SECONDARY SCHOOL STUDENTS' MISCONCEPTIONS IN ALGEBRA

by

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Abstract

This study investigated secondary school students' errors and misconceptions in algebra with a view to expose the nature and origin of those errors and to make suggestions for classroom teaching. The study used a mixed method research design. An algebra test which was pilot-tested for its validity and reliability was given to a sample of grade 11 students in an urban secondary school in Ontario. The test contained questions from four main areas of algebra: variables, algebraic expressions, equations, and word problems. A rubric containing the observed errors was prepared for each conceptual area. Two weeks after the test, six students were interviewed to identify their misconceptions and their reasoning. In the interview process, students were asked to explain their thinking while they were doing the same problems again. Some prompting questions were asked to facilitate this process and to clarify more about students' claims.

The results indicated a number of error categories under each area. Some errors emanated from misconceptions. Under variables, the main reason for misconceptions was the lack of understanding of the basic concept of the variable in different contexts. The abstract structure of algebraic expressions posed many problems to students such as understanding or manipulating them according to accepted rules, procedures, or algorithms. Inadequate understanding of the

uses of the equal sign and its properties when it is used in an equation was a major problem that hindered solving equations correctly. The main difficulty in word problems was translating them from natural language to algebraic language. Students used guessing or trial and error methods extensively in solving word problems.

Some other difficulties for students which are non-algebraic in nature were also found in this study. Some of these features were: unstable conceptual models, haphazard reasoning, lack of arithmetic skills, lack or non-use of metacognitive skills, and test anxiety. Having the correct conceptual (why), procedural (how), declarative (what), and conditional knowledge (when) based on the stage of the problem solving process will allow students to avoid many errors and misconceptions. Conducting individual interviews in classroom situations is important not only to identify errors and misconceptions but also to recognize individual differences.

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"If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more headaches, and more sleepless nights than any other school subject."
(NCTM yearbook, 2008, p. 3)

Chapter 1

Introduction

1.1 Statement of the Problem

Algebra is one of the most abstract strands in mathematics. Once largely limited to the secondary school curriculum, algebra is now commonplace in middle school mathematics (Ministry of Education, 2005). At the same time, enrollment in community college algebra courses is burgeoning. According to Greens and Rubenstein (2008), until relatively recently, the study of algebra was reserved for college-bound students. After a widespread push by NCTM and teachers nationwide, algebra is now a required part of most curricula including in the US and Canada. However, many attempts to better prepare students for algebra have not resulted in greater achievement in first-year algebra. Students in grades 8 and 9 are still struggling with algebraic concepts and skills (Greens & Rubenstein, 2008). Many are discontinuing their study of higher-level mathematics because of their lack of success in algebra.

The demand for algebra at more levels of education is increasing. WikiAnswers (2010), one of the world's leading questions and answers websites, lists some of the uses of algebra in today's world. Algebra is used in companies to figure out their annual budget which involves their annual expenditure. Various stores use algebra to predict the demand of a particular product and subsequently place their orders. Algebra also has individual applications in the form of calculation of annual taxable income, bank interest, and installment loans. Algebraic expressions and equations serve as models for interpreting and making inferences about data. Further, algebraic reasoning and symbolic notations also serve as the basis for the design and use of computer spreadsheet models. Therefore, mathematical reasoning developed through algebra is necessary all through life, affecting decisions we make in many areas such as personal finance, travel, cooking and real estate, to name a few. Thus, it can be argued that a better understanding of algebra improves decision making capabilities in society.

More analysis is necessary in order to develop a clear understanding of what factors help students to be successful in algebra and how schools and other systems can assist in achieving this goal. We already know that even very basic mathematical concepts such as addition of whole numbers involve complicated cognitive processes. Since teachers are already very familiar with those basic concepts, this leads them to ignore or underestimate the complexity by taking a naïve approach to teaching those concepts (Schoenfeld, 1985). Without adequate knowledge about students' learning of basic mathematics concepts or operations, teachers could underestimate the complexity of the individual learning process of mathematics.

Teachers or experts in the field often have differences of opinions about students' conceptions and misconceptions. This is not only because the amount of quantitative reasoning that experts use is greater than what novices use in a problem solving situation. It is also because of the qualitative nature of the reasoning that experts use in a situation. Frequently, experts do not realize that this quality is important to disseminate to their students. Students should be allowed to use this information that is sometimes not in the textbooks. For experts, this knowledge is structured in their heads as informal, imagistic, metaphoric, and heuristic forms (Kaput, 1985). The problem is that this knowledge is not properly represented in the modern curricula. If this happens, students will be the beneficiaries.

Although there are many causes of student difficulties in mathematics, the lack of support from research fields for teaching and learning is noticeable. If research could characterize students' errors and misconceptions, it would be possible to design effective instructions to avoid those situations. Research on student errors and misconceptions is a way to provide such support for both teachers and students. Problems of this nature are particularly worthy of investigation as there is still a lack of robust research in identifying students' misconceptions for more than one conceptual area collectively. The existing research is mostly about identifying and explaining causes for a particular misconception. If researchers can identify students' difficulties collectively in more than one area, it will be easier to identify the systematic patterns of errors (if there are any) that spread through the areas and make suggestions for remediation.

Another point is that there is a methodological shift in modern research from classical studies in mathematics education, which were statistical statements about populations, to a closer observation of individuals doing mathematical tasks. In this context, this study is significant because it addresses the errors made by Grade 11 students and their misconceptions in algebraic problem solving tasks. I hope that addressing this issue will reduce the distance between research and the real classroom leading to more practically applicable findings.

The theoretical framework to this study is based on Piaget's epistemology (Piaget, 1970, 1977, 1985), which implies that learners actively organize their experiences by constructing mental schemas to accommodate new knowledge and connecting it with existing knowledge. In this view, learning mathematics is seen as a continuous process through abstraction of relationships between actions and reflections. During this process, students construct schemas and modify and/or apply them intentionally to achieve their goals. Careful analysis of these actions will allow the researcher or the teacher to identify student conceptions or misconceptions wherever they may lead.

1.2 Background of the researcher

Many years ago, when I was a secondary school teacher, I observed many students in my class struggling to cope with learning algebra. They had a good arithmetic background, and they could solve a problem using lengthy arithmetic procedures that they came up with themselves, but were hesitant to use algebraic methods. I always tried to use algebraic methods on my own to motivate them. However, my attempts were not very successful as students used their own lengthy arithmetic procedures or rather failed in using algebraic methods.

By observing the students informally, I found that they have some misconceptions that were persistent. Sometimes, they repeatedly made the same error. Also, through discussion with my fellow teachers, I realized that their explanations for these types of behaviors were surprisingly consistent with mine. However, one thing was clear to me. These misconceptions were neither inborn nor were they instantaneous. Rather, students have acquired those misconceptions during their learning process for yet unknown reasons. Whatever the reasons may be, there should be a way to identify and remedy these problems.

Several years later, I left my secondary school teaching and joined a university where I did not have further opportunities to pursue this area. However, to my surprise, I observed that many university students also lack some basic understanding of algebra. Sometimes, they commit the same mistakes as their secondary school counterparts. I also observed that these students memorized only a few facts, formulas, and algorithms without understanding them conceptually, even though they could manipulate those limited number of facts in a correct or incorrect manner. Their lack of conceptual understanding prevented them from applying mathematical knowledge to new contexts in a flexible way. This was one of my own explanations for the reasons of student errors in algebra.

When I later started to teach in community colleges, the problem resurfaced again. What I observed from my teaching was that even college students commit the same mistakes as secondary school students. By then, I realized that this problem is common to many education systems in the world. Up to this time, I had seen student errors on paper when they did classroom work or answered the tests. I did not have extensive opportunities to listen to students for their explanations except for a few informal talks with them. However, I always thought that there should be a systematic way of studying the errors and to see what students have to admit about their own mistakes. Thinking along this line, I formed my main research question: What personal mathematical constructs cause secondary school students to make errors or to have misconceptions in algebra?

1.3 Significance of the problem

There are a considerable number of studies on students' errors and misconceptions in arithmetic. Comparatively, few studies address the issue of students' misconceptions in algebra. They too pay attention to some isolated topics such as variables, equations, or expressions. Little or no attempt has been made to study the interrelated nature of the misconceptions in more than one conceptual area. One should taste the whole sandwich in order to get a real sense of its ingredients. Tasting each item in the recipe separately will not give a complete sense.

More detailed exploration of the misconceptions is a crucial prerequisite for any further attempt to improve the quality of mathematics education and the levels of student achievement. Considering these issues, the results of this study will inform teachers, curriculum planners, textbook writers, and other stakeholders to broaden their understanding of how errors and misconceptions in algebra can be identified and thoughtfully engaged.

Thus, if researchers can know and describe the ways of students' understanding in algebra in a detailed way, it will be easier for teachers and researchers to design effective methods to improve students' understanding. I addressed the above issues in this study by inquiring into students' misconceptions in the basic building blocks in algebra: variables, expressions and equations. Word problems introduce a context where the above three components can link to a solution model. Hence, the objectives of my research are twofold: one is to determine sources of errors and misconceptions so that these sources can be eliminated through properly organized instructional methods, and to gain insights into the students' thinking processes which will reveal their knowledge schemas in algebra.

1.4 Research Questions

The study will address the following research questions:

1. What are secondary school students' categories of errors and misconceptions in solving problems related to variables?
2. What are secondary school students' categories of errors and misconceptions in solving problems related to algebraic expressions?
3. What are secondary school students' categories of errors and misconceptions in solving equations?
4. What are secondary school students' categories of errors and misconceptions in solving word problems?
5. Do the existing theoretical explanations account for the errors and misconceptions observed in this study?
6. What can be learned from students' problem solving processes and reasoning in algebra?

1.5 Key terms

Cognition

In this study, I will use the definition for cognition given by Matlin (2005). It defines cognition as an “action of knowing” (p. 1) and “cognition or mental activity describes requisition, storage, transformation, and use of knowledge” (p. 2).

A strategy

A strategy is considered as a goal-directed, domain specific procedure employed to facilitate task performance. It is used to facilitate both knowledge acquisition and utilization. Hence, throughout this study, a strategy is viewed as a goal-directed procedure that facilitates both problem solution and acquisition of domain-specific knowledge. A strategy is also seen as potentially conscious and controllable (English, 1996).

Conceptions and misconceptions

Student beliefs, their theories, meanings, and explanations will form the basis of the term student conceptions. When those conceptions are deemed to be in conflict with the accepted meanings in mathematics, then a misconception has occurred (Osborne & Wittrock, 1983).

There are various terms in the literature that have been used in relation to the discussion of student misconceptions. Some of the terms used are misconceptions, preconceptions, alternative conceptions, naïve beliefs, naïve theories, alternative beliefs, flawed conceptions, buggy algorithms and so on (Smith et al., 1993). Each of these terms conveys an epistemological or a psychological position and some of them even carry the same or similar meanings. While recognizing the substantial theoretical diversity of meanings, I define two overarching terms in my study -- errors and misconceptions that contain many of the above theoretical underpinnings.

Errors

Generally, an error means a simple lapse of care or concentration which almost everyone makes at least occasionally. In mathematics, an error means the deviation from a correct solution of a problem. In this study, an error is regarded as a mistake in the process of solving a mathematical problem algorithmically, procedurally or by any other method. Errors could be found in wrongly answered problems which have flaws in the process that generated the answers (Young & O'Shea, 1981).

Schema

A schema is a mechanism in human memory that allows for the storage, synthesis, generalization, and retrieval of similar experiences (Marshall, 1995). A schema allows an individual to organize similar experiences in such a way that the individual can easily recognize additional similar experiences. Schemas are triggered when an individual tries to comprehend, understand, organize, or make sense of a new situation (Greeno et al., 1996). In knowledge construction, there is always a base structure from which to begin construction and this is called a structure of assimilation. The process of continual revision of structures is called accommodation (Noddings, 1990).

Diagnosis

The term "diagnosis" means the identification and characterization of errors or misconceptions of students while they are involved in the mathematical problem solving process (Brueckner & Bond, 1955).

Metacognition

In this study, I use the term “metacognition” to refer to students’ individual awareness of their own thinking; their evaluation of that thinking, and their regulation of that thinking (Wilson & Clarke, 2004).

Word problems

Some of the word problems used in this study lead to the process of inquiry, in which students had to develop methods for exploring unfamiliar situations (Ministry of Education, 2007). In these problems, students had to consider real world situations and represent them in mathematical form. They could be termed “non-routine” problems (Ministry of Education, 2005, p.13). The other kind was “routine problems” (Ministry of Education, 2005, p.13) that could be described as the problems given to students at the end of a lesson or after teaching a particular concept. In other words, these problems are routine classroom exercises that do not require students to develop new or adapted solution processes.

1.6 Organization of the thesis

I have organized this thesis into five chapters. Chapter 1 describes a brief statement of the research problem and its evolution, the significance of the problem, research questions, and the definitions of some key terms used in the study.

As a theoretical basis to the study and to examine the previous work done in this area, a literature review was carried out as the next step. This review appears in chapter 2. This chapter is organized into several areas starting from an introduction. Different views of studying cognition and different notions of constructivism are discussed next. The nature of algebra and its problem solving process was also discussed referring to the importance of metacognition in problem solving. Chapter 2 ends with a classification of various error types in the literature pertaining to the four topics under investigation in the research.

Chapter 3 is devoted to discuss the methodological constructs of the research. Since this research belongs to the mixed methods tradition, the first part of the chapter describes the quantitative phase of the study, a mathematics test as the main research instrument and its reliability and validity issues. In the qualitative phase, the interview method was discussed as part of the case study method. The chapter ends with a brief discussion about the sample, data analysis methods, and ethical considerations.

In chapter 4, the main findings pertaining to both stages of the mixed method design are discussed. This is started with a quantitative analysis of data followed by a qualitative analysis. A rubric containing error types was prepared for each conceptual area under the study. Individual percentages for each error type were given by categorizing errors into various groups. Further, a detailed analysis of errors was carried out using student interview protocols. At the end of the chapter, six cases were discussed to illuminate the explanations through student reasoning. Since the goal of this study is to identify students' misconceptions underlying their errors, I justified, whenever necessary, how students' wrong responses expose their misconceptions.

In chapter 5, I discussed the errors and misconceptions that I found under the four areas together with students' reasons relating them to similar discussions of students' errors in the literature. I made an attempt to identify all possible error categories whether they were algebraic or non-algebraic. This discussion was combined with other theoretical discussions in the literature on debates about students' errors and misconceptions.

Chapter 2

Review of literature

2.1 Introduction

In this chapter, I will discuss the psychological view of studying cognition as a basis for my theoretical foundation followed by a discussion of its relatedness to constructivism. I also will elaborate on error analysis literature based on constructivist viewpoints by ending the discussion with a focus on specific discussions on misconceptions under the four main areas of the study: variables, expressions, equations, and word problems. The organization of this chapter addresses three aspects: (1) Psychological approaches to the study of human cognitive activities (2) Constructivist models used to analyze student errors in mathematics; and (3) Student errors related to the four areas in this study.

2.2 Contemporary psychological views on studying cognitive activities

Psychology plays an important role in mathematics education especially in cognitive analyses of students' mathematical thinking. The disciplines of psychology and cognitive science have a considerable influence on how mathematics is learned and taught. The contemporary psychology starts with the concept of an abstract mind (Cole, 1996). Local conditions like settings and cultures are external to the mind and detached from it. Contemporary psychology believes that mind can be studied explicitly with the use of appropriate psychometric methods. This paradigm further assumes that the field and its practitioners are making no a priori value judgments in seeking for what's going on in the mind. In sum, cognitive science and cognitive psychology generally imply that complex psychological activity can be reduced to primitive functions (reducibility). Psychological entities such as representations, skills, or memory can be characterized independently from context. This stance focuses on individuals' construction of knowledge based on their previous experiences.

2.3 My approach to cognition

Since my intention in this research is to find out students' reasoning for their errors and misconceptions in algebra, I will inquire into the current thinking processes of students without investigating the way they achieved these concepts. Thus, I assume that individuals could explain their thinking based on what they have constructed through their experiences in the classroom and other learning processes. This insistence on the personal nature of constructions does not deny the fact that persons are also the same in many basic ways sharing the same modes of thought. Despite shared social influences or commonalities in thinking, everyone's constructed world is to some extent personal and distinct. Thus, entering the child's mind requires methods that do not impose on the child our world view or our meanings (Piaget, 1970).

In the forthcoming sections, I will discuss and detail the aspects of constructivism that seem to me the most compatible, consistent, and appropriate ways for studying student cognition under the psychological approach. I will also discuss how research results in the literature may generate feedback for my own research that may require modifications or elaborations to my theoretical foundation. This is true to the spirit of constructivism, in which concepts are shaped and reshaped in the process of learning, particularly in light of the fact that research itself is fundamentally a learning process.

2.4 The notion of constructivism

The constructivist perspective, derived as a part from the work of Piaget asserts that conceptual knowledge cannot be transferred from one person to another (Piaget, 1970). Rather, it must be constructed by each person based on his/her own experience. Piaget, an epistemologist, explained the term genetic epistemology as discovering the roots of varieties of knowledge including the scientific knowledge (Piaget, 1970). He was especially interested in the development of qualitative knowledge and explained knowledge development as a process of equilibration using two main concepts in his theory: assimilation and accommodation, as belonging not only to biological interactions but also to cognitive ones (Wikipedia, 2010). Piaget studied about children's conceptions. However, this idea initially came to him when he was examining children's' wrong answers in tests. Piaget's theoretical research program contained four broad areas: the social model of development, the biological model of intellectual development, the logical model of intellectual development, and the study of figurative thought (Wikipedia, 2010).

The language of constructivism is omnipresent in many aspects of modern pedagogical theory and practice. This perspective on learning has been central to much of the recent empirical and theoretical work in education (Steffe & Gale, 1995; Von Glasersfeld, 1991). As a result, it has contributed to shaping mathematics reform efforts (NCTM, 2000; NCTM, 2007). Constructivism is a term with many shades of meanings. Neither its proponents nor its opponents necessarily agree on what constitutes constructivism. Although terms such as "radical constructivism" and "social constructivism" provide some orientation for the discussion, there is a diversity of epistemological perspectives even within these categories (Steffe & Gale, 1995). Since the guiding principles of my inquiry rest on individual construction of knowledge, I will set forth a definition of constructivism that will serve as a theoretical foundation of my work into students' mathematical misconceptions.

Constructivism applies both in learning theory (how people learn) and epistemology (nature of knowledge). Hein (1991) asked the question, "What is meant by constructivism?" He said that the term refers to the idea that learners construct knowledge by constructing meaning individually and/or socially when they are in the learning process. This view has two fundamental aspects. First, we have the learner as the main person constructing knowledge in the learning process. Second, there is no knowledge independent of the meaning attributed to the experience by the learner or a community of learners. Simply, this knowledge depends on peoples' experiences and their thinking. Therefore, constructivism as a theory has shown how useful it is, in allowing researchers to make sense of others' experiences. According to Steffe and Gale (1995), when our experiences differ from the expected or intended, then disequilibrium occurs and our adaptive learning process is triggered. Reflection on successful adaptive operations which is called the "reflective abstraction" leads to new or modified concepts.

Constructivism has been characterized as both a cognitive position and a methodological position. As a methodological perspective, constructivism assumes that human behavior is mainly purposive and human organisms have a highly developed capacity for organizing knowledge (Noddings, 1990). These assumptions suggest methods such as ethnography and clinical interviews to study human cognition. On the other hand, as a cognitive perspective, constructivism holds that all knowledge is constructed. To some constructivists, these cognitive structures are innate (Von Glaserfeld, 1993). For other constructivists like Piaget, cognitive constructs are products of development. This position is held by many other constructivists in mathematics education (Noddings, 1990).

Essential to the constructivist view is the recognition that the way of knowing the real world is not directly through our senses, but through our material or mental actions. Thus, the parts of child's actions should be recognized in relation to the child's current forms of operations and actions. Von Glasersfeld and Steffe (1991) perceived constructivism as the acquisition of knowledge with understanding. They argued that there are two general approaches to educational research and practice: mechanistic and organismic. The mechanistic approach is exemplified by traditional learning theory and the basic assumption of this approach is that the learner passively receives information from the environment. This could be most notably from the teacher.

The latter organismic approach is illustrated by the development of memory representations for specific conceptual structures. This development assumes that any such changes in overt behavior or mental representations generally occur without the child conceptually understanding the material (Cobb et al., 1992). In my view, constructivist ideas reject both of these notions. The constructivist perspective offers an alternative to the organismic approach. This alternative to the organismic approach is exemplified by the theories of Piaget. Here, the basic assumption is that children are active learners and must construct knowledge for themselves. For example, in mathematics, in order to completely understand the material, the child must rediscover basic mathematical principles.

In sum, the basic stance that underlies constructivism is the view that all learning involves the interpretation of phenomena, situations, and events, including classroom instruction, from the perspective of the learner's existing knowledge. Constructivism emphasizes the role of prior knowledge in learning. Students interpret tasks and instructional activities involving new concepts in terms of their prior knowledge. Misconceptions are characteristic of initial phases of learning because students' existing knowledge is inadequate and supports only partial understandings (Smith et al., 1993). As their existing knowledge is recognized to be inadequate to explain phenomena and solve problems, students learn by transforming and refining that prior knowledge into more sophisticated forms.

2.4.1 Radical constructivism

Researchers from a variety of disciplines make reference to some of the basic forms of constructivism. According to Simon (1995), constructivism derives from a philosophical position that we have no access to an objective reality which is independent of our experiences. We construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge. Learning is the process by which human beings adapt to their experiential world. From a radical constructivist perspective, we have no way of knowing whether a concept matches an objective reality.

This perception of knowledge acquisition is referred to as “viability” (Von Glasersfeld, 1987, 1995) which resonates in the biological model of learning as “adaptation” developed by Piaget (1970). To elaborate this further, radical constructivists discard the word “truth” and replace it with the notion of “viability”. Truth, according to them, is what works at least well enough for now. Simon (1995) further clarified the notion of viability. He said that a concept is viable when it does what we need it to do. That is to make sense of our perceptions or data, to make an accurate prediction, to solve a problem, or to accomplish a personal goal.

Therefore, in constructivism, it is an illusion to believe that what someone can do paints a picture of the real world. It describes our experiential reality as we happened to be experiencing it at that moment. What we experience is shaped by the conceptual relationships we have at the moment. Therefore, radical constructivism gives up the traditional conception of knowledge as independent of the experience of the knower. That is why radical constructivism does not claim the knowledge constructed by the learner at one stage as the ultimate truth, but a viable truth.

The term “radical constructivism” was proposed by Von Glasersfeld and he dismissed those who fall short of his standards of radical constructivism as “trivial constructivists” (Von

Glasersfeld, 1993, 1998). In his view, radical constructivism represents a break with the traditional role of epistemology. It is a theory of knowing rather than a theory of knowledge because it eschews the usual connection between knowledge and the real world. The notion that knowledge is the result of a learner's activity rather than the passive reception of information or instruction goes back to Socrates and today embraced by all who call themselves as constructivists (Von Glasersfeld, 1991). However, radical constructivism in the present form has seriously been pioneered mainly by Jean Piaget. For Piaget, the partners that the child interacts with are part of the environment. This is no more and also no less than any of the relatively permanent objects the child constructs within the range of its lived experience (Von Glasersfeld, 1995).

The transformation from a theory of knowledge to a theory of knowing has redefined the concept of knowledge as an adaptive function. This means that the results of our cognitive efforts have the purpose of helping us to cope in the world of our experiences. This is opposed to the traditional goal of furnishing an objective representation of the world as it might exist apart from our experience. The idea of an adaptive function was further elaborated by Greer (1996). He said, "the learner should reinvent mathematizing rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalizing rather than formulas; algorithmizing rather than algorithms; verbalizing rather than language" (p. 183).

The above discussion shows that constructivism in general and radical constructivism in particular are evolving processes rather than end products. Constructivism acknowledges the value of knowledge that is not universal but individual, personal, and subjective. The theory posits that reality resides in the mind of each person. "An individual makes sense of events according to his or her own experiences, beliefs, and knowledge" (Wilson, 1996, p. 95).

Learning takes place when individuals make use of their existing knowledge and experience to make sense of new material. Learning materials are structured around problems, questions, and situations that may not have one correct answer (Wilson, 1996). This is in conformity with the radical constructivist view that says that knowledge is always a result of a constructive activity. It cannot be transferred to a passive receiver. It has to be actively constructed by each individual learner.

2.4.2 Social constructivism

Social constructivism focuses on knowledge as taken-as-shared (Cobb, Yackel, & Wood, 1989). In a classroom context, "taken-as-shared" indicates that members of the classroom community, having no direct access to each other's understanding, achieve a sense that some aspects of knowledge are shared but have no way of knowing whether the ideas are in fact shared (Cobb, Yackel, & Wood, 1992; Streeck, 1979). According to Wood, Cobb, & Yackel (1995), it is useful to see mathematics as both a cognitive activity constrained by social and cultural processes, and as a social and cultural phenomenon that is constituted by a community of actively cognizing individuals. They referred to this coordination of psychological and sociological analyses as "social constructivism". Ernest (1991) claimed that a central thesis of social constructivism is the unique subjective meanings and theories constructed by individuals that are developed to "fit" the social and physical world.

Social constructivism is a development started by some who claim radical constructivism does not take into account the role of social interaction in the construction of knowledge (Von Glasersfeld, 1993). Glasersfeld further claims that constructivists from Piaget's tradition have always maintained that social interaction is a powerful influence in the construction of knowledge. But neither Piaget nor recent constructivists have actually specified a detailed model of how social interaction works from the constructivist point of view. The fundamental difference between social constructivism and radical constructivism, according to Von Glasersfeld (1993), is that social constructivists tend to consider the society's influence on individual knowledge construction which is opposed to the view of radical constructivists.

The two positions discussed above are not too apart from each other. In a nutshell, the difference arises in the mode of knowledge construction where one is individual to the learner and the other is through social mediation. Steffe (1990) said that mathematical learning is considered as a social enterprise in the notion of social constructivism. Thus, social disagreements about the meaning of mathematical materials or concepts provide the gist for mathematical development. These disagreements provide the impetus to change or accommodate one's understanding of such concepts. Any such change serves to make the child's understanding of mathematics more consistent with the understanding of the larger social community (Steffe, 1990). With the development of appropriate social mathematical environments, "it is possible for students to construct for themselves the mathematical practices that, historically, took several thousand years to evolve" (Cobb et al., 1992, p. 28).

2.4.3 Radical constructivism versus social constructivism

Radical constructivists consider knowledge as an individual construction while social constructivists believe that knowledge production is a result of social interactions. The difference between the two positions seems to depend on the focus of the observer. The radical constructivist position focuses on the individual's construction, thus taking a cognitive or psychological perspective. Although social interaction is seen as an important context for learning in this perspective, the focus is on the resulting reorganization of individual cognition. On the other hand, social constructivists see higher mental processes as socially mediated. In this view, sociocultural processes are given analytical priority when understanding individual mental functioning (Wertsch & Toma, 1995). From a social perspective, knowledge resides in the society, which is a system that is greater than the sum of its parts. Therefore, the difference lies between the individual construction of knowledge and the knowledge constructed by socially-mediated processes.

Whether it is radical constructivism or the social form of it, constructivism in general could be termed as an educational paradigm. As such, it might be represented by:

> 1. An ontology: a theory of existence concerning the status of the world and what populates it.
> 2. An epistemology comprising: (a) a theory of the nature, genesis, and warranting of subjective knowledge, including a theory of individual learning and (b) a theory of the nature, genesis, and warranting of knowledge (understood as conventional or shared human knowledge), as well as a theory of "truth."
> 3. A methodology: a theory of which methods and techniques are appropriate and valid to use to generate and justify knowledge, given the epistemology.
> 4. A pedagogy: a theory of teaching, the means to facilitate learning according to the epistemology. (Ernest, 1996, p. 337)

The above paradigm elaborates the fundamental links between the two constructivist traditions and emphasizes that both individual construction as well as social construction of knowledge are important if we are to think of a combined and complete notion of constructivism.

2.4.4 Constructivism and students' conceptual models

Constructivism is an epistemological position. The emerging picture of learning in constructivism is that learners actively construct their own understanding by looking for regularity and order in the events of the world. According to Piaget (1970, 1977, 1985), the process of constructing knowledge has to undergo two main stages -- assimilation and accommodation. He explained that no experience is ever the same as another in the absolute sense. By disregarding certain differences, repetition and subsequent regulation of experiences can be obtained. This active construction implies both a base structure from which to begin (a structure of assimilation) and a process of transformation or creation. The process of continual revision of structures is called the accommodation.

Hence, learning can happen only by relating the unknown to what is already known. Thus, all learning depends on the prior knowledge of the learner, which serves as a format, or schema, into which the new information is fitted. When the existing schemas are not adequate to absorb the new knowledge, they are extended or new schemas are constructed by the learner during the learning process (Skemp, 1987).

Wittrock (1986) explained the personal nature of schemas in conformity with the constructivist view. He pointed out that learning involves the active generation of new links between new information and existing knowledge by the learner. Since knowledge schemata are personal and individual, learners generate unique links between new and old information. Therefore, it is not surprising to find that different learners construct alternative conceptions of the same phenomena.

In mathematics, the persistently fixed nature of incorrect schemas in students' mind makes them to formulate wrong rules. This is termed "degenerate formalism" (Krygowska, 1977 (in Polish) cited in Demby, 1997, p. 46; Cwik, 1984 (in Polish) cited in Demby, 1997, p. 46). Degenerate formalism consists of a detachment of the rules of manipulating symbols from their meaning. It has two features. First, referring to the meaning is no longer a way of checking the correctness of the computation since this is the individual's incorrect perception. Second, the student formulates his/her own wrong rules. These wrong rules are often persistently fixed in the students' mind, or may be used ad hoc, based on various associations, but neither on the meaning of symbols nor on formal deduction. Sometimes, there are thoughtless, slapdash manipulation of symbols which was totally different from correct formalism and they were not in accordance with strictly applied rules.

Gourgey (2001) suggested that some students do not behave according to the models that researchers set up as a normative framework. Often, students do not pay careful attention to sense-making and clarification. This often leads to impulsive and illogical attempts at solutions. When asked to solve a math problem, they immediately performed operations without thinking carefully about what the problem was asking for and whether the operations were appropriate. As a result, their answers frequently did not make sense. Students often had to be forced to examine their reasoning and to connect the problem with their concrete experience before they can see their errors (Gourgey, 2001). According to Schoenfeld (1987), novice students quickly select a solution strategy and then spent all their time executing it, rarely stopping to evaluate their work to see if it was leading to the goal. Lacking self-monitoring and self-regulation, they waste much time on “wild goose chases” (p. 193). Even when they had adequate mathematical knowledge to solve the problem, they were unable to activate it constructively.

Constructivist theories suggest that in order for students to be successful in solving a problem, they should select and apply the correct solving schema. There are situations where students apply incorrect schemas while having the correct ones in their heads. One possible explanation for this stems from the neural network theory of mind. It indicates that students probably had the correct methods in their long-term memory but they could not recall the information (Martindale, 1991; Matlin, 2005). The theory further says that the students probably had both the correct and wrong schemas in their long-term memory but recalled the wrong information. Despite the existence of correct information, the reason for recalling wrong information was that the correct information was covered or inhibited.

2.4.5 Criticisms to constructivism

A major criticism about constructivism is its subjective nature of knowledge. Some critics argue that constructivism is a stance that denies reality (Kilpatrick, 1987; Kitchener, 1986). They argue that if everyone has a different experiential world, no one could agree on anything and, above all, no one could even communicate. Constructivists reply to this argument by saying that, even though we do agree on certain things and that we can communicate does not prove that what we experience has objective reality. If we as a society look at something using different lenses and agree on what we see does not make what we see any more real. It merely means that we can build up a consensus in certain areas under the guidance of our subjective experiential worlds. These areas, according to Von Glaserfeld (1991), are called "consensual domains". The models that we construct about something are our constructs that are accessible to us. However, this does not mean that we cannot use our cognitive ability to construct a view that turns out to be more compatible with what we perceive of the other's actions.

Another similar criticism is that, if everyone is a captive of their own constructs, this means that no appeal to an external reality can be made to assess the quality of those constructs. Then everyone's constructs must be equally valid. Constructivists reply to this argument in two ways. First, they say constructive process does not happen in isolation. It is subject to social influences. Our acceptance of a level of rigor for a solution is both social and individual. Therefore, there are always common elements that shape the individual constructions. Second, people try to assess the congruency between their constructs through the use of languages, choice of references, and selection of examples. Moreover, people try to assess the strengths of other people's constructs by considering the level of internal consistency of those constructs.

2.5 The nature of mathematical understanding

There are different views about the nature and content of mathematics within the research community. In one study, lecturers of mathematical sciences and mathematics education in some Canadian universities were asked to define mathematics (Radipotsane, 1996). Fourteen different themes were emerged. Some of them were: study of formal axiomatic systems, application of laws and rules, a set of notations and symbols, problem solving, a science, truth, and a socially constructed artifact. Lerman (1990) said that the multiplicity of philosophies of mathematics can be identified as two competing programs which he calls as Euclidean and quasi-empirical. The former group attempts to base all of mathematics on universal absolute foundations. The latter group accepts the uncertainty of mathematical knowledge as part of the nature of mathematics.

Another two views discussed by Lerman (1990) were the platonic and fallibilist approaches. The former view explains mathematics as certain, absolute, and value-free knowledge which has connections to the real world. The latter view describes mathematics as a social construction and its results are relative to time and space. The mathematics curricula in the USA, Canada, and many other European countries are based on problem solving with students' active involvement in their own learning. This notion reflects the ideas of constructivism as a method of learning. In many Asian countries like Japan, Korea, and India, the mathematics curricula are based on disseminating knowledge by the teacher. Therefore, it seems reasonable to employ research methodologies based on constructivism to study student conceptions and/or misconceptions in a North American country like Canada.

