is twice differentiable and satisfies Equation (8.1.1).***

EXAMPLE 8.1.4

1. e^x and e^{-x} are solutions of $y'' - y = 0$.
2. $\sin x$ and $\cos x$ are solutions of $y'' + y = 0$.

We now state an important theorem whose proof is simple and is omitted.

THEOREM 8.1.5 (Superposition Principle) ***Let y_1 and y_2 be two given solutions of***

$$p(x)y'' + q(x)y' + r(x)y = 0, \quad x \in I. \tag{8.1.2}$$

Then for any two real number $c_1, c_2,$ the function $c_1y_1 + c_2y_2$ is also a solution of Equation (8.1.2).

It is to be noted here that Theorem 8.1.5 is not an existence theorem. That is, it does not assert the existence of a solution of Equation (8.1.2).

DEFINITION 8.1.6 (Solution Space) ***The set of solutions of a differential equation is called the solution space.***

For example, all the solutions of the Equation (8.1.2) form a solution space. Note that $y(x) \equiv 0$ is also a solution of Equation (8.1.2). Therefore, the solution set of a Equation (8.1.2) is non-empty. A moments reflection on Theorem 8.1.5 tells us that the solution space of Equation (8.1.2) forms a real vector space.

Remark 8.1.7 *The above statements also hold for any homogeneous linear differential equation. That is, the solution space of a homogeneous linear differential equation is a real vector space.*

The natural question is to inquire about its dimension. This question will be answered in a sequence of results stated below.

We first recall the definition of Linear Dependence and Independence.

DEFINITION 8.1.8 (Linear Dependence and Linear Independence) ***Let I be an interval in $\mathbb{R}$ and let $f, g : I \longrightarrow \mathbb{R}$ be continuous functions. we say that f, g are said to be linearly dependent if there are real numbers a and b (not both zero) such that***

$$af(t) + bg(t) = 0 \quad \text{for all} \quad t \in I.$$

The functions $f(\cdot), g(\cdot)$ are said to be linearly independent if $f(\cdot), g(\cdot)$ are not linear dependent.

To proceed further and to simplify matters, we assume that $p(x) \equiv 1$ in Equation (8.1.2) and that the function $q(x)$ and $r(x)$ are continuous on I.

In other words, we consider a homogeneous linear equation

$$y'' + q(x)y' + r(x)y = 0, \quad x \in I, \tag{8.1.3}$$

where q and r are real valued continuous functions defined on I.

The next theorem, given without proof, deals with the existence and uniqueness of solutions of Equation (8.1.3) with initial conditions $y(x_0) = A, \; y'(x_0) = B$ for some $x_0 \in I$.

THEOREM 8.1.9 (Picard's Theorem on Existence and Uniqueness) ***Consider the Equation (8.1.3) along with the conditions***

$$y(x_0) = A, \; y'(x_0) = B, \;\; \text{for some} \;\; x_0 \in I \tag{8.1.4}$$

where A and B are prescribed real constants. Then Equation (8.1.3), with initial conditions given by Equation (8.1.4) has a unique solution on I.

A word of Caution: NOTE THAT THE COEFFICIENT OF #MATH4130# MATHEND000# IN EQUATION ( ) IS MATHEND000# BEFORE WE APPLY THEOREM , WE HAVE TO ENSURE THIS CONDITION.

An important application of Theorem 8.1.9 is that the equation (8.1.3) has exactly 2 linearly independent solutions. In other words, the set of all solutions over $\mathbb{R},$ forms a real vector space of dimension 2.

THEOREM 8.1.10 ***Let q and r be real valued continuous functions on I. Then Equation (8.1.3) has exactly two linearly independent solutions. Moreover, if y_1 and y_2 are two linearly independent solutions of Equation (8.1.3), then the solution space is a linear combination of y_1 and y_2.***

Proof. Let y_1 and y_2 be two unique solutions of Equation (8.1.3) with initial conditions

$$y_1(x_0) = 1,\ y_1'(x_0) = 0, \quad \text{and} \quad y_2(x_0) = 0,\ y_2'(x_0) = 1 \ \text{ for some } \ x_0 \in I. \tag{8.1.5}$$

The unique solutions y_1 and y_2 exist by virtue of Theorem 8.1.9. We now claim that y_1 and y_2 are linearly independent. Consider the system of linear equations

$$\alpha y_1(x) + \beta y_2(x) = 0, \tag{8.1.6}$$

where α and β are unknowns. If we can show that the only solution for the system (8.1.6) is $\alpha = \beta = 0$, then the two solutions y_1 and y_2 will be linearly independent.

Use initial condition on y_1 and y_2 to show that the only solution is indeed $\alpha = \beta = 0$. Hence the result follows.

We now show that any solution of Equation (8.1.3) is a linear combination of y_1 and y_2. Let ζ be any solution of Equation (8.1.3) and let $d_1 = \zeta(x_0)$ and $d_2 = \zeta'(x_0).$ Consider the function ϕ defined by

$$\phi(x) = d_1 y_1(x) + d_2 y_2(x).$$

By Definition 8.1.3, ϕ is a solution of Equation (8.1.3). Also note that $\phi(x_0) = d_1$ and $\phi'(x_0) = d_2.$ So, ϕ and ζ are two solution of Equation (8.1.3) with the same initial conditions. Hence by Picard's Theorem on Existence and Uniqueness (see Theorem 8.1.9), $\phi(x) \equiv \zeta(x)$ or

$$\zeta(x) = d_1 y_1(x) + d_2 y_2(x).$$

Thus, the equation (8.1.3) has two linearly independent solutions. height6pt width 6pt depth 0pt

Remark 8.1.11

1. Observe that the solution space of Equation (8.1.3) forms a real vector space of dimension 2.
2. The solutions y_1 and y_2 corresponding to the initial conditions

$$y_1(x_0) = 1,\ y_1'(x_0) = 0, \quad \text{and} \quad y_2(x_0) = 0,\ y_2'(x_0) = 1 \ \text{ for some } \ x_0 \in I,$$

are called a FUNDAMENTAL SYSTEM of solutions for Equation (8.1.3).