Skemp (1987) defined two different categories of mathematical understanding as relational and instrumental. Instrumental understanding is the knowledge of rules and how to apply and carry out a procedure in mathematics without necessarily understanding the reasoning behind those rules. Relational understanding deals with the knowledge of what to do and why. It is the ability to deduce specific rules or procedures from more general mathematical relationships. Hiebert and Carpenter (1992) defined two similar categories: conceptual and procedural. Conceptual understanding is the acquisition of knowledge that is equated with connected networks. This is the knowledge that is rich in relationships and it is a concept oriented, relational approach. It includes both knowing how and knowing why. In contrast, procedural understanding is a rule-oriented, instrumental approach. It is to know how but not knowing why. Further, mathematical understanding can be characterized by the kinds of relationships or connections that have been constructed between ideas, facts, procedures, and so on. According to Hiebert et al. (1997), there are two processes which help to make such connections: Reflection is central for individual cognition and communication is central for social cognition. Communication works together with reflection to produce new relationships and connections. Understanding and skills can and should develop together but the primary goal of mathematics instruction is conceptual understanding.

2.6 The nature of algebra

There are many conceptions about algebra in the literature. Many historically developed algebraic concepts can be observed in many of the current secondary school algebra curricula throughout the world. These inclusions show how various structural features of algebra are connected together to form broader conceptions within algebra in conformity with its historical development. It further shows how algebra is related to the other branches of mathematics. These ideas are especially useful when selecting and including algebraic concepts in a test for a certain grade level in the secondary school.

Usiskin (1988) described four fundamental conceptions of algebra. The first conception considers algebra as generalized arithmetic. In this conception, a variable is considered as a pattern generalizer. The key instructions for students in this conception are “translate and generalize”. For example, the arithmetic expressions such as $-1\times 5=-5$ and $-2\times 5=-10$ could be generalized to give properties such as $-x\times y=-xy$.

There is a close relationship between the cognitive processes involved in the learning of school algebra and the historical development of algebra as a symbol system (Kieran et al., 1990). Historically, algebra has been transformed into many other forms of mathematics such as analytic geometry and calculus because of the power of algebra as generalized arithmetic. The debate among one group of British mathematicians in the first half of the 19^{th} century about the nature of algebra (Pycior, 1984; Wheeler, 1989; Wu, 2001) also drew attention to the important epistemological problem of algebra as universal arithmetic. They believed that it deals with quantities and the permissible operations on quantities and its rules are dictated by the well-known properties of quantitative arithmetic. However, according to them, this attitude put into question the legitimacy of the algebraic use of negative, irrational, and imaginary numbers since these numbers cannot be interpreted as measures of quantity.

The second conception suggests that algebra is a study of procedures for solving certain kinds of problems. In this conception, we have to find a generalization for a particular question and solve it for the unknown. For example, if we consider the problem “When 3 is added to 5 times a certain number, the sum is 40. Find the number.” (Usiskin, 1988, p. 12). The problem translated into algebraic language will be an equation of the form “$5x+3=40$” with a solution of $x=7.4$. Therefore, in this conception, variables are either unknowns or constants. The key instruction here is “simplify and solve”.

In the third conception, algebra is considered to be the study of relationships among quantities. Here, variables really tend to vary. For example, a formula for the area of a rectangle is $A = L.W$. This is a relationship among three quantities. There is no feeling of an unknown here. Instead all *A*, *L*, and *W* can take many values. In such an example, no solution process is involved.

The fourth conception accepts algebra as the study of structures. Under this notion, the variable is little more than an arbitrary symbol. The variable will become an arbitrary object in a structure related by certain properties. This is the view of variable found in abstract algebra. As an example, when factorizing the problem "$3x^2 + 4ax - 132a^2$", the conception of a variable represented in here is not the same as any previously discussed notions. The variable neither acts as an unknown nor is it an argument.

2.7 Problem solving and students' mental models

Students' construction of knowledge in mathematical problem solving is reflected in their use of strategies as they attempt to master a problem situation. Various stages of the solving process will bring different sets of challenges to them. It is the construction of cognitive structures that are enabling, generative, and proven successful in problem solving (Confrey, 1991). Confrey presented a simple model to describe the construction of cognitive structures in problem solving. As shown in figure 1, students begin by identifying their problems, acting on them, and then reflecting on the results of those actions to create operations. This is followed by checks to determine whether those problems were resolved satisfactorily by reflecting on the problems again, thereby making the process cyclic.

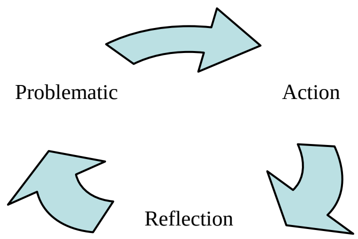

Figure 1: Stages of problem solving (Source: Confrey ,1991, p. 119)

“If this process proven successful, it is repeated in other circumstances to create a scheme, a more automated response to a situation” (Confrey, 1991, p. 120).

Over time, these schemes emerge from assimilations of experience to ways of knowing. They have duration and repetition, and they are more easily examinable than isolated actions (Confrey, 1994). “Assimilating an object into a scheme simultaneously satisfies a need and confers on an action, a cognitive structure” (Thompson, 1994, p. 182). By listening to student explanations, teachers can decode student thinking patterns thereby allowing teachers to identify not only the reasons behind their particular actions but also their misconceptions. Hence, analyzing student data can prompt re-examination of one’s mathematical understanding and their mathematical meaning.

According to Polya (1957), problem solving is a stage-wise procedure. Polya (1957) presented a four-phase heuristic process of problem solving. The stages under this model are: understanding the problem, devising a plan, carrying out the plan, and looking back. Schoenfeld (1983) devised a model for analyzing problem solving that was derived from Polya's model. This model describes mathematical problem solving in five levels: reading, analysis, exploration, planning/implementation, and verification. In applying this framework, Schoenfeld discovered that expert mathematicians returned several times to different heuristic episodes. For example, in one case, an expert engaged in the following sequence of heuristics: read, analyze, plan/implement, verify, analyze, explore, plan/implement, and verify. Therefore, according to Schoenfeld (1983), the model is cyclic rather than linear.

English (1996) reviewed the steps on children's development of mathematical models. According to her empirical findings, children first examine the problem for cues or clues that might guide the retrieval from memory of a relevant mental model of a related problem or situation. After retrieving a model, they attempt to map the model onto the problem data. This mapping may involve rejecting, modifying, or extending the retrieved model or perhaps replacing it with another model. If there is a correspondence between the elements of the mental model and the data of the problem, the model is then used to commence the solution process.

However, retrieving an appropriate mental model may not be automatic or easy for children. English (1996) further said that, as children progress on the problem, they may recycle through the previous steps in an effort to construct a more powerful model of the problem situation and its solution process. This construction process is considered responsible for the development of new mathematical ideas. This model is very similar to Confrey's cyclic model. English's model also provides important clues about ongoing metacognitive activity during or after each cycle.

Comparing and contrasting the above models, it is evident that, although the number of steps in the solving process is different for each model, almost all the models contain similar basic aspects. For example, Schoenfeld's categories of reading, analysis, and exploration taken together could be considered as "understanding" in Polya's model. Exploration was not specified in the Garofalo and Lester framework, although they indicated the distinctive metacognitive behaviors that may be associated with each category. What is *sine qua non* is that, in any kind of problem solving activity, mental mapping, constructing schemes, and metacognition play important roles during various stages of the process.

2.8 Some philosophical underpinnings of algebraic concepts and their influence on problem solving

One of the debates about algebra learning is whether algebra should be presented as generalized arithmetic governed by the laws of those concerning computations on plain numbers (Kilpatrick & Izsak, 2008). The other side of this argument is that instead of working with specific numbers, the letters which represent numbers of algebra should be treated as a separate symbolic system based on formal rules. There are arguments for and against these two views.

There is a connection between arithmetic and algebraic concepts (Norton & Irvin, 2007; Stacey & Chick, 2004; Stacey & MacGregor, 1999; Wu, 2001). For example, manipulating algebraic expressions having integers (operating with negative integers) and over generalization of cancelling procedures (fraction errors) have their roots in arithmetic misconceptions, and incomplete understandings and failure to transfer arithmetic understandings to algebraic contexts (Norton & Irvin, 2007; Stacey & Chick, 2004; Stacey & MacGregor, 1999). Wu (2001) reinforced this idea and said that students who are not comfortable computing with numbers will be less disposed to manipulate symbols because computational procedures with fractions provide a natural entrée into symbol use.

Many algebraic problems are difficult for students, because solving them may require an understanding of the conceptual aspects of fractions, decimals, negative numbers, equivalence, ratios, percentages, or rates (Norton & Irvin, 2007; Stacey & Chick, 2004; Stacy & Macgregor, 1999). Conceptual understanding consists of knowing the structure or rules of algebra or arithmetic such as the associativity, commutativity, transitivity, and the closure property. For an example, students should understand that $\frac{1+3}{5}$ can be separated as $\frac{1}{5}+\frac{3}{5}$ in the same way as they understand the reverse process. Stavy and Tirosh (2000) also perceived a connection between arithmetic and algebra. According to them, students sometimes assume incorrect rules when solving algebra problems. One such rule implies that although the quantities A and B are equal, students incorrectly assume that "more A implies more B". As an example, when they were asked "what is larger, smaller, or equal: $\frac{16y}{8}$ or $2y$?" , they say that $\frac{16y}{8}$ is larger because it has larger quantities.

Lee and Wheeler (1989) perceived the worlds of algebra and arithmetic as different having no connection. They suggested that lack of numerical support for algebraic reasoning is a plausible reason for why some students perceive the world of algebra and the world of arithmetic to be disconnected. According to Wheeler (1989), when we come to consider algebra in school, we find that algebra is derived overtly from arithmetic. But the pedagogy of secondary school algebra is not consistent with this notion. There are many covert signs in secondary school that algebra has its own rules, not necessarily deducible from the rules of arithmetic. For example, the two digits in number 23 have their own place values whereas in algebra xy means x times y. The consequence of this confusion, according to Wheeler (1989), is that it leaves many students unsure of the grounds that justify particular algebraic transformations.

Booth (1984) distinguished some properties of arithmetical strategies, which hamper the development of algebraic understanding. He said that arithmetical strategies are intuitive, primitive, and context-bound. They involve little or no symbolism and usually involve only whole numbers. They are based on basic operations. According to Booth (1984), arithmetical problems are "connected", so that the student can reason from the known to the unknown directly. On the contrary, algebraic problems are labeled as “disconnected” because they require reasoning with unknowns. Hence, arithmetical and algebraic reasoning appear to be essentially different and this could cause serious obstacles for the passage from arithmetic to algebra.

The rules used to solve the problems in algebra are closely associated with the procedural and conceptual (structural) aspects of algebra. According to Kieran (1992), substituting different values for the variable in a simple equation until the correct value is found is a process that is procedural. This is akin to a trial-and-error process. A student does not need to understand the underlying principles of the structure of algebra to solve such problems. Examples of the conceptual category include applying characteristics such as commutative or distributive laws or equivalency relationships to solve algebraic equations. Students have to understand how and why these rules or properties work in order for them to explain the application of these rules. Most often, students fail to explain the rationale behind applying these rules because of their lack of conceptual understanding.

2.9 Problem solving and metacognition

Metacognitive beliefs are an internal part of any mathematical problem solving process. Cognitive researchers point out that, in any problem solving endeavor, the processes other than reading, understanding, and planning/implementing used by the solvers are monitoring, evaluation, and overseeing. The term commonly used in the psychological literature for these cognitive processes is “metacognition” (Flavell, 1981; Jacobs & Paris, 1987; Wilson & Clarke, 2004). Flavell (1981) defined metacognition as the knowledge or cognition that takes as its object or regulates any aspect of cognitive endeavor. The term metacognition derives from this quality of cognition about cognition. This definition implies that metacognition includes reflection on cognitive activities as well as decisions to modify these activities at any time during a given cognitive enterprise. According to Flavell (1981), people develop cognitive actions or strategies for monitoring cognitive progress. This is a metacognitive strategy.

Schoenfeld (1985) explained the connection between problem solving and metacognitive skills. He said that skilled problem solving appears to require metacognitive processes in addition to fundamental algebraic skills such as knowing how to correctly execute basic rules and procedures and knowing how to represent the problem. Metacognitive processes involve knowing which problem-solving options are available, evaluating the potential usefulness of these options, and then choosing the most efficient route to the goal. In the midst of problem solving, students sometimes keep tabs on how well things are going. If things appear to be progressing well, they continue along the same path. When things appear to be problematic, they take stock and consider other options (Schoenfeld, 1992). Students use metacognition to explain the paths that they take when solving a problem and rely on "mathematical memory" rather than memorization (Ebdon, Coakley & Legnard, 2003). Metacognitive processes help to deepen conceptual knowledge and to consolidate it.

Metacognitive skills also involve knowing one's own skills and limitations within the domain, as well as knowing when to give up a chosen path through the problem space and try something else (Geary, 1994). Problem space refers to all of the procedures and rules that the student knows about a particular type of a problem, as well as all of the different ways that the problem can be solved. Therefore, the processes involved in choosing the best overall problem solving strategy are also metacognitive processes (Clement, 1982; Mayer, 1982). By training students in metacognitive skills, the students can be made aware of the impact of rigid associations, inadequate models or an inadequate handling of models, inadequate intuitive beliefs, and incorrect generalizations to control their impact (Fischbein, 1990).

Although one can conceptually distinguish the nature of cognitive and metacognitive actions, operationally the distinction is often blurred. For example, cognition is implicit in any metacognitive activity, and metacognition may or may not be present during a cognitive act and perhaps this presence may not be very obvious. For this reason, it is extremely difficult to categorize a certain act as purely cognitive or purely metacognitive. Flavell (1981) provided some hints to distinguish between cognitive and metacognitive actions. According to him, cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done. For example, metacognitive behaviors can be exhibited by statements made about the problem or statements made about the problem solving process. Cognitive behaviors can be exhibited by verbal or nonverbal actions that indicate actual processing of information.

Artzt and Armour-Thomas (1992) provided action verbs classified by predominant cognitive level. They classified reading as cognitive, understanding as metacognitive, analyzing as metacognitive, exploring as both cognitive and metacognitive, planning as metacognitive, implementing as both cognitive and metacognitive, and verifying as both cognitive and metacognitive. According to them, actions of planning would be evidenced by statements made about how to proceed in the problem-solving process. Further, actions of understanding were categorized as predominantly metacognitive, because they occur only when students make comments that reflected attempts to clarify the meaning of the problem.

Although it is true that some of the things that one does to understand a problem are cognitive, when we rely only on the verbal comments of students, it is impossible to decipher the understanding that is being derived during the actual doing of the problem. Behaviors coded as reading were categorized as predominantly cognitive, because they exemplify instances of doing.

When exploration is guided by the monitoring of oneself, that behavior can be categorized as exploration with monitoring or exploration with metacognition, thereby keeping the exploration controlled and focused. The same analysis applies for implementation and verification, which can occur with or without monitoring and regulation.

2.10 A general discussion of algebraic errors and misconceptions

The foundation for research on student conceptions comprises three major traditions. Each tradition has its own epistemological assumptions (Confrey, 1990). These epistemological positions are: Piagetian studies in the tradition of genetic epistemology, applications of the philosophy of science in the tradition of conceptual change, and research on systematic errors. According to Confrey (1990), research in the first two traditions tends to be on student conceptions in science and mathematics, whereas research in the third area focuses on mathematics and computer programming.

These three categories are not exhaustive, nor are they mutually exclusive. Among these traditions, the first and the third are closely related to my research. Piagetian work on student conceptions examined the development of student understanding of particular mathematical and scientific concepts over time. Piaget’s fundamental assumption was that knowledge is a process, not a state. Hence knowledge needs to be examined in relation to its developmental associations. In line with this thinking, Piaget studied about conceptions, not misconceptions.

Researchers in the tradition of systematic errors have documented that students hold mini-theories about scientific and mathematical ideas. Numerous studies have shown that students have many naive theories, preconceptions, or misconceptions about mathematics that interfere with their learning (Posamentier, 1998). Because students have actively constructed their misconceptions from their experiences, they are very attached to them. They find it very difficult to give them up.

Radatz (1979) proposed that student errors could be categorized by following through problem solving stages. According to Radatz, various causes of errors in mathematics can be identified by examining the mechanisms used in obtaining, processing, retaining, and reproducing the information in mathematical tasks. He identified four error categories. Those are (1) errors due to processing iconic representations, (2) errors due to deficiencies of mastery prerequisite skills, facts, and concepts, (3) errors due to incorrect associations or rigidity of thinking leading to inadequate flexibility in decoding and encoding new information and the inhibition of processing new information, and (4) errors due to the application of irrelevant rules or strategies. Barrera et al. (2004) reported that errors caused by a lack of meaning can be differentiated into three different stages: algebra errors originating in arithmetic, use of formulas or procedural rules inadequately (procedural errors), and errors due to the properties themselves of algebraic language (structural errors).

Fischbein and Barash (1993) developed a theory in their seminal analysis of students' mathematical performances. This theory is related to three components of knowledge: algorithmic, formal, and intuitive. According to them, algorithmic knowledge is the ability to use theoretically justified procedures. This is the ability to activate procedures in solving problems and understand why these procedures work. The formal aspect refers to axioms, definitions, theorems, and proofs (Fischbein, 1994). This relates to rigor and consistency in deductive reasoning and it is free from the constraints imposed by concrete and practical situations. The intuitive knowledge is described as immediate, self-evident cognition imparting the feeling that no justification is required.

Sometimes, these three components converge. Usually, in the process of learning, understanding, and problem solving, conflictual interactions will appear (Fischbein, 1994). Often, intuitive background knowledge manipulates and hinders the formal interpretation or the use of algorithmic procedures (Fischbein & Barash, 1993). For example, students' misinterpretations of $(a+b)^5$ as a^5+b^5 or $3(a+b)^2$ as $3a^2+3b^2$ can be categorized as evolving from the application of the distributive law intuitively. Sometimes, a solving schema is applied inadequately because of superficial similarities in disregard of formal similarities. Other times, a solving schema deeply rooted in the student's mind is mistakenly applied despite a potentially correct, intuitive understanding. The three components, according to Fischbein and Barash (1993), are inseparable and they play a vital role in students' mathematical performance.

Usually, it is the intuitive interpretation based on a primitive, limited, but strongly rooted individual experience that annihilates the formal control or the requirements of the algorithmic solution, and thus distorts or even blocks a correct mathematical reaction (Fischbein, 1994). The solving procedures, acting as overgeneralized models, may sometimes lead to wrong solutions in disregard of the corresponding formal constraints. As an example, students often write $\sin(a+b)=\sin a+\sin b$, or $\log(a+b)=\log a+\log b$. Obviously, the property of distributivity of multiplication over addition [$m(a+b)=ma+mb$] does not apply in the above situations (Fischbein & Barash, 1993). The formal distributive property of multiplication over addition is deeply deposited in their mind so that they intuitively misapply the rule in similar situations. This is an example where intuitive component overtakes the formal component.

Matz (1980) extended the research on students' error behaviors in rule-based problems with a view to building a generative theory that accounts for as many common errors as possible that students make in problem solving. The theory states two extrapolation mechanisms for generating algebra errors. They are the use of a known rule in a new situation where it is inappropriate, and incorrectly adapting a known rule so that it can be used to solve a new problem. The examples for these categories again emanated from the overgeneralization of the distributive law (Matz, 1980; Matz, 1982; Kaput, 1982; Kirshner, 1985). Kirshner (1985) said that overgeneralization of rules is common in almost every student up to a certain stage. Even successful students tend to go through a phase of overgeneralizing distributivity before achieving fluency in manipulative skills.

Errors are logically consistent and rule based rather than random (Ben-Zeev, 1998). "Investigating errors, therefore, presents an opportunity for uncovering the mental representations underlying mathematical reasoning" (Ben-Zeev, 1998, p. 366). In preparing a taxonomy of errors, Ben-Zeev (1998) discussed the need to have a clearer distinction among various stages of the problem solving process such as execution errors and encoding errors.

2.11 A discussion of errors pertaining to the four conceptual areas

In this study, I decided to consider the study of students' errors and misconceptions in four conceptual areas in algebra. These are: variables, algebraic expressions, algebraic equations, and word problems. In the next sections, I will elaborate the literature pertaining to students' misconceptions and the reasons that is specifically relevant to these four conceptual areas.

2.11.1 Student difficulties in comprehending variables in algebra

Letters represent different meanings in different contexts. When letters are present in algebraic entities, this is a seeming difficulty for students. Kieran et al. (1990) explained an example. In arithmetic, 12 m can mean 12 meters, that is, 12 times 1 meter. But in algebra, 12m can mean 12 times some unknown number of meters. Therefore, the letter carries two different meanings depending on the context. Davis (1975) provided another dilemma of using the same expression to express two different things in the same context. According to him, a + b represents both the procedure of adding a and b and the object a + b taken as one quantity. This is characterized as the process-product dilemma. In algebra there is no clear cut distinction between these two entities.

Philipp (1999) used seven categories to group variables with examples to illustrate the uses of them. They were: letters as labels as f and y in $3f = 1y$ to denote 3 feet in 1 yard; as constants π, e, and c; as unknowns to denote x in $5x - 9 = 11$; as generalized numbers to denote a, b in $a + b = b + a$; as varying quantities to denote x, y in $y = 9x - 2$; as parameters to denote m, b in $y = mx + b$; and as abstract symbols to denote e, x in $e * x = x$.

A detailed classification about children's' interpretation of letters was given by Kuchemann (1981) reporting from the program Concepts in Secondary Mathematics and Science (CSMS). He administered a 51-item, paper-and-pencil test to 3000 British secondary school students. Using a category originally developed by Callis in 1975, Kuchemann categorized each item in the test to six levels: letter evaluated, letter ignored, letter as an object, letter as a specific unknown, letter as a generalized number, and letter as a variable. An example in the first category was "What can you say about a if $a + 5 = 8$?" Examples in the second and third categories were "If $n - 246 = 762$, then what is $n - 247$" and "Simplify $2a + 5b + a$". For the fourth and fifth categories, the examples were "Add 4 onto 3n" and "What can you say about c, if $c + d = 10$ and c is less than d?". For the last category, one of the given examples was "Which is larger $2n$ or $n + 2$?".

The results of the CSMS project indicated that student's interpretations of letters were partly depended on the nature and complexity of the question. Based on the hypothesized six levels of interpretation, a very small percentage of 13 to 15 year old students interpreted the letter as a generalized number although they were exposed to generalizing number patterns in classrooms. A greater number of students interpreted letters as specific unknowns. Comparatively, fewer students interpreted letters as variables. Nevertheless, 73% of 13 year olds, 59% of 14 year olds, and 53% of 15 year olds either treated letters as concrete objects when they are not or they ignored the existence of the letters completely. Kuchemann's levels represent a hierarchy. Even though Kuchemann named these categories as 'levels of understanding', it should have been more appropriate to name them as 'students' understanding of letters'.

Both classifications used by Philipp (1999) and Kuchemann (1981) were instances of different uses of letters in different situations. Philipp's category is broader in the sense that it includes some of the Kuchemann's categories. The variety of meanings that a single letter can take indicates the complexity of identifying and using them in different contexts, especially for students.

Other studies also have found that the majority of students up to age 15 could not interpret algebraic letters as generalized numbers or even as specific unknowns. They simply ignored the letters and replaced them with numerical values, or regarded the letters as standing for shorthand names. Macgregor and Stacey (1997) claimed that the principal explanation given in the literature for this type of error has a general link to levels of cognitive development. However, they provided alternative explanations for specific origins of misinterpretation that have been overlooked in the literature which may or may not be associated with cognitive level. According to them, these origins are: intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials. Macgregor and Stacey (1997) stated that the Roman Numeral System is an example for the "analogies with familiar symbol systems" category. In the ancient Roman Numeral System VI means '1 more than 5' and IV means '1 less than 5' which indicates that the position and the value of one numeral will change the value of the other numeral. This analogy causes students to apply their experiences in one number system to a different system where it is inapplicable.

2.11.2 Student difficulties in dealing with algebraic expressions

Letters are used to build up algebraic expressions. Either one letter or a combination of letters could be used in an expression. Therefore, there is a close relationship of understanding the meaning of letters in the context of an expression. Agnieszka (1997) commented on some misleading instances where students use objects for symbols or they often refer letters to real life objects. For example, sometimes students interpret the algebraic expression 8a as short for "8 apples". Such procedures are efficient in the case of simple tasks such as transforming $2a+3a$ as two apples plus three apples. These interpretations are categorized as lower forms of understanding and they are not sufficient for somewhat more difficult tasks. Agnieszka (1997) provided an example of an expression such as $3a-b+a$, where such low-level procedures cannot be used but both younger and older students still use the same object such as an apple to represent both a and b.

A similar explanation for conjoining is the duality of mathematical concepts as processes or objects, depending on the problem situation and on the learner's conceptualization. One of the most essential steps in learning mathematics is objectification: making an object out of a process. This is reflected in the mathematics curriculum as to develop operational thinking, that is, thinking about a process in terms of operations on objects (Dreyfus et al., 1990). Due to this dual nature of mathematical notations as processes and objects (Davis, 1975; Sfard, 1991; Tall and Thomas, 1991), students encounter many difficulties. For example, $3x+2$ stands both for the process ‘add three times x and two’ and for an object as $3x+2$. This dual conception causes students to confuse between $3x+2$ as a process or as an object. They simplify $3x+2$ as $5x$ when $3x+2$ is actually an object (for example, in a final answer).

Hallagan (2006) commented on a teacher model in which students were asked to visually represent an algebraic expression given in four different forms. The same expression was given in four different forms: $4(s+1)$, $s+s+s+s+4$, $2s+2(s+2)$, $4(s+2)-4$. A square pool with measurements $s \times s$ and a small square with measurements 1×1 were given as pictures to illustrate the border of a square pool in four different ways related to the above four expressions. There were four main conclusions. First, transition from arithmetic to algebra takes time for students. Second, students preferred numerical answers and to conjoin algebraic terms. Third, on a positive note, visual representations helped students to understand the algorithms in algebra. Fourth, students could not understand the concept of a variable clearly.

Conjoining letters in algebra is to connect together the letters meaninglessly. Researchers have differences of opinions about reasons for this error. Due to similar meanings of ‘and’ and ‘plus’ in natural language, students may consider ab to mean the same as $a+b$ (Tall & Thomas, 1991; Stacey & MacGregor, 1994). Students may erroneously draw on previous learning from other subjects that do not differentiate between conjoining and adding. For example, in chemistry, adding oxygen to carbon produces CO_2.

Booth (1988), Collis (1975), and Davis (1975) explained this tendency as a difficulty in accepting the lack of closure property of algebraic letters. Students perceive open algebraic expressions as ‘incomplete’ and try to ‘finish’ them by oversimplifying. For example, they consider an answer such as $a+b$ as incomplete and try to simplify it to ab. A typical explanation for this misconception is the tendency in many arithmetic problems to have a final single-digit answer (Booth, 1988; Tall & Thomas, 1991) or to interpret a symbol such as '+' as an operation to be performed, thus leading to conjoining of terms (Davis, 1975).

Similarly, Tirosh and Almog (1989) said that students in higher grade levels feel reluctant to accept $3+2i$ as a complete number. According to Macgregor and Stacey (1997), some conjoiners believe that, if a coefficient is of the left on the letter, it indicates subtraction and if it is on the right, it indicates addition. For example, they write $h10$ to mean 'add 10 to h' and $1y$ to mean 'take away 1 from y'. Interestingly, this is somewhat similar to the previous discussion about the place value system of the Roman numerals.

Many common errors in simplifying algebraic expressions seem to be instances of the retrieval of correct but inappropriate rules (Matz, 1980). For example, students incorrectly misapply $\frac{ax}{bx}=\frac{a}{b}$ into expressions like $\frac{a+x}{b+x}$ to get $\frac{a+x}{b+x}=\frac{a}{b}$. This is an application of a known rule to an inappropriate situation by incorrectly perceiving the similarities of the two situations. Schoenfeld (1985) said that an inappropriate use of arithmetical and algebraic procedures is called an algebraic bug. Bugs are procedures that are correct in some situations, but are incorrect if applied to other situations. As an example, Schoenfeld (1985) described that students sometimes write $\mathrm{x}(\mathrm{yz})=xy+xz$ by considering the transformation $x(y+z)=xy+xz$. The application of the distributive law is incorrect when the parenthetical values are multiplied. Lack of understanding of the structural features of algebra causes this type of misuse.

2.11.3 Student difficulties in solving equations

When two algebraic expressions combine together with an equal sign, it is called an equation. To solve an equation correctly, one must know the application of rules of simplifying algebraic expressions. An equal sign is used to express the equivalence between the two sides of the equation. This is an additional burden to students. Arithmetic and algebra share many of the same symbols and signs, such as the equal sign and the addition and subtraction signs.

The interpretation given to the equal sign by students is sometimes different from its accepted meaning. There are two interpretations attributed to the equal sign. The symmetric relation indicates that the quantities on both sides of the equal sign are equal. The transitive relation indicates that a quantity on one side can be transferred to the other side using rules. Kieran et al. (1990) said, in elementary school, the equal sign is used more to announce a result than to express a symmetric or a transitive relation. An example is: "Daniel went to visit his grandmother, who gave him $1.50. Then he bought a book costing $3.20. If he has $2.30 left, how much money did he have before visiting his grandmother?" (p. 98). Sixth graders will often write the answer as 2.30 + 3.20 = 5.50 – 1.50 = 4.00. The symmetric property of the equal sign is violated here. Kieran et al. further claimed that the equal sign is perceived by students as "it gives," that is, as a left-to-right directional signal rather than a structural property. In other words, students perceive the equal sign as a symbol inviting them to do something rather than a relationship (Kieran, 1992; Weinberg, 2007; Foster, 2007; Falkner, Levi, & Carpenter, 1999). This error type is extensively elaborated in the literature (Kieran et al., 1990; Foster, 2007; Herscovics & Linchevski, 1994).

Weinberg (2007) said that instead of uniquely denoting sameness, the equals sign seems to be a "Swiss army knife" (p. 170) of symbols, representing a ratio, the co-existence of unequal sets, or an undefined relationship between two objects, ideas, or symbols. This variety of meanings causes students problems. Kieran (1992) further elaborated the sources of errors for the misuse of the equal sign. She said that students' tendency to interpret the equal sign as a command to compute an answer suggests that aspects of arithmetic instruction were contributing to their difficulties in algebra. When students use the equal sign as a 'step marker' to indicate the next step of the procedure, they do not properly consider the equivalence property of it.

One other explanation for the use of the equal sign as to do something is attributed to the fact that the equal sign mostly "comes at the end of an equation and only one number comes after it" (Falkner et. al., 1999, p. 3). Another possible origin of this misconception is the '=' button on many calculators, which always returns an answer. Foster (2007) said that, in the United States, although students use the equal sign early in their school careers, they often use it to mean that the answer follows. When used in an equation, the equals sign indicates that the expressions on the left and right sides have the same value. This is a stumbling block for students who have learned that the equal sign means 'the answer follows'.

The procedures required to solve some equations involve transformations that are different from normal operations that students are used to employ. The procedure for equation solving rest on the principle that adding the same number to or subtracting the same number from both sides of the equation conserves the equality (Filloy & Rojano, 1984; Filloy, Rojano, & Solares, 2003; Filloy, Rojano, & Puig, 2007). This principle is equally applicable to multiplying or dividing both sides by the same number. Equations that have the variable on one side such as $x+a=b$, $ax=b$, $ax+b=c$ can be solved by those methods. However, according to Filloy and Rojano (1984), the rupture occurs with equations of the form $ax+b=cx+d$. The procedures required to solve equations of this type involve transformations that are different, such as subtracting ax or cx from both sides.

Similarly, students usually have difficulties in solving linear systems of equations with two unknowns (Filloy, Rojano & Solares, 2003; Filloy, Rojano & Puig, 2007). In the two-unknown linear system: $y = 2x + 3$; $y = 4x + 1$, despite the unknown is being represented by a letter (the y), it has also been represented by an expression that involves another unknown (the x). Therefore, students will have to operate the unknowns with a second level representation (Filloy, Rojano & Puig, 2007). This second level representation of the variables brings additional difficulties to them.

2.11.4 Student difficulties in solving word problems

It is argued that word problems have traditionally been the nemesis of many algebra students. The primary source of difficulty for students in solving algebraic word problems is translating the story into appropriate algebraic expressions (Mayer, 1982; Bishop, Filloy, & Puig, 2008). This involves a triple process: assigning variables, noting constants, and representing relationships among variables. Among these processes, relational aspects of the word problem are particularly difficult to translate into symbols. Bishop et al. (2008) further claimed that students' difficulties in translating from natural language to algebra and vice versa is one of the three situations that generally arise when students have just completed elementary education and are beginning secondary education.

According to Mayer (1982), the specifics of algebraic translation errors have not been examined as closely as the translation errors associated with arithmetic word problems. He further said that it is reasonable to assume that algebraic translation errors result from the semantic structure and memory demands of the problem. Hinsley et al. (1977) showed that the translation of algebraic word problems is guided by schemas. These schemas are mental representations of the similarities among categories of problems. Translation errors frequently occur during the processing of relational statements. This is confirmed by Newman (1977). In her study, the majority of errors occurred in the processing stage or the stages before that. Usually, the translation of words into algebraic language occurs in these stages.

To emphasize student difficulties in translating relational statements into algebraic language, Clement (1982), Clement, Lochhead, and Monk (1981), and Kaput (1985) extensively discussed the famous “student-professor” problem. The problem reads as, “there are six times as many students as professors at this university” (p. 17) and students were asked to write an algebraic expression for the relationship. Many researchers found that there was a translation error such as **“**6S = P” where S and P represent the number of students and the number of professors respectively (Clement, 1982; Clement, Lochhead, & Monk, 1981; MacGregor & Stacey, 1993; Weinberg, 2007; Rosnick & Clement, 1980).

According to Clement (1982), there appears to be two reasons for this type of a translation error. First, students have literally translated the syntax of the relational statement into an algebraic expression without considering the magnitude of the relationship. Second, they have used 6S to represent the group of students and P to represent the group of professors. For those who committed this error, the “$=$” symbol did not mean to represent a mathematical relationship. Instead, for them, it simply separated the two groups (Clement, 1982). Rosnick and Clement (1980) noted that not only does the reversal error appear in many situations, but it has also proven difficult to remediate.

MacGregor and Stacey (1993) commented on the reasons for students to write additive totals such as $6s + p$ as the answer to the “student professor” problem. They said, in such answers, students do not match the symbols with the words but were expressing features of some underlying cognitive model of an invisible mathematical relationship. Weinberg (2007) described this strategy as operative reasoning. In that, students performed hypothetical operations on two quantities to equalize the totals.

Not all the errors that occur while solving algebraic word problems result from difficulties in representing and translating problem statements. Once the problem has been translated, problem solving errors can and do still occur and these errors are often due to bugs (Lewis, 1981). For example, errors that occur during the manipulation of algebraic expressions typically involve the inappropriate use, or misapplication, of an algebraic or arithmetical procedure.

Sometimes, students get confused when they try to formulate a solution for an algebraic word problem. Kieran et al. (1990) said that, to solve a problem such as “When 4 is added to 3 times a certain number, the sum is 40”, students would subtract 4 and divide by 3 using arithmetic. But solving the problem using algebra would require setting up something like $3x + 4 = 40$. To set up the equation, students must think precisely the opposite way they would solve it using arithmetic. Therefore, two different kinds of thinking are involved in these two contexts which would sometimes confuse students. In arithmetic, students think of the operations they use to solve the problem whereas in algebra, they must represent the problem situation rather than the solving operations.

This dilemma could be interpreted in another way as the interference from previously learned arithmetical procedures hindering the development of subsequent algebraic concepts. Therefore, apart from the difficulties encountered by students when translating word problems into algebraic language, there are other barriers such as interferences from other systems, not understanding the equal sign as a relationship, and other misconceptions in simplifying algebraic expressions.

2.12 Identification of misconceptions through student interviews

Diagnose before you dose is a rule that is applied to medicine. However, this is equally important in mathematics as well. There are several procedures to diagnose student errors in mathematics. Observation of a student at work, careful scrutiny of the written product of a student to understand the logic behind the thinking that led to an error, think aloud protocols, and diagnostic interview procedures are the most common among them. Brueckner (1955) stated that there are three levels of diagnosis: general diagnosis, analytical or differential diagnosis, and case-study procedures. General diagnosis focuses on the use of survey tests and other evaluative procedures to examine the general level of performance of students. Analytical diagnosis uses systematic procedures for locating specific weaknesses or shortcomings. In case-study procedures, clinical diagnostic methods are applied to study the performance of an individual in detail. This procedure would also help to pinpoint the nature of the problem and its root causes. The techniques used in case studies are clinical in nature. To determine the nature of the error, several different methods are used. Faulty thought processes are detected by observation of behaviors, analysis of written work, analysis of oral responses and interviewing or questioning.

Booth (1988) pointed out that, “one way of trying to find out what makes algebra difficult is to identify the kinds of errors students commonly make in algebra and then to investigate the reasons for these errors” (p. 20). In mathematics, it is sometimes difficult to prove a problem using a direct method of proof. However, it could be easy to use a method such as “contra-positive proof”. In this method, we consider the opposite of what is to be proved and arrive at a conclusion that is opposite to the result. Likewise, if the reasons that students misunderstand mathematical concepts can be well understood, it is helpful to design remedial measures to avoid the misconceptions. To investigate the reasons behind misunderstandings, we have to inquire deep into students' minds.

Some errors are persistent, so that they will occur due to flawed conceptual knowledge (misconceptions) which are amenable to analysis, rather than the random errors that merely occur due to human fallibility. Therefore, examining deep into student thinking and their beliefs is necessary to find reasons for them to make these misconceptions. Since student reasoning is based on student beliefs for their errors, one can argue that these beliefs are wrong because they led to incorrect mathematical results. However, these belief systems possess students' own sort of integrity and robustness as intellectual constructs and they may have survived in the learning process for a long time. In that sense, inquiring into student belief systems will provide an insight into the actual reasons behind these erroneous beliefs. Hence, looking at these beliefs in depth will undoubtedly provide an insight into student misconceptions that led to misinterpretations.

One of the most well-known methods for analyzing verbal data used to be protocol analysis (Ericsson & Simon, 1984), which focuses on processes of problem solving aiming to create computer-model simulation. In protocol analysis, the subject is an individual who undertakes a sequence of problem states as (s)he applies permissible operators. Computer-model simulation is the ultimate goal of protocol analysis, indicating that human behaviors can be represented as step-by-step predictable processes. Verbal Analysis (Chi, 1997) reflects a theoretical shift in Cognitive Science from a search for a cybernetic view of mental processes to a quest for mental representations. The aim is not to predict behavior on problem solving but to investigate mental models and representations that explain human behavior.

The clinical interview method mainly developed by Piaget is one of the widely used methods for diagnosing errors. Actually, Piaget's method was both clinical and experimental. In the interview process, he used a method called "protocol analysis" (Inhelder, 1958). Piaget's two main protocol approaches were "thinking aloud" and "clinical interviewing". By thinking aloud, he encouraged the student to reflect on his/her own thoughts thereby allowing metacognition. Piaget used oral questioning extensively (Inhelder, 1958). He followed the student's thought processes and asked spontaneous questions based on their earlier responses. Careful planning of the interview questions is a necessary precondition in this process. Successful interviews invite useful communication despite the method used. This process always depends on the competent interviewer.

The following sample interview demonstrates how probing and conflicting questions could be used to get insights into students' thinking processes.

T – Teacher S – Student

T – Which of these expressions will always produce odd numbers: $\frac{x}{3}$, $x-3$, or $2x-1$.
S – I think the second one, because you're taking away an odd number.
T – How could you check?
S – I could substitute some values for $x = 4-3$ is an odd number, so is 8 – 3 and 10 – 3.
T – Does this mean it will work for all values?
S – All the numbers I tried were even. If I use odd numbers for x like 5 or 7, the result is an even number.
T – How can you use what you've learned from the second expression to know whether or not the third expression will always produce odd numbers?
S - $2x$ will always be an even number, so an even number minus 1 will always produce an odd number (Elchuck, et al., 1997, p. 20).

The interviewer has used "how" questions to get more information from the student. These questions have instigated the student to think more about his/her thinking and activate metacognitive skills. For example, checking the answers and giving numerical examples shows that these actions have erupted from the skillful questioning of the interviewer. Further, the last question would cause the student to think more deeply and resolve any conflicts in the answering process.

According to Ginsburg (1997), the interviewer has the freedom to vary questions as necessary in the clinical interview process. There are unplanned as well as unstandardized questions. On-the-spot hypothesis making and testing is basic to the interview process. By doing this, the interviewer attempts to uncover the thoughts and concepts underlying students' verbalizations. Therefore, clinical interview seems to provide rich data that could not be obtained by other means. The interviewer's behavior seems to be dominated by at least one broad goal, that is to understand the child's thinking (Ginsburg, 1997). But as the interview evolves, the interviewer may develop various sub goals or purposes in order to put the child at ease or to explore or clarify.

Newman (1977) suggested an error classification and some guiding interview questions to analyze mathematical problems. In this classification, Newman suggested that errors occur in the interaction between the question and the person who is attempting to solve the problem. She classified the sources of errors into a five element hierarchy: reading, comprehension, transformation, process, and encoding. Other general sources of errors out of this classification include but not limited to carelessness and lack of motivation (Figure 2). Newman also provided a list of guiding questions pertaining to each stage of the problem solving process (Appendix 4).

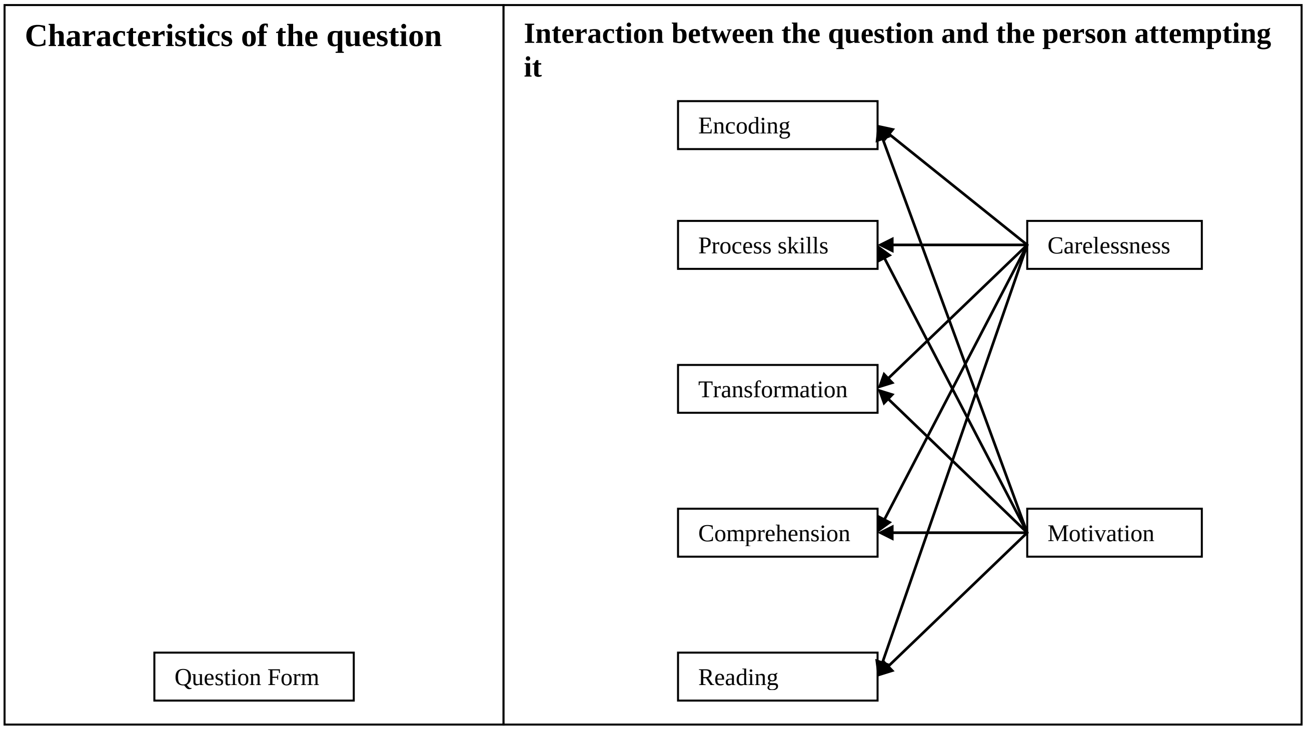

Figure 2: The Newman hierarchy for one-step verbal mathematical problems (Clements, 1980, p. 4).

The importance of Newman's model is that it provides a comprehensive stage-wise procedure to analyze mathematical problem solving tasks. Using this framework, Newman found that 47% of her population of low-achievers in grade 6 made errors prior to the process stage (of which 12% were at the transformation stage). The Newman (1977) model was later adopted by Casey (1978) and Clements (1980) for their studies. For a different group of pupils in grade five, six, and seven, Clements (1980), found that fewer errors were made at the two lower levels; one-quarter of the errors were at the transformation stage. Clements inferred that failure in the early stages of problem solving can lead to selection of incorrect processes later.

The unique feature of the Newman model is that it is well suited for word problems although there is no restriction of its use in other contexts as well. The left block of the diagram represents the difficulties of comprehending or understanding the question which is named as the “question form”. This emphasizes the necessity of providing appropriate questions. Even though the right questions are provided, some students may interpret them differently from the implied meaning. The right block represents the five stages of the problem solving process.

Basically, the model depicts when a student produces an incorrect answer to a question, the error resulting in that answer may have occurred at one of several stages in the process of solving that problem. The student may have misread the question (reading error), or may have misinterpreted it (comprehension error). Although the student has correctly comprehended the problem, s/he may incorrectly transform it into mathematical language. Alternatively, despite a correct transformation, an incorrect method may have been used to solve the problem (process error). Even though all the above steps are correct, the answer may have been wrongly encoded (encoding error). Still, the student may have a conflict with explaining or verifying the answer (verification error).

There could, however, be other possibilities as well such as the possibility of any combination or interaction of the above errors. Also, there could possibly be psychological factors rather than mathematical factors such as low attention to the task, anxiety, carelessness, or lack of motivation. Anxiety is a common feature in any problem solving situation. This erodes confidence and interferes with thinking. According to Posamentier (1998), there are two different components of math anxiety: intellectual or cognitive and emotional or affective. The intellectual component primarily involves worrying about failure and its consequences. The emotional component involves fear, feeling nervous, and being uncomfortable. The emotional component has a stronger and more negative relationship to children’s math performance (Posamentier, 1998).

There may also have influences from the student's lack of academic self-concept. This involves a feeling of not having confidence in one's ability to achieve, no self-reliance, and not recognizing of one's strengths and weaknesses. Errors caused by students' affective attitudes are of different types. Lack of concentration is sometimes caused by over-confidence, blockages, or forgetfulness.

2.13 Summary

I started this chapter with a discussion of the psychological approach that will be used as a theoretical foundation to study human cognition in this study. Under this approach, various forms of constructivism and their merits and demerits were discussed with an emphasis on my theoretical stance of radical constructivism as the basis for constructivism. The nature of mathematical understanding in general and the algebraic thinking in particular were discussed with special references to problem solving and metacognition. To identify better on students' consistent as well as inconsistent errors and misconceptions, think aloud protocols and student interviews were discussed as methods of exposing student reasoning. Next, I explained the error categories under the four main areas of this research. Finally, I discussed the interview method as a form of identifying students' difficulties and their reasoning.

The main cognitive obstacles that students encounter in solving algebraic problems related to the four main areas under discussion were: difficulties in transitioning from arithmetic to algebra, difficulties to understand the procedural and structural aspects of algebra, use of incorrect mini-theories or buggy algorithms, difficulties in processing iconic representations, difficulties in understanding the syntax and language of algebra, interferences from other subject areas such as English language and Chemistry, deficiencies of pre-requisite skills, inadequate decoding, encoding procedures, and application of irrelevant rules. There are other non-mathematical factors such as anxiety, over-confidence, lack of motivation, carelessness, and lack of attention to the task to hinder student progress.

In the past, research on student misconceptions has been limited to the study of isolated conceptions in algebra such as variables, equations, inequalities, or word problems to name some of them. Some researchers have attempted to construct hierarchies of errors under one area. Some others have attempted to provide specific causes for errors through student reasoning processes. However, comparatively fewer attempts have been made to understand the combined effects of misconceptions and their interrelatedness pertaining to a number of areas. As in any other area of mathematics, algebraic concepts are also interconnected. Basic algebraic concepts in secondary school algebra are closely linked so that student misconceptions could better be viewed if we could study those concepts together in a study and examine the interrelationship among error patterns. Otherwise, it may not provide a global view of student misconceptions.

Chapter 3

Research methodology

3.1 Introduction

In this chapter, I explain the main methodological constructs that were employed in various stages of the study and later unite them together to create an overall summary of the methodology. This discussion includes a review of the methods that were used in different stages of the study and their validity and reliability, sampling procedures, the pilot study, the main study, data collection instruments, data analysis methods, ethical issues, and a chapter summary.

3.2 Research traditions

Research involves systematic investigations undertaken to discover resolutions to a problem. According to Brew (2001), the general purpose of research is to contribute to the body of knowledge that shapes and guides academic and/or practice disciplines. There are two main approaches to research: scientific and naturalistic. Synonyms for the scientific approach are the objectivist or the positivist. In the scientific approach, quantitative research methods are employed in an attempt to establish general laws or principles (Burns, 2000). This approach assumes that social reality is objective and external to the individual.

Alternatively, synonyms for the naturalistic approach are the subjectivist or the anti-positivist. This method emphasizes the importance of the subjective experience of individuals with a focus on qualitative analysis. In this approach:

> Social reality is regarded as a creation of individual consciousness with meaning. Qualitative descriptions can play the important role of suggesting possible relationship(s), causes, effects, and even dynamic processes in social settings. (Burns, 2000, p. 3)

The paradigmatic division between quantitative and qualitative research is still prevalent. At the same time, mixed methods research is drawing increasing attention in educational circles. This paradigm systematically combines ideas from both quantitative and qualitative methods. Mixed methods researchers believe that they can get richer data and strong evidence for knowledge claims by mixing qualitative and quantitative methods rather than using a single method (Johnson & Christensen, 2008; Creswell, 1998; Gay, Mills & Airasian, 2006). This idea is further reinforced by the belief that social phenomena are extremely complex and in order to understand them better, we need to employ multiple methods. Johnson and Christensen (2008) listed five major purposes to select a mixed method design (Table 1).

Table 1
Purposes of mixed methods research

Purpose	**Explanation**
Triangulation	Seeks convergence, corroboration, correspondence of results from different methods
Complementarity	Seeks elaboration, enhancement, illustration, clarification of the results from one method with the results from the other method
Development	Seeks to use the results from one method to help develop or inform the other method, where development is broadly construed to include sampling and implementation, as well as measurement decisions
Initiation	Seeks the discovery of paradox and contradiction, new perspectives of frameworks, the recasting of questions or results from one method with questions or results from the other method
Expansion	Seeks to extend the breadth and range of inquiry by using different methods for different inquiry components

Source: Johnson & Christensen (2008, p. 451)

There is no doubt that all of the above five methods will improve the focus of research. I addressed all of these five components in my study. Triangulation is the term used to indicate the use of multiple pieces of evidence to claim a result with confidence. This increases the credibility or trustworthiness of the findings (Johnson & Christensen, 2008). For example, I used students' written work, interview transcripts, and researcher notes to triangulate the data and arrive at valid conclusions about student misconceptions. The term complementarity is used to elaborate and understand the overlapping and different facets of a phenomenon. For example, to clarify and further elaborate the results of student answers in the test, I used interviews which informed and enriched the data. In addressing the developmental purpose of mixed method inquiry, I used the quantitative phase to inform the qualitative phase. For example, I selected students for interviews based on the test results. The two phases were integrated together to get better explanations about my main focus of this study on students' misconceptions.

According to Johnson and Christensen (2008), a sequential design is necessary, if development is an objective of the research design. My study employed a sequential design which flowed from quantitative to qualitative methods. Finally, the word "expansion" promotes the breadth and range of inquiry by using different methods for different inquiry components. In my study, the quantitative part helped me to understand student errors and misconceptions numerically while the qualitative part helped me to deepen my focus to explain more about those errors through student reasoning processes.

3.3 Research design

The purpose of my study was to identify student errors and misconceptions in algebra pertaining to variables, expressions, equations, and word problems. I employed a sequential explanatory design which is characterized by the collection and analysis of quantitative data followed by the collection and analysis of qualitative data (Creswell, 1998). In the quantitative phase, I used a test instrument to identify and classify student errors. Based on my hypotheses on student misconceptions at this stage, I used interviews to expose student reasoning for their misconceptions and errors in the qualitative phase of the study.

Typically, the purpose of a sequential explanatory design is to use qualitative results to assist in explaining and interpreting the findings of a primarily quantitative design. The initial quantitative phase of the study may be used to characterize individuals along certain traits of interest related to the research questions. These quantitative results can then be used to guide the purposeful sampling of participants for a primarily qualitative study.

The findings of the quantitative study determine the type of data to be collected in the qualitative phase (Gay, Mills & Airasian, 2006). There are four main stages in the sequential explanatory study. The following schematic diagram indicates these stages (figure 3).

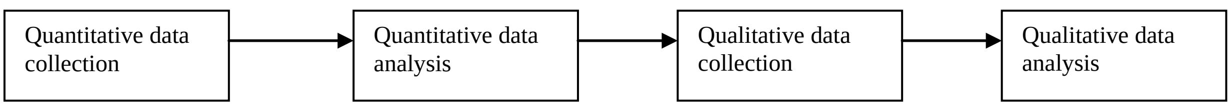

Figure 3: Schematic diagram representing the various stages of the design

Maxwell (2005) identified five particular research purposes for which qualitative studies are especially suited. They are: to understand the meaning of the events, situations and actions involved, to understand the particular context within which the participants act, to identify unanticipated phenomena and to generate new grounded theories, to understand the process by which events and actions take place, and to develop causal explanations. Sometimes, more than one of the above purposes would likely be achieved in one study.

In my study, the qualitative data were used for explanatory as well as for exploratory purposes. First, qualitative data were used to explain the quantitative data. Second, qualitative data were used to explore the quantitative data deeply. My study on students' errors and misconceptions addressed at least three of the above five purposes. The context and the process in which actions took place were not examined since I have used a psychological approach in this study.

Creswell (1998) argued that there are eight compelling reasons to undertake a qualitative study. I addressed four of them in my research. Creswell's first rationale was to select a qualitative study because of the nature of the research questions. For example, qualitative researchers often start with how or what questions while quantitative researchers start with why questions. The second rationale was to choose a qualitative study when variables cannot be easily identified or theories are not available to explain the behavior of the population. The third one was to choose a qualitative approach in order to study individuals in their natural setting. In the fourth rationale, Creswell said, "employ a qualitative approach to emphasize the researcher's role as an active learner who can tell the story from the participant's view rather than as an expert who passes judgment on participants" (p. 18). These purposes are not entirely mutually exclusive. However, I paid attention to all of these four areas.

Case study procedure is one of the main research methodologies in a qualitative research design (Goldin, 2008). According to Wiersma and Jurs (2005), a case study is a detailed examination of a specific event, an organization, or a school system. Much of the data of case studies come from observations, documents, and interviews. Case study research can be used to address exploratory, descriptive, and explanatory research questions (Yin, 2003; Johnson & Christensen, 2008). Robert (1994) said that case studies employ multiple sources of information to represent the case but not the world. I used more than one source to obtain data whenever possible.

Since the overall design of my research is more exploratory in nature, I used the case study method considering individual students as cases. I used multiple data sources such as students' written work, student interview transcripts, and researcher's notes to triangulate data.

The reason that individual interviewing was necessary was that from a constructivist point of view, reflective ability is a major source of knowledge on all levels of mathematics. Students should be allowed to articulate their thoughts and to verbalize their actions which will ensure insights into their thinking processes. During such mental operations, insufficiencies, contradictions, or irrelevancies are likely to be spotted. Students' thoughts opened up a way to explain why a particular misconception occurred.

As a general framework for interviews, I adopted the interview format elucidated by Newman (1977), Casey (1978), and Clements (1980). The questions in this format were divided into three main areas: input, process, and output. In the input stage, the components were: reading the problem, interpreting it, and selecting a strategy to solve it. The process stage contained solving the problem using the selected strategy. The components in the output stage were encoding the answer and answering the consolidating or verification questions from the interviewer (Newman, 1977; Casey, 1978; Clements, 1980). A series of questions could be asked in each stage depending on the previous answers of the student to gain insights into students' constructions, interpretations, and reasoning processes (Appendix 4). Communicating one's rationale and reasoning processes to another simultaneously shapes and transforms one's reflective thinking and schemes of internalized actions (Confrey, 1994). Therefore, this process promotes more self-reflection and a stronger approach to knowledge construction for the student.

I did not strictly follow all the stages or questions in the Newman et al. format in my study. I used interviews to explore students' thinking processes. However, these interviews were not clinical in nature. Instead, I provided students a chance to elaborate their thinking such as in think aloud methods. I prompted them with “explain more”, “go ahead” and “how” or “why” questions whenever necessary. Sometimes, I asked further questions or provided examples for further explanations but this was limited to the cases that needed more elaboration. Therefore, my method of interviewing was a mix of think aloud procedures with a lighter version of interview questions. I used the term “interviews” for this dual methodology in this study.

3.4 The pilot study

A pilot study is essential to refine instruments and to identify any other problems in the design. I conducted a two-phase pilot study to gather information about the test instrument. The first phase involved administering and evaluating the test instrument. In this stage, I found that some structural changes should be made to the test instrument. Therefore, I organized a second phase to include the changes and re-evaluate the test instrument. Further, I used this phase to conduct practice interviews with students.

3.4.1 Pilot study - Phase 1

It was necessary to pilot the test instrument to make it reliable and valid. I selected the questions under the four main areas of the study using the Ontario Grade 10 mathematics curriculum (Ministry of Education, 2005) as a guide. These areas were variables, expressions, equations, and word problems. Since the test was administered to grade 11 students at the beginning of the academic year, I decided that the content of the test would be based on grades 9 and 10 algebraic concepts in the Ontario secondary school curriculum.

The test was prepared to obtain data about student understanding of the concepts by including a variety of items. Test items were prepared by considering two main aspects. There were some items that were directly related to the conceptual understanding of algebra. In them, students had to explain some basic properties in algebra or they had to identify patterns or relationships and represent or interpret them algebraically. Some of them contained algebraic manipulations. Problems without a specific context pertaining to simplification of algebraic expressions, evaluating expressions, and solving equations were the examples of this group.

The next type of questions was the word problems that students needed to represent algebraically in order to solve them. These items usually appear in day-to-day life. Most of them were contextual problems. In some of the short-answer problems in the test, students had to provide and justify their answers by using mathematical language or other representations. In this way, the lapses of their conceptual understanding were identified.

Features such as the overall structure of the test, suitability of the items, item coherence, their appropriateness, and other features such as the face validity of the test were discussed with two subject experts and two teachers. This discussion was aimed to improve the validity of the test instrument. In phase 1 of the pilot test, the test contained 22 items under the 4 main areas (Appendix 1). Each question belonged to one of the four categories: variables, expressions, equations, and word problems. The items in these categories were not mutually exclusive in the sense that an item could belong to more than one category. For example, a word problem may contain the concepts of variables, expressions, or even equations. However, the major concept that was expected to test using the item was considered as the one that makes up that category.

An item was considered as a word problem when it had a word format using everyday language or mathematical language. By including these items, I expected to test whether the student could read the item, understand the structure of the problem, translate it into algebraic language, and solve it. In other words, these items mainly assessed whether students could translate word problems into symbolic representations and solve them. Table 2 shows the categorization of problems into the four main areas of the study.

Table 2
Classification of questions into categories

Category	**Sub category**	**Question number**
Variables	Variable as a specific unknown	1(a), 1(b)
	Variable as a generalized number	2, 3, 16
	Non-variable	1(c)
Expressions	Evaluating expressions	5
	Equivalent expressions	6
	Simplifying expressions	7, 8, 13
	Comparing expressions	12
	Building expressions	9, 10, 11
Equations	Simultaneous equations (different formats)	19, 20, 21, 22
Word problems	Context – Everyday language	4, 14, 17, 18
	Context – Mathematical language	15

3.4.1.1 The facility value

The first phase of the pilot study was conducted with a group of thirty students in an urban public school. The items were marked and the facility index for each item was calculated using the formula: Facility index = $\frac{C}{N}$, where C is the number of students who answered an item correctly and N is the total number of students in the sample (Nunnally, 1972; Alderson et al. 1995; McAlpine, 2002). The items that were too easy to answer will give fewer student errors. The response rate will be low for the items that are difficult to answer. Therefore, a reasonable facility index in between 0.3 and 0.8 was selected for the items that were to be included in the second trial.

3.4.1.2 Reliability of the test

Ensuring reliability is a prerequisite of constructing a good test. If a test is reliable, all the items should correlate with one another. If the items are highly correlated with each other, the whole test then should correlate highly with an alternate form (Nunnally, 1972; Alderson et al. 1995; McAlpine, 2002). Measurements are reliable if they reflect the true aspects but not the chance aspects of what is going to be measured (Gilbert, 1989). Thus, internal consistency of a test is essential for it to serve its purpose.

There are several forms of reliability measures described in the literature. Nunnally (1972) suggested that three methods exist: alternate-form reliability, retest reliability, and split-half reliability. Alternate-form reliability is the most comprehensive measure which correlates the scores of students obtained by administering two alternate forms of the same test to the same group of students. The retest method gives the same test on two occasions. The split-half method needs the same test to be administered on one occasion only. In this method, the test is divided into two parts and the correlation between these two parts is calculated.

When an alternate form of the test is not available and the retest method is cumbersome, good reliability estimates can be obtained from the split-half method. I used the split-half method in my study to obtain the reliability measure. In this method, the test scores were divided into two halves: scores for odd-numbered items and scores for even-numbered items. Then the correlation between the two halves was determined. The following Spearman-Brown prophecy formula (Gay, Mills & Airasian, 2006) was used to calculate the reliability coefficient of the whole test. $r_{totaltest} = \frac{2r_{split-half}}{1+r_{split-half}}$

The split-half reliability coefficient for the preliminary trial was 0.66 and the reliability coefficient for the whole test using the above formula was 0.8. Since this shows an adequate level of reliability, the test was considered to be reliable.

3.4.1.3 Validity of the test

The validity of a test instrument is equally important as its reliability. If a test does not serve its intended function well, then it is not valid. According to Remmers (1965), there are four main types of validity: content, concurrent, predictive, and construct. Content validity addresses how well the content of the test samples the subject matter. Concurrent validity measures how well test scores correspond to already accepted measures of performance. Predictive validity deals with how well predictions made from the test are confirmed by subsequent evidence. This type of validity is not directly relevant to the current study. Construct validity is about what psychological qualities a test measures. This type of validity is primarily used when the other three types are insufficient.

In order to preserve content validity, the content of the test was prepared by consulting the Ontario mathematics curriculum: Grades 9 and 10 (Ministry of Education, 2005) as a basis. The content of the test was discussed with two subject experts and two mathematics teachers and their suggestions were included prior to the first administration. Also, similar test construction procedures in the literature were consulted when preparing the test items.

3.4.1.4 Selection of students for interviews

Another task of the pilot test was to determine ways to identify students for interviews. To get an adequate and a manageable number of students, I decided to take one student from every five students who wrote the test. In order to select students for interviews, I followed the theoretical sampling strategy while analyzing student answers in the test. Charmaz (2006) describes theoretical sampling as starting with the data, constructing tentative ideas about the data, and then examining these ideas through further empirical inquiry. Therefore, I selected my interview participants by thoroughly examining their answers to the test.

Two error categories were distinguished from student responses to the test: non-systematic errors and systematic errors. In the non-systematic category, student errors had no apparent patterns to be identified and they were not connected to other concepts. It was hypothesized that these errors were random or they were made due to some other reasons such as forgetfulness, stress, or carelessness. However, this was only a hypothesis and there could be other reasons for these errors. There are no clear cut methods to deduce the reasons for students making these errors other than listening to their reasoning. Two examples of the random category are discussed below.

Case 1: Anton

Anton is a grade 11 student in the college/university mathematics stream. He had answered the first 17 questions in the test and he left the last five questions unanswered. There were no specific identifiable patterns in the answers. The following examples show some of his work.

1. Simplify: $\frac{r}{4} - \frac{(6-s)}{2}$

$$= \frac{r}{4} - \frac{2(6-s)}{4} = \frac{r}{4} - \frac{12+s}{4} = \frac{r-12+s}{4}$$

Although there seems to be some confusion with the minus sign in the second step, Anton got the correct sign for each term in the final answer. However, his elimination of brackets was incorrect as he did not multiply the second term by 2. His answers to the following questions show that the same multiplication error was not committed in these problems.

2. Multiply $e+2$ by 3.

$3e+6$

3. I thought of a number, I added 7 to this number, and then I multiplied the result by 3. I got 36. What was the number I thought about?

$3(n+7)=36$; $3n+21=36$

Anton seems to have some problems applying the distributive property when he gets a complex expression with several other terms. He seems comfortable with applying the property with a simple, single statement without any attached terms.

Case 2: Laksha

Laksha is a grade 11 student in the college/university mathematics stream. She answered many questions in the test correctly except the last three questions. Again, there were no specific identifiable patterns in her answers. The following examples show some of her work.

1. Simplify : $\dfrac{xa+xb}{x+xd}=\dfrac{x^2ab}{x^2d}$ 2. Simplify : $\dfrac{A}{B}+\dfrac{A}{C}=\dfrac{2A}{BC}$

In the first example, Laksha seems to have multiplied the two terms both in the denominator and the numerator although there is a plus sign in between the terms. However, in the second example, she may have taken the common denominator or may have multiplied the two terms in the denominator. She may have added the two terms in the numerator as there is a plus sign in between them. Of course, there could be other possibilities as well. I only can hypothesize reasons for student errors at this stage. Still, there could be totally different reasoning other than mine from the student’s point of view. However, this is worth further examination by considering such cases as potential candidates for interviews.

The second category of errors was the same or similar errors that were noticed repeatedly in many questions for the same student. This designates a student conception that produces a systematic pattern of errors. For example, Nesher (1987) explained them as systematic errors and described them as a line of thinking that causes a series of errors all resulting from an incorrect underlying premise, rather than sporadic, unconnected, and non-systematic errors. The assumption is that, when the same error or similar errors occur more than once in different situations, then it is possible that the student may have a misconception. Therefore, they are worth analyzing. Some of the examples in this category appear below.

Case 1: Amanda

Amanda is a grade 11 student in the university academic mathematics stream. She shows a good understanding of variables but a poor understanding of simplification of algebraic expressions. The following examples show some of her work.

1. Simplify: $A\times\frac{1}{A}$

$=\frac{\cancel{A}}{A^{\cancel{2}}}$

$=A$

2. Simplify: $x\left(\frac{a}{b}\right)$

$=\frac{ax}{bx}$

3. Solve the following linear systems. Explain why you chose this method.

$\frac{x}{2}-\frac{2y}{3}=\frac{7}{3}$

$\frac{3x}{2}+2y=5$

Multiply (1) by 3,

$3\left(\frac{x}{2}\right)-\cancel{3}\left(\frac{2y}{\cancel{3}}\right)=\cancel{3}\left(\frac{7}{\cancel{3}}\right)$

$=\frac{3x}{6}-2y=7$

Multiply (2) by 3,

$3\left(\frac{3x}{2}\right)+3(2y)=3(5)$

$=\frac{9x}{6}+6y=15$

$$\frac{3x}{6}+\frac{9x}{6}-2y+6y=15+7$$
$$\frac{12x}{6}+4y=22$$
$$=2x+4y=22$$

The above three examples suggest that Amanda has a misconception when she has to multiply an algebraic fraction by an unknown (in examples 1 and 2), an algebraic fraction by a constant (in example 3), or a numeric fraction by a constant (in example 3). She seems to be multiplying both the denominator and the numerator of the algebraic (or numeric) fraction by the unknown (or the number). It was further evident from her answers that this did not occur when the constant was the same as the denominator of the algebraic (or numeric) fraction since the numbers get cancelled out in simplification.

She seems to apply two different rules when simplifying algebraic and numeric fractions. Instead of applying the same rule to algebraic fractions in example 1, she may have done an incorrect cross multiplication. This is an uncertainty which may have occurred when there was no visible denominator. Anyway, she has a misconception regarding multiplying algebraic or numeric fractions. Therefore, this was hypothesized as a systematic error leading to a misconception which is worthwhile to be examined further.