3. Note that the fundamental system for Equation (8.1.3) is not unique.

Consider a 2×2 non-singular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$. Let $\{y_1, y_2\}$ be a fundamental system for the differential Equation 8.1.3 and $\mathbf{y}^t = [y_1,\ y_2]$. Then the rows of the matrix $A\mathbf{y} = \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix}$ also form a fundamental system for Equation 8.1.3. That is, if $\{y_1, y_2\}$ is a fundamental system for Equation 8.1.3 then $\{ay_1 + by_2, cy_1 + dy_2\}$ is also a fundamental system whenever $ad - bc = \det(A) \neq 0$.

EXAMPLE 8.1.12 ***$\{1, \mathbf{x}\}$ is a fundamental system for $y'' = 0$.***

Note that $\{1 - \mathbf{x}, 1 + \mathbf{x}\}$ is also a fundamental system. Here the matrix is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

EXERCISE 8.1.13

1. State whether the following equations are SECOND-ORDER LINEAR or SECOND-ORDER NON-LINEAR equaitons.
    1. $y'' + y \sin x = 5$.
    2. $y'' + (y')^2 + y \sin x = 0$.
    3. $y'' + yy' = -2$.
    4. $(x^2 + 1)y'' + (x^2 + 1)^2 y' - 5y = \sin x$.
2. By showing that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of

$$y'' - y = 0$$

conclude that $\sinh x$ and $\cosh x$ are also solutions of $y'' - y = 0$. Do $\sinh x$ and $\cosh x$ form a fundamental set of solutions?

3. Given that $\{\sin x, \cos x\}$ forms a basis for the solution space of $y'' + y = 0$, find another basis.

More on Second Order Equations

In this section, we wish to study some more properties of second order equations which have nice applications. They also have natural generalisations to higher order equations.

DEFINITION 8.2.1 (General Solution) ***Let y_1 and y_2 be a fundamental system of solutions for***

$$y'' + q(x)y' + r(x)y = 0, \; x \in I. \tag{8.2.1}$$

The general solution y of Equation (8.2.1) is defined by

$$y = c_1 y_1 + c_2 y_2, \; x \in I$$

where c_1 and c_2 are arbitrary real constants. Note that y is also a solution of Equation (8.2.1).

In other words, the general solution of Equation (8.2.1) is a 2-parameter family of solutions, the parameters being c_1 and c_2.

Second Order equations with Constant Coefficients

DEFINITION 8.3.1 ***Let a and b be constant real numbers. An equation***

$$y'' + ay' + by = 0 \tag{8.3.1}$$

is called a SECOND ORDER HOMOGENEOUS LINEAR EQUATION WITH CONSTANT COEFFICIENTS.

Let us assume that $y = e^{\lambda x}$ to be a solution of Equation (8.3.1) (where λ is a constant, and is to be determined). To simplify the matter, we denote

$$L(y) = y'' + ay' + by$$

and

$$p(\lambda) = \lambda^2 + a\lambda + b.$$

It is easy to note that

$$L(e^{\lambda x}) = p(\lambda)e^{\lambda x}.$$

Now, it is clear that $e^{\lambda x}$ is a solution of Equation (8.3.1) if and only if

$$p(\lambda) = 0. \tag{8.3.2}$$

Equation (8.3.2) is called the CHARACTERISTIC EQUATION of Equation (8.3.1). Equation (8.3.2) is a quadratic equation and admits 2 roots (repeated roots being counted twice).

Case 1: Let λ_1, λ_2 be real roots of Equation (8.3.2) with $\lambda_1 \neq \lambda_2$. Then $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are two solutions of Equation (8.3.1) and moreover they are linearly independent (since $\lambda_1 \neq \lambda_2$). That is, $\{e^{\lambda_1 x}, e^{\lambda_2 x}\}$ forms a fundamental system of solutions of Equation (8.3.1).

Case 2: Let $\lambda_1 = \lambda_2$ be a repeated root of $p(\lambda) = 0$. Then $p'(\lambda_1) = 0$. Now,

$$\frac{d}{dx}(L(e^{\lambda x})) = L(xe^{\lambda x}) = p'(\lambda)e^{\lambda x} + xp(\lambda)e^{\lambda x}.$$

But $p'(\lambda_1) = 0$ and therefore,

$$L(xe^{\lambda_1 x}) = 0.$$

Hence, $e^{\lambda_1 x}$ and $xe^{\lambda_1 x}$ are two linearly independent solutions of Equation (8.3.1). In this case, we have a fundamental system of solutions of Equation (8.3.1).

Case 3: Let $\lambda = \alpha + i\beta$ be a complex root of Equation (8.3.2).
So, $\alpha - i\beta$ is also a root of Equation (8.3.2). Before we proceed, we note:

LEMMA 8.3.2 ***Let*** $y = u + iv$ ***be a solution of Equation (8.3.1), where*** u ***and*** v ***are real valued functions. Then*** u ***and*** v ***are solutions of Equation (8.3.1). In other words, the real part and the imaginary part of a complex valued solution (of a real variable ODE Equation (8.3.1)) are themselves solution of Equation (8.3.1).***
Proof. exercise. height6pt width 6pt depth 0pt

Let $\lambda = \alpha + i\beta$ be a complex root of $p(\lambda) = 0.$ Then

$$e^{\alpha x}(\cos(\beta x) + i\sin(\beta x))$$

is a complex solution of Equation (8.3.1). By Lemma 8.3.2, $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = \sin(\beta x)$ are solutions of Equation (8.3.1). It is easy to note that y_1 and y_2 are linearly independent. It is as good as saying $\{e^{\lambda x}\cos(\beta x), e^{\lambda x}\sin(\beta x)\}$ forms a fundamental system of solutions of Equation (8.3.1).