Case 2 : Navin

Navin is a grade 11 student in the university/college mathematics stream. He showed a good understanding of variables and simplifying algebraic expressions. Navin has a misconception when he forms algebraic expressions from word sentences. He refuses to accept algebraic expressions as final answers in some occasions. The following examples illustrate this misconception.

1. Add 3 to $5y$.

$$0 = 5y + 3$$
$$= 5y + 3 - 3$$
$$\frac{-3}{5} = \frac{\cancel{5}y}{\cancel{5}}$$
$$\therefore y = \frac{-3}{5}$$

2. Subtract $2b$ from 7.

$$2b - 7 = 0$$
$$2b - \cancel{7} + \cancel{7} = 0$$
$$\frac{\cancel{2}b}{\cancel{2}} = \frac{7}{2}$$
$$b = \frac{7}{2}$$

3. Multiply $e + 2$ by 3.

$$(e+2)3 = 0$$
$$3e + 6 = 0$$
$$3e + \cancel{6} - \cancel{6} = 0$$
$$\frac{\cancel{3}e}{\cancel{3}} = \frac{-6}{3}$$

$$e = -2$$

In addition to the main misconception of treating algebraic expressions as equations, Navin made another error in solving the equations (third line of examples 1 and 2 and fourth line of example 3). Is this common for him in other situations where he gets an algebraic expression as the answer? His following answers suggest that it is not common to all situations.

Simplify where it is possible.

4. $2x+5y+9z$	5. $7+3x$	6. $p+p+2c+5p$
Not possible	Not possible	$= p+p+5p+2c$
		$= 7p+2c$

One of the possible hypotheses for Navin's misconception is that he used to have this misconception when the problem is given in a word format. Whatever the reason is, it is worthwhile to examine this situation.

Case 3: Jasmine

Jasmine is a grade 11 student in the university/college mathematics stream. She showed a poor understanding of almost all the concepts on the test. The following examples show her poor understanding of the simplification of algebraic expressions. Particularly, Jasmine has difficulties in distinguishing variables and constants. Her responses indicate this problem.

1. Antonio sells y donuts. Maria sells three times as many donuts as Antonio. A donut costs 25 cents.

a) Name a variable in this problem.
y = donuts

b) Name another variable in the problem.
m = money

c) Name something in the problem that is not a variable.
A = Antonio

2. What does $5y$ mean? Write your answer in words.
The amount of donuts Antonio sells

3. What does yz mean? Write your answer in words.

z is not stated

It seems that Jasmine has a problem of understanding the difference between variables and constants. There seems to be a difficulty for her to distinguish variables in different situations: variables as specific unknowns, variables as generalized numbers, and constants.

3.5 Pilot study – Phase 2

A perfectly reliable and valid test may contain items that have to be revised or eliminated. The test items for each of the four areas under study should be roughly balanced in terms of their weights. As an example, if there are many word problems, this would affect the students who are weak in reading. This situation then may not provide sufficient number of interviewees from that area. In addition, if there are new items that are considered to serve a special purpose, these items should be included. Also, wording of the questions should be changed if necessary. Considering these new ideas, I made some changes to the test in the second phase.

For example, item numbers 14, 17, and 19 (Appendix 2) were included to examine the differences between student solutions when the same problem is given in three different formats: symbolic, word format without a context, and word format with a context. The items that are revised or eliminated after the first trial and the reasons for their elimination or revision are given in table 3.

Table 3

Deleted or revised item numbers and the reasons

Item no. in the first trial	**Reason(s) for elimination or revision**	**Additions/changes in the second trial**
2	Similar concept was tested in item 1(b)	Deleted
5(a), 6, 7, 9, 13(a), 13(b), 13(c), 15	Facility value was greater than 0.8	Deleted
16	Facility value was greater than 0.8	Changed to item 9

14	Facility value was greater than 0.8	Changed to item 10
5(h)	Facility value was 0	Deleted
12	Expert opinion	Amended as item 8
17	Facility value was 0	Changed to item 12
18	Many students had used non-algebraic methods to solve the problem	Changed to item 13
21, 22	Facility value was below 0.2	Deleted

More word problems were added in the second trial with diversified objectives. There were five new additions to the second phase of the pilot study. These were items 11, 14, 17, 18, and 19. Item 11 was added to test students' understanding of variables further. Question 18 was a word problem to convert an everyday situation into a mathematical form and provide reasons for the answers. Items 14, 17, and 19 were included to compare whether students have difficulties in understanding the different structural forms of the same question. The amended version contained 19 items (Appendix 2).

3.5.1 Administration of the second trial

The second version of the test was administered to 30 Grade 11 students in the university/college stream in another urban secondary school. Since there were new items added, I decided to conduct a second trial to refine the test items further and to obtain a valid and reliable test instrument. Facility values were again calculated for each item and the same formula was used to calculate the reliability coefficient. The split-half reliability coefficient for the second trial was 0.78 and the reliability coefficient for the whole test using the above formula was 0.88.

3.5.2 Practice interviews

After administering the second trial, I selected two students randomly and conducted practice interviews with them. This exercise was for me to understand the right kind of questions to be asked and to decide on a suitable pace for interviewing students. These interviews were tape recorded. By listening to the interviews, I decided to make some adjustments to my questioning patterns. I decided to provide the students with more time to explain rather than me asking lengthy questions. Second, I understood that my pace was too quick and I should allow them to have more time to think and answer rather than hurriedly moving from one question to another.

3.5.3 Rubric construction

After the administration of the two trials of the test, I decided to categorize the errors and prepare a rubric of errors as a practice exercise so that it would give an idea of its structure and content. The incorrect answers in the “variables” category were grouped and each of them was given a name. One error was categorized into only one error group. Sometimes, there was more than one error in a single answer. Then, the major error was selected. Two secondary school teachers helped me in this process. Based on our own hypotheses, we compared the categories with each other to find similarities and dissimilarities or to arrive at consensus.

As a result of this comparison, we reduced the number of categories to a few describing students’ errors. One challenge was when one error type belonged to more than one category. When this happened, we included them into only one category through consensus. It was not possible to categorize all the errors due to their large numbers. However, all major types were taken into account with a description of them. This was conducted in three cycles looking for any new emerging categories every time. A detailed categorization for “variables” appears in Appendix 8.

3.6 The main study

After the first two administrations, the final version of the test was prepared with a total of 19 items (Appendix 3). Since the reliability of the test in the second trial was at a reasonable value, no major structural changes were made to the final version except for a few word adjustments. For example, in question 4(c), the word 'evaluate' was changed to 'expand'. The purpose of question 18 was to invite students to extend their thinking beyond the given situation and generalize the situation. Students were given instructions in the test to use algebraic methods to solve all the problems. The wording of problem 12 was changed from a 'true-false' item to a 'written response' item since this will give more information about student work and their thinking. Also, a slight adjustment was made to question 9, since its facility value was greater than 0.8 in the second trial. Table 4 shows the composition of items in each category.

Table 4
Composition of questions in different categories in the test

Category	**Item number**
Variables	1, 3, 9, 11
Expressions	4, 5, 6, 7, 8
Equations	15, 16; 14, 17, 19 (special formats)
Word problems	2, 10, 12, 13, 18

3.6.1 Administration of the final test

The final version of the test was administered to 30 students in an urban secondary school. They were in grade 11 mathematics college/university stream. The test was administered with the help of the grade 11 mathematics teacher. Later, the test papers were marked and categorized for errors. New emerging categories were always added to the existing error categories and some categories were combined and renamed whenever necessary.

3.6.2 Rubric construction

To analyze students' errors and misconceptions, I developed four rubrics containing error groups for each of the four conceptual areas. The creation of the rubrics was mainly drawn from the experience I had with the analysis of pilot data. Students' answers from the final test were classified into error categories. For this, students' answers were carefully examined and they were grouped into various error types as this was done in the pilot stage. One question was categorized into only one error group. However, I assembled the same error that appeared in different questions into one category with their percentages.

For each error category, I calculated the percentage of occurrence of a particular error in that category. For this, the number of students who made this error was divided by the total number of students who attempted the question. When the same error appeared in different questions, I calculated the percentages separately for each item. I used these percentages later to calculate the mean number of errors for each conceptual area. I used this mean number as one of my criteria to select students for interviews.

Later, some of these error groups were combined together to form broader groups when it was necessary. For example, the oversimplification category contains answers that were oversimplified in many different ways. There were some common errors that were sufficiently significant and they warranted special attention. The four rubrics were developed based on these identifiable error categories that formed different groups. For a reliability check, the classification was discussed with two secondary school teachers and necessary amendments were made when there were inconsistencies.

Apart from the rubric construction, there were two other components to the quantitative analysis: the mean percentage error responses for each conceptual area and the four highest percentages of errors for each conceptual area. The detailed calculation of the mean percentage of errors appears in Appendix 7. The four highest percentages for each conceptual area were chosen as a criterion for selecting students for interviews using the individual error percentages.

3.6.3 Student interviews

In my study, I used interviews to explore students' thinking processes. However, these were not clinical interviews. Instead, students were given a chance to elaborate their thinking such as in think aloud methods. Some prompts such as, "explain more", "go ahead" or prompting questions such as "how", "why" were asked whenever necessary. Asking further questions or providing examples for explanation were also used but this was limited to the cases that needed more elaboration. Therefore, the method was a mix of think aloud procedures with a lighter version of interview questions. Hence, I used the term "interviews" for this dual methodology in this study.

When selecting the six students for interviews, a number of criteria were used. As explained in the pilot study section, I mainly searched for students who made systematic and non-systematic errors. For example in the systematic category, I chose students who made conjoining errors on many answers. Also in this category, there were students who tried to convert an algebraic expression into an equation. Apart from the two main criteria, I decided to interview students as much as possible to represent the four highest categories of errors that I identified under each conceptual area. This will be discussed further in section 4.7. It was challenging to find students who fulfilled all the above criteria together. Therefore, I interviewed one student for more than one question.

During the interview process, participants were encouraged to explain what they were doing as they attempted to solve the problem. Corrective feedback was not provided during the process. However, some short intervening questions were asked during the process to understand their thinking more thoroughly. Each interview lasted between 20 to 30 minutes. The interviews were tape-recorded and later transcribed. I had my own notes during the process to obtain a written account of the participant's overt problem solving and other activities.

At the start of each interview, I explained to the participants about the objectives of the study and what was expected from them. The initial script was:

> I am going to ask you to work out some of the problems from the test again. I would like you to try and explain to me your thinking while you are working. What I am interested in is how you are going to arrive at your answers. Sometimes I may ask some short questions from you for further clarifications.

Further, I made clear that it was quite acceptable to make mistakes in the interviews, and what was more important for me than the answer was the thinking that underlies it. Further, I made them understand that the primary goal of this exercise was not to evaluate them and to offer them a mark. At the end of the session, I thanked them and their written work was collected.

In the analysis stage, I looked at students' written work and their answers to the test as a cartoon, making meaning of every line drawn and every pencil mark made. I listened to the interviews and looked at the written work simultaneously. Every pencil mark of the written work was, therefore, important for me to interpret the interviews accurately. Interview transcripts were considered as an episode. Student reasoning patterns were discovered with the help of semantic features in the script, such as ideas, argument chains, use of examples, or impasses while solving problems. During this process, I also examined whether certain hypotheses I made about student misconceptions were actually reasonable or not.

3.7 Schematic diagram of the main study

A schematic diagram was prepared to depict the connection between data collection methods and data sources in the two-stage main study (Figure 4). In the diagram, the one-way arrow from the quantitative phase to the qualitative phase indicates that the research process is sequential. However, in the data analysis process, I had to move back and forth between qualitative and quantitative data sources in order to clarify and/or verify information and obtain a complete understanding of the situation. The two-way arrow indicates this phenomenon.

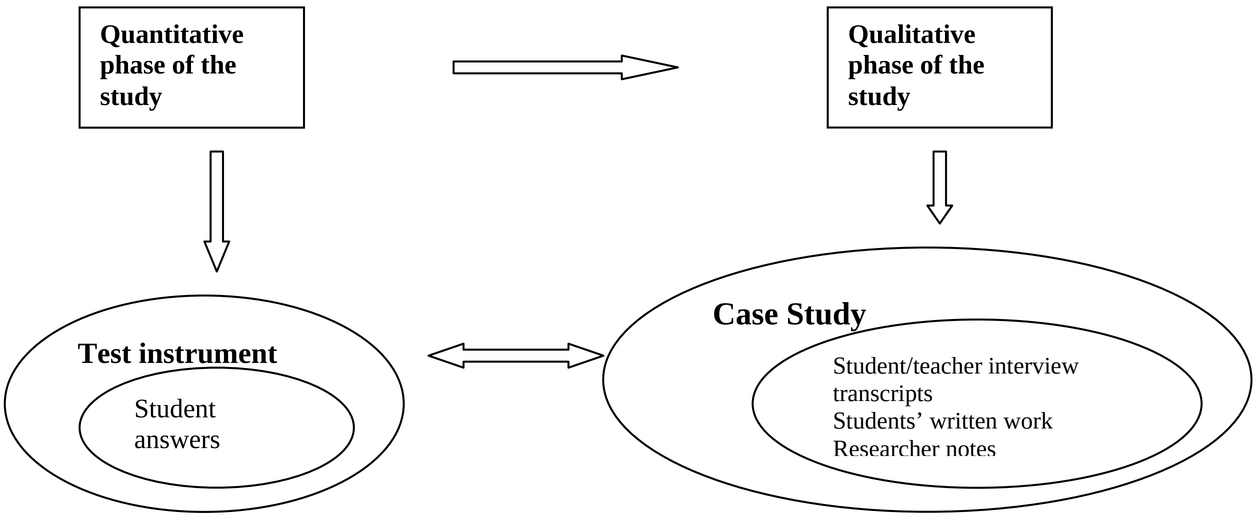

Figure 4: Schematic diagram representing the connections between the two stages of the study

3.8 Ethical issues

Prior to conducting the research, I obtained the approval from the University of Toronto Ethics Review Office. Approvals from the two school districts were also obtained to conduct research in their schools. Informed consent of principals and parents/guardians were obtained using relevant documentation (Appendices 5 and 6). These documents include informed invitation letters to the principals to conduct the research in their schools, informed invitation letters to students for their participation and consent forms to parents/guardians for their children's participation in the study. Only the students whose parents/guardians had granted permission were tested and interviewed. Participation was voluntary and participants had the right to withdraw from the study at any time.

During the reporting and discussion of data, none of the participants, schools, or communities were identified (pseudonyms were used) and participants were not judged or evaluated on their participation or non-participation. All the data that was collected had the names removed prior to analysis and reporting. By introducing myself to the students prior to the test and the interviews, I assumed that they would feel more comfortable during the interviews by knowing that they could communicate freely with me. In the debriefing, I told them that if they felt uncomfortable at any stage of the interview, they had the right to withdraw.

3.9 Summary

This research used mixed methods as the overall design and case studies as the main method in the qualitative phase. It attempted to expose Grade 11 students' errors and misconceptions in algebra. A sequential explanatory design was used and this is characterized by the collection and analysis of quantitative data followed by the collection and analysis of qualitative data. The main research instrument in the quantitative phase was a test instrument while interviews served the purpose of the main research instrument in the qualitative phase. The main study was conducted after two pilot trials. Students' answers to the test, their written work during the protocols, their interview transcripts, and researcher notes were simultaneously used as multiple data sources to arrive at valid conclusions about student misconceptions.

Chapter 4

Findings

4.1 Introduction

In this chapter, I will explain the main findings pertaining to both stages of the mixed method design. First, the detailed data analysis of the quantitative phase will be discussed with an emphasis on the construction of the rubrics for each conceptual area under the study: variables, expressions, equations, and word problems. Second, the detailed qualitative analysis will be discussed with an emphasis on student interviews.

I used the methods elucidated by Johnson and Onwuegbuzie (2004) for mixed method research data analysis. This prescription contained three stages in the analysis process: discovery of patterns (induction), testing of patterns and hypotheses (deduction), and uncovering the best set of explanations for constructing meaning related to findings (abduction). My focus was mainly on students' conceptions, procedures, errors, misconceptions, and their reasoning processes. Since the goal of this study was to identify students' misconceptions underlying their errors, I justified, whenever necessary, how students' wrong responses expose their misconceptions.

4.2 Mean percentage errors for each category

As a first step, mean percentage errors for each conceptual area were calculated. There were two steps to this process. First, the percentage number of error responses for each question under each conceptual area was calculated. Second, the overall mean percentage for that conceptual area was obtained by calculating the average of the mean percentages (Appendix 7). The bar chart (Figure 5) represents this information graphically.

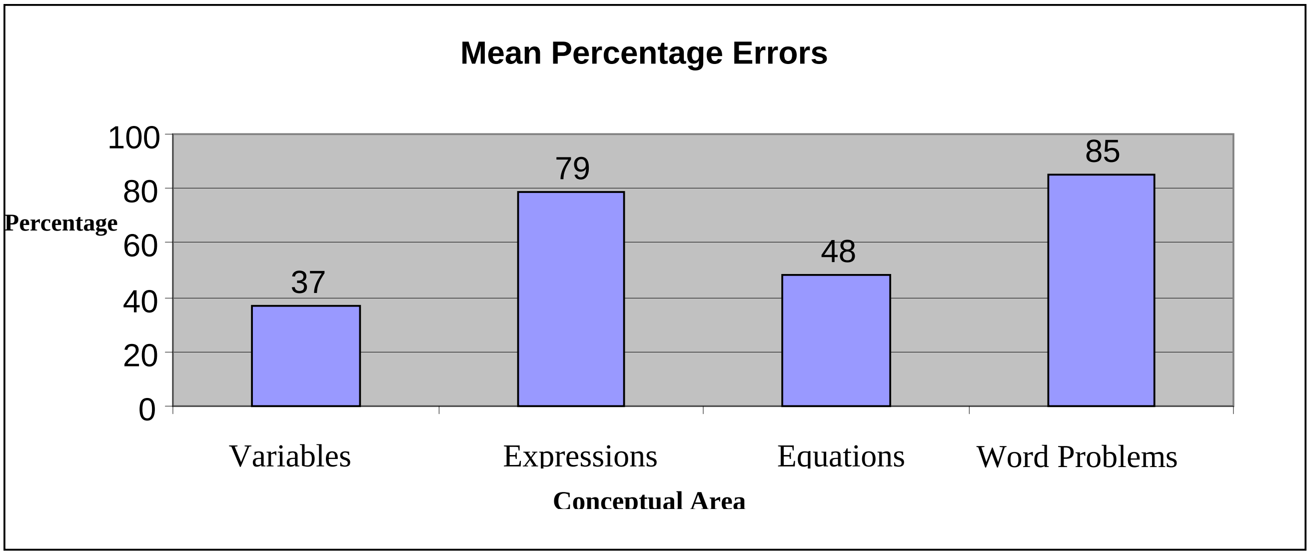

Figure 5: Mean percentage errors for each category

Word problems and expressions had the highest percentage of errors followed by equations and variables. Solving word problems could be difficult for students because there were many steps involved in the solving process. These steps include but are not limited to reading the question given in English language, understanding it, formulating a method, and solving it using an algorithm or any other method.

Many questions under "expressions" were abstract in the sense that there was not much context attached to them. The problems were in symbolic forms and the most challenging part for students was to find the correct method of solution or algorithm. Students had to choose the correct method from a wide range of possible strategies which include but are not limited to determining common denominators, factoring, expansions, building up expressions, simplifications, and comparisons. Many of the incomplete answers that were observed in students' responses bear evidence that they could not select the correct strategy or the strategy that they selected was inadequate to solve the problem. Some students obtained the correct answers, but they oversimplified them to reach incorrect final answers. In a way, this shows their lack of confidence of the solving process leaving them unaware of when to end the solving process.

Equations were relatively easier for the students compared to variables and expressions. The questions on equations were mainly on solving simple or linear systems of equations in which students had to use a particular algorithm. They had to use elimination, substitution, or working backward methods that have a prescribed procedure. One exception was question 15 where they had to decide the answers with or without carrying out the algorithm. The questions under “variables” were easier and solving many of them did not need to follow algorithms. Some of these problems did not require deeper thinking strategies. The questions were mostly about students’ knowledge of basic definitions. The low incorrect response rate in this area indicated that these questions were relatively easier than the other areas.

When students misconstrue or misuse the standard mathematical practices, then we say that an error has occurred. At this stage, we only can hypothesize whether this error had really occurred due to a robust misconception or it simply was a momentarily lapse of concentration on the part of the student. Sometimes, errors seem to point towards having all the features of a misconception. However, this is a hypothesis which can be tested by listening to students. For example, some errors disappeared when students were asked to work out the problems again in the interviews. Therefore, it could be assumed that these errors may have occurred due to a simple lapse of concentration, forgetfulness, or any other reason other than due to a deeply-rooted misconception. In the next sections, I will explain students’ errors under each conceptual area.

4.3 Variables

For "variables", students' answers for questions 1, 3, 9, and 11 were grouped into a number of tables before constructing the rubric (Appendix 8). The next step was to combine all incorrect responses and construct coherent error groups or possible misconception groups with their individual percentages (rubric of errors or possible misconceptions). For this, 'correct answers' and 'no answers' were eliminated. There were different forms of incorrect answers. Sometimes, there were no visible reasons for some of the incorrect responses which were classified into a separate group. Finally, the error groups were carefully examined again to combine similar groups together or separate different groups. The percentages for each error type were calculated based on the number of students who answered the question (Table 5).

Table 5
Rubric of errors or possible misconceptions for variables

Type of error or possible misconception	Students' answers and their percentages
Assigning labels, arbitrary values, or verbs for variables	**Q.1**: 3 times **(8%)**, 3**(8%)**, cost 25 cents **(8%)**, cents **(4%)**, 25**(4%)**, donuts **(4%); Q.11**: It represents buying 3 shirts and 2 pants, $3s$ represents 3 shirts and $2p$ represents 2 pants **(3%),** 3 shirts plus 2 pairs of pants as well as 3 dollars plus 2 dollars equals 5 dollars **(3%)**
Assigning labels for constants	**Q.1**: Donuts **(11%)**, Antonio **(7%)**, cents **(4%)**, Maria or Antonio **(4%)**
Misinterpreting the product of two variables	**Q.3**: yz means a variable **(11%)**, yz means the variable represents part of the question **(6%),** The number of people per row **(6%),** A variable to represent something and another variable to change the value of the first variable **(6%)**
Misjudging the magnitudes of variables	**Q.9**: t has a bigger value beside it **(11%)**, t is larger because you can multiply by 2 **(4%),** None is larger **(4%)**
Lack of understanding of variables as generalized numbers	**Q.9**: I don't know, they are both variables **(4%)**
Lack of understanding of the unitary concept when dealing with variables	**Q.11**: It means you bought 3 shirts for s dollars and 2 pairs of pants for p dollars **(3%)**
Forming incorrect equations as	**Q.1**: $3y + y = 25$ **(4%)**, $y \times 3y = 25$ **(4%)**

answers when they are not necessary	

4.3.1 Assigning labels, arbitrary values, or verbs for variables and constants

Some students misinterpreted a variable as a ‘label’, as a ‘thing’, or even as a verb such as ‘buying’. They really did not perceive the correct interpretation of the variable as the ‘number of a thing’. It was difficult for them to distinguish between variables and non-variables in terms of the varying and non-varying quantities in the question. Often, they were confused with viewing variables as constants or vice versa. This error type was observed in other questions too. It is noticeable that, when students were asked to name something in the problem that is not a variable (Question 1), the answers such as ‘Antonio’, ‘cents’, ‘donuts’ were given. In a general sense, these answers may be considered as correct. Sometimes, the words ‘donuts’ and ‘cents’ could be considered as symbols representing variables in some contexts. However, these answers were considered as incorrect in the context of the given problem since there was a variable or a number attached to these words. Therefore, these words have meanings in the given context when they were taken together with those variables or numbers.

4.3.2 Misinterpreting the product of two variables

Students who made the above error had difficulties to perceive the product of two variables as two separate variables combined together by a sign. They viewed the product as one variable. The highest percentage for this category was 11%. Sometimes, this misconception led the students to assign a constant for the product. For example, 6% of the students misinterpreted the product as a constant by connecting it to the previous question (Question 2). One such answer was ‘the number of girls per row’. Another 6% of the students misinterpreted the product of two variables as the second variable to change the value of the first variable. This indicates that these students could perceive the product as two separate variables, but they incorrectly perceived an interaction between the two variables. This is a typical property of some numeral systems such as the ancient Roman numeral system but it is not a property of algebraic variables.

4.3.3 Misjudging the magnitudes of variables and lack of understanding of variables as generalized numbers

Some students judged the magnitude of two variables by examining their coefficients when they are in an equation such as $y = 2t + 3$. Since t has a larger value beside it, they thought that $2t$ is larger than y in the equation. This comparison is correct when comparing two like terms such as $2t$ and t but it is inapplicable when comparing unlike terms and also when they are related to each other in an equation with different coefficients. The highest percentage in this category was 11%.

Not realizing that variables take many values in some contexts was another problem for some students. In an equation such as $y = 2t + 3$, these students recognized that both y and t are variables. However, they did not realize that these variables can take more than one value. Further, they focused only on the domain of positive numbers when substituting values for y and t to examine which is larger. For this question, it is necessary to substitute numbers from the negative number domain as well, and this will lead to a different result. Another misconception that I found here was that some students considered y as the answer obtained by doing the operations to the right hand side of the equation. In other words, they perceived the equal sign as “to do something” to the right hand side of the equation to get the answer on the left hand side. This misconception will be further discussed under equations.

4.3.4 Lack of understanding of the unitary concept when dealing with variables

Another possible misconception for some students was their difficulties in understanding the unitary concept when multiplying a variable with a constant. When the price of a shirt is s dollars and when they have to find out the price of 3 shirts, they should understand that the unit price s has to be multiplied by 3. This is a basic arithmetic concept. The only difference in this question is that the price was given as a variable. They interpreted the term '$3s$' as '3 shirts for s dollars'. Here again, it is evident that, in addition to the incorrect calculation, they considered s as the label for 'shirts', rather than the unit price of a shirt and at the same time considered s as the item price.

4.3.5 Forming incorrect equations as answers when they are not necessary

In question 1, students were asked to name a variable and a non-variable. One student's answers were both in the form of equations ($3y + y = 25$, $y \times 3y = 25$). There was no meaning attached to these equations, and they indicate a false relationship between the variables and constants in the problem. This student may have assumed that a hypothetical relationship exists between the variables. It is difficult to predict any theoretical attachment of the answer with the question. Although this is only one student, I decided to interview him to see whether this misconception has a connection with his other misconceptions.

4.4 Algebraic expressions

In this study, algebraic expressions had the longest list of student errors. Before constructing the rubric, students' answers to questions 4, 5, 6, 7, and 8 were separately categorized into error groups with their frequencies (Appendix 9). After many regroupings, the errors were then classified into nine major groups with their severity (Table 6).

Table 6

Rubric of errors or possible misconceptions for algebraic expressions

Type of error or possible misconception	Students' answers and their percentages
Incomplete simplification	**Q.4a**: $\frac{1A}{A}$ **(11%)**, $\frac{A}{A}$ **(22%)**, $\frac{A1}{1A}$ **(6%); Q.4b**: $0A$ **(4%);** **Q.4c**: $(A+B)(A+B)$ **(8%); Q.5b**: $x(a \div b)$ **(6%);** **Q.7**: $3(e+2)$ **(12%)**
Incorrect cross multiplication	**Q.4a**: $\frac{1}{A^2}$ **(11%)**, $\frac{1A}{A^2}$ **(6%); Q.5a**: $r-\frac{4(6-s)}{2}$ **(10%); Q.5b**: $\frac{ax}{bx}$ **(50%)**, $\frac{a}{xb}$ **(6%)**
Converting algebraic expressions as answers into equations	**Q.4a**: $A \times A \times 1 = A$ **(6%)**, $A = 0$ **(4%); Q.5b**: $xa = b$ **(6%);** **Q.6**: $b = 3.5$ **(8%)**, $b = 5$ **(4%); Q.7**: $e = -6$ **(8%)**
Miscellaneous forms of incorrect answers	**Q.4a**: $A \times A \times 1$ **(6%)**, $\frac{1}{2A}$ **(6%)**, $A \times A + 1$ **(6%)**, $1A$ **(6%)**, A **(4%)**, A^0 **(4%); Q.5a**: $\frac{r}{4}-\frac{(6-s)}{4}$ **(10%)**, $sr-12$ **(10%);** **Q.5c**: $xa+xb$ **(11%); Q.5d**: $\frac{A}{BC}$ **(20%)**, $\frac{A}{C}+\frac{A}{C}$ **(7%); Q.6**: You cannot subtract **(4%)**, 7**(4%)**
Oversimplification	**Q.5a**: $\frac{r}{4}-\frac{6s}{2}$ **(10%)**, $\frac{r}{4}(6-s)$ **(10%); Q.5c**: $\frac{x^2ab}{x^2d}$ **(41%)**, $\frac{a+b}{d}$ **(12%)**, $\frac{x^2ab}{x^2d}=\frac{ab}{d}$ **(6%)**, $2x^2abd$ **(6%)**, $\frac{xab}{xd}$ **(6%)**, $\frac{2xab}{2xd}$ **(6%); Q.5d**: $\frac{A^2}{BC}$ **(27%)**, $\frac{2A}{BC}$ **(20%); Q.7**: $6e$ **(8%)**, $3\times 2(e)$ **(12%)**, $2e \times 3 = 5e$ **(4%),** $e + 2 \times 3 = 2e \times 3 = 6e$ **(8%);** **Q.6**: $7-2b=5$ **(4%)**, $7-2b=5b$ **(13%)**, $-5b$ **(8%)**, $5b$ **(8%)**
Invalid distribution	**Q.4c**: A^2+B^2 **(20%)**, $A^2+A^2B^2+B^2$ **(4%)**, $A^2+B^2+AB^2$ **(4%)**, $A^2+B^2+AB^2$ **(4%)**, A^2+B^2+AB **(4%),** A^2B^2 **(16%)**, $(AB)^2$ **(4%)**, AB^2 **(4%);** **Q.5a**: $\frac{r-12-2s}{4}$ **(10%)**, $0.25r-3-s$ **(10%); Q.7**: $(e+2)3=3e+2$ **(4%)**, $e+2(3)=e+6$ **(4%)**, $(e+2)(e+2)(e+2)=e^3+2^3=e^3+8$ **(4%)**, $3\times e+2$ **(4%)**, $e+6$ **(8%)**, $(e+2)^2=e^2+4$ **(4%)**
Incorrect common denominator	**Q.5a**: $\frac{r-6-s}{2}$ **(10%)**, $\frac{r-(6-s)}{2}$ **(10%);**

	Q.5d: $\frac{A^2}{B+C}$ **(7%)**, $\frac{2A}{B+C}$ **(7%)**
Reversal error	**Q.6**: $2b-7$ **(17%)**
Incorrect quantitative comparisons	**Q.8**: $\frac{1}{n+1} > \frac{1}{n}$ **(48%)**

4.4.1 Incomplete simplification

An answer was categorized as incomplete when some students terminated the simplification of the algebraic expression somewhere in the middle of the process without reaching the final answer. In the students' point of view, these answers are final but they are incomplete when compared to standard algebraic procedures. Another possibility is that these students probably may not know how to proceed further. Some of them wrote the problem again in another form as the answer or they terminated the procedure abruptly without completion.

4.4.2 Incorrect cross multiplication

Some invalid cross multiplications were observed during the categorization of errors for algebraic expressions. When these students multiplied an algebraic fraction with a letter [$x\left(\frac{a}{b}\right)$], they often multiplied both the denominator and the numerator of the fraction by that letter ($\frac{ax}{bx}$). Sometimes, they may have assumed that there is no denominator to the letter. Often this happens to students when there is no visible denominator. They seem to have difficulties in realizing that a single letter can be represented by an algebraic fraction by making the denominator as 1. Because of this lack of understanding, students tend to assume that both the denominator and the numerator of the fraction should be multiplied by the letter.

4.4.3 Converting algebraic expressions in answers into equations

In this category, some students formed invalid equations from the answers in the form of algebraic expressions. These students proceeded further to solve these equations. There were two varieties to this error. First, when simplifying algebraic expressions, students connected the variables in the problem in a meaningless way to form an equation. Second, they were reluctant to accept an algebraic expression as the final answer and came up with a solution by solving the forged algebraic equation. Connected to this error, I found one student previously who also formed a fake relationship with the letters under “variables”.

4.4.4 Oversimplification

This was the largest category of errors. One of the interesting features in this group was that the students conjoin, connect, or even put together the terms without even considering the operations that are to be carried out on these terms. Addition, subtraction, division, and multiplication signs were left out to form a single bundle of strings. These students disregarded the fractional forms of expressions and reduced them into one letter or an array of letters.

4.4.5 Invalid distribution

Invalid distribution is a kind of misuse of the distributive property in algebra. The distributive property states that $a(b+c)=ab+ac$. This implies that we can either do the addition first, and then multiply, or multiply first and then add. It makes no difference. However, when unlike terms are inside the brackets, it is impossible to add them. Students have to multiply the brackets by the letter outside of the parenthesis. Actually, the distributive property helps us to simplify algebraic quantities by allowing us to replace terms containing parenthesis with equivalent terms without the parenthesis anymore.

Under ‘invalid distribution’, I found many forms of incorrect use of this property. The most common form occurred when raising a binomial to a power. Students mistakenly distributed exponentiation over addition as $(A+B)^2 = A^2 + B^2$ and they even proceeded further to oversimplify the answers [A^2B^2, $(AB)^2$ etc.].

Another subcategory of misusing the distributive property is 'incomplete distribution'. Sometimes, students began to apply the distributive property correctly, but failed to complete the process leaving incorrect answers [$(e+2)3 = 3e+2$, $e+2(3) = e+6$]. Incomplete distribution also occurred when there is a minus sign in front of the parenthesis.

[$\frac{r-2(6-s)}{4} = \frac{r-12-2s}{4}$ **(10%)**, $0.25r-3-s$ **(10%)**].

4.4.6 Incorrect common denominator

Two different error types were detected in this group. They were incorrect calculation of the common denominator for two numbers or two letters. When calculating the common denominator of two numbers, some students incorrectly chose the smaller number as the common denominator. This left the rest of the procedure incorrect. On the other hand, when the fractions were algebraic, students considered the sum of their denominators as the common denominator instead of taking their product.