EXERCISE 8.3.3

1. Find the general solution of the follwoing equations.
    1. $y'' - 4y' + 3y = 0.$
    2. $2y'' + 5y = 0.$
    3. $y'' - 9y = 0.$
    4. $y'' + k^2 y = 0,$ where k is a real constant.
2. Solve the following IVP's.
    1. $y'' + y = 0,\ y(0) = 0,\ y'(0) = 1.$
    2. $y'' - y = 0,\ y(0) = 1,\ y'(0) = 1.$
    3. $y'' + 4y = 0,\ y(0) = -1,\ y'(0) = -3.$

4. $y'' + 4y' + 4y = 0,\ y(0) = 1,\ y'(0) = 0.$

3. Find two linearly independent solutions y_1 and y_2 of the following equations.
   1. $y'' - 5y = 0.$
   2. $y'' + 6y' + 5y = 0.$
   3. $y'' + 5y = 0.$
   4. $y'' + 6y' + 9y = 0.$ Also, in each case, find $W(y_1, y_2).$
4. Show that the IVP

$$y'' + y = 0,\ y(0) = 0 \ \text{ and } \ y'(0) = B$$

   has a unique solution for any real number $B.$

5. Consider the problem

$$y'' + y = 0,\ y(0) = 0 \ \text{ and } \ y'(\pi) = B. \tag{8.3.3}$$

6.

7. Show that it has a solution if and only if $B = 0.$ Compare this with Exercise 4. Also, show that if $B = 0,$ then there are infinitely many solutions to (8.3.3).

Non Homogeneous Equations

Throughout this section, I denotes an interval in $\mathbb{R}$. we assume that $q(\cdot), r(\cdot)$ and $f(\cdot)$ are real valued continuous function defined on I. Now, we focus the attention to the study of non-homogeneous equation of the form

$$y'' + q(x)y' + r(x)y = f(x). \tag{8.4.1}$$

We assume that the functions $q(\cdot), r(\cdot)$ and $f(\cdot)$ are known/given. The non-zero function $f(\cdot)$ in (8.4.1) is also called the non-homogeneous term or the forcing function. The equation

$$y'' + q(x)y' + r(x)y = 0. \tag{8.4.2}$$

is called the homogeneous equation corresponding to (8.4.1).

Consider the set of all twice differentiable functions defined on I. We define an operator L on this set by

$$L(y) = y'' + q(x)y' + r(x)y.$$

Then (8.4.1) and (8.4.2) can be rewritten in the (compact) form

$$L(y) = f \tag{8.4.3}$$

$$L(y) = 0. \tag{8.4.4}$$

The ensuing result relates the solutions of (8.4.1) and (8.4.2).

THEOREM 8.4.1

1. Let y_1 and y_2 be two solutions of (8.4.1) on I. Then $y = y_1 - y_2$ is a solution of (8.4.2).
2. Let z be any solution of (8.4.1) on I and let z_1 be any solution of (8.4.2). Then $y = z + z_1$ is a solution of (8.4.1) on I.

Proof. Observe that L is a linear transformation on the set of twice differentiable function on I. We therefore have

$$L(y_1) = f \quad \text{and} \quad L(y_2) = f.$$

The linearity of L implies that $L(y_1 - y_2) = 0$ or equivalently, $y = y_1 - y_2$ is a solution of (8.4.2).

For the proof of second part, note that

$$L(z) = f \quad \text{and} \quad L(z_1) = 0$$

implies that

$$L(z + z_1) = L(z) + L(z_1) = f.$$

Thus, $y = z + z_1$ is a solution of (8.4.1). height6pt width 6pt depth 0pt

The above result leads us to the following definition.

DEFINITION 8.4.2 (General Solution) ***A general solution of (8.4.1) on I is a solution of (8.4.1) of the form***

$$y = y_h + y_p, \; x \in I$$

where $y_h = c_1 y_1 + c_2 y_2$ is a general solution of the corresponding homogeneous equation (8.4.2) and y_p is any solution of (8.4.1) (preferably containing no arbitrary constants).

We now prove that the solution of (8.4.1) with initial conditions is unique.

THEOREM 8.4.3 (Uniqueness) ***Suppose that $x_0 \in I$. Let y_1 and y_2 be two solutions of the IVP***

$$y'' + qy' + ry = f, \;\; y(x_0) = a, \;\; y'(x_0) = b. \tag{8.4.5}$$

Then $y_1 = y_2$ for all $x \in I$.

Proof. Let $z = y_1 - y_2$. Then z satisfies

$$L(z) = 0, \;\; z(x_0) = 0, \;\; z'(x_0) = 0.$$

By the uniqueness theorem 8.1.9, we have $z \equiv 0$ on I. Or in other words, $y_1 \equiv y_2$ on I. height6pt width 6pt depth 0pt

Remark 8.4.4 *The above results tell us that to solve (i.e., to find the general solution of (8.4.1)) or the IVP (8.4.5), we need to find the general solution of the homogeneous equation (8.4.2) and a particular solution* y_p *of (8.4.1). To repeat, the two steps needed to solve (8.4.1), are:*

1. compute the general solution of (8.4.2), and
2. compute a particular solution of (8.4.1).

   Then add the two solutions.

Step $1.$ *has been dealt in the previous sections. The remainder of the section is devoted to step* $2.$, *i.e., we elaborate some methods for computing a particular solution* y_p *of (8.4.1).*

EXERCISE 8.4.5

1. Find the general solution of the following equations:
   1. $y'' + 5y' = -5.$ (You may note here that $y = -x$ is a particular solution.)
   2. $y'' - y = -2\sin x.$ (First show that $y = \sin x$ is a particular solution.)
2. Solve the following IVPs:
   1. $y'' + y = 2e^x,\ y(0) = 0 = y'(0).$ (It is given that $y = e^x$ is a particular solution.)
   2. $y'' - y = -2\cos x,\ \ y(0) = 0,\ y'(0) = 1.$ (First guess a particular solution using the idea given in Exercise 8.4.5.1b)
3. Let $f_1(x)$ and $f_2(x)$ be two continuous functions. Let y_i's be particular solutions of

   $$y'' + q(x)y' + r(x)y = f_i(x),\ \ i = 1, 2;$$

   where $q(x)$ and $r(x)$ are continuous functions. Show that $y_1 + y_2$ is a particular solution of $y'' + q(x)y' + r(x)y = f_1(x) + f_2(x).$

Variation of Parameters

In the previous section, calculation of particular integrals/solutions for some special cases have been studied. Recall that the homogeneous part of the equation had constant coefficients. In this section, we deal with a useful technique of finding a particular solution when the coefficients of the homogeneous part are continuous functions and the forcing function $f(x)$ (or the non-homogeneous term) is piecewise continuous. Suppose y_1 and y_2 are two linearly independent solutions of

$$y'' + q(x)y' + r(x)y = 0 \tag{8.5.1}$$

on $I,$ where $q(x)$ and $r(x)$ are arbitrary continuous functions defined on I. Then we know that

$$y = c_1y_1 + c_2y_2$$

is a solution of (8.5.1) for any constants c_1 and c_2. We now ``vary'' c_1 and c_2 to functions of $x,$ so that

$$y = u(x)y_1 + v(x)y_2, \quad x \in I \tag{8.5.2}$$

is a solution of the equation

$$y'' + q(x)y' + r(x)y = f(x), \quad \text{on } I, \tag{8.5.3}$$

where f is a piecewise continuous function defined on I. The details are given in the following theorem.

THEOREM 8.5.1 (Method of Variation of Parameters) ***Let $q(x)$ and $r(x)$ be continuous functions defined on I and let f be a piecewise continuous function on I. Let y_1 and y_2 be two linearly independent solutions of (8.5.1) on I. Then a particular solution y_p of (8.5.3) is given by***

$$y_p = -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx, \tag{8.5.4}$$

where $W = W(y_1, y_2)$ ***is the Wronskian of*** y_1 ***and*** y_2. ***(Note that the integrals in (8.5.4) are the indefinite integrals of the respective arguments.)***

Proof. Let $u(x)$ and $v(x)$ be continuously differentiable functions (to be determined) such that

$$y_p = uy_1 + vy_2, \quad x \in I \tag{8.5.5}$$

is a particular solution of (8.5.3). Differentiation of (8.5.5) leads to

$$y_p' = uy_1' + vy_2' + u'y_1 + v'y_2. \tag{8.5.6}$$

We choose u and v so that

$$u'y_1 + v'y_2 = 0. \tag{8.5.7}$$

Substituting (8.5.7) in (8.5.6), we have

$$y_p' = uy_1' + vy_2', \text{ and } y_p'' = uy_1'' + vy_2'' + u'y_1' + v'y_2'. \tag{8.5.8}$$

Since y_p is a particular solution of (8.5.3), substitution of (8.5.5) and (8.5.8) in (8.5.3), we get

$$u\big(y_1'' + q(x)y_1' + r(x)y_1\big) + v\big(y_2'' + q(x)y_2' + r(x)y_2\big) + u'y_1' + v'y_2' = f(x).$$

As y_1 and y_2 are solutions of the homogeneous equation (8.5.1), we obtain the condition

$$u'y_1' + v'y_2' = f(x). \tag{8.5.9}$$

We now determine u and v from (8.5.7) and (8.5.9). By using the Cramer's rule for a linear system of equations, we get

$$u' = -\frac{y_2 f(x)}{W} \quad \text{and} \quad v' = \frac{y_1 f(x)}{W} \tag{8.5.10}$$

(note that y_1 and y_2 are linearly independent solutions of (8.5.1) and hence the Wronskian, $W \neq 0$ for any $x \in I$). Integration of (8.5.10) give us

$$u = -\int \frac{y_2 f(x)}{W} dx \quad \text{and} \quad v = \int \frac{y_1 f(x)}{W} dx \tag{8.5.11}$$

(without loss of generality, we set the values of integration constants to zero). Equations (8.5.11) and (8.5.5) yield the desired results. Thus the proof is complete. height6pt width 6pt depth 0pt

Before, we move onto some examples, the following comments are useful.

Remark 8.5.2

1. The integrals in (8.5.11) exist, because y_2 and $W(\neq 0)$ are continuous functions and f is a piecewise continuous function. Sometimes, it is useful to write (8.5.11) in the form

$$u = -\int_{x_0}^{x} \frac{y_2(s) f(s)}{W(s)} ds \quad \text{and} \quad v = \int_{x_0}^{x} \frac{y_1(s) f(s)}{W(s)} ds$$

   where $x \in I$ and x_0 is a fixed point in I. In such a case, the particular solution y_p as given by (8.5.4) assumes the form

$$y_p = -y_1 \int_{x_0}^{x} \frac{y_2(s) f(s)}{W(s)} ds + y_2 \int)_{x_0}^{x} \frac{y_1(s) f(s)}{W(s)} ds \tag{8.5.12}$$

   for a fixed point $x_0 \in I$ and for any $x \in I$.

2. Again, we stress here that, q and r are assumed to be continuous. They need not be constants. Also, f is a piecewise continuous function on I.
3. A word of caution. While using (8.5.4), one has to keep in mind that the coefficient of y'' in (8.5.3) is 1.

EXAMPLE 8.5.3

1. Find the general solution of

$$y'' + y = \frac{1}{2 + \sin x}, \quad x \geq 0.$$

Solution: The general solution of the corresponding homogeneous equation $y'' + y = 0$ is given by

$$y_h = c_1 \cos x + c_2 \sin x.$$

Here, the solutions $y_1 = \sin x$ and $y_2 = \cos x$ are linearly independent over $I = [0, \infty)$ and $W = W(\sin x, \cos x) = 1.$ Therefore, a particular solution, $y_h,$ by Theorem 8.5.1, is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2}{2 + \sin x} dx + y_2 \int \frac{y_1}{2 + \sin x} dx \\ &= \sin x \int \frac{\cos x}{2 + \sin x} dx + \cos x \int \frac{\sin x}{2 + \sin x} dx \\ &= -sinx \ \ln(2 + \sin x) + \cos x \ (x - 2 \int \frac{1}{2 + \sin x} dx). \end{aligned} \quad (8.5.13)$$

So, the required general solution is

$$y = c_1 \cos x + c_2 \sin x + y_p$$

where y_p is given by (8.5.13).