4.4.7 Reversal error

Incorrect word order matching led to a reversal error when forming algebraic expressions from a word sentence. When the subtrahend was an algebraic term and the minuend was a number in a word sentence, students carried out the operation in the reverse order by exactly matching the letters in the given word order. This error was observed in word problems as well and it will be discussed further later.

4.4.8 Incorrect quantitative comparisons

A high percentage (48%) of the answers to question 8 was attributed to an incorrect quantitative comparison to compare two algebraic fractions. These students substituted numbers to the algebraic expressions in order to compare them. After the substitution, they only compared the magnitudes of the denominators instead of comparing the whole fractions thereby arriving at faulty conclusions. They did not realize that the reciprocal of a number is smaller than the number itself under certain conditions. Some others incorrectly separated the fraction $\frac{1}{n+1}$ as $\frac{1}{n}+1$. I asked students to provide reasons for their answers in the test. Some of them wrote:

> $\frac{1}{n+1}$ is more because if $n=20$ then $\frac{1}{n}=\frac{1}{20}$ and $\frac{1}{n+1}=\frac{1}{20+1}$. $\therefore$ no matter $\frac{1}{n+1}$ will always be +1 more than $\frac{1}{n}$.
>
> $\frac{1}{n+1}$ is more because whichever number n is, it will be one number higher because it's adding 1 to it.
>
> $\frac{1}{n+1}$ because you add one to the number n represents.

4.4.9 Miscellaneous forms of incorrect answers

The real reasons for the errors in this category are not very obvious. Students somehow manipulated the symbols to form an answer. It was hard to guess the reasons behind these incorrect answers as each answer was different from another. Some of the answers were laid out without apparent justifications. It was obvious that students in this category may have used their own wrong rules. All that can be said is that each answer is very unique and student choices of the selection of their methods are very personal.

In general, it can be observed that these students executed mathematical procedures without properly understanding them. It seems that they did not have an understanding of the structural features of algebra to choose a correct method. The manipulation of symbols was not in accordance with accepted rules so that referring to the answer was no longer a way of understanding their reasoning. Students may have used their own wrong rules that were persistently fixed in their mind, or they may have used ad-hoc rules depending on the problem.

4.5 Algebraic equations

There were five questions in the test for algebraic equations involving building up and/or solving equations. Question 15 was about solving a system of linear equations. Questions 14, 17, and 19 were the same problem with three different formats: algebraic format, word format without a day-to-day context, and a word format with a day-to-day context. It is important to mention that some error types appeared more than once in the same question and in different questions. For example, the error type "Add when the equations have to be subtracted or vice versa" appeared in questions 15(a), 15(b), 15(c), and 16. The grouping of answers appears in Appendix 10. The final categorization of seven error types and their percentages appear below (Table 7).

Table 7
Rubric of errors or possible misconceptions for algebraic equations

Type of possible misconception	**Sample answers and percentages**
Numbers as labels	**Q.14**: $4x+25=73$, when $x=8$, 48+25=73 **(3%)**
Misinterpreting the elimination method in equation solving	**Q.15**: Add when the equations have to be subtracted or vice versa **(48%),** You don't get the same solution when you add or subtract the equations **(76%)**
Wrong operations in the substitution method	**Q.16**: $2m+n-2=3m-2n-3$ **(22%)**
Oversimplification	**Q.16**: $n=2-2m$, $\frac{n}{0}=\frac{0m}{0}$ **(11%)** ; $3-2n=3$, $n=3$ **(11%)**
Misuse of the "change-side, change-sign" rule	**Q.16**: $-7n=0$, $n=7$ **(11%)**
Interference from previously learned methods	**Q.16**: Use solving methods for quadratic equations to solve linear systems **(11%)**
Misreading the problem	**Q.19**: $\frac{48}{3}=x$, $x=16$ **(7%)**, $\frac{73}{4}=\$18.25$ **(12%),** $73-25=\frac{48}{3}=\$16$ **(15%)**

4.5.1 Numbers as labels

One student made this error and it was a different form of the same error discussed under 'variables as labels'. This student used a number as a label to replace or substitute a variable. Solving for x in $4x+25=73$, this student wrote $x=8$ by pasting the number 8 into the position of x to get 48. This student had understood the property of equivalence as he pasted the correct number to make the equivalence work, although he did not follow the normal equation solving procedures. This error may have occurred due to students' previous knowledge of number equations where students had to insert a number to satisfy a numeric equation. Similarly, this student may have used the number as a label for a letter to satisfy the equation numerically.

4.5.2 Misinterpreting the elimination method in equation solving

When eliminating a variable from a system of linear equations, some students misjudged the operations to be performed. Some of them chose the reverse operation, for example, adding when it had to be subtracted or vice versa. Probably, this misunderstanding came from their fragile understanding of simplifying integers and manipulating signs. Their difficulties were aggravated when the variables in the two equations had opposite signs $(+b,-b)$. The data showed that 87% of the students answered the problem correctly when these signs were the same, while 52% answered correctly when the signs were opposite. In addition, 26% said that both adding and subtracting have to be performed to eliminate a variable, while 4% of the students said that the exact operation depends on the equation.

A large number of the students (76%) did not seem to be comfortable in explaining what is meant by a solution of a linear system. In the test, students were asked to provide explanations for their decisions. These reasons show that many of them did not have a proper understanding of the meaning of a solution. They were asked, “Will you obtain the same solution if you add or subtract the two equations? Explain”. Some of the answers were:

> No, because you have two different numbers (referring to the constants on the right hand side of the equation).
> No, one is positive and one is negative.
> No, you will not because the signs of b are different.
> No, because by adding you get rid of b and subtracting you get rid of a.

These answers show that some of these students thought that the constants (5 and 7) would change the solution while others thought that the signs of the variable in the two equations would have an influence on the solution. These students seem to have a poor understanding of the solution a system of two linear equations. Interestingly, the solution of an equation also appears to be another equation. Probably, some of the above students may have thought that the solution is the two different equations that they get as answers.

4.5.3 Wrong operations in the substitution method

In this study, students used two methods to solve a linear system: the substitution method and the elimination method. In the substitution method, students had to isolate a variable from one equation and substitute its value in the second equation. Frequently, they isolated the same variable from both equations and equalized them. However, 22% of the students made the right hand side of both equations as zero and equalized them. This method will not work unless one of the variables has the same coefficient and the same sign in both equations. Otherwise, they will get a single equation with two variables which is insolvable or it will have infinite solutions. This is a wrong application of the substitution method.

One of the common deficiencies identified in equation solving was that some students had difficulties in applying short-cut methods to solve problems. They used longer methods that were not actually required. For example, the following student used a lengthy substitution method to solve the linear system.

Question: $x+y=4$; $y=2x+4$ (Pilot test - stage 1, question 21)

$x+y=4$	$y=2x+4$
$y=4-x$	$2x=-4+y$
$x=4-y$	$x=\frac{-4+y}{2}$

The student correctly isolated x's from both equations to equalize them. For some reason, she did not proceed further with the solution and this was considered as an incomplete answer. One of the objectives of this question was to identify that y is already being isolated in equation 2, and it can be substituted in the first equation right away to find x which is easier than the above method. Also, the student did not have to solve complex equations such as the above by using this method. However, a majority of students did not identify or use this short-cut method.

4.5.4 Oversimplification

The above error was observed under 'equations' when students used to oversimplify algebraic terms in an illegal manner. They operated directly on numbers separating them from adjacent items. This separation sometimes led to bizarre situations where the answers were in undefined forms ($n=2-2m$, $\frac{n}{0}=\frac{0m}{0}$). These errors usually occurred during the last steps of the equation solving process. One of the possible explanations for this error was students' lack of understanding of the closure property. If they knew that numbers from two different systems (2 and $2m$) cannot be subtracted to get another number in either system, they would have not committed this error.

4.5.5 Misuse of the "change- side, change-sign" rule

This misconception was observed in the last steps of the equation solving process. Some students carried over the terms to the other side of the equation without properly changing the signs or without executing proper operations ($-7n = 0$, $n = 7$). This error may have happened because the student has assumed that there is a plus sign in between the coefficient and the letter. In other words, instead of the multiplication sign in between the number and the letter, the student may have assumed a plus sign. Because of this incorrect assumption, he carried over the number to the other side of the equation.

4.5.6 Interference from previously learned methods

This specific misconception originated from one student's answer and he mistakenly chose a previously learned method which is not applicable for solving linear equations. He used a method for solving quadratic equations to solve linear equations. The detailed solution was:

$$\begin{aligned} &2m + n = 2 \\ &2m + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + n = 2 \\ &(2m+1) - 1 + n = 2 \\ &(2m+1) + n = 2 + 1 \\ &(2m+1) + n = 3 \end{aligned}$$

$$\begin{aligned} &3m - 2n = 3 \\ &3m + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 2n = 3 \\ &(3m+2.25) - 2.25 - 2n = 3 \\ &(3m+2.25) - 2n = 3 + 2.25 \\ &(3m+2.25) - 2n = 5.25 \end{aligned}$$

Interestingly, this student did not realize that he would end up with the original equation when he uses this algorithm to solve linear equations. The method only works for finding solutions to quadratic equations. It is not useful to use this algorithm for linear equations since this will furnish an alternate form of the same equation rather than a way of solving it.

4.5.7 Misreading the problem

This is an error that occurs when students have difficulties in comprehending a word problem. Usually, students in Grade 11 can read the questions correctly. Therefore, I assumed the above error as emanating from a comprehending difficulty. In this error, students perceived the number of friends as 3 (Question 19) instead of 4. Some of the other students left out the balance money ($25) and divided the cost of the basketball by 4. They may have misread the question or may have misinterpreted it.

4.6 Word problems

In the past, many empirical studies indicated that students face difficulties in translating algebra word problems that state relationships between two or more variables into a symbolic form. In my study, there were five word problems which consisted of mainly word sentences. Students had to read the problems, convert them into algebraic forms, and solve them. Some of these problems contained relational proportions (Question 12). In some questions, students had to provide reasons for their answers. Among others, there are two main processes involved in solving a word problem. One is the translation process, which is to read and translate the words of the problem into an algebraic representation. The solution process is to apply legal rules of algebra in order to arrive at a solution.

Several types of errors were seen from the careful analysis of answers (Appendix 11) and they were categorized into six major error groups (Table 8). One observation was that a considerable number of students used arithmetic methods rather than algebraic methods to solve word problems. For example, 'working backward' and 'trial and error' methods were prominent.

Table 8
Rubric of errors or possible misconceptions for algebraic word problems

Type of possible misconception	**Sample answers and percentages**
Reversal error	**Q.2**: $n = \frac{8}{r}$ **(4%); Q.12**: $4B = 5R$ **(48%)**
Miscellaneous forms of incorrect answers	**Q.2**: $8n$ **(22%)**, $x = 8n$ **(4%)**, $y = 8n$ **(4%)**, $8n + n$ **(4%)**, $y = nx + 8$ **(4%)**, $y = n + 8$ **(4%)**, $n + 8$ **(4%)**, 8 **(4%)**, $n = 8$ **(4%)**, n^8 **(4%)**, $8 \times n = 0$ **(4%)**, $x = n8$ **(4%)**
Guessing without reasoning	**Q. 10:** Answers such as 11 years, 26 to 27 years, 68 years, 26 years**,** Never, 156 years **(overall 71%); Q.13:**

	37 stamps to each child* **(57%)**, **Q. 18:** Yes (no reasoning given) **(41%)**
Forming additive or multiplicative totals from proportional relationships	**Q. 12:** $Total = 4B + 5R$ **(28%)**, $Total = (4B)(5R)$ **(8%)**, $B = R + 1$ **(8%)**, $R = B + 1$ **(4%)**
Difficulties in grasping the relationship between two or three varying quantities	**Q. 10:** $40 - 14 = 26$, $41 - 15 =$ (not continued)* **(5%);** **Q.13:** $T = 5J$ (not continued)* **(7%)**
Incorrect reasoning	**Q. 18:** Yes, it will because the price is a flat rate and probably won't change* **(65%)**

* This is only one answer. There were other incorrect answers in this group. The percentage given is the overall percentage.

4.6.1 Reversal error

Two different forms of reversal errors were observed in the answers to questions 2 and 12. In question 2, students were asked to write an algebraic expression for the number of rows in the parade and the correct answer was $\frac{n}{8}$. The answer was considered as a reversal error when it was $\frac{8}{n}$. If students could not understand and use 'n' as representing 'the number of girl scouts', it is difficult for them to write a correct algebraic expression representing the 'the number of rows'. Further, the problem could be difficult for them because the dividend is a variable, not a number. Another possible cognitive obstacle is that students are more familiar with multiplying a variable with a given number but dividing may not that easy for them. In other words, it could be relatively easy for them to calculate the total number of girls when the number of rows is given as a variable and the number of girls in each row is given as a number.

For question 12, only 4% of the students perceived the given relationship as a relational proportion although they did not form the correct equation. Again, the most common error was the reversal error: $4B = 5R$ (48%). The majority of students (84%) used the equal sign to denote equality without considering the proportional relationship of the variables. Some of them used the letters as labels instead of a varying quantity (B for blue cars and R for red cars).

The majority of the students did not match the correct symbols with the words. Instead, they considered symbols as labels and formed the equation by mapping the sequence of words directly into the corresponding sequence of literal symbols. It was difficult for them to understand the underlying notion of mutual variation of the two quantities. This led to a wrong syntactical representation of the variables.

4.6.2 Guessing without reasoning

Errors that resulted when students apparently solved a problem by guessing-that is, when there was no overt evidence that the stated information was the result of a mathematical operation. We note that it is possible the student may have performed a mental operation; hence, we call these unsubstantiated outcomes rather than guessing.

Guessing is a common phenomenon when students answer mathematical problems. Frequently, there are some acceptable reasons behind guessing although the answers are incorrect. Sometimes, it is possible that the student may have performed a mental operation. However, when there was no overt evidence that the stated answer was the result of a mathematical operation, then this answer was considered as a guess.

In this study, there were instances where students did not make educated guesses. For example, in question 10, students provided answers such as 68 years or 156 years for the son to become half of the father's age. Another answer was that the father will never reach double the son's age. These answers were given without any methodological justifications or explanations. Since there were no explanations for the answers, I assumed that they were mere guesses.

Further, these students did not verify the realness of their answers. They did not use metacognitive abilities such as verification or looking back.

4.6.3 Forming additive or multiplicative totals from proportional relationships

In this cluster of errors, students attempted to connect the two variables in an equation as an additive total (Question 12). In this question, $B : R$ represents the ratio of blue cars to red cars. Students misinterpreted these ratios as the actual number of cars produced in the factory and built up equations to represent incorrect totals [$Total = 4B + 5R$, $Total = (4B)(5R)$]. Not only for the total number of cars, but they built up equations for other relationships as well. ($B = R + 1$, $R = B + 1$). This misuse may have occurred due to poor understanding of the relational proportion of the two quantities involved in the problem.

4.6.4 Difficulties in grasping the relationship between two or three varying quantities

In questions 10 and 13, students were expected to understand the relationships among the variables, form equation(s), and solve them. Many of the answers indicated that students used arithmetic methods, working backwards, or guessing to find solutions rather than algebraic methods. Only 5% (Question 10) and 14% (Question 13) of the students used algebraic methods to solve the problems. However, the correct use of the algebraic method was seen in only 0% and 7% of the answers for the two problems respectively. This shows that students may not have used the algebraic methods or they may have difficulties in applying algebraic methods to solve word problems.

There were marvelous ‘working backwards’ methods in answers to question 10. Unlike question 10, there are no simple ways to use arithmetic methods in question 13 other than guessing or using trial and error. One noticeable feature in the answers was that students especially had difficulties in comprehending the relationship among three varying quantities.

Since the total number of stamps were not given, it was hard for them to formulate equations especially when one variable is varying with respect to another variable.

4.6.5 Incorrect reasoning

Question 18 was a word problem with a familiar context which is very similar to the current cellular telephone plans. In order to solve the problem, students have to think beyond the given data. When they use algebraic methods, they have to construct an equation from the given data and prove that the rule does not always work by explaining the relation between variables. There were no students who used this method.

Some students thought beyond the given data to produce a counter example and to illustrate that the rule does not always work. Some of them (14%) used only the given data to arrive at incomplete or wrong conclusions. In this problem, the rule works for 100 and 300 minutes. However, this does not mean that it always works as it does not work beyond 300 minutes. The following quotes show some explanations given by the students.

> Yes, because he has a flat rate of $10 month. So if he continues to talk 3 times as long (300 mins.), then it will only cost him 2 times as much.
>
> Kevin’s rule will always work because he has a flat rate of $10 that he has to pay each month, and as long as long distance cost $10 each minute used, the amount of money paid for the amount of time being used will remain the same.
>
> Yes, because he pays $10 for whatever minutes he use for long distance calls.

Some of the answers were based on students’ everyday experiences or guessing. The arguments offered were not entirely founded on the given mathematical bases, but included appeals to social and personal factors. Some students relied on their personal experiences to verify their statements by referring to irrelevant arguments. The following responses demonstrate this thinking pattern:

> It is not 3 times because the money is changed for [every] 60 seconds.

Kevin's rule will only work if he makes long distance calls only. It probably won't work if he used both long distance calls and regular phone calls.
No, because it won't be constant.
Yes, it will because the price is a flat rate and probably won't change within time. In this case, it is not even Kevin's loss.

These students seem to have poor deductive reasoning abilities. They have difficulties proving something by using many concrete situations. Even when using concrete situations, they faced problems using them adequately to express whether the given rule always works or not. Building up a single algebraic relationship to satisfy the conditions was so hard for them. None of the students were successful in this. Further, their arguments were illogical and incoherent.

There was an overall tendency to use trial and error methods or working backward methods instead of using algebraic procedures. In the above question, 14% of the students extended their thinking beyond the given data but they could not grasp the relationship between the two variables (usage and cost) at the same time. This group concentrated only on the change of one variable at a time (either usage or time). It was difficult for them to understand the changing relationship between two variables at the same time. This again points to the fact that they lack proportional or relational reasoning.

4.6.6 Miscellaneous forms of incorrect answers

A large variety of incorrect answers were observed in question 2 and, as a result, I formed the above error category. Among the answers given, $8n$ was significant (22%). As mentioned previously, this shows the students' tendency to misinterpret the operation as a multiplication when it is actually a division. There were other forms of answers to this question such as additions, exponential forms, conjoined forms, constants, and equations. Some of them even have formed incorrect equations as answers ($8 \times n = 0$). Interestingly, each variety in the miscellaneous category appeared relatively in small percentages. However, the diversity of those answers was relatively expansive.

4.7 Highest incorrect responses categories

What are the highest occurrences of errors in each conceptual area? Inside an error category, there were clusters of answers with different percentages. Out of these numbers, the four highest percentages in a conceptual area were selected. One advantage of this selection is that this will provide an opportunity to separate the most frequent errors and analyze them deeply in the qualitative phase. A graphical representation of these percentages (Figure 6) shows some patterns in the data.

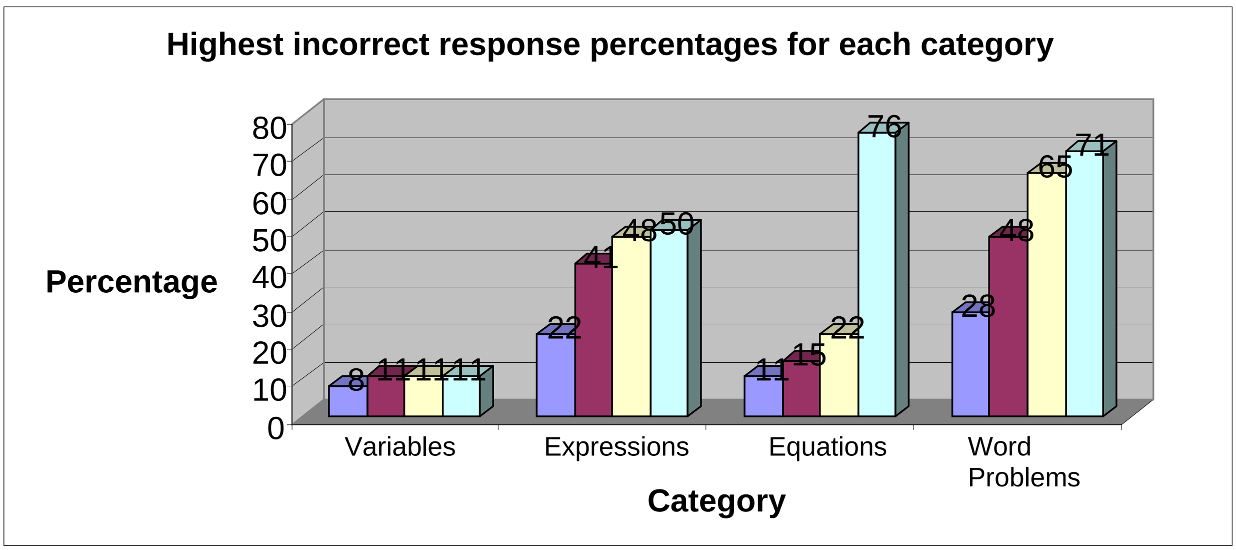

Figure 6: Highest incorrect responses for each conceptual area

Under variables, most of the percentages are similar or close to each other. The vertical bars represent the error types from the lowest to the highest percentage. They are: assigning labels, arbitrary values, or verbs for variables (8%), assigning labels for constants (11%), misinterpreting the product of two variables (11%), and misjudging the magnitudes of variables (11%). These percentages are relatively smaller compared to other conceptual areas. This indicates that the students answered more questions correctly in this category than the other categories.

The pattern for expressions indicates that the two highest values are closer to each other. The vertical bars from the lowest to the highest represent: incomplete simplification (22%), oversimplification (41%), incorrect quantitative comparisons (48%), and incorrect cross multiplication (50%). The numbers show that close to half of the students have difficulties in three areas under algebraic expressions.

For equations, there is a wide percentage difference between the highest and the lowest bars. The bars from the lowest to the highest represent: interference from previously learned methods (11%), Misreading the problem (15%), wrong operations in the substitution method (22%), and misinterpreting the elimination method in equation solving (76%). The majority of students had difficulties answering the problems under this area.

There are two high percentages in this group. The rectangular bars from the lowest to the highest represent: forming additive or multiplicative totals from proportional relationships (28%), reversal error (48%), incorrect reasoning (65%), and guessing without reasoning (71%).

4.8 The six cases

I obtained data for this study in several forms: students' answers to the test, student interviews, teacher interviews, and my notes during the research process. In this section, I will use the data from all these sources to discuss students' errors and misconceptions further. In a way, this will establish the link between qualitative and quantitative data. In the next section, I will describe the interviews with six students. These students were chosen by using the criteria already discussed in chapter 3.

4.8.1 The case of Rashmi

Rashmi is a Grade 11 mathematics student. She is 16 years old and was born in Canada. She is studying in the college/university stream. She demonstrates a good understanding of algebra with a few possible misconceptions. Some of her answers indicated that she may have misconceptions in algebraic expressions and equations. The following quote describes Rashmi's description of a problem to simplify polynomials:

Question 5c: Simplify: $\frac{xa+xb}{x+xd}$; Her answer was: $\frac{a+b}{d}$

> I – Can you read the question and do it for me, please?
> R – So, $xa+xb$ over $x+xd$. Em….yeah…so…you multiply by this, right? (pointing to xa) So I divided this, because there's an x over, and divided by another x .So this cancels off with this one ($\frac{\cancel{x}a+xb}{\cancel{x}+xd}$) and I got this one also off with this one ($\frac{\cancel{x}a+\cancel{x}b}{\cancel{x}+\cancel{x}d}$). So that, there are really no x s anymore. So I'll write $a+b$ over d .
> I – (pointing to the first x in the denominator) Is there anything remaining here after you cancelled out the x s?
> R – No, nothing.
> I – Ok, can you do this problem for me, please? ($\frac{3\times7+3\times8}{3+3\times2}$)
> R – Ok, so…Do it just like this or do it in a different way?
> I – Whatever the way you like.
> R – Ok, so… if I do like that the 3 becomes x s, right? So I cross these off ($\frac{\cancel{3}\times7+\cancel{3}\times8}{\cancel{3}+\cancel{3}\times2}$). Oh….I don't think, it's right. If I did it normally, I would do it like…I would multiply and then divide. So 3 times 7 plus 3 times 8… you know, I do it like that.
> I – If you use your previous method, I mean crossing out the 3 s, do you still get the same answer?
> R – Em….(thinking)….. I don't know….. I'm not sure.
> (Rashmi, Interview 1)

It reveals that the way Rashmi cancelled out the x 's were problematic. She first cancelled out the first x 's in the denominator and the numerator and then cancelled out the second x 's in the denominator and the numerator. This would not have been exposed by merely looking at her answer in the test. Her method of cancellation indicates that she separated the algebraic fraction into two fractions although this separation and cancellation of x's is illegal. Her cancellation technique does not obey the formal rules.

Her second misconception was assuming a zero in the denominator when an x is divided by another x. I provided her with a numerical example to see whether she will follow the same method. My assumption was that, by tailoring a numerical example and some follow-up questions, she will explain more on her inconsistencies. I presumed that this will also provide evidence on possible arithmetic-algebraic connections.

When Rashmi chose a different method to do the numerical example, I stopped questioning her as this was heading in another direction. In this case, my intention to elaborate her reasoning by giving a numerical example failed. Of note here is that this could be my "expert blind spot", where I made a wrong assumption about the student's thinking. It did not come up to my mind that the student will use a different method in the numeric example. However, providing the right kind of numerical examples to show the arithmetic-algebraic connection would be good to uncover students' conceptions/misconceptions for a classroom teacher when students have problems with figuring out answers algebraically.

Since my intention was not to teach the student, I did not proceed further in that direction. However, I suggested a solving method to see whether she knows the method.

I – Do you know any other way to do this problem ($\frac{xa+xb}{x+xd}$)?
R – (Long pause)
I – By taking out common factors or something like that.
R – Yeah, I can do it like this. $x(a+b)$ over $x(1+d)$ and then this piece gets cancelled off (cancels the x's), ok? So…yeah.. $a+b$ over $1+d$.
I – So, do you get the same answer as your previous one?
R – No, there's a 1 here (pointing to the first x in the denominator $\frac{\cancel{x}a+\cancel{x}b}{\cancel{x}+\cancel{x}d}$ …(laughing).
I – What do you think now?

R – Yeah, there's a 1 in here. Because, it's times 1 right? So, it's x times 1 because there's a 1. So if I take out the x, 1 is the multiplier. So it's left out…right? (laughing)
I – So why did you use a different method in the test?
R – I don't know…… I was confused.
(Rashmi, Interview 2)

This time, Rashmi did the problem correctly and she also explained her thinking correctly. This is an indication that the student had the correct solving schema for this problem in her head but for some reason, it did not occur to her in the previous occasion. Instead, she applied an incorrect method believing that it was correct. One possible explanation for this behavior is that the incorrect solution method could have been deeply-rooted in her mind overtaking the correct method. Often, this happens to students although it is difficult for them to explain what made them to choose an incorrect method in a previous occasion. Another important point is that sometimes students do have difficulties in executing algorithms even though they select the correct solving method. Not only this, they do have problems in recalling the correct algorithm too.

Another observation is that students should be asked proper questions before arriving at conclusions about their thinking. This questioning will expose students' misconceptions as well as the teacher's or the researcher's invalid assumptions about student thinking. Because I followed up with questioning, I realized that this student knew the correct answer. Otherwise this would have never known. Therefore, allowing students to explain their thinking in detail will expose not only their misconceptions but also the other ideas that were covered up in their heads.

My next interview with Rashmi further confirmed the fact that the correct schema was covered up with a deeply-rooted misconception. She explained her mistake in this time too.

Question 16: Solve the following linear system of equations: $2m+n=2$, $3m-2n=3$

I – Rashmi, can you read this question and do it for me? (She read the question correctly and eliminated m from both equations and wrote the answer for n as $n=0$)

I - Ok, that day you got $n = 7$ as the answer, right? Can you explain? (pointing to the last two lines of her previous answer: $-7n = 0$, $n = 7$)
R – Ok, I tried to do it in my head. I just did it…..I tried to figure it out in my head without putting the actual steps down and so I just got mixed up.
I – So, what did you actually do there?
R – I added it. So, I tried to…. like the negative 7 as if it was a number and not a variable. So I added it to both sides. I added negative 7 to both sides.
I – Is that correct?
R – Or, no, no, I added 7. That is what I did. Instead of dividing it, I added it.
I – Ok, if you add 7 to this side what will happen? (pointing to the left side of $-7n = 0$)
R – Because I like….I assumed that…..I thought it as being -7, plus n, that's how I thought about it.
I – Did you assume that there is a plus sign in between -7 and n.
R – Yeah, that is how I wanted to get it finished. (laughing)
(Rashmi, Interview 3)

Assuming a plus sign instead of a multiplication sign between a number and a variable is a common problem. The multiplication sign in between the constant and the variable is not well established in students' heads especially when the variables and constants are connected in an equation. In these situations, students often misapply the "change side – change sign" rule. Another observation is that it was difficult for Rashmi to explain exactly what caused her to make an error in the first time. Her reasons were unclear. Making incorrect hasty decisions in problem solving is very often among students. This could also be such a situation. Or again, this could be a situation where the correct schema was covered up with a misconception.

4.8.2 The case of Kathy

Kathy is a 16 year old, Grade 11 mathematics student. She completed her elementary education in her country of origin and currently in the college/university stream in her secondary school in Canada. She shows a poor understanding of dealing with symbolic statements and prefers to use "working backwards" methods to solve equations. She used very long "trial and error" methods to come up with some solutions. For example, she used 12 different values to obtain the solution for question 17.

Kathy's answer to the question $x\left(\frac{a}{b}\right)$ was $\frac{ax}{bx}$. I wanted to test whether she would use the same method with a numerical example. I gave her the problem $2\left(\frac{4}{3}\right)$. She did it correctly by putting a 1 as the missing denominator for 2 and her answer was $\frac{8}{3}$. Then I asked her to do the algebraic problem and this time her work was $\frac{x}{1}\left(\frac{a}{b}\right)=\frac{xa}{b}$. In this case, the numerical example helped the student to uncover the correct schema from her head. However, what is not apparent is that whether she will commit the same mistake next time with another similar algebraic problem although she understood the arithmetic-algebraic connection of the problem at least for this time.

I asked Kathy to explain her thinking for question 18 as she seemed to be confused with proportional reasoning.

> Question 18: Kevin has a phone plan. He pays $10.00 each month plus $0.10 each minute of long distance calls. One month, Kevin made 100 minutes of long distance calls and his bill was $20.00.In the next month, he made 300 minutes of long distance calls and his bill was $40.00.Kevin said, "If I talk 3 times as long it only costs me 2 times as much!" Will Kevin's rule always work? Explain your reasoning?

Her answer was:

> Yes, the rule will always work.
> 100 minutes = $ 20.00
> 300 minutes = $ 40.00
> 500 minutes = $ 60.00
> 700 minutes = $ 80.00

Kathy calculated the cost for equal time intervals (200 minutes). She did not consider the other side of the argument. In other words, the proportional relationship between the cost and time was not considered. Her reasoning was: "So, it is doubled by 20 from the start". She mistakenly considered the $20 gaps as doubles. I asked her to take another value which is three times of the number of minutes and asked her to check whether the rule works. This time the answer was "2100 minutes = $ 160.00" and she said the rule works because the cost is doubled. This time, she mistakenly doubled the previous amount by 2 ($80x2=$160) by applying the rule to the previous amount.

Kathy's reasoning clearly shows that she did not concentrate on two increasing amounts in the problem at the same time. She concentrated only on one side of the argument at a time (number of minutes or cost). Another problem was her failure to transform the word argument into a correct numerical form. Question 18 is a classic example to investigate students' proportional reasoning. In the test, many grade 11 students did not display this reasoning well.

Other examples of Kathy's work illustrated the fact that she frequently changed her methods and used different incorrect methods to solve the same problem. She was uncertain about her selection of methods and oscillated between methods when she was asked to choose the one that she thinks is correct. I asked Kathy to explain her reasoning for question 5(c): $\frac{xa+xb}{x+xd}$. Her answer in the test was $\frac{x^2ab}{x^2d}$ and in the interview, it was $\frac{x^2+ab}{x^2+d}$.

I – Kathy, how did you get x^2 ($\frac{x^2+ab}{x^2+d}$).
K – Add x and x.
I - ab?
K – Add a and b.
I – This x^2?
K - x and x…. And the d …there's no value there. So I just put it.
I – How did you get this answer? (showing her answer in the test).
K – I think I multiplied and added. x by x and then ab and then x^2 and then d.
I – Which answer do you think is correct?
K – I think this one (pointing to $\frac{x^2+ab}{x^2+d}$). It makes more sense.
I - Why?
K – Because you have to have the plus signs. That makes more sense.

I – Why?
K – I do not know. That's the way it is.
(Kathy, Interview 1)

In both occasions, Kathy oversimplified the answers by using two different methods. For me, it was difficult to verify her reasoning with standard mathematical practices. She sometimes operated directly on variables without considering the effect of adjacent letters or signs. Overall, her explanations of the selection of methods were very unstable.

Directly operating on variables without considering their attachment to signs or other variables was further evident from Kathy's other work too. I asked her reasoning to the question $\frac{A}{B}+\frac{A}{C}$. Her answer in the test was $\frac{A}{BC}$.

K – So…I'll simplify….. A over BC.
I – How did you get BC?
K – I add B and C. And for A….I simplify… so I just put it as one A.
I – Did you add?
K – Yes.
I – If you add A to A, what will you get?
K - A^2.
I – Then, why did you put A?
K – No…no…it is A^2.
(Kathy, Interview 2)

When Kathy answered, she was quick to respond. She showed no signs of hesitation even though some of her answers were wrong. There were no backup procedures, reflection, or self monitoring to check the answers. Many of the conjoined answers show her lack of understanding of the structural features (properties, laws, rules) of algebra. She mostly thought that letters can be combined without considering the meanings attached to them. The rules of algebra were of little importance to her. She frequently changed her thinking. Lack or non-use of metacognitive skills aggravated her problems.