2. Find a particular solution of

$$x^2 y'' - 2xy' + 2y = x^3, \ x > 0.$$

Solution: Verify that the given equation is

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x$$

and two linearly independent solutions of the corresponding homogeneous part are $y_1 = x$ and $y_2 = x^2$. Here

$$W = W(x, x^2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = x^2, \quad x > 0.$$

By Theorem 8.5.1, a particular solution y_p is given by

$$\begin{aligned} y_p &= -x \int \frac{x^2 \cdot x}{x^2} dx + x^2 \int \frac{x \cdot x}{x^2} dx \\ &= -\frac{x^3}{2} + x^3 = \frac{x^3}{2}. \end{aligned}$$

The readers should note that the methods of Section 8.7 are not applicable as the given equation is not an equation with constant coefficients.

EXERCISE 8.5.4

1. Find a particular solution for the following problems:
    1. $y'' + y = f(x), \;\; 0 \leq x \leq 1$ where $f(x) = \begin{cases} 0 & \text{if } \; 0 \leq x < \frac{1}{2} \\ 1 & \text{if } \; \frac{1}{2} \leq x \leq 1. \end{cases}$
    2. $y'' + y = 2\sec x$ for all $x \in (0, \frac{\pi}{2})$.
    3. $y'' - 3y' + 2y = -2\cos(e^{-x}), \;\; x > 0.$
    4. $x^2 y'' + xy' - y = 2x, \;\; x > 0.$
2. Use the method of variation of parameters to find the general solution of
    1. $y'' - y = -e^x$ for all $x \in \mathbb{R}$.
    2. $y'' + y = \sin x$ for all $x \in \mathbb{R}$.
3. Solve the following IVPs:
    1. $y'' + y = f(x), \;\; x \geq 0$ where $f(x) = \begin{cases} 0 & \text{if } \; 0 \leq x < 1 \\ 1 & \text{if } \; x \geq 1. \end{cases}$ with $y(0) = 0 = y'(0)$.

2. $y'' - y = |x|$ for all $x \in [-1, \infty)$ with $y(-1) = 0$ and $y'(-1) = 1$.

Higher Order Equations with Constant Coefficients

This section is devoted to an introductory study of higher order linear equations with constant coefficients. This is an extension of the study of 2^{nd} order linear equations with constant coefficients (see, Section 8.3).

The standard form of a linear n^{th} order differential equation with constant coefficients is given by

$$L_n(y) = f(x) \quad \text{on} \quad I, \tag{8.6.1}$$

where

$$L_n \equiv \frac{d^n}{dx^n} + a_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + a_{n-1} \frac{d}{dx} + a_n$$

is a linear differential operator of order n with constant coefficients, $a_1, a_2, \ldots, a_n$ being real constants (called the coefficients of the linear equation) and the function $f(x)$ is a piecewise continuous function defined on the interval I. We will be using the notation $y^{(n)}$ for the n^{th} derivative of y. If $f(x) \equiv 0$, then (8.6.1) which reduces to

$$L_n(y) = 0 \quad \text{on} \quad I, \tag{8.6.2}$$

is called a homogeneous linear equation, otherwise (8.6.1) is called a non-homogeneous linear equation. The function f is also known as the non-homogeneous term or a forcing term.

DEFINITION 8.6.1 ***A function y defined on I is called a solution of (8.6.1) if y is n times differentiable and y along with its derivatives satisfy (8.6.1).***

Remark 8.6.2

1. If u and v are any two solutions of (8.6.1), then $y = u - v$ is also a solution of (8.6.2). Hence, if v is a solution of (8.6.2) and y_p is a solution of (8.6.1), then $u = v + y_p$ is a solution of (8.6.1).
2. Let y_1 and y_2 be two solutions of (8.6.2). Then for any constants (need not be real) c_1, c_2,

$$y = c_1 y_1 + c_2 y_2$$

   is also a solution of (8.6.2). The solution y is called the superposition of y_1 and y_2.
3. Note that $y \equiv 0$ is a solution of (8.6.2). This, along with the super-position principle, ensures that the set of solutions of (8.6.2) forms a vector space over $\mathbb{R}$. This vector space is called the SOLUTION SPACE or space of solutions of (8.6.2).

As in Section 8.3, we first take up the study of (8.6.2). It is easy to note (as in Section 8.3) that for a constant λ,

$$L_n(e^{\lambda x}) = p(\lambda) e^{\lambda x}$$

where,

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \tag{8.6.3}$$

DEFINITION 8.6.3 (Characteristic Equation) ***The equation $p(\lambda) = 0,$ where $p(\lambda)$ is defined in (8.6.3), is called the CHARACTERISTIC EQUATION of (8.6.2).***

Note that $p(\lambda)$ is of polynomial of degree n with real coefficients. Thus, it has n zeros (counting with multiplicities). Also, in case of complex roots, they will occur in conjugate pairs. In view of this, we have the following theorem. The proof of the theorem is omitted.

THEOREM 8.6.4 $e^{\lambda x}$ ***is a solution of (8.6.2) on any interval $I \subset \mathbb{R}$ if and only if λ is a root of (8.6.3)***

1. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct roots of $p(\lambda) = 0,$ then

$$e^{\lambda_1 x}, e^{\lambda_2 x}, \ldots, e^{\lambda_n x}$$

   are the n linearly independent solutions of (8.6.2).

2. If λ_1 is a repeated root of $p(\lambda) = 0$ of multiplicity $k,$ *i.e.*, λ_1 is a zero of (8.6.3) repeated k times, then

$$e^{\lambda_1 x}, x e^{\lambda_1 x}, \ldots, x^{k-1} e^{\lambda_1 x}$$

   are linearly independent solutions of (8.6.2), corresponding to the root λ_1 of $p(\lambda) = 0.$

3. If $\lambda_1 = \alpha + i\beta$ is a complex root of $p(\lambda) = 0,$ then so is the complex conjugate $\overline{\lambda_1} = \alpha - i\beta.$ Then the corresponding linearly independent solutions of (8.6.2) are

$$y_1 = e^{\alpha x}\big(\cos(\beta x) + i\sin(\beta x)\big) \quad \text{and} \quad y_2 = e^{\alpha x}\big(\cos(\beta x) - i\sin(\beta x)\big).$$

   These are complex valued functions of $x.$ However, using super-position principle, we note that

$$\frac{y_1 + y_2}{2} = e^{\alpha x}\cos(\beta x) \quad \text{and} \quad \frac{y_1 - y_2}{2i} = e^{\alpha x}\sin(\beta x)$$

are also solutions of (8.6.2). Thus, in the case of $\lambda_1 = \alpha + i\beta$ being a complex root of $p(\lambda) = 0,$ we have the linearly independent solutions

$$e^{\alpha x}\cos(\beta x) \quad \text{and} \quad e^{\alpha x}\sin(\beta x).$$

EXAMPLE 8.6.5

1. Find the solution space of the differential equation

   Solution: Its characteristic equation is

   $$p(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

   By inspection, the roots of $p(\lambda) = 0$ are $\lambda = 1, 2, 3.$ So, the linearly independent solutions are e^x, e^{2x}, e^{3x} and the solution space is

   $$\{c_1 e^x + c_2 e^{2x} + c_3 e^{3x} \ : \ c_1, c_2, c_3 \in \mathbb{R}\}.$$

2. Find the solution space of the differential equation

   $$y''' - 2y'' + y' = 0.$$

Solution: Its characteristic equation is

$$p(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = 0.$$

By inspection, the roots of $p(\lambda) = 0$ are $\lambda = 0, 1, 1.$ So, the linearly independent solutions are $1, e^x, xe^x$ and the solution space is

$$\{c_1 + c_2 e^x + c_3 x e^x \ : \ c_1, c_2, c_3 \in \mathbb{R}\}.$$

3. Find the solution space of the differential equation

$$y^{(4)} + 2y'' + y = 0.$$

Solution: Its characteristic equation is

$$p(\lambda) = \lambda^4 + 2\lambda^2 + 1 = 0.$$

By inspection, the roots of $p(\lambda) = 0$ are $\lambda = i, i, -i, -i.$ So, the linearly independent solutions are $\sin x, x \sin x, \cos x, x \cos x$ and the solution space is

$$\{c_1 \sin x + c_2 \cos x + c_3 x \sin x + c_4 x \cos x \ : \ c_1, c_2, c_3, c_4 \in \mathbb{R}\}.$$

From the above discussion, it is clear that the linear homogeneous equation (8.6.2), admits n linearly independent solutions since the algebraic equation $p(\lambda) = 0$ has exactly n roots (counting with multiplicity).

DEFINITION 8.6.6 (General Solution) ***Let*** $y_1, y_2, \ldots, y_n$ ***be any set of*** n ***linearly independent solution of (8.6.2). Then***

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$$

is called a general solution of (8.6.2), where $c_1, c_2, \ldots, c_n$ ***are arbitrary real constants.***

EXAMPLE 8.6.7

1. Find the general solution of $y''' = 0$.
   Solution: Note that 0 is the repeated root of the characteristic equation $\lambda^3 = 0$. So, the general solution is

$$y = c_1 + c_2 x + c_3 x^2.$$

2. Find the general solution of

$$y''' + y'' + y' + y = 0.$$

   Solution: Note that the roots of the characteristic equation $\lambda^3 + \lambda^2 + \lambda + 1 = 0$ are $-1, i, -i$. So, the general solution is

$$y = c_1 e^{-x} + c_2 \sin x + c_3 \cos x.$$

EXERCISE 8.6.8

1. Find the general solution of the following differential equations:
   1. $y''' + y' = 0$.
   2. $y''' + 5y' - 6y = 0$.
   3. $y^{iv} + 2y'' + y = 0$.
2. Find a linear differential equation with constant coefficients and of order 3 which admits the following solutions:
   1. $\cos x, \sin x$ and e^{-3x}.
   2. e^x, e^{2x} and e^{3x}.
   3. $1, e^x$ and x.
3. Solve the following IVPs:
   1. $y^{iv} - y = 0, \;\; y(0) = 0, y'(0) = 0, y''(0) = 0, y'''(0) = 1$.
   2. $2y''' + y'' + 2y' + y = 0, \;\; y(0) = 0, y'(0) = 1, y''(0) = 0$.
4. *Euler Cauchy Equations:*
   Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ be given constants. The equation

$$x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0, \quad x \in I \qquad (8.6.4)$$

5.

6. is called the homogeneous Euler-Cauchy Equation (or just Euler's Equation) of degree n. (8.6.4) is also called the standard form of the Euler equation. We define

$$L(y) = x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y.$$

7.

8. Then substituting $y = x^\lambda,$ we get

$$L(x^\lambda) = \big(\lambda(\lambda-1)\cdots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + a_0\big)x^\lambda.$$

9.

10. So, x^λ is a solution of (8.6.4), if and only if

$$\lambda(\lambda-1)\cdots(\lambda-n+1) + a_{n-1}\lambda(\lambda-1)\cdots(\lambda-n+2) + \cdots + a_0 = 0. \qquad (8.6.5)$$

11.

12. Essentially, for finding the solutions of (8.6.4), we need to find the roots of (8.6.5), which is a polynomial in λ. With the above understanding, solve the following homogeneous Euler equations:
    1. $x^3 y''' + 3x^2 y'' + 2xy' = 0.$
    2. $x^3 y''' - 6x^2 y'' + 11xy' - 6y = 0.$
    3. $x^3 y''' - x^2 y'' + xy' - y = 0.$

    For an alternative method of solving (8.6.4), see the next exercise.