4.8.3 The case of Tony

Tony is 16 years old and he was born in Canada. He is in the Grade 11 mathematics college/university stream. He has some problems with correctly identifying the variables, simplifying expressions, and solving equations and seems to have the habits of oversimplification, incorrect distribution, and some incorrect reasoning processes.

As many other algebra students, Tony also has problems with simplifying binomials. His answer to the expansion $(A+B)^2$ was A^2+B^2. His remarks were as follows.

> I – Can you explain your answer to me please while you are doing the problem?
> T – So what I think I can do is…..em…..I think I am going to use the power of 2 add to these powers. So, A^2+B^2. That's it.
> I – Can you repeat?
> T – I put the power to each.
> I – What do you mean?
> T – I multiplied by 2. So it's $2A+2B$.
> I – Ok, now you got a different answer. Did you do anything differently?
> T – No, I did the same thing.
> I – Why two answers?
> T – Oh, yeah…hold down…ok… so I applied it to the 1s here in front of A and B. That's the way I saw it. Ok, so, there's an invisible 1 on both of these [writes $(A^1+B^1)^2$]. This means like multiply these by 2. So it's A^2+B^2.
> I – Now, which answer do you think is correct?
> T - A^2+B^2 , I think.
> (Tony, Interview 1)

This discussion shows that Tony's conceptual models are unstable. Despite his misconception of incorrect distribution of the power, he is confused with the meanings of multiplying the coefficients of A and B by 2 and multiplying the powers of A and B by 2. He implies that both of these operations are the same. It is noticeable that this unstable thinking pattern resurfaced even at the last stage of the solving process when he says, "so I applied it to the 1s here in front of A and B". However, he admits this mistake by saying, "that's the way I saw it" although he still could not identify the distribution error.

Tony incorrectly applied the distributive law ($A(B+C)=AB+AC$) to expand $\left(A+B\right)^2$ and to get the answer $2(A+B)$. This is his first misconception. Further, he generalized the property $(A^1+B^1)^1=A+B$ to get $\left(A+B\right)^2=A^2+B^2$. He contradicted his own statements continuously and tends to see a misleading correlation between a particular rule or a property in algebra (distributive law) with an erroneous application of it. This forged correlation in his head led him to contradict his own statements.

Despite this problem, Tony's answers further indicate that he has some basic problems with distinguishing between variables, expressions, and equations. He does not properly understand the actual difference between algebraic expressions and equations. According to him, a variable can exist only in an equation. Moreover, he says that a variable can take any value. I asked him about his understanding of some basic concepts.

> I – What is an equation?
> T – They are like with letters and numbers.
> I – Is that all?
> T – Yes.
> I – Do they have equal signs?
> T – They don't have to have equal signs.
> (Tony, Interview 2)

Tony's above misunderstanding echoes very well in his answers in the test.

Subtract $2b$ from 7 (Question 6)

$=7-2b$

$b=7-2$

$b=5$

Multiply $e+2$ by 3 (Question 7)

$=e+2\times 3$

$\frac{-e}{-1}=\frac{6}{-1}$

$e=-6$

The first line of both answers indicates neither an equation nor an expression. However, Tony assumed that it is an equation and proceeded to solve it starting from the second line afterwards. Tony used the equal sign to indicate "the answer is" or, in other words, he used the equal sign as a step marker in the first line. This misuse led him to misconstrue the expression as an equation. I asked Tony why he cannot leave the answers as $7-2b$ or $e+2\times 3$. He said that, since they are equations, they have to be solved. The unwanted equal sign in the first line may also have contributed toward his misconception. However, this contradicts his previous statement that equations do not have to have equal signs. This partial understanding of the concept has led him to convert algebraic expressions in the answers into equations. He seems to believe that a variable can only exist in an equation and the value of the variable is obtained by solving the equation. The following quote shows his interpretation of a variable.

> I – What is a variable?
> T – A variable is a letter that represents part of the equation like you know what the answer is.
> I – Can a variable take values?
> T – Yeah, you substitute it.
> I – Any other example?
> T – (no answer).
> (Tony, Interview 3)

Some of Tony's other answers reflect this thinking pattern. For example, in question 1, his answers were equations not letters. This is further evidence that he may think that variables can only exist in equations.

Name a variable in this problem	Name another variable in the problem
$3y+y=25$	$y\times 3y=25$
(Question 1)	(Question 1)

Tony has a lack of understanding of the existence of variables outside the context of equations. Finally, I asked him what values can y and t take in the equation $y=2t+3$ (Question 9) and he said that each y and t can take only one value.

The answers show that this student has many misconceptions. He has a weak perception of the differences of variables, expressions, and equations. His misconceived notions are interconnected since they came up in many situations. Tony does not seem to believe that a variable can exist in isolation or it can take one value or many values depending on the circumstances. He lacks the understanding of a variable as a generalized number. Some of these misconceptions led him to commit other serious errors such as extending the answers unnecessarily to create faulty solutions.

Tony committed another error when he solved the system of linear equations in question 16. The following extracts were taken from the final stage of his solution.

$$
\begin{aligned}
&3-2n=3\\
&\frac{1n}{1}=\frac{3}{1}\\
&n=3
\end{aligned}
$$

In the interview, he blamed himself for his oversight and moved quickly to correct the error. He said, “oh, no…no…I can remember my answer [in the test]. It is wrong. I know how to do that”. I asked him to explain the reasons for his previous error and he said he deducted 2 from 3 to get a 1. This is again a common feature of directly operating on numbers which is an indication of students’ lack of understanding of the closure property. They often separate the numbers from the letters. However, addition is not closed when numbers are together with algebraic terms or in other words, when the numbers are from two different systems.

4.8.4 The case of Colin

Colin is 16 years old and was born in Canada. He is in the Grade 11 mathematics college/university stream. The majority of his answers to the test were incorrect. Basically, he has problems with understanding the differences between variables and constants. It is also difficult for him to distinguish between variables as specific numbers and variables as generalized numbers. Apart from that, he has some misconceptions in solving equations as well. I interviewed him for question 16.

Question16: Solve the following linear system of equations.

$2m + n = 2$

$3m - 2n = 3$

His answer:

$$2m + n = 2$$

$$2m + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + n = 2$$

$$(2m+1) - 1 + n = 2$$

$$(2m+1) + n = 2 + 1$$

$$(2m+1) + n = 3$$

I – Why did you choose this method?
C – Because, that is the way [my teacher] told us.
I – Is this the only method you know to solve these equations?
C – I am not sure.
I – Do you know any other method?
C – No.
I – What do you mean by solving the equations?
C – You're probably going to have to find the roots of the equation.
(Colin, Interview 1)

Although Colin could not provide exact reasons for his selection of method, it is clear that he mistakenly applied a method for solving quadratic equations to solve the linear system. The clue for this is from the solving algorithm itself and from the phase "roots of the equation" in the interview. He had assumed a forged relationship between methods of solving a quadratic equation and solving a system of linear equations. When he was asked to do the problem again in the interview, he did it using the same method. He applied the algorithm without referring to the meaning of it. Lack of monitoring of the solving process was another mistake because he applied the same method to both linear equations. At the end, when he was even stuck with the solution, he was unaware that he had chosen a wrong method.

I interviewed Colin for his error in question 11. His reasoning reveals his misconception of considering letters as objects.

Question 11: Shirts cost s dollars each and pants cost p dollars a pair. If I buy 3 shirts and 2 pairs of pants, explain what 3s + 2p represents?

> I – Can you work out question 11 please?
> C – So there's $3s+2p$. $3s$ would equal to 3 shirts and $2p$ would equal to 2 pairs of pants. So this would represent the total amount of items he bought.
> I – Did you say that s stands for shirts?
> C – Yeah.
> I – What does p stand for?
> C – Pair of pants.
> I – What is $3s+2p$?
> C – So in total, there would be 5 items and each item costs a dollar.
> I – How?
> C – It would be $5 in total. Three shirts represent 3 dollars and 2 represents….. $2p$ represents 2 dollars and in total it used 5 dollars.
> (Colin, Interview 2)

Naming s stands for shirts and p stands for a pair of pants is a clear indication of perceiving letters as objects but not as the cost of each item. Interestingly, this information is given in the problem although the student did not grasp it. Colin has some other misconceptions. Despite assuming that the letters stand for objects, he incorrectly assumed that each item costs one dollar and ultimately concluded that the number of items in the problem represents the cost of that item. This is because he already assumed the letters that represent the costs as objects.

For question 12, Colin said that the right answer is $4B=5R$. This was a mere syntactic representation of the problem without understanding the meaning of the corresponding ratios. His explanation indicated an exact matching of word order with the letters in the problem which has caused a reversal error. There was no indication that he knew how to write the relationship as a ratio or a proportional relationship. Further, I asked him, "What does B represent?" and his answer was "B stands for blue cars". This again shows his misuse of the letter as an object rather than a "number of something".

4.8.5 The case of Ann

Ann is 19 years old and she came to Canada in the beginning of 2009. She is in the Grade 11 mathematics college/university stream. The majority of her answers to the test were incorrect. Basically, she seems to have problems with understanding the concept of a variable. Many of her other answers were in oversimplified forms. Her reluctance to use algebraic methods was evident when she attempted to answer question 13 by guessing values. For questions 17 and 19, she used "working backwards" methods. I interviewed Ann for her errors in question 2 and the following quote shows her reasoning.

Question 2: There are n girl scouts in a parade. There are 8 girls in each row. Write an algebraic expression to find out how many rows of girl scouts are marching in the parade.

I – Can you do question number 2 please?
A – So…yeah… n girls… n is not given….. and since there are 8 girls in every row…. n is a variable…and since there are 8 girls in every row… (long pause, no answer).
I – Would you like to draw a diagram?
A – (Draws 8 small circles in a single column)
I – What are these? (pointing to the circles)
A – They are rows.
I – How many rows?
A – Eight.
I – What does that mean?
A – There are 8 girls in each row and depending on how much rows are….. there will be 8 girls in each row.
I – For each row?
A – Yeah.
I – So, where's n here? (pointing to the diagram)
A – I don't know. It's a variable. It can be anything.
I – Okay…then what's the answer?
A - $8n$.
I – How do you know?
A – Oh… hold on…. $\frac{n}{8}$.
I – Why?
A – Because, say it is like 64 girls…. n divided by 8 equals how much rows are.
I – Was it easier for you?
A – Yeah.
(Ann, Interview 1)

At the beginning, Ann seems to be confused with the question. Her main concern was how to start the problem since n is unknown. I asked her to draw a diagram if possible. I thought that a geometric representation of the problem would help her to perceive the problem better although I assumed that without a starting number, this could also be difficult for her. While accepting that it is difficult to draw a diagram with an unknown total number of girls in the parade, Ann chose whatever the number given in the problem (8) to represent the number of rows. This shows her difficulty in starting the problem with a variable.

However, after many unsuccessful attempts, Ann instantaneously selected a numerical example by herself. This made the algebraic problem easier for her. So, in this case, the numerical example helped the student to gain conceptual understanding of the algebraic problem. To understand Ann's thinking of variables further, I next moved onto question 9.

Question 9: Which is larger y or t in $y = 2t + 3$? Explain.

Her reasoning to this question in the test was the following.

> t is just a variable. It can be anything. However, since the whole equation is $y = (2t + 3)$, I'd say y is much more bigger in this case.

I – Which is larger, y or t?
A – I think y is larger because t is whatever a number… a variable that is inside y.
I – What values can t take?
A – Any value.
I – Zero?
A – Yeah.
I – Negative values?
A – Yeah.
I – What values can y take?
A – Zero (hesitantly).
I – Negative values? Positive values?
A – It's just the same….. zero…. like it has to balance the equal… so they are equal.
I – Then…
A - y has to take that outcome whatever it is. So if t was 2… 2 times 2 is four plus 3 seven. So y would have to equal 7.
I – Did you say that y cannot be greater than t?

A – Yeah.
I – Always?
A – Actually, y is equal to whatever it is in this side (pointing to the right hand side of the equation)
I – Can y and t have equal values?
A – No.
(Ann, Interview 2)

Ann's answers indicate that her reasoning is not consistent. What happens in this problem is that y can be larger than t ($t = 2$), smaller than t ($t = -6$), or even equal to t ($t = -3$) depending on the t value. Ann's answer to the test shows that she had partially understood the problem. This is because she only considered positive values for t ($t = 2$). This indicates her partial or lack of understanding of variables as generalized numbers.

Apart from that, Ann seems to understand the equal sign from a right to left perspective. She thought that whatever it is on the right hand side should be calculated to get the answer on the left hand side. She seems to know the "equality" property of the equal sign, but at the same time she thinks that the equal sign represents as 'to do something'. When I asked "what values can y take?", this was not immediately obvious to her. It was relatively easier for her to answer the question "what values can t take?" because her explanation was to substitute values for t in order to get the values of y.

Ann was one of the few students who attempted to solve question number 13 by using the backwards method. Actually, her answer contained a mix of methods: algebraic, working backwards and trial and error. Her answer in the test was as follows.

22 stamps for J & R
-4 from 22 = 18 stamps for T
T $\div$ 5 = 3.6; $\therefore$ 3 stamps for Javier.
3 – 22 =19 = R's stamps
19 – 4 = Teresa's = 15; 15 $\div$ 5 = 3
15 stamps for Teresa
3 for Javier
19 for Raul

In line 2, Ann made a mistake by using the previous incorrect answer. However, by chance, she approximated 3.6 as 3 to get the correct number of stamps for Javier. Fortunately, she never used the incorrect answer (18) for Teresa in the next steps. Instead, she used the other relationships given in the problem to get the correct answer. In the interview, Ann said that she was confident with her answer because it satisfies the given relationships. So, she used ‘verifying’ or ‘looking back’ as a metacognitive function. However, she could not discover her mistake. I asked her whether she could use “working backwards” methods to solve this problem and her answer was in affirmative. This is an indication to her selection of methods.

In this problem, Ann used trial and error, working backwards, and using the given relationships together with some chance aspect to reach the correct answer. However, what she did not realize was that working backwards method cannot be used for a problem with 3 variables. This shows that although Ann knows how to use certain methods, she does not properly know when and why these methods could be used. To be a successful problem solver, it is important to have all of these kinds of knowledge.

4.8.6 The case of Joshua

Joshua was born in Canada. He is 18 years old and studying in the Grade 11 mathematics college/university stream. There were a lot of blank spaces without answers in his test paper. Interestingly, he worked out the problems correctly under equations with one variable. He was one of the few students who used algebraic methods to solve problems 17 and 19. Surprisingly, he seemed to have a poor understanding of manipulating algebraic expressions. Joshua mostly oversimplified the answers in conjoined forms. The following interview illustrates his understanding of simplifying algebraic expressions.

Question 5b: Evaluate: $x\left(\frac{a}{b}\right)$

Joshua’s work in the test: $x\left(\frac{a}{\cancel{b}}\right)\times b = x(ab) = ax + bx$. His work during the interview:

$x\left(\frac{a}{\cancel{b}}\right)\times b = x(a\times b) = x(ab) = (xa)(xb) = x^2 + xb + ax + ab$.

> I – Can you please do this question for me?
> J – You divide……I mean you multiply b besides a . So the denominators get cancelled out. So it is just equal to $x\left(\frac{a}{\cancel{b}}\right)\times b$ which is equal to $x(ab)$.
> I – Can you explain it again?
> J – You just….because you want to like cancel out. So to make it easier you just multiply….you divide……you multiply this numerator. So this is cancelled and it gets simplified and you get….
> I – How did you get $(a\times b)$?
> J – Em…when you multiply…no, wait. You multiply these two together which will be like that and then you put them in here and then you get x times a which is xa and x times b which is xb . [writes $(xa)(xb)$]
> I – Is there any sign in between xa and xb ?
> J – Well, it is the multiplication sign there in between the brackets, yeah. I think that’s how you do it. And then if you want to simplify, I think you do….so this is two binomials and so you multiply this first value into the second (drawing arrows from the first x onto both x and b) and then you multiply this into the next (drawing arrows from a onto both x and b). So it would be $x^2 + xb + ax + ab$.
> I –How did you get the plus signs?
> I – Because you’re simplifying it. So even though it is multiplied in here, you just add. Yeah, that’s it.
> (Joshua, Interview 1)

Analytical examination of Joshua’s answers shows that he has many misconceptions despite his poor knowledge of the structural features of algebra. He did not provide correct logical reasons for his actions and rather seemed to be confused. He manipulates the symbols haphazardly. It is interesting to note that Joshua conjoins the answers first and then expands those answers again using inapplicable laws. Joshua’s another misconception was the incorrect use of the distributive law. He misused the distributive law only once in his answer to the test but he did it twice in the interview. He did not seem to accept a product containing letters as the final answer. Instead, he incorrectly applied the distributive law to simplify the answer further. This is actually in addition to his illegal cancellation of the b's in the first step.

In the interview, he mentioned that $(xa)(xb)$ is a binomial. Misidentification of this product as a binomial led him to commit many other errors afterwards. He knew that the product of two binomials can be simplified by applying the distributive law twice. This was evident from his answer to the next question [$\left(A+B\right)^2 = A^2 + B^2 + 2AB$]. However, he did not identify the characteristics of a binomial which is a core concept. Therefore, the automated application of the distributive law occurred when he encountered an expression that looks similar to a binomial.

Joshua had some other misconceptions in solving linear systems of equations. He did not properly understand solving two linear equations by using elimination or substitution methods. He tried to use the substitution method but stopped prematurely ($a+b-5 = a-b-7$). When using the elimination method, he focused on only one equation at a time and did not use the second equation together with the first one. Further, he was confused with many rules of the elimination method. I interviewed Joshua for question 15.

Question 15: Consider solving the linear system: $a+b=5$, $a-b=7$
a) To eliminate a from both equations, do you add or subtract the two equations?
b) To eliminate b from both equations, do you add or subtract the two equations?
c) Will you obtain the same solution if you add or subtract the two equations? Explain.

I – How do you answer part (a).
J - You would subtract a.
I – Why?
J – I don't know. I think because it is positive here (pointing to a in the first equation), you subtract it from there….and for this one (pointing to b in the second equation), you would have to add instead of subtracting to make it equal to zero.
I – So, what is the answer for part (a)?
J – I think you would have to do both.
I - What is the answer for part (b)?
J – You do the same. You would add and subtract. For this one (pointing to b in the first equation), you would subtract and for this one (pointing to b in the second equation), you would add.

I – Can you explain more?
J – For this b (pointing to b in the first equation), since you have to eliminate it, you would have to subtract it in order to equal to zero and for this one (pointing to b in the second equation), you would have to add.
I – What is the answer for part (c)?
J – You would not obtain the same solution.
I – Can you explain?
J – Because the sum of it is different. Five is not equal to seven. In this (pointing to the first equation) a and b have a different value than this (pointing to the second equation). So, that's why the sum is different.
I – Do you have a way to verify your answer?
J – Like in a math way.
I – Any way you like.
J – You can just tell because the sum of it… 5 is just not equal to 7… you can just tell from that. That's why you can't get the same solution. Because a and b in this case (pointing to the first equation) have a different value than a and b in this case (pointing to the second equation).
I – Can you show me how to solve this linear system of equations?
J - (He wrote up to the step: $a+b-5=a-b-7$ and said that he does not know how to proceed further).
I – What do you mean by solving these two equations?
J – It is called substitution… it is for linear equations.
I – When you were asked to solve the equations, what are you going to find out?
J – You have to find x and y.
I – In this problem?
J - a and b.
(Joshua, Interview 2)

Joshua has two main misconceptions. When answering questions on the elimination of variables from a system of linear equations, he separated the system into single equations. Sometimes, he meant to operate directly on variables to remove them, which is not an accepted procedure. This is a misconception as there is no such method to eliminate variables. Second, he used to believe that the constants on the right hand side of the two equations would decide the values of the variables. This is another misconception.

When I asked him to solve the equations, he used the substitution method but failed to proceed after a few steps. He did not indicate any knowledge of the elimination algorithm although he mentioned the word "eliminate" in the interview. He did not realize that both equations should be used to eliminate variables. As a whole, Joshua did not have the correct conceptual model to apply and solve the two equations using either the elimination or the substitution method. Only a partial understanding of the principles of substitution and elimination led him to arrive at incoherent or faulty conclusions.

Generally, Joshua had a low response rate to word problems in the test. I asked him to solve question 10 during the interview and explain his thinking.

Question 10: Sachin is 14 years old now and his father is 40 years old. How many years will it be until Sachin's father is twice as old as Sachin?

> I – Can you do question 10 please?
> J – Ok, this one….(long pause)…..14 plus 14 is 28… right? If I add another 14, I will get 42. So it is 2 years.
> I – For what?
> J – Sachin's father to get as twice as Sachin's age.
> I – After how many years?
> J – After 2 years and then Sachin will become 16. Yeah, it is 2 years.
> I – After 2 years what is Sachin's age?
> J – 16.
> I – What is father's age then?
> J – Oh…yeah…no…no… it is after 4 years, right?
> I – How?
> J – Oh, no…no…it is 2 years.
> I – Because?
> J – It is not working my way. I think it is not 2 years. I am so confused.
> (Joshua, Interview 3)

Joshua's answer has two main errors. First, instead of doubling Sachin's age, he tripled it. This confusion may have come from the word "twice" as Joshua added 14 twice to 14. Second, he did not realize that father's age also would go up with Sachin's age. In other words, he could not grasp the correct proportional relationship between the two ages. He did not notice that Sachin would overtake his father's age according to his argument.

Like many other students, Joshua did not bother to check the validity of his answers. He would have used his common sense to double check his answers since this problem is related to a real life incident. There were no reflections on or monitoring of the answering process. In the interview, he realized that he made a mistake but could not figure it out. He would have identified at least some of his mistakes if he used some metacognitive processes such as monitoring the process or verifying the solution.

4.9 Summary

In this mixed method research design, I identified student errors and misconceptions pertaining to four main areas in algebra: variables, expressions, equations, and word problems. The data analysis contained three stages: discovery of patterns (induction), testing of students' and the researcher's assumptions (deduction), and uncovering the best set of explanations for the findings (abduction). My focus was on students' conceptions, procedures, algorithms, possible misconceptions, and their reasoning. Since the goal of this study was to identify students' misconceptions underlying their errors, I justified, whenever necessary, how students' wrong responses expose their misconceptions.

The quantitative analysis of the data showed that the students had most difficulties in answering questions on word problems with a mean error percentage of 85% followed by expressions (79%). Equations and variables were the next two sections with mean error percentages of 48% and 37% respectively. A rubric containing the error types for each conceptual area was constructed. In the qualitative phase, six cases were discussed with detailed student reasoning. This analysis showed that students had misconceived notions due to a variety of reasons. Among them, misuse of rules, confusion with previously learned concepts, problems with the syntax of algebra, problems with the structure of algebra, not identifying arithmetic-algebraic connections, not knowing the core concepts, and lack of metacognitive skills were prominent.

Chapter 5

Conclusions and Discussion

5.1 Introduction

The overarching objective of this research was to explore students' errors and misconceptions in four main conceptual areas of algebra and to expose student reasoning for them. By doing this, I assumed that a better understanding of students' algebra learning could be obtained. Despite the difficulty of directly accessing students' mathematical thinking and reasoning behind their actions, it is true that we can have access to their thinking through other methods such as using interviews. In the forthcoming sections, I will discuss the errors/misconceptions that I found under the four conceptual areas together with students' reasoning relating to them. Finally, I will discuss what lessons can be learned from the analysis of students' errors and misconceptions and how we can make suggestions to incorporate these findings into classroom teaching.

5.2 Research Questions

In this thesis, I explored the following six research questions:

1. What are secondary school students' categories of errors and misconceptions in solving problems related to variables?
2. What are secondary school students' categories of errors and misconceptions in solving problems related to algebraic expressions?
3. What are secondary school students' categories of errors and misconceptions in solving equations?
4. What are secondary school students' categories of errors and misconceptions in solving word problems?

5. Do the existing theoretical explanations account for the errors and misconceptions observed in this study?

6. What can be learned from students' problem solving processes and reasoning in algebra?

All the errors I found were not simply the absence of correct answers or the result of unfortunate accidents. Many of them were robust misconceptions partly because they occurred on more than one occasion. Interestingly, they were the consequences of definite processes whose nature must be discovered. That is why it was important to analyze these errors and misconceptions in a way that would expose the underlying reasoning. Further, this process would expose the individual differences among students who commit the same error or different errors. I explained the error types in four rubrics in the previous chapter. In the next section, I will elaborate those errors under each conceptual area by relating them to various existing theories in the literature. Whenever possible, I will explain how students' errors will allow us to determine their misconceptions and beliefs.

5.2.1 What are secondary school student's categories of errors and misconceptions in solving problems related to variables?

In this study, I found four categories of students' misconceptions that are related to variables. Mainly, these misconceptions emanated from the lack of understanding of the basic building block of algebra -- the variable, in different situations. In this section, I will explain these four interrelated categories of misconceptions. They are assigning labels, arbitrary values, or verbs for variables and constants, misinterpreting the product of two variables, lack of understanding of variables as generalized numbers and forming incorrect equations as answers.

5.2.1.1 Assigning labels, arbitrary values, or verbs for variables and constants

This error type contains a number of subcategories. First, students tend to misinterpret a variable as a “label”, as a “thing” or even as a verb such as “buying” rather than as the “number of a thing”. For example, when the price of a shirt is s dollars, students in this group thought that $3s$ stands for a label for “3 shirts” thereby misinterpreting the meaning of the algebraic term (Case of Colin, Interview 2). Philipp (1999) explained a similar use of letters as labels when f and y is used in $3f = 1y$ to denote 3 feet equals 1 yard. In this interpretation, f and y stand for ‘feet’ and ‘yard’ respectively. The letter was used to denote the name of the unit in this context. These different interpretations of letters in different contexts may cause students to mix up and misinterpret the use of variables. Another instance was when students labeled B for blue cars and R for red cars rather than taking them representing the number of cars from each color (Case of Colin). Further use of letter as a label was found in a different context. That was when one student solved $4x + 25 = 73$ (Question 14) by pasting number 8 as a label for x but not by substituting it. This is similar to finding a number to satisfy a number equation in arithmetic.

Second, it was difficult for some students to distinguish between variables and non-variables in an algebraic way (Question 1). They often provided names of persons, things, or letters for non-variables. In a general sense, some of these answers are correct; however, they are unacceptable under algebraic interpretations. Also, I found that students sometimes assign verbs (buying) for variables.

Misinterpreting letters as labels is a basic misconception which will lead to many other errors in algebra. In the famous Student-Professor Problem (Clement, 1982; Clement, Lochhead, & Monk, 1981; Kaput, 1985), college students pursued similar interpretations of variables. In that problem, students used p to represent professors rather than the number of professors and s

to represent students rather than the number of students. The result was that a reversal error (writing the equation as $6s = p$) has occurred. This error would not have been so high if the sample of college students had known the correct interpretations of p and s (Clement, 1982; Clement, Lochhead, & Monk, 1981; Kaput, 1985). I found evidence to this claim in my study (Question 12). One student who correctly identified B as the number of blue cars and R as the number of red cars subsequently wrote the correct ratio of the two quantities.

5.2.1.2 Misinterpreting the product of two variables

There were three varieties of this error. They were viewing the product of two variables as one variable, assigning a single value for the product, and viewing a numerical relationship between the two variables in the product (Question 3). The first two errors are more about perceiving the product yz as a single variable. Students do not take note of the multiplication sign in between the letters and they simply think that yz is similar to a number such as 12. In the third type, students viewed the product as two variables but interpreted it in a way that one variable would change the value of the other variable. As discussed in the literature review, Macgregor and Stacey (1997) termed this misconception as an analogy with other symbol systems. Some of the students interpreted yz as two separate variables but thought that one variable would change the value of the other variable as same as in the Roman numeral system.

Another source for this misconception which is related to another type of misconception was found in this study. That is the conjoining of letters or letters and numbers to form a single string of answers which is also termed as oversimplification. As Macgregor and Stacey (1997) stated some conjoiners believe that $h10$ is 'add 10 to h' and '$1y$ is take away 1 from y'. In a similar way, students in my study who said that one variable changes the value of the other variable in yz may have seen yz as a conjoined answer.

5.2.1.3 Lack of understanding of variables as generalized numbers

In the equation $y = 2t + 3$ (Question 9), many students recognized both y and t as variables. However, they did not realize that these variables can have more than one value in the equation (Case of Tony; Case of Ann, Interview 2). Further, students focused only on the domain of positive numbers to substitute values for y and t and to decide on the larger variable. They even did not think of zero as a value for substitution. This is a problem which occurs partly when students cannot view variables as generalized numbers that can take more than one value in some situations.

A variable as a generalized number is an important concept but many students seem to be misunderstanding this concept. Philipp (1999) categorized seven situations where variables are used and one of these categories was the variable as a generalized number. He said that, if it is difficult to understand that a variable can take many values in certain situations, then it is also difficult to understand the concept of functions because, most often, a variable in a function can have more than one value. In the categorization of literal terms, Kuchemann (1981) categorized letters as generalized numbers. According to Kuchemann (1981), there was a very small percentage of 13 to 15 year old students who interpreted the letter as a generalized number although they were exposed to generalizing number patterns in classrooms. Comparatively, a greater number of students interpreted letters as specific unknowns which is another common use of the variable in equations.

Like many other students, Tony and Ann in my study were not aware of the existence of a variable as a generalized number (Case of Tony; Case of Ann, Interview 2). In the interviews, they could not perceive the three situations ($y < t$, $y = t$, and $y > t$) which is an indication of their lack of understanding of the variable as a generalized number. Macgregor and Stacey

(1997) also addressed this issue and claimed that the principle explanation given in the literature for this type of error has a general link to the levels of cognitive development. However, this is not the only reason. Macgregor and Stacey (1997) provided alternative explanations for specific origins of misinterpretation that have been overlooked in the literature which may or may not be associated with a cognitive level. These origins are: intuitive assumptions and pragmatic reasoning about a new notation, analogies with familiar symbol systems, interference from new learning in mathematics, and the effects of misleading teaching materials.

5.2.1.4 Forming incorrect equations as answers

In this study, I found a student (Tony) whose answers to question 1 were in the form of equations. When he was asked to name a variable in the problem, his answer was $3y + y = 25$ and for a non-variable, his answer was $y \times 3y = 25$. These equations as answers are meaningless in the context of the problem and they indicate a false relationship between the variables and constants in the problem. They are mere syntactic representations not depending on the meaning of the question. Tony did not believe the existence of a variable outside of the context of an equation. This is a misconception for which I did not find any supporting evidence from the literature. However, lack of understanding of basic concepts is a possible reason for this misconception. What is of note here is that Tony's responses to the interview questions were contradictory to each other. He continuously seemed confused and could not provide good explanations.

5.2.2 What are secondary school student's categories of errors and misconceptions in solving problems related to algebraic expressions?

A large number of misconceptions found in this study were under the category of algebraic expressions. Obviously, in the initial analysis there were some other types of errors under this category mostly interconnected to the nine misconceptions discussed below. They all were included and renamed to come up with the following categories: incomplete simplification, incorrect cross multiplication, converting algebraic expressions as answers into equations, oversimplification, invalid distribution, reversal error, incorrect common denominator, incorrect quantitative comparisons, and miscellaneous forms of incorrect answers.

5.2.2.1 Incomplete simplification

This was a common error observed in simplifying algebraic expressions. Students usually start the problem and proceed with one or two steps to terminate the process without arriving at the final answer most probably not knowing how to proceed. Some of the students actually do think that they have reached the final answer. For these incomplete answers, further simplification was possible to reach the final answer. This error is actually the opposite of oversimplification. Since there were many varieties to this error type, I did not attempt to interview individual students for every single error type. However, what was apparent was the students' lack of knowledge or lack of confidence in the solving process.

5.2.2.2 Incorrect cross multiplication

The structure of algebra is so subtle that students often get confused with different procedures when they are very similar to each other. When an algebraic fraction has to be multiplied by a letter, students often use cross multiplication although this is inappropriate (Questions 4a, 5a, and 5b). It seems that they are confused with the arrangement of literal terms. This error especially occurred when there was no visible denominator in one of the terms. Students in my study lack the experience of making 1 as the denominator in such situations. However, Kathy answered the symbolic problem correctly after she did a numerical example.

This is the evidence to believe that sometimes numerical examples help students to solve algebraic problems with a similar structure.

5.2.2.3 Converting algebraic expressions as answers into equations

This is a common error mostly seen when there was an algebraic expression as an answer. I observed this error in questions 1, 4a, 4b, 5b, 6, 7, and in the case of Tony. This error mostly happened in situations where students used the equal sign to indicate to mean "the next step is" or, in other words, when the equal sign was used as a step marker. This additional equal sign at the beginning of the answer leads them to misconstrue the algebraic expression in the answer as an equation and they follow through the procedure to solve it. The reasons for this problem will be discussed later with other error types.

5.2.2.4 Oversimplification

This is another common error which occurs when simplifying algebraic expressions. In this group of errors, students conjoin, connect, or even put together the terms against the accepted algebraic manipulations. Addition, subtraction, division, and multiplication commands were left out forming bundles of strings. I observed this misconception in many answers including the problems on equation solving (Case of Kathy, Interview 1; Case of Joshua, Interview 1; Case of Tony). There are a number of discussions in the literature for this error.

Sometime, students oversimplify algebraic expressions by illegal cancellations and divisions of terms. They often tend to misuse factorization and cancellation procedures (Case of Rashmi, Interview 1). This interview reveals that the error has happened since she used an illegal cancellation procedure as a short-cut method. She used the correct application of the rule $\frac{ax}{bx} = \frac{a}{b}$ (Matz, 1980) but incorrect separation of terms in the algebraic expression made the cancellation incorrect.

The reasons given for oversimplification in the literature are due to similar meanings of ‘and’ and ‘plus’ in natural language (Tall & Thomas, 1991), erroneously drawing on previous learning from other areas that do not differentiate between conjoining and adding (Stacey & MacGregor, 1994), conceiving open algebraic expressions as ‘incomplete’ and try to ‘finish’ them by oversimplifying (Booth, 1988; Collis, 1975; Davis, 1975), the tendency in arithmetic to get a final single-termed answer (Booth, 1988; Tall & Thomas, 1991), to interpret symbols such as ‘+’ as an operation to be performed, thereby leading to conjoin the terms (Davis, 1975) or due to the dual nature of mathematical notations as processes and objects (Davis, 1975; Sfard, 1991; Tall & Thomas, 1991).