13. Consider the Euler equation (8.6.4) with $x > 0$ and $x \in I$. Let $x = e^t$ or equivalently $t = \ln x$. Let $D = \frac{d}{dt}$ and $d = \frac{d}{dx}$. Then
    1. show that $xd(y) = Dy(t),$ or equivalently $x\frac{dy}{dx} = \frac{dy)}{dt}$.
    2. using mathematical induction, show that $x^n d^n y = \big(D(D-1)\cdots(D-n+1)\big)y(t).$
    3. with the new (independent) variable t, the Euler equation (8.6.4) reduces to an equation with constant coefficients. So, the questions in the above part can be solved by the method just explained.

We turn our attention toward the non-homogeneous equation (8.6.1). If y_p is any solution of (8.6.1) and if y_h is the general solution of the corresponding homogeneous equation (8.6.2), then

$$y = y_h + y_p$$

is a solution of (8.6.1). The solution y involves n arbitrary constants. Such a solution is called the GENERAL SOLUTION of (8.6.1).

Solving an equation of the form (8.6.1) usually means to find a general solution of (8.6.1). The solution y_p is called a PARTICULAR SOLUTION which may not involve any arbitrary constants. Solving (8.6.1) essentially involves two steps (as we had seen in detail in Section 8.3).

Step 1: a) Calculation of the homogeneous solution y_h and

b) Calculation of the particular solution y_p.

In the ensuing discussion, we describe the method of undetermined coefficients to determine y_p. Note that a particular solution is not unique. In fact, if y_p is a solution of (8.6.1) and u is any solution of (8.6.2), then $y_p + u$ is also a solution of (8.6.1). The undetermined coefficients method is applicable for equations (8.6.1).

Method of Undetermined Coefficients

In the previous section, we have seen than a general solution of

$$L_n(y) = f(x) \quad \text{on} \quad I \tag{8.7.6}$$

can be written in the form

$$y = y_h + y_p,$$

where y_h is a general solution of $L_n(y) = 0$ and y_p is a particular solution of (8.7.6). In view of this, in this section, we shall attempt to obtain y_p for (8.7.6) using the method of undetermined coefficients in the following particular cases of $f(x)$;

1. $f(x) = ke^{\alpha x}$; $k \neq 0, \alpha$ a real constant
2. $f(x) = e^{\alpha x}\big(k_1 \cos(\beta x) + k_2 \sin(\beta x)\big)$; $k_1, k_2, \alpha, \beta \in \mathbb{R}$
3. $f(x) = x^m$.

Case I. $f(x) = ke^{\alpha x}$; $k \neq 0, \alpha$ a real constant.

We first assume that α is not a root of the characteristic equation, *i.e.*, $p(\alpha) \neq 0.$ Note that $L_n(e^{\alpha x}) = p(\alpha)e^{\alpha x}.$ Therefore, let us assume that a particular solution is of the form

$$y_p = Ae^{\alpha x},$$

where $A,$ an unknown, is an undetermined coefficient. Thus

$$L_n(y_p) = Ap(\alpha)e^{\alpha x}.$$

Since $p(\alpha) \neq 0,$ we can choose $A = \dfrac{k}{p(\alpha)}$ to obtain

$$L_n(y_p) = ke^{\alpha x}.$$

Thus, $y_p = \dfrac{k}{p(\alpha)}e^{\alpha x}$ is a particular solution of $L_n(y) = ke^{\alpha x}.$

Modification Rule: If α is a root of the characteristic equation, *i.e.*, $p(\alpha) = 0$, with multiplicity r, (*i.e.*, $p(\alpha) = p'(\alpha) = \cdots = p^{(r-1)}(\alpha) = 0$ and $p^{(r)}(\alpha) \neq 0$) then we take, y_p of the form

$$y_p = Ax^r e^{\alpha x}$$

and obtain the value of A by substituting y_p in $L_n(y) = ke^{\alpha x}$.

EXAMPLE 8.7.1

1. Find a particular solution of

$$y'' - 4y = 2e^x.$$

Solution: Here $f(x) = 2e^x$ with $k = 2$ and $\alpha = 1$. Also, the characteristic polynomial, $p(\lambda) = \lambda^2 - 4$. Note that $\alpha = 1$ is not a root of $p(\lambda) = 0$. Thus, we assume $y_p = Ae^x$. This on substitution gives

$$Ae^x - 4Ae^x = 2e^x \implies -3Ae^x = 2e^x.$$

So, we choose $A = \frac{-2}{3}$, which gives a particular solution as

$$y_p = \frac{-2e^x}{3}.$$

2. Find a particular solution of

$$y''' - 3y'' + 3y' - y = 2e^x.$$

Solution: The characteristic polynomial is $p(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$ and $\alpha = 1$. Clearly, $p(1) = 0$ and $\lambda = \alpha = 1$ has multiplicity $r = 3$. Thus, we assume $y_p = Ax^3e^x$. Substituting it in the given equation,we have

$$Ae^x\left(x^3+9x^2+18x+6\right) - 3Ae^x\left(x^3+6x^2+6x\right) + 3Ae^x\left(x^3+3x^2\right) - Ax^3e^x = 2e^x.$$

Solving for $A,$ we get $A=\frac{1}{3},$ and thus a particular solution is $y_p=\frac{x^3e^x}{3}.$

3. Find a particular solution of

$$y''' - y' = e^{2x}.$$

Solution: The characteristic polynomial is $p(\lambda)=\lambda^3-\lambda$ and $\alpha=2.$ Thus, using $y_p=Ae^{2x},$ we get $A=\frac{1}{p(\alpha)}=\frac{1}{6},$ and hence a particular solution is $y_p=\frac{e^{2x}}{6}.$

4. Solve $y'''-3y''+3y'-y=2e^{2x}.$

EXERCISE 8.7.2 ***Find a particular solution for the following differential equations:***

1. $y''-3y'+2y=e^x.$
2. $y''-9y=e^{3x}.$
3. $y'''-3y''+6y'-4y=e^{2x}.$

Case II. $f(x)=e^{\alpha x}\left(k_1\cos(\beta x)+k_2\sin(\beta x)\right);\ \ k_1,k_2,\alpha,\beta\in\mathbb{R}$