In my study, Kathy’s oversimplification of algebraic expressions does not carry a significant theoretical explanation attached to it as her conjoined answers are different in some occasions. This shows that her conceptual models are very inconsistent. Her reasoning changes from situation to situation. Radical constructivism is basically situated on two fundamental beliefs: the action and the reflection on action. One possible explanation for Kathy’s inconsistent behavior is that after the answer, she may have reflected on it and realized that her answer was not the best form of suitable action.

Tony’s answers show that he commits this error due to his lack of understanding of the basic properties in algebra. The failure of differentiating between variables, expressions, and equations has caused him to commit the error. His perception of an algebraic expression as an equation led him to oversimplify the expression. This is more related to the process-product dilemma as he sees the answer as a process rather than a product. Joshua has the same misapplication of the rules in algebra to oversimplify a product by perceiving it as a process.

Further, his thinking was inconsistent as he also provided different explanations in different occasions.

5.2.2.5 Invalid distribution

Under "invalid distribution", I found several forms of incorrect use of the distributive property. The most common among them was when raising a binomial to a power (Case of Tony, Interview 1; Case of Joshua, Interview 1). Another connected feature is the oversimplification of answers after applying the property. The examples were: $(A+B)^2 = A^2 + B^2$ and then proceeding further to oversimplify the answers as A^2B^2, $(AB)^2$ and so on. Another subcategory of misusing the distributive property is incomplete distribution. In this category, students applied the multiplication on one term leaving the second term unattended. The examples were answers such as $(e+2)3 = 3e+2$ or $e+2(3) = e+6$.

Overgeneralization of a correct rule to misapply it in another situation is a result from explicit, declarative knowledge gained from the curriculum (Matz, 1980; Matz, 1982; Kaput, 1985). Matz (1980) termed this phenomenon as the "(mis)application of extrapolation techniques" (p. 95) and said that students incorrectly apply the correct rule because of the similarity of the two situations. Another explanation for the misuse of the distributive law is that these errors have their roots in arithmetic misconceptions. Lack or incomplete understandings of arithmetical concepts or the failure to transfer arithmetic understandings to algebraic contexts are the leading factors (Norton & Irvin, 2007; Stacey & Chick, 2004; Stacey & MacGregor, 1999).

5.2.2.6 Reversal error

Under algebraic expressions, some students committed the reversal error in question 6 (17%) where they formed the expression in the reverse order. In this question, students had to read the word sentence and translate it into an algebraic form. The reverse order of the answers indicated that these students have matched the word order given in the problem into the algebraic form rather than understanding the correct relationship among variables. This is a common error often comes with word problems and this will be discussed further under word problems.

5.2.2.7 Incorrect common denominator

There were two different kinds of errors detected in this category. They were choosing the smaller number as the common denominator between two numbers and taking the sum of the letters in the two denominators as the common denominator instead of taking their product (Questions 5a, 5d). This is an indication that students thought differently depending on whether there are numbers or letters in the denominator. After taking the sum of the letters as the common denominator, they simply added or multiplied the two letters or whatever the numbers in the two numerators without applying the proper algorithm to simplify them. This is an indication to a manipulation of symbols in a haphazard manner without following the correct algorithm.

5.2.2.8 Incorrect quantitative comparisons

In this category, students compared the magnitudes of two algebraic fractions by examining their denominators only (question 8). In other words, they perceived $\frac{1}{n+1}$ as more than $\frac{1}{n}$. They explicated that since $\frac{1}{n+1}$ has a larger quantity in the denominator, it is more than $\frac{1}{n}$. As discussed in chapter 2, students' wrong assumption of "more A implies more B" (Stavy & Tirosh, 2000) is applicable to this kind of a scenario. Students' reasons indicate that they perceived $\frac{1}{n+1}$ is more because it has a larger denominator. They incorrectly perceived that since the denominator is larger in $\frac{1}{n+1}$, the whole fraction is greater than $\frac{1}{n}$. In other words, a larger denominator implies a greater fraction; this is a misconception. Intuitively, they may have used a rule which is similar to “more A implies more B”. In spite of this, some of the students who used a numerical example to compare the two fractions later identified the magnitudes of algebraic fractions correctly. This is another example where a numerical example helped to identify and establish the arithmetic-algebraic connection.

5.2.2.9 Miscellaneous forms of incorrect answers

This name suggests that these answers are diverse and each answer deserves a different explanation. For example, there were different forms of answers for the same question especially for simplifying algebraic expressions. Parts of these answers contained other error types such as oversimplification, incomplete answers, and incorrect use of rules. To explain these errors thoroughly, individual answers should be considered separately. It seems that students have executed random manipulation of symbols not in accordance with accepted rules. They may have used their own wrong rules that are persistently fixed in their minds, or they may have used ad hoc rules depending on the situation.

5.2.3 What are secondary school student’s categories of errors and misconceptions in solving equations?

There were seven categories of misconceptions under equation solving. They were numbers as labels, misinterpreting the elimination method in solving equations, wrong operations in the substitution method, misuse of the “change- side, change-sign” rule, interference from previously learned methods, misreading the problem, and misinterpreting the equal sign.

5.2.3.1 Numbers as labels

As mentioned in chapter 4, only one student made this error. This is in a different context and a number was used as a label for a letter. Solving for x in $4x + 25 = 73$, this student used $x = 8$ by pasting the number 8 into the position of x to get 48. The answer shows that the student understands the equivalence property as he pasted the correct number to make the equivalence work, although he did not follow the normal equation solving procedures. The most visible reason for the error is that he may have used his previous knowledge of number equations to insert a number to satisfy the numeric equation. If this is the case, this error could also be considered in the category of "interference from other learning". Since I did not interview this student, this is only a hypothesis.

5.2.3.2 Misinterpreting the elimination method in solving equations

In this study, I found a considerable number of students who misconstrue the elimination method when solving a system of linear equations (Question 15). They often misjudged the operations to be performed and chose a reverse operation (Case of Joshua, Interview 2). There were three types of misunderstandings emerged when students were questioned about their solutions. They were: 1) considering only one equation in the system, 2) selecting wrong operations to be performed, or 3) considering only the constants on the right hand side of the equations. All of these incorrect methods indicate students' incomplete understanding of the elimination method. This is further evidenced from their reluctance to solve the equations using the elimination method because a majority of them used the substitution method to answer question 16.

I observed that some students always attempted to concentrate on variables in one equation rather than taking both equations simultaneously. Substituting for a variable in terms of the other variable in a system of two-unknown linear equations is called second level representation. It was difficult for the students to overcome the methods that they have used for solving equations with one variable. This may be because of students' additional difficulties in operating with the second level representations in a two-unknown linear system than operating with a single-variable equation (Filloy et al., 2003; Filloy et al., 2007). Also, there were instances where the elimination method was used not to eliminate variables but to combine the two equations to form a single equation with two variables. One of their main problems was the lack of understanding of how to start with the solving algorithm.

Another observation was the students' difficulty in arriving at conclusions intuitively. It was difficult for them to deduce whether the solution was the same in the two solving methods. Three out of four students who gave the correct answer solved the equations to arrive at their conclusions.

5.2.3.3 Wrong operations in the substitution method

The incorrect substitution method used by students for question 15 and 16 (for example: $2m+n-2=3m-2n-3$) showed that apart from the misunderstanding of the substitution method, there was lack of monitoring of the solution process (Case of Joshua, Interview 2). If students examined their solution processes carefully, they would have realized that substituting for a zero will not eliminate any variable from the equations. Instead, it will give a single equation with two variables. Similar to this, there was another observation in the pilot test, stage 1 (Question 21). In this question too, there is a chance that careful examination of the question before answering would suggest short-cut methods.

5.2.3.4 Misuse of the "change- side, change-sign" rule

This is a common error in solving equations in algebra. Sometimes, students forget to change the sign when they carry over terms to the other side of the equation or apply wrong operations to the terms. In this study, I found this error in more than once in response to question 16. At one stage of the solving process, the student misused the "change-side, change-sign rule. (Case of Rashmi, Interview 3). She gave a clear indication as to why she made the error ($-7n = 0$, $n = 7$). This happened to her since she attempted to separate the letters and constants in an algebraic term. The main reason for this is the lack of understanding of the basic structural features of algebra such as the properties of an algebraic term. Even if they understand those properties, sometimes students commit these errors unknowingly.

5.2.3.5 Interference from previously learned methods

Students mistakenly modify and apply a previously learned rule or an algorithm to a new problem situation (Question 16). This interference often occurs to them and what is noticeable is that they do not realize that a misuse has been occurred (Case of Colin, Interview 1). I discussed earlier a similar situation in the literature (Macgregor & Stacey, 1997) where previously learned Roman Numeral System could have interfered with the students' understanding of variables. Further, I discussed the instances where retrieval of correct but inappropriate rules occur (Matz, 1980). Based on these reasons, it is appropriate to say that students misuse the previously learned procedures, rules, or algorithms in novel situations where they are inappropriate.

5.2.3.6 Misreading the problem

Misreading and misinterpreting problems often happens to students. The frequency of occurrence of this error is very high in word problems since students have to read, interpret, and convert English words into algebraic language (Question 19). One of the reasons for this difficulty in translating from natural language to algebra and vice versa comes from elementary education (Bishop et al., 2008). In this study, I found that this is a common situation which will lead to committing many other errors such as reversal errors because of the misinterpretation of the problem situation.

5.2.3.7 Misinterpreting the equal sign

Two misinterpretations of the equal sign were found in this study. They were using the equal sign as a step marker (Case of Tony) and interpreting the equal sign as a signal to do something (Case of Ann, Interview 2). In the test, I found that some other students used the equal sign as a 'step marker' to indicate the next step of the procedure when actually it was not necessary. These students sometimes violated the equivalence property of the equal sign by equalizing statements that were not equal to each other.

The sources of the misuse of the equal sign were discussed in chapter 2. Both Tony and Ann believed that the equal sign means to do something to one side of the equation to get the answer on the other side. The misuse of the equal sign to interpret it as a command to compute an answer suggests that aspects of arithmetic instruction were contributing to this difficulty (Kieran, 1992). However, Tony's misconceptions were complicated. He did not have a clear idea of how to distinguish between the various characteristics of variables, expressions, and equations. These erroneous conceptual models would have contributed further to his misinterpretations.

Ann's interview suggests that she has the understanding of the equal sign as an equivalence relationship. However, at the same time she assumes that it means to do the operations to the right hand side of the equation to get the answer on the left. Kieran (1992) said that students' tendency to interpret the equal sign as a command to compute an answer suggests that aspects of arithmetic instruction were contributing to their difficulties in algebra. One other explanation for the use of equal sign as to do something is attributed to the fact that the equal sign mostly “comes at the end of an equation and only one number comes after it” (Falkner et. al., 1999, p. 3). Another possible origin of this misconception is the ‘=’ button on many calculators, which always returns an answer (Foster, 2007).

5.2.4 What are secondary school student’s categories of errors and misconceptions in solving word problems?

Many of the difficulties that students face in solving word problems mainly emanated from their failure to translate the word problems into algebraic language. Many of the errors that were identified under word problems in chapter 4 were connected to each other and they came from the above source. Students were more comfortable when they had to do less reading in a word problem. One of the other problems they had was to understand the varying relationships between variables. Sources of the other errors in word problems were discussed previously with the other three conceptual areas. In the following section, I will elaborate the errors and their sources for the categories: reversal error, guessing without reasoning, and incorrect or lack of understanding of proportional relationships.

5.2.4.1 Reversal error

In this study, reversal error appeared in many forms in different situations (Questions 2, 6, and 12). Mainly, it happened in word problems where students map the sequence of words directly into the corresponding sequence of literal symbols in the problem, thereby forming a reverse relationship (Case of Colin). This is consistent with previous studies (Clement, Lochhead, & Monk, 1981). In that, reversal errors occur due to student difficulties when translating from natural language to algebra.

One of the unique difficulties for students in question 12 was that the actual number of cars produced from each color was not given. It was difficult for them to understand the numbers given in ratios. Many students followed the word order in the problem to construct the equation rather than referring to the actual ratio in the problem. There was another misconception that was embedded in the answers. That was the use of letters as objects. Their difficulties were amplified since they used B and R for objects standing for blue cars and red cars.

In the 'student- professor problem', students could make an educated guess using their common sense. Often there are more students than professors in a college and, therefore, they have a clue in the problem. However, in the car problem, it is difficult to guess the magnitude of actual numbers. The reversal error is even more common when both variables in the linear relationship have coefficients other than 1 (Clement, Lochhead, & Monk, 1981). Clement et al. found that the reversal error arises not only when students construct an equation based on words, but also when they attempt to construct an equation based on a table of values or a diagram.

For question 12, I also found additive totals as answers. ($Total = 4B + 5R$, $Total = (4B)(5R)$, $B = R + 1$, $R = B + 1$). Some reversal errors are visible even in these additive totals. As MacGregor and Stacey (1993) noted, these students do not match the symbols with the words in an answer like this but were expressing features of some underlying cognitive model of an invisible mathematical relationship. This error is further consistent with the comments of Weinberg (2007) who described that students performed hypothetical operations on two quantities to equalize the totals. One hypothetical operation performed in the problem was to multiply B by 4 and R by 5 and add (or multiply) them together to get the total number of cars. The assumption could be that there were four blue cars and five red cars. Another operation was that they equalized the two ratios thinking that they were the actual number of cars.

5.2.4.2 Guessing without reasoning

Guessing is common in any problem solving situation. I observed this phenomenon especially in questions 10, 13, and 18. Students mainly used this procedure when there was a situation where they had to build up and solve equations on complex relationships (Questions 10 and 13). In question 10, I noticed that students made unrealistic guesses. They did not use any back up procedures to check the validity of their answers. To see whether a guess is valid, it is essential to check it out for internal coherence. This counts as a metacognitive action as well. However, in my study, I observed that very few students used this procedure. On the other hand, those guesses were not educated guesses. An important characteristic of an educated guess is that the guess will improve every time based on previous guesses.

5.2.4.3 Incorrect or lack of understanding of proportional relationships

One of the findings of this study was students' lack of understanding of relational equivalence between two quantities. Especially, they had problems with understanding the relationship between two variables when their actual quantitative relationship was not given; instead, the relational proportion of the two quantities was given. One of the misconceptions resulting from this lack of understanding was discussed under the "reversal error".

Lack of proportional reasoning affects equation solving too. The use of the equal sign to denote a relationship between the two sides of the equation was somewhat confusing for the students. The large number of incorrect responses to question 12 (97%) shows that many grade 11 students could not build equations based on proportional relationships (Case of Kathy). It was difficult for them to understand the conceptual basis behind a ratio and construct a proportional relationship from that ratio (Case of Colin).

This weakness emerged when they had to build up equations based on complex algebraic relationships (Questions 10 and 13). In such situations, they mostly used the trial and error or other arithmetical methods. Question 10 was a problem that demands the understanding of the relationship between two varying quantities. Students have to form an equation having the variable on both sides of the equation. Question 13 was a problem that involves the understanding of three varying quantities. In both questions, I noticed that the majority of students had difficulties in understanding the relationships and forming the equations. As mentioned in the previous category, they used arithmetic methods (working backwards), trial-and-error methods, or guessing unsuccessfully. In general, this is really not an error or a misconception but it is a difficulty of forming and solving algebraic equations with more than one variable. This difficulty has caused them to form incorrect relationships of variables.

5.2.5 Summary of algebraic errors and misconceptions

In this chapter, I discussed 21 error types under the four conceptual areas. Out of this number, 13 error types were already established in the literature. There were eight new error types that I found in this study. They were misinterpreting the product of two variables, giving answers in the form of equations, incomplete simplification, incorrect cross multiplication, incorrect quantitative comparisons, numbers as labels, misinterpreting the elimination method when solving equations, and forming additive totals as answers.

The symbols in algebra have different meanings and interpretations in different situations. Students have incorrect and incomplete perceptions about the letters, numbers, and signs. The overall image that emerged from the findings was that students have difficulties in understanding various uses of letters and signs in different situations. The misunderstanding of the concept of the variable did have a clear bearing on their errors and misconceptions.

With regard to algebraic expressions, students' problems increased due to their lack of understanding of the basic concept of the variable. I found that problems with algebraic expressions were the second most difficult ones for grade 11 college/university students. I observed three features in this conceptual area. First, there was a lack of understanding of the structural features of algebra which led students to use many illegal procedures. To understand algebraic expressions, students have to have a good understanding of the structure and the properties of algebra. This is because many algebraic expressions are made of letters and signs and most often they do not involve words. When letters, numbers, and signs are put together to produce expressions, these entities should be manipulated according to accepted procedures. Students made many errors during this symbol manipulation stage. In other words, they mostly made these errors in the processing stage of Newman's error categorization.

Second, students modify or misapply rules or procedures which are inappropriate in certain situations. Most often, the similarity of the two situations caused this misapplication. Quite apart from that, the third observation was students' haphazard reasoning that is unaccounted for any accepted rules or procedures. To understand or explain these thinking procedures, in my opinion, a different kind of analysis is necessary.

With regard to equation solving, students' problems were mostly interrelated to the two previous conceptual areas. Apart from this, some other difficulties were the reasons of other errors and misconceptions. First, the misuse of the equal sign out of its accepted meaning was obvious. They mostly used the equal sign in a single sense, that is, to do the operation to the left and get the answer on the right or vice versa. Others misused the equal sign as a step marker in inapt situations. Second, students had problems with understanding the given relationships and build up equations. To avoid building up algebraic equations, they used other methods such as arithmetical methods or trial and error. Third, their solutions to the systems of equations were not accurate. They often misused the elimination and substitution methods. They had some misconceptions about the solution of a linear system. Looking back at the solving procedures, verifying the solutions, and using other metacognitive processes were also missing in many situations.

"Word problems" was the most problematic area for students. They had difficulties in managing the details of a problem given in a word format. In solving a word problem, students had to pass through a number of stages. These stages were deciphering the problem given in English language, translating it into a mathematical form, solving it, and interpreting the results. Students need to do some additional work in solving word problems other than solving problems in the other three conceptual areas : variables, expressions, and equations, since a word problem may contain concepts related to one or more of the above three areas:. This is one of the reasons that word problems were harder for the students. Sometimes, there were no visible clear-cut methods to solve word problems. Using guess and check methods was, therefore, difficult in such a scenario. Inquiry and reasoning skills are important in arriving at a correct solution. Also, in this study, I found that students did not stop to think carefully and reflect during problem solving. Lack or non-use of metacognitive skills would also have played a role.

One of the main misconceptions I found under word problems was students' attempts to match the word order in natural language with letters. Obviously, this was the reason for committing the "reversal error". Misunderstandings of proportional reasoning led them to arrive at faulty solutions. In this study, I did not find any Grade 11 student who could not read the problems. Understanding the problem and translating it were two of their major problems. When they could not understand the problems properly, students resorted mostly to trial and error or guessing procedures. However, most often, these were not educated guesses.

5.2.6 Do the existing theoretical explanations account for the errors and misconceptions observed in this study?

There are numerous explanations given in error analysis literature for errors and misconceptions to occur in many conceptual areas. Most of them refer to a particular error type in one conceptual area. In the foregoing sections, I related these ideas from the literature to the discussion of errors and misconceptions that I found in this study. Apart from that, there are no universal explanations or one theory that explains all student errors or misconceptions in algebra.

However, as mentioned in chapter 2, there are three broad theories that disclose some of the reasons for some errors and/or misconceptions. The work exemplified by Matz (1980) examined students' error behaviors in rule-based problems with a view to building a generative theory that accounts for as many common errors as possible in problem solving. The theory states two extrapolation mechanisms for generating algebra errors. They are the use of a known rule in a new situation where it is inappropriate, and incorrectly adapting a known rule so that it can be used to solve a new problem. I found both of these situations in my study. For example, misuse of the distributive law (Case of Tony; Case of Joshua) was an example for the first type and the use of solving methods in quadratic equations to solve two linear equations was an example (Case of Colin) for the second type among other examples. Further, I found other evidence for the first type when students misapply the distributive property or the property, $(A+B)^1 = A^1 + B^1$ to oversimplify $(A+B)^2$ as $A^2 + B^2$. The external similarities of the situations may have caused the students to misapply these rules.

Another theory elaborated by Martindale (1991) and Matlin (2005) states reasons for students not using the correct method the first time was the neural network theory. In my study, I found evidence to this theory. Some students did not use the correct method when they solved the problem the first time, but they solved it correctly in a later occasion. Therefore, one possibility is that they could not retrieve the correct information from their memory the first time although they had this information in their long term memory. The evidence for this was the retrieval of correct information later without any outside help.

Constructivist theories propose that people continuously repair or renovate their previous experiences based on new information. Both of the above theories indicated that students have retrieved the wrong information although they had the correct information in their heads. One possible explanation for this behavior is that correct information may have been stored in a new location in the brain without overwriting the wrong information. Mistakenly, the wrong information has been retrieved in the first time, probably caused by a cognitive conflict. Test anxiety or other pressures in an examination situation would also have caused this problem. It could be argued that students were more relaxed or more comfortable when they answered the same question again in a non-testing situation.

The third theory that is helpful to analyze student errors and misconceptions in algebra was given by Fischbein (1994) and it can also be applied to other branches of mathematics as well. As explained in chapter 2, this theory takes into account three components: the formal, the algorithmic, and the intuitive in analyzing students' mathematical behavior. Sometimes a solving schema is applied inadequately because of superficial similarities in disregard of formal constraints. Sometimes, a solving schema, deeply rooted in the student's mind, is mistakenly applied despite a potentially correct, intuitive understanding. But, usually, it is the intuitive interpretation based on a primitive, limited, but strongly rooted individual experience that annihilates the formal control or the requirements of the algorithmic solution, and thus distorts or even blocks a correct mathematical reaction. This view confirms the constructivist views as it states that persistently fixed errors will produce misconceptions. Also, strongly rooted individual experiences are the ones that take over weaker experiences.

The solving procedures, acting as overgeneralized models would sometimes lead to wrong solutions disregarding the corresponding formal constraints. As another example, a solving technique that does not obey the formal rules and thus wrongly applied is:

$\frac{an}{an+by} = \frac{\cancel{an}}{\cancel{an}+by} = \frac{1}{by}$. A similar situation in my study was the simplification of $\frac{xa+xb}{x+xd}$ as $\frac{a+b}{d}$ by using an illegal cancellation.

Fischbein said that in order to overcome many errors, students need to gain a fuller understanding of the relationships between the formal and the algorithmic components in mathematics. Students have to understand the formal basis (definitions and theorems) that justifies an algorithm. It is the blind learning of algorithms that leads to misuse of them. It is arguable that this claim is not always true since I found some evidence against this. For example, students who did not understand the formal definitions of variables or algebraic terms manipulated such symbols effectively. Hence, it may not always be required to know the formal basis to justify the use of other components in the model.

Findings of Kirshner (1985) and Demby (1997) also have different views on Fischbein’s claim. They too suggest that it is not compulsory to know the formal basis of an algorithm before applying it. However, Fischbein argued that the interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified and understood. Theoretical analyses, attentive observations, and experimental research have to collaborate in revealing the multiple sources of mistaken attitudes in a mathematical activity.

It is often quite difficult to make a universal theory comprising all the errors and misconceptions and an explanation of the behavior of these errors. There are close interactions among different components explained in the above theories. The same problem can give rise to errors from different sources, and the same error can arise from different problem solving processes. Therefore, a definite classification or hierarchy of all error/misconception types with their reasons seems impossible to achieve. Despite these and other practical problems, I think that the error analysis illustrated in this study provided satisfactory answers to my research questions and some help for teachers with regards to individualizing instruction and becoming sensitive to the effects of their own instruction.

5.2.7 What can be learned from students' problem solving processes and reasoning in algebra?

The research literature consistently indicates that misconceptions are deeply-seated and not easily dislodged; in many instances, students appear to overcome a misconception only to have the same misconception resurface later. This is probably a result of the fact that, when students construct learning, they become attached to the notions they have constructed. Therefore one important requirement in eliminating those misconceptions is that students must actively participate in the process of overcoming their misconceptions. This is not a process that is entirely dependent on the teacher. However, the teacher also has to play an important role in completely eliminating the misconceptions. All that is said here is important if misconceptions were found in a later stage. However, it is of utmost importance that in teaching these concepts teachers provide students with classroom learning environments that help them develop both conceptual and procedural knowledge so that they construct correct conceptions right from the start.

The results in this study reinforce the view that a self-explanation procedure is potentially a powerful technique for exposing student misconceptions. If students are expected to write or tell about what they do well, what they struggle with, and why they believe that is the case, this process supports students' understanding as well. This process is collaborative, student centered, and by students' own admission. By tailoring the follow-up questions to allow students to explain their thinking and struggle with inconsistencies, the teacher will get better insights into students' thinking and the students will get better opportunities to re-arrange their own (mis)understandings. I believe that this is more meaningful when it is done individually than in large groups. When teachers listen to their students, they will develop sophisticated schemas for understanding the diversity of student thinking. At the same time, students will revise and refine their own mathematical thinking. This latter action shifts classroom practice significantly from the role of the teacher as evaluator of student ideas to the role of students as self-evaluators of their emerging ideas.

NCTM (1991) pointed out the importance of employing diagnostic interview methods in everyday classroom situations. It is because it fulfills at least three objectives of teaching mathematics: problem solving, reasoning, and communication. "Adjusting to student's understanding is central to constructivist methods and the fact that good teachers *calibrate* their instruction based on student needs makes detailed classroom observations and interviews imperative (Woodward, 2004, p. 11).

Attempting to uncover the hidden misconceptions by rewinding the student's memory and finding the schema where the deep-seated misconception lies has another benefit. By this revelation, we can decide methods to change the misconception and reorganize/repair the incorrect schema. The correction process could be carried out during or after the interview. It is important that the teacher should explicitly try to assess the misconception of the student and engage with the student in a way in which (s)he rearranges the concept. One of the suggestions in this direction is to take students' incorrect answers for a discussion in the classroom. This way, students will get a better understanding of their own mistakes and the mistakes of their peers.

The four areas that were examined in this study were variables, expressions, equations, and word problems. The idea was not only to study the errors and misconceptions in each of these individual areas but also to observe the combined nature of the misconceptions. One of the current debates in the US mathematics reforms is whether the priority should be given to mathematical processes or concepts in teaching (Klein, 2007). This is an ongoing debate and some researchers argue that both procedures and concepts are equally important. I found some evidence in my pilot study and the main study that there were instances where students made similar procedural errors in more than one conceptual area. Also, I observed that some students did not answer the questions on conceptual knowledge correctly when this knowledge was related to a wrong procedure that they carried out in another area. However, my study was not especially designed to collect specific information about this interrelatedness, although there was some evidence in this direction.

The above finding leads to the claim that it is important for students not only to have procedural knowledge (how procedures and algorithms work) but they should also develop conceptual knowledge and be able to explain why the procedures and algorithms work. Gaining both conceptual and procedural knowledge would lead to understanding the interconnectedness of these two types of knowledge. Consequently, students would be more likely to use the correct procedures in solving algebraic problems. Equally important for them is to have proper conditional (when to use) and declarative (what -- to explain the concepts, facts, and principals) knowledge. This study showed that all of these types of knowledge are important to prevent many errors and misconceptions. Further, when teaching a new concept, providing examples as well as non-examples is extremely important. In this way, students might get a better understanding of the concepts, facts, procedures, and principles. The profound connection between meaning and skills is a basic condition for productive and efficient mathematical reasoning.

One of the other notable features of the interview process was that students achieved new ways of thinking, sometimes giving up their previous erroneous methods. Occasionally, students reflected on their use of methods and identified that mistakes were made. Sometimes, they reflected on their previous mistakes and corrected them during the interviews. To a considerable extent, the interview questions helped the researcher to direct the students to explain more or get alternative explanations for the same phenomenon. The errors and misconceptions, therefore, served the purpose of constructive and adaptive tools for promoting understanding.

When searching for the origins of the errors, students reached a better understanding of their own mathematical reasoning. This is in agreement with the claim that interviewing improves students' reasoning abilities (NCTM, 1991). As Smith, diSessa, and Roschelle (1993) said, student errors do not have to be a hindrance to the mathematical learning process. By committing errors and looking to understand their origins, students may achieve a stronger conceptual basis for reasoning correctly than if they have never committed the errors in the first place. Matz (1980) further reinforced this idea by saying that rational errors should not have to be a hindrance to the mathematical learning process. They can also serve as constructive and adaptive tools for promoting understanding. In the process of correcting or searching for the origins of errors, students may reach a better understanding of their own mathematical reasoning.

This study further shows that posing conflictive questions and providing numerical examples make students reveal more about their thinking. Activities that produce states of cognitive conflict are certainly desirable and conducive to conceptual change. It is not always guaranteed that by learning a new method, old ones have been unlearned or modified. For the student, to unlearn deeply-rooted misconceptions is not easy. For the teacher, it is also not easy to make this happen. In this study, I did not make an extensive attempt to identify the exact causes for students' misconceptions. In my view, it is difficult to judge the exact error causes by looking at one-time student answers to a test. A series of carefully planned problems under each concept followed by subsequent interviews should be conducted to identify specific error causes properly.

The errors and misconceptions found in this study belong to two categories. First, students' lack of understanding or misuses of algebraic concepts tends to initiate some errors and misconceptions. Second, some common deficiencies that could happen in any problem solving situation initiated some errors. For example, students' hurriedness to start solving a problem without properly reading or understanding it, using incorrect short-cut methods, lack or non-use of metacognitive skills such as not monitoring the solution process, not verifying the answers, and not being aware of the validity of the answers were obstacles not only to find the correct solutions but they could also initiate errors. I will discuss these difficulties next.

Another area that needs to have attention is students' lack of arithmetic skills. In my study, there were students who solved some problems using arithmetic methods (Questions 10, 17, 18, and 19). The answers showed that some of these students struggled with arithmetic manipulations associated with equivalence, operations with negative integers, and fractions (Questions 8 & 9). Sometimes, they made errors in the algebraic solutions because of their incorrect arithmetic manipulations. Interestingly, I observed that these students preferred to use arithmetic methods especially when solving equations. Sometimes, they elected numerical examples to prove their arguments. However, some of these attempts were not successful because of their weaknesses in arguing with numbers or manipulation of numbers (example: what is greater? $\frac{1}{2}$ or $\frac{1}{2+1}$). Teachers in this study also indicated that one of the reasons for student failures in algebra is their poor arithmetic skills.

Many researchers have confirmed that some error patterns associated with manipulating algebraic expressions have their roots in arithmetic. For example, manipulating algebraic expressions having integers (operating with negative integers) and over generalization of cancelling procedures (fraction errors) have their roots in arithmetic misconceptions, and incomplete understandings and the failure to transfer arithmetic understandings to algebraic contexts (Norton & Irvin, 2007; Stacey & Chick, 2004; Stacey & MacGregor, 1999). Students who are not comfortable computing with numbers will be less disposed to manipulate symbols because computational procedures with fractions provide a natural entrée into symbol use (Wu, 2001). Considering those facts, I believe that having poor arithmetic skills is a factor that contributes to algebraic errors.

Some students' whose answers were wrong in the test recovered in the interviews and provided the correct answers to the same questions. If we assume that no new learning occurred during the test and the interview, then “Why did they behave differently in the two situations?” is a key question. When students were asked to explain what made them answer incorrectly in the test, their answers were: “I was confused, I was lost, or I was nervous”. One of the explanations for this behavior is test anxiety. Anxiety is a factor that makes students solve problems incorrectly in a test situation. This erodes their confidence and interferes with their thinking in that particular situation.

There are two different components of math anxiety: intellectual or cognitive and emotional or affective (Posamentier, 1998). The intellectual component primarily involves worrying about failure and its consequences. The emotional component involves fear, feeling nervous, and being uncomfortable. In my study, students were not required to worry about the consequences of failing since the test was non-evaluative. However, the emotional component may have played a role. Also, the emotional component has a stronger and more negative impact on students’ mathematical performance (Posamentier, 1998).

Another possible explanation stems from the neural network theory of mind discussed in chapter 2. It indicates that the students probably had the correct methods in their long-term memory but they could not recall the information (Martindale, 1991; Matlin, 2005). Students probably had both the correct and wrong information in their long-term memory but recalled the wrong information in the first time. The correct information may have been covered or inhibited with the wrong information.

Cognitive skills are important in problem solving but they need to be accompanied by metacognitive skills. During the interview process, I was constantly paying attention to students’ metacognitive processes such as self-monitoring, verification, and awareness of the reasonableness of their answers. I observed that, in general, students did not generate sufficient self-explanations, verify their answers, or monitor their problem solving processes regularly. Many students did not demonstrate such important metacognitive skills. This may not be the entire fault of the student. Lack of familiarity with these procedures could be a reason.

Allowing students to utilize their metacognitive skills is very important in any problem solving situation. Extending the results to the academic environment, there are instructional techniques that are effective for enhancing metacognition. Constructivists believe that students construct schema by acting and reflecting on an action. In my view, teachers ought to be good listeners who analytically attend to students' reasoning processes and their products. Therefore, it is necessary to give ample opportunities to reflect after successful performance so that students can acquire concepts in more connected and logical forms (Steffe & Gale, 1995).

Comparison of the test answers and the interview results showed that some students did not use uniform mental mechanisms when solving problems. In particular, the lack of uniformity between their strategies in the written test and their strategies during the interview points to the instability of their thinking processes. They had many potential models and the impetus for using a particular one depended not only on the specific task, but also on the situation. Since some students did not perform consistently, predicting a model to explain their misconceptions was difficult.

Another noticeable feature in my study was the sizable amount of errors that cannot be explained using a uniform theory. I categorized them as "miscellaneous forms of incorrect answers" and those were the answers that seemed to have obtained by manipulating symbols, procedures, or algorithms in a way that is best known to the student. There could be some formal methods that may have followed, but these methods were not easily visible. Sometimes, there were thoughtless, slapdash manipulation of symbols which was totally different from correct formalism and they were not in accordance with strictly applied rules. These wrong rules could be persistently fixed in the students' mind, or they may have used them ad hoc. I believe that analysis of those behaviors requires a different theoretical framework.