We first assume that $\alpha+i\beta$ is not a root of the characteristic equation, *i.e.*, $p(\alpha+i\beta)\neq 0.$

Here, we assume that y_p is of the form

$$y_p=e^{\alpha x}\left(A\cos(\beta x)+B\sin(\beta x)\right),$$

and then comparing the coefficients of $e^{\alpha x}\cos x$ and $e^{\alpha x}\sin x$ (why!) in $L_n(y)=f(x),$ obtain the values of A and $B.$

Modification Rule: If $\alpha + i\beta$ is a root of the characteristic equation, *i.e.*, $p(\alpha + i\beta) = 0,$ with multiplicity $r,$ then we assume a particular solution as

$$y_p = x^r e^{\alpha x}\big(A\cos(\beta x) + B\sin(\beta x)\big),$$

and then comparing the coefficients in $L_n(y) = f(x),$ obtain the values of A and $B.$

EXAMPLE 8.7.3

1. Find a particular solution of

$$y'' + 2y' + 2y = 4e^x \sin x.$$

   Solution: Here, $\alpha = 1$ and $\beta = 1.$ Thus $\alpha + i\beta = 1 + i,$ which is not a root of the characteristic equation $p(\lambda) = \lambda^2 + 2\lambda + 2 = 0.$ Note that the roots of $p(\lambda) = 0$ are $-1 \pm i.$

   Thus, let us assume $y_p = e^x\left(A\sin x + B\cos x\right).$ This gives us

$$(-4B + 4A)e^x \sin x + (4B + 4A)e^x \cos x = 4e^x \sin x.$$

   Comparing the coefficients of $e^x \cos x$ and $e^x \sin x$ on both sides, we get $A - B = 1$ and $A + B = 0.$ On solving for A and $B,$ we get $A = -B = \frac{1}{2}.$ So, a particular solution is $y_p = \frac{e^x}{2}\left(\sin x - \cos x\right).$

2. Find a particular solution of

$$y'' + y = \sin x.$$

Solution: Here, $\alpha = 0$ and $\beta = 1.$ Thus $\alpha + i\beta = i,$ which is a root with multiplicity $r = 1,$ of the characteristic equation $p(\lambda) = \lambda^2 + 1 = 0.$

So, let $y_p = x\left(A\cos x + B\sin x\right).$ Substituting this in the given equation and comparing the coefficients of $\cos x$ and $\sin x$ on both sides, we get $B = 0$ and $A = -\frac{1}{2}.$ Thus, a particular solution is $y_p = \frac{-1}{2} x \cos x.$

EXERCISE 8.7.4 ***Find a particular solution for the following differential equations:***

1. $y''' - y'' + y' - y = e^x \cos x.$
2. $y'''' + 2y'' + y = \sin x.$
3. $y'' - 2y' + 2y = e^x \cos x.$

Case III. $f(x) = x^m.$

Suppose $p(0) \neq 0.$ Then we assume that

$$y_p = A_m x^m + A_{m-1} x^{m-1} + \cdots + A_0$$

and then compare the coefficient of x^k in $L_n(y_p) = f(x)$ to obtain the values of A_i for $0 \le i \le m.$

Modification Rule: If $\lambda = 0$ is a root of the characteristic equation, *i.e.*, $p(0) = 0,$ with multiplicity $r,$ then we assume a particular solution as

$$y_p = x^r\left(A_m x^m + A_{m-1} x^{m-1} + \cdots + A_0\right)$$

and then compare the coefficient of x^k in $L_n(y_p) = f(x)$ to obtain the values of A_i for $0 \le i \le m.$

EXAMPLE 8.7.5 ***Find a particular solution of***

$$y''' - y'' + y' - y = x^2.$$

Solution: As $p(0) \neq 0,$ ***we assume***

$$y_p = A_2x^2 + A_1x + A_0$$

which on substitution in the given differential equation gives

$$-2A_2 + (2A_2x + A_1) - (A_2x^2 + A_1x + A_0) = x^2.$$

Comparing the coefficients of different powers of x ***and solving, we get*** $A_2 = -1,\ A_1 = -2$ ***and*** $A_0 = 0.$ ***Thus, a particular solution is***

$$y_p = -(x^2 + 2x).$$

Finally, note that if y_{p_1} is a particular solution of $L_n(y) = f_1(x)$ and y_{p_2} is a particular solution of $L_n(y) = f_2(x),$ then a particular solution of

$$L_n(y) = k_1f_1(x) + k_2f_2(x)$$

is given by

$$y_p = k_1y_{p_1} + k_2y_{p_2}.$$

In view of this, one can use method of undetermined coefficients for the cases, where $f(x)$ is a linear combination of the functions described above.

EXAMPLE 8.7.6 ***Find a particular soltution of***

$$y'' + y = 2\sin x + \sin 2x.$$

Solution: We can divide the problem into two problems:

1. $y'' + y = 2\sin x.$
2. $y'' + y = \sin 2x.$

For the first problem, a particular solution (Example 8.7.3.2) is

$$y_{p_1} = 2\,\frac{-1}{2}x\cos x = -x\cos x.$$

For the second problem, one can check that $y_{p_2} = \frac{-1}{3}\sin(2x)$ ***is a particular solution.***

Thus, a particular solution of the given problem is

$$y_{p_1} + y_{p_2} = -x\cos x - \frac{1}{3}\sin(2x).$$

EXERCISE 8.7.7 ***Find a particular solution for the following differential equations:***

1. $y''' - y'' + y' - y = 5e^x \cos x + 10e^{2x}.$
2. $y'' + 2y' + y = x + e^{-x}.$
3. $y'' + 3y' - 4y = 4e^x + e^{4x}.$
4. $y'' + 9y = \cos x + x^2 + x^3.$
5. $y''' - 3y'' + 4y' = x^2 + e^{2x}\sin x.$
6. $y'''' + 4y''' + 6y'' + 4y' + 5y = 2\sin x + x^2.$