5.3. Reflection

The intention of my study was to explore students' errors and misconceptions in algebra. In my view, a mixed method approach proved to be entirely appropriate for this purpose. In particular, interviewing of students provided rich and insightful information. The quantitative analysis further supplemented the qualitative data. This reaped a wealth of data which may not have been tapped via other methods. My visit to schools established a personal connection with the subjects of the study encouraging them to give expansive responses. Students' written work in the interviews also added a wealth of additional information which would not have emanated through other processes. As a whole, I think the validity and the reliability of the findings were enhanced as a result of the methodology undertaken.

The interview process was exhaustive and demanding. There were pitfalls as well. Students sometimes make careless or capricious errors. From a constructivist perspective, I mainly sought to address the systematic qualities of errors which are typically grounded in the conceptions of the student. In this way, I thought that I could understand the sensibleness of students' approaches from their point of view. Second, I had to be very cautious when students' responses deviated from my own assumptions. These deviations were very important to me in the sense that they possessed the seeds of alternative approaches which were ideal to be examined. I was always ready to challenge my own assumptions. Third, when students proceeded in accord with my expectations, I assumed agreement with caution because students' conceptual models did not always fit with my own hypotheses. In such a scenario too, I tried to gain insights into their perspectives by establishing or rejecting my own hypotheses.

I always tried to show the whole picture by relying on the data and presenting other explanations as well. Although my perceptions, interpretations, and my own assumptions played a key role in identifying students' misconceptions, my attempt was always to validate my assumptions by regularly moving back to the data.

Finally, there is a debate about the generalizability of the results in qualitative studies. Since I used the case study method to examine students' misconceptions, it is arguable that these results are valid only for that particular group of students. Goldin (2008) said that when a clinical researcher focused on the elaboration of a single case, a different form of generalizability results from it than from studies with large samples. Clinical researchers feel that they can generalize from a study of a single case to some other individual cases because they have seen a given phenomenon in one situation in sufficient detail and know its essential workings to recognize it when they encounter it in another situation. Although my study was not a clinical study, I believe that the above remarks are applicable to my study as well.

The case study approach that I used in this study is equally or more important than other empirical investigation methods in education based on controlled experimentation. In theory, no two individuals or populations could have the same characteristics. Hence, in general, it is difficult to replicate research on human beings. However, in essence, my results can safely be replicated for a population with the same or similar characteristics. Therefore, I believe that this study can be replicated to any urban secondary school, Grade 11 mathematics university/college classroom with a mixed group of students.

Considering the nature and origins of errors and misconceptions would give specific help to mathematics teachers by allowing them to integrate their knowledge of curriculum content with their knowledge of individual differences in students. However, one should note some difficulties in the present state of research into students' errors. It is often quite difficult to make a sharp separation among the possible origins of errors because there is such a close interaction among those origins. The same problem can give rise to errors from different sources, and the same error can arise from different problem solving processes. A definite classification and hierarchy of all the origins of errors would be impossible to achieve.

Finally, it is appropriate to discuss the limitations of this study. First, I assumed that the implementation of the secondary mathematics curriculum in Ontario classrooms honors the inquiry approach and students were given adequate opportunities to learn using constructivist approaches. However, if students learned the necessary skills traditionally without using problem solving and inquiry approaches and in isolation from context, then my assumption could be inappropriate. Second, since I did not observe the real classroom teaching in this study, the findings could not be extended to suggest specific classroom teaching procedures. In other words, I mostly addressed the areas of measuring and observing the outcomes of the classroom activities as they existed in the chosen classrooms under the study.

Despite those limitations, error analysis appears to provide both information about basic research questions in mathematics learning and practical help for teachers with regard to individualizing instruction, the importance of using individual interviewing, and becoming sensitive to the needs of the individual learner. Finally, the overall research process was a demanding as well as an illuminating experience for me and in many ways a good experience for working as a potential researcher in the future.

5.4 Future research

As a researcher, my journey will not end with this work. There were a number of issues that came up during this expedition. I hope to explore this fascinating arena further. I will extend the ideas that emanated from this research into three main branches. First, it is difficult to elaborate the actual error causes without having a deep examination into each one of them. This needs a proper identification and a micro analysis of individual errors. Hence, as a first step, I will plan and administer a set of carefully planned questions to identify a specific error/misconception under a given concept.

Second, any identification of errors is worthless unless we make suggestions to overcome them. Based on my findings, I will prepare lesson plans and test them in real classroom situations. Research studies suggest that, if students can visualize abstract algebraic concepts, it will help them immensely to understand them. Using technology to explain concepts, procedures, and algorithms is one way of achieving this. Therefore, as a second step, I will use dynamic worksheets to help students visualize the concepts and purge their misconceptions.

No lesson plan is perfect until we are satisfied with its results. In the third step, I will conduct pre- and post tests before and after administering these new lesson plans. This will, in part, make the items more valid and provide insights into what should be altered.

5.5 Summary

In this chapter, I articulated a number of student errors and misconceptions based on the findings in chapter 4. To answer the six research questions set out in the study, I explained in detail the nature and, whenever possible, the origin of these errors and misconceptions. Under the four main conceptual areas in the study, I elaborated the main errors and misconceptions and their nature and origins. Later, these explanations were related to the existing literature on misconceptions in order to connect them with broader theoretical arguments. Finally, the implications of the findings were discussed with suggestions for classroom teaching.

References

Agnieszka, D. (1997). Algebraic procedures used by 13-to-15-year-olds. *Educational Studies in Mathematics, 33*, 45–70.

Alderson, J. C., Clapham, C. & Wall, D. (1995). *Language test construction and evaluation*. UK: Cambridge University Press.

Anderson, C.W., & Smith, E.L. (1987). Teaching science. In V. Richardson-Koehler (Ed.), *Educators' handbook: A research perspective* (pp. 84-111). New York: Longman.

Artzt, A. F. & Armour-Thomas, E. (1992). Development of a cognitive-metacognitive framework for protocol analysis of mathematical problem solving in small groups. *Cognition and Instruction*, *9*(2), 137-175.

Barrera, R. R., Medina, M. P., & Robayna, M. C. (2004). Cognitive abilities and errors of students in secondary school in algebraic language processes. In D. E. McDougall & J. A. Ross (Eds.), *Proceedings of the Twenty-sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1*, (pp. 253-260). Ontario Institute for studies in Education, University of Toronto, Canada.

Ben-zeev, T. (1998). Rational errors and the mathematical mind. *Review of General Psychology*, *2*(4), 366-383.

Bishop, A., Filloy, E., & Puig, L. (2008). *Educational algebra: A theoretical and empirical approach*. Boston, MA, USA: Springer.

Bogdan, R. C. & Biklen, S. K. (1998). *Qualitative research for education: An introduction to theory and methods, Third edition.* Allyn and Bacon.

Booth, L. R. (1984). *Algebra: Children's strategies and errors*. Windsor: NEFR-Nelson.

Booth, L. R. (1988). Children's difficulties in beginning algebra. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K-12* (pp. 20-32). Reston, VA: NCTM.

Brew, A. (2001). *The nature of research inquiry in academic contexts.* New York: Routledge Falmer.

Brueckner, L. J. & Bond, G. L. (1955). *The diagnosis and treatment of learning difficulties.* New York: Appleton-Century-Crofts.

Burns, R. B. (2000). *Introduction to research methods, Fourth edition.* London: Sage.

Casey, D. P. (1978). Failing students: A strategy of error analysis. In P. Costello (Ed.), *Aspects of motivation* (pp. 295-306). Melbourne: Mathematical Association of Victoria.

Charmaz, K. (2006). *Constructing grounded theory: A practical guide through qualitative analysis.* London: Sage Publications.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, *13*(1), 16-30.

Clement, J., Lochhead, J., & Monk, G. (1981). Translation difficulties in learning mathematics. *The American Mathematical Monthly, 8,* 286-290.

Clements, M. A. (1980). Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics*, *11*, 1-21.

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. *For the Learning of Mathematics, 9*, 32-42.

Cobb, P., Yackel, E., & Wood, T. (1989). Young children's emotional acts while doing mathematical problem solving. In D. McLeod & V. Adams (Eds.), *Affect and mathematical problem solving: A new perspective* (pp. 117-148). New York: Springer-Verlag.

Cobb, P., Yachel, F., & Wood, T. (1992). A constructive alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education, 23,* 2-33.

Cole, M. (1996). *Cultural psychology: A once and future discipline.* London: The Belknap Press of Harvard University Press.

Cole, M. (2005). Introduction: Studying cognition in flux: A historical treatment of fu in the shifting structure of Oksapmin mathematics. *Mind, culture, and activity*, *12*(3 & 4), 169-170.

Collis, K. E. (1975). *The Development of Formal Reasoning*. University of Newcastle, Newcastle, Australia.

Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. *Review of Research in Education, 16*, 3-56.

Confrey, J. (1991). Learning to listen: A student's understanding of the powers of ten. In E. Von Glaserfeld (Ed.), *Radical constructivism in mathematics* (pp. 111-138). Dordrecht, The Netherlands, Kluwer.

Confrey, J. (1994). Splitting, similarity and rate of change: A new approach to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp. 293-330). NY: SUNY.

Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions.* Thousand Oaks, CA:Sage.

Davis, R. B. (1975). Cognitive processes involved in solving simple algebraic equations. *Journal of Children's Mathematical Behavior*, *1*(3), 7-35.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education.* London: Croom Helm.

Demby, A. (1997). Algebraic procedures used by 13- to15-year olds. *Educational studies in mathematics*, *33,* 45–70.

Dreyfus, T., Artigue, M., Eisenberg, T., Tall, D., & Wheeler, D. (1990). Advanced mathematical thinking. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition* (pp. 113-134). Cambridge University Press.

Ebdon, S. A., Coakley, M. M., & Legnard, D. (2003). Mathematical mind journeys: Awakening minds to computational fluency. *Teaching Children Mathematics, 9*(8), 486-493.

Elchuck, L., Hope, J., Scully, B., Scully, J., Small, M. & Tossell, S. (1997). *Interactions 7: Professional assessment handbook.* Scarborough: Prentice Hall Ginn.

English, L. D. (1996). Children's construction of mathematical knowledge in solving novel isomorphic problems in concrete and written form. *Journal of Mathematical Behavior, 15,* 81-112.

Ericsson, K. A., & Simon, H. A. (1993). *Protocol analysis: verbal reports as data*. Cambridge, MA: MIT Press.

Ernest, P. (1991). *The Philosophy of Mathematics Education.* The Falmer Press.

Ernest, P. (1996). Varieties of constructivism: A framework for comparison. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds), *Theories of mathematical learning*. Lawrence Erlbaum Associates, Publishers.

Falkner, K. P., Levi, L. & Carpenter, T. (1999). *Building a Foundation for Learning: Algebra in the elementary grades*. Retrieved 24 April, 2010 from http://www.wcer.wisc.edu/ncisla.

Filloy, E., & Rojano, T. (1984). From an arithmetical to an algebraic thought. In J. M. Moser (Ed.), *Proceedings of the sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 51-56). Madison: University of Wisconsin.

Filloy, E., Rojano, T., & Puig, L. (2007). *Educational algebra: A theoretical and empirical approach*. Berlin Heidelberg, New York: Springer.

Filloy, E., Rojano, T., & Solares, A. (2003). Two meanings of the equal sign and senses of substitution and comparison methods. In N. A. Paterman, B. Dogherty, & J. Zilliox (Eds.), *Proceedings of the Twenty-seventh Annual Conference of the International Group for the Psychology of Mathematics Education, 4*, (pp. 223-229), Honolulu, Hawaii, USA.

Fischbein, E. (1990). Introduction. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 1-13). Cambridge University Press.

Fischbein, E. & Barash, A. (1993). Algorithmic models and their misuse in solving algebraic problems. *Proceedings of the Seventeenth Annual Conference of the International Group for the Psychology of Mathematics Education,* 1 (pp. 162-172). Tsukuba, Japan.

Fischbein, E. (1994). The interaction between the formal, the algorithmic, and the intuitive components in a mathenmatical activity. In R. Biehler, R. W. Scholz, R. Straber, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp.231-245). Dordrecht:Kluwer.

Flavell, J. H. (1981). Cognitive monitoring. In W. P. Dickson (Ed.), *Children's oral communication skills* (pp. 35-60). New York: Academic.

Foster, D. (2007). Making meaning in algebra: Examining students' understandings and misconceptions. In A. H. Schoenfeld (Ed.), *Assessing mathematical proficiency* (pp. 163-176). New York: Cambridge University Press.

Garofalo, J., & Lester, F. K. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education**,** 16**,*** 163-176.

Gay, J. & Cole, M. (1967). *The new mathematics and an old culture: A study of learning among the Kpelle of Liberia.* Holt, Rinehart and Winston, Inc.

Gay, L. R., Mills, G. E. & Airasian, P. (2006). *Educational research: Competencies for analysis and applications, Eighth edition*. NJ: Pearson Education Inc.

Geary, D. C. (1994). *Children's mathematical development: Research and practical applications.* Washington, DC: Americal Psychological Assosciation.

Gilbert, S. (1989). *Principles of educational and psychological measurement and evaluation, Third edition*. CA : Wardsworth.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Goldin, G. (2008). Perspectives on representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education*, second edition. (pp. 176-201), New York: Routledge.

Gourgey, A. F. (2001). Metacognition in basic skills instruction. In H. J. Hartman (Ed.), *Metacognition in learning and instruction: Theory, research and practice* (pp 17-32), USA: Kluwer.

Greeno, J. G., Collins, A. M. & Resnick, L. (1996). Cognition and learning. In D. C. Berliner & R. Calfee (Eds.), *Handbook of Educational Psychology*. NY: Macmillan, pp. 14-46.

Greens, C. E. & Rubenstein, R. (2008). *Algebra and algebraic thinking in school mathematics, Seventieth yearbook*, NCTM.

Greer, B. (1996). Theories of mathematics education: The role of cognitive analyses. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds), *Theories of mathematical learning.* Lawrence Erlbaum Associates, Publishers.

Hallagan, J. E. (2006). The case of Bruce: A teachers's model of his students' algebraic thinking about equivalent expressions. *Mathematics Education Research Journal, 18(1)*, 103-123.

Hammer, D. (1996). Misconceptions or P-Primes: How may alternative perspectives of cognitive structure influence instructional perceptions and intentions? *The Journal of The Learning Science*, *5*, 97-127.

Hatano, G. (1996). A conception of knowledge acquisition and its implications for mathematics education. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin & B. Greer (Eds), *Theories of mathematical learning.* Lawrence Erlbaum Associates, Publishers.

Hein, G. E. (1991). *The museum and the needs of people.* CECA (International Committee of Museum Educators) Conference, Jerusalem, Israel, 15-22, Retrieved November 2, 2008 from http://www.exploratorium.edu/ifi/resources/museumeducation.html.

Herscovics, N & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. *Educational Studies in Mathematics, 27*, 59-78.

Hiebert, J., & Carpenter, T. (1992). Learning and Teaching with Understanding. In D. Grouws. (Ed.), *Handbook of Research on Mathematics Teaching and Learning.* (pp. 65-97). NY: Macmillan Publishing Company.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K. C., Wearne, D., Murray, H., Olivier, A. & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*, Heinemann: Portsmouth.

Hinsley, D. A., Hayes, J. R. & Simon, H. A. (1977). From words to equations: Meaning and representation in algebra word problems. In M. A. Just & P. A. Carpenter (Eds.), *Cognitive processes in comprehension* (pp. 89-106). New York: Wiley.

Inhelder, B. & Piaget, J. (1958). *The growth of logical thinking from childhood to adolescence: An essay on the construction of formal operational structures.* New York: Basic Books.

Jacobs, J., & Paris, S. (1987). Children's metacognition about reading: Issues in definition, measurement, and instruction. *Educational Psychologist, 22***,** 255-278.

Johnson, B. & Christensen, L. (2008). *Educational research: Quantitative, qualitative, and mixed approaches. Third edition*, LA: Sage.

Johnson, R.B. & Onwuegbuzie A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher*, *33*(7), 14-26.

Kaput, J. (1985). Research in the learning and using of mathematics: some genuinely new directions. In D. J. Albers, S. B. Rodi, & A. E. Watkins (Eds.), *New directions in two-year college mathematics: Proceedings of the Sloan Foundation Conference on two-year college mathematics*. July 11-14, Menlo College, Atherton, California, (pp. 313-340). New York: Springer – Verlag.

Kieran, C., Booker, G., Filloy, E., Vergnaud, G., & Wheeler, D. (1990). Cognitive processes involved in learning school algebra. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 96-112). Cambridge University Press.

Kieran, C. (1992). The learning and teaching of school algebra. In D. A. Grouws. (Ed.), *Handbook of Research on Mathematics Teaching and Learning.* (pp. 390-419), NY: Macmillan Publishing Company.

Kilpatrick, J. (1987). What constructivism might be in mathematics education. In J. C. Bergeron, N. Herscovics & C. Kieran (Eds.), *Proceedings of the Eleventh Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 1,* (pp. 3-27). Montreal, Canada.

Kilpatrick, J. & Izsak, A. (2008). A history of algebra in the school curriculum. In C. E. Greens & R. Rubenstein (Eds), *Algebra and algebraic thinking in school mathematics, Seventieth yearbook* (pp. 3-18), NCTM.

Kirshner, D. (1985). Linguistic and mathematical competence. *For the Learning of Mathematics 5* (2), 31–33.

Kitchener, R. F. (1986). *Piaget's theory of knowledge: Genetic epistemology and scientific reason*. New Haven: Yale University Press.

Klein, D. (2007). A quarter century of US 'math wars' and political partisanship. *Journal of the British society for the history of mathematics, 22*(1), 22 - 33.

Kuchemann, D. (1981). Algebra. In K. M. Hart (Ed.), C*hildren's understanding of mathematics, 11-16* (pp. 102-119). London: Murray.

Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*, 29-63.

Lancy, D. F. (1983). *Cross-cultural studies in cognition and mathematics.* Academic Press.

Lane, S. (1993). The conceptual framework for the development of a mathematics performance assessment instrument. *Educational Measurement: Issues & Practices, 12*(2), 16-23.

Lee, L., & Wheeler, D. (1989). The arithmetic connection. *Educational Studies in Mathematics, 20*, 41-54.

Lerman, S. (1990). Alternative perspectives of the nature of mathematics and their influence on the teaching of mathematics. *British Educational Research Journal, 16*(1), 53-61.

Lewis, C. (1981). Skills in algebra. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition* (pp. 85-110). Hillsdale, NJ:Erlbaum.

MacGregor, M. & Stacey, K. (1993). Cognitive models underlying students' formulation of simple linear equations. *Journal for Research in Mathematics Education, 24*(3), 217-232.

Macgregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33,* 1–19.

Marshall, S. P. (1995). *Schemas in Problem Solving*. Cambridge University Press, Cambridge.

Martindale, C. (1991). *Cognitive Psychology: A Neural-network Approach*. USA: Brooks/Cole Publishing Company.

Matlin, M. W. (2005). *Cognition*. USA: John Wiley & Sons.

Matz, M. (1980). Towards a computational theory of algebraic competence. *Journal of Mathematical Behavior*, *3*(1), 93-166.

Matz, M. (1982). Towards a process model for school algebra error. In D. Sleeman and J. S. Brown (Eds.), *Intelligent tutoring systems* (pp. 25-50). New York: Academic Press.

Maxwell, J. A. (2005). *Qualitative research design: An interactive approach*, Second edition. Thousand Oaks, CA:Sage.

Mayer, R. E. (1982). Memory for algebra story problems. *Journal of Educational Psychology, 74*, 199-216.

McAlpine, M. (2002). *Principles of assessment*. Bluepaper No. 1, CAA centre, University of Luton. Retrieved April 7, 2010 from http://www.caacentre.ac.uk/dldocs/Bluepaper1.pdf

Ministry of Education (2005). *The Ontario Curriculum: Grades 9 and 10 mathematics*. Queen's Printer for Ontario.

Nathan, M. J. & Koedinger, K. R. (2000). An investigation of teachers' beliefs of students' algebra development. *Cognition and instruction*, *18*(2), pp. 209-237.

National Council of Teachers of Mathematics (NCTM) (1991). *Professional standards for teaching mathematics*, Reston, VA.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA.

National Council of Teachers of Mathematics (2007). *Mathematics teaching today: Improving practice, improving student learning.* Reston, VA.

Newman, M. (1977). An analysis of sixth-grade pupils' errors on written mathematical tasks. In M. A. Clements & J. Foyster (Eds.), *Research in Mathematics Education in Australia, 1*, 239-258.

Noddings, N. (1990). Constructivism in mathematics education, In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics* (pp. 7-18). NCTM.

Norton, S. & Irvin, J. (2007). *A concrete approach to teaching symbolic algebra*. Retrieved March 24, 2010 from www.merga.net.au/documents/RP502007.pdf.

Nunnally, J. C. (1972). *Educational measurement and evaluation, Second edition.* NY: McGraw-Hill.

Osborne, R. J., & Wittrock, M. C. (1983). Learning science: A generative process. *Science Education*, *67*(4), 498-508.

Philipp, R. A. (1999). The many uses of algebraic variables. In B. Moses (Ed.), *Algebraic thinking, Grades K-12: Readings from NCTM's school-based journals and other publications*. (pp. 157-162), Reston, VA: NCTM.

Piaget, J. (1970). *Genetic epistemology*. New York: Columbia University Press.

Piaget, J. (1977). *Psychology and epistemology: Toward a theory of knowledge*. NY: Penguin books.

Piaget, J. (1985). *The equilibrium of cognitive structures*. Cambridge: Harvard University Press.

Polya, G. (1957). *How to solve it.* Garden City, NY: Doubleday**.**

Posamentier, A. S. (1998). *Tips for the mathematics teacher: Research-based strategies to help students learn.* CA: Corwin Press, Inc.

Pycior, H. M. (1984). Internalism, externalism and beyond: 19^{th} century British algebra. *Historia Mathematica, 11*, 424-441.

Radatz, H. (1979). Error analysis in mathematics education. *Journal for Research in Mathematics Education, 10(3)*, 163-172.

Radipotsane, B. J. (1996). *A proposal for an in-service programme for mathematics teachers in Botswana in view of the new policy in education*. Unpublished MA dissertation, Institute of Education, University of Sussex, UK.

Remmers, H. H. (1965). *A practical introduction to measurement and evaluation, Second edition.* NY: Harper & Row.

Roberts, E. S. (2000). Case studies. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research, Second edition* (pp. 435-454). Thousand Oaks, CA:Sage.

Rosnick, P., & Clement, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *Journal of Mathematical Behavior, 3*(1), 3-27.

Salomon, G. (1993). *Distributed cognitions: Psychological and educational considerations*. Cambridge University Press.

Samo, M. A. (2009). Students' perceptions about the symbols, letters, and signs in algebra and how do these affect their learning of algebra: A case study in a government girl's secondary school, Karachi. *International Journal for Mathematics Teaching and Learning*. Retrieved April 14, 2010 from http://www.cimt.plymouth.ac.uk/journal/samo.pdf

Saxe, G. B. (1991). *Culture and cognitive development: Studies in mathematical understanding.* Lawrence Erlbaum Associates, Publishers

Schoenfeld, A. H. (1983). Episodes and executive decisions in mathematical problem solving. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes,* (pp. 345-395). New York: Academic.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Orlando: Academic Press.

Schoenfeld, A.H. (1987). What's all the fuss about metacognition? In A.H. Schoenfeld (Ed.), *Cognitive science and mathematics education.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. Grouws. (Ed.), *Handbook of Research on Mathematics Teaching and Learning.* (pp. 334-370). NY: Macmillan Publishing Company.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114 - 145.

Skemp, R. R. (1982). Symbolic Understanding. *Mathematics Teaching, 99*, 59-61.

Skemp, R. R. (1987). *The psychology of learning mathematics*. USA: Erlbaum Associates.

Smith, J. P., diSessa, A. A. & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition, *The Journal of the Learning Sciences, 3*(2), 115-163.

Stacey, K. & Chick, H. (2004). Solving the problem with algebra. In K. Stacey, H. Chick, & M. Kendal (Eds.), *The Future of Teaching and Learning of Algebra*. *The 12th ICMI Study* (pp. 1-20). Boston: Kluwer.

Stacey, K. & MacGregor, M. (1994). Algebraic sums and products: Students' concepts and symbolism. In J. P. da Ponte & J. F. Matos (Eds.), *Proceedings of the 18th International Conference for the Psychology of Mathematics Education*, Vol. 43, Lisbon, Portugal, pp. 289-296.

Stacey, K., & MacGregor, M. (1999). Taking the algebraic thinking out of algebra. *Mathematics Education Research Journal, 1,* 24-38.

Stavy, R. & Tirosh, D. (2000). *How students (Mis-)Understand science and mathematics: Intuitive rules*. NY: Teachers College Press.

Steffe, L. P. (1990). Inconsistencies and cognitive conflict: A constructivist's view. *Focus on Learning Problems in Mathematics*, *12*, 99-109.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4,* 259-309.

Steffe, L., & Gale, J. (1995). *Constructivism in education*. Hillsdale, NJ: Lawrence Erlbaum

Strauss, A. L. & Corbin, J. (1994). Grounded theory methodology: An overview. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (pp.273-285). Thousand Oaks, CA: Sage.

Streeck, J. (1979). Good for you. In J. Dittman (Ed.), *Arbeitenzur konversations analyse* (pp.235-257). Tubingen, Germany: Niemeyer.

Tall, D. O & Thomas, M. O. J. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*, 125-147.

Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning* (pp.181-234). Albany, NY: SUNY.

Tirosh, D., & Almog, N. (1989). Conceptual adjustments in progressing from real to complex numbers. In G. Vergnaud, J. Rogalski, and M. Artigue (Eds.), *Proceeding of the 13th International Conference for the Psychology of Mathematics Education, Vol. 3*, Paris, France, pp. 221-227.

Usiskin, Z. (1988). Conceptions of school algebra and uses of variables. In A. F. Coxford & A. P. Shulte (Eds.), *The ideas of algebra, K – 12: 1988 yearbook* (pp. 8-19). National Council of Teachers of Mathematics.

Van de walle, J. & Folk, S. (2008). *Elementary and middle school mathematics: Teaching developmentally.* Second Canadian Edition, Toronto: Pearson.

Von Glaserfeld, E. (1987). Learning as a constructive activity. In C. Janvier (Ed.), *Problems of representation in the teaching and learning of mathematics* (pp. 3-17). Hillsdale, NJ: Lawrence Erlbaum.

Von Glaserfeld, E. (1991). *Radical constructivism in mathematics education*. Dordrecht, The Netherlands: Kluwer.

Von Glaserfeld, E. (1993). Questions and answers about radical constructivism. In K. Tobin (Ed.), *The practice of constructivism in science education* (pp. 23-38). Hillsdale, NJ: Lawrence Erlbaum Associates.

Von Glaserfeld, E. (1995). A constructivist approach to teaching. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 3-16). Hillsdale, NJ: Lawrence Erlbaum.

Von Glaserfeld, E. (1998). Constructivism reconstructed: A reply to Suchtig. In M. Matthews (Ed.), *Constructivism in science education: A philosophical examination* (pp. 93-112). Boston: Kluwer Academic Publishers.

Von Glaserfeld, E., & Steffe, L. P. (1991). Conceptual models in educational research and practice. *Journal of Educational Thought*, *25*, 91-103.

Weinberg, A. (2007). New perspectives on the student-professor problem. In T. Lamberg, & L. R. Wiest (Eds.), *Proceedings of the Twenty-ninth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (pp. 164-170). NV: University of Nevada, Reno. Retrieved February 7, 2010 from http://www.allacademic.com/one/www/research/index.php?click_key=1&PHPSESSID=5ac1081359d51c80d655fd6b3aae93bd

Wertsch, J., & Toma, C. (1995). Discourse and learning in the classroom: A sociocultural approach. In L. Steffe & J. Gale (Eds.), *Constructivism in education*. Hillsdale, NJ: Lawrence Erlbaum.

Wheeler, D. (1989). Contexts for research on the teaching and learning of algebra. In S. Wagner & C. Kieran (Eds.), *Research agenda for mathematics education: Vol.4, Research issues in the learning and teaching of algebra* (pp. 278-287). NJ:Erlbaum.

Wiersma, W. & Jurs, S. G. (2005). *Research methods in education: An introduction, Eighth edition*. Boston: Pearson.

WikiAnswers (2010). *What is the importance of algebra in today's world?* Retrieved 17 September, 2010 from http://wiki.answers.com/Q/What_is_the_importance_of_algebra_in_today's_world.

Wikipedia (2010). *Jean Piaget*. Retrieved 14 October, 2010 from http://en.wikipedia.org/wiki/Jean_Piaget

Wilson, B. G. (1996). *Constructivist learning environments.* Englewood Cliffs, New Jersey: Educational Technology Publications.

Wilson, J. & Clarke, D. (2004). Towards the modeling of mathematical metacognition. *Mathematics Education Research Journal, 16*(2), 25-48.

Wittrock, M. (1986). *Handbook of research on teaching*, Third Edition, New York: Macmillan.

Wood, T., Cobb, P., & Yackel, E. (1995). Reflections on learning and teaching mathematics in elementary school. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (pp. 401-422). Hillsdale, NJ: Lawrence Erlbaum.

Woodward, J. (2004). Mathematics education in the United States: Past to present. *Journal of Learning Disabilities, 37*, 16-31.

Wu, H. (2001). How to prepare students for algebra. *American Educator, 25* (summer), 10-17.

Yin, R. K. (2003). *Case study research: Design and methods.* Thousand Oaks, CA: Sage.

Young, R & O'Shea, T. (1981). Errors in children's subtraction. *Cognitive Science, 5*, 152-177.

Appendices

Appendix 1

Test Instrument - Pilot Study – Stage 1

Student Name**:**

This is a non-evaluative assessment. Your performance in this assessment will have no bearing on your grades or evaluations in the course. The assessment is designed to help you with algebra, by helping your teacher understand the mistakes you make, as well as why you make them.

Instructions:

1. Answer all questions.
2. Use algebraic methods to solve all the problems.
3. Time: one hour

1) Antonio sells y donuts. Maria sells three times as many donuts as Antonio. A donut costs 25 cents.
a) Name a variable in this problem.
b) Name another variable in the problem.
c) Name something in the problem that is not a variable.

2) What does $5y$ mean? Write your answer in words.

3) What does yz mean? Write your answer in words.

4) There are n girl scouts in a parade. There are 8 girls in each row. Write an algebraic expression to find out how many rows of girl scouts are marching in the parade.

5) Simplify: a) $\frac{2x}{2x}$ b) $A \times \frac{1}{A}$ c) $0 \times A$ d) $(A+B)^2$

e) $\frac{r}{4} - \frac{(6-s)}{2}$ f) $x\left(\frac{a}{b}\right)$ g) $\frac{xa+xb}{x+xd}$ h) $\sqrt{-x}.\sqrt{-y} =$

6) Write in another form : $\frac{x-3}{2x}$

7) Compute : $-(3x-y)$

8) Simplify : $\frac{A}{B}+\frac{A}{C}$

9) Add 3 to $5y$.

10) Subtract $2b$ from 7.

11) Multiply $e+2$ by 3.

12) The letter n represents a natural number. What is more, $\frac{1}{n}$ or $\frac{1}{n+1}$? How do you know?

13) Simplify where it is possible.

a) $2x+5y+9z$ b) $7+3x$ c) $p+p+2c+5p$

14) Gill is exactly three years older than Bill. Let G stands for Gill's age and B stands for Bill's age. Write an equation to compare Gill's age to Bill's age.

15) I thought of a number, I added 7 to this number, and then I multiplied the result by 3. I got 36. What was the number I thought about?

16) Write 3 pairs of values for a and b to make $a=b+2$ a true statement.

17) Write an equation using the variables d and c to represent the following statement: "At Mindy's restaurant, for every four people who order doughnuts, there are five people who order coffee." Let d represent the number of doughnuts and c represent the number of coffee ordered.

18) A bookcase has three divisions. A certain amount of books is located in first division there; in the second division there are 13 more books than in the first one and in third one there are 19 more books than in the second one. The whole amount of books is 96, how many books are there in each division?

19) Consider solving the linear system:

$$\begin{aligned} a+b&=5 \\ a-b&=7 \end{aligned}$$

a) To eliminate a, do you add or subtract the two equations?
b) To eliminate b, do you add or subtract the two equations?
c) Will you obtain the same solution if you add or subtract the two equations? Explain.

20) Solve the following linear system of equations.

$$\begin{aligned} 2m+n&=2 \\ 3m-2n&=3 \end{aligned}$$

21) Solve the following linear system of equations.

$$\begin{aligned} x+y&=4 \\ y&=2x+4 \end{aligned}$$

22) Solve the following linear systems. Explain why you chose this method.

$$\frac{x}{2}-\frac{2y}{3}=\frac{7}{3}$$

$$\frac{3x}{2}+2y=5$$

Appendix 2

Test Instrument - Pilot Study – Stage 2

Student Name**:**

This is a non-evaluative assessment. Your performance in this assessment will have no bearing on your grades or evaluations in the course. The assessment is designed to help you with algebra, by helping your teacher understand the mistakes you make, as well as why you make them.

Instructions:

1. Answer all questions.
2. Use algebraic methods to solve all the problems.
3. Time: one hour

1) Antonio sells y donuts. Maria sells three times as many donuts as Antonio. A donut costs 25 cents.
a) Name a variable in this problem.
b) Name another variable in the problem.
c) Name something in the problem that is not a variable.

2) There are n girl scouts in a parade. There are 8 girls in each row. Write an algebraic expression to find out how many rows of girl scouts are marching in the parade.

3) What does yz mean? Write your answer in words.

4) Simplify: a) $A\times\frac{1}{A}$ b) $0\times A$ c) $\left(A+B\right)^2$

d) $\frac{r}{4}-\frac{(6-s)}{2}$ e) $x\left(\frac{a}{b}\right)$ f) $\frac{xa+xb}{x+xd}$

5) Simplify : $\frac{A}{B}+\frac{A}{C}$

6) Subtract $2b$ from 7. 7) Multiply $e+2$ by 3.

8) The letter n represents a natural number. What is more, $\frac{1}{n}$ or $\frac{1}{n+1}$? How do you know?

9) Which is larger y or t in $y+2t+3$. Explain.

10) Sachin is 14 years old now and his father is 40 years old. How many years will it be until Sachin's father is twice as old as Sachin?
11) Shirts cost s dollars each and pants cost p dollars a pair. If I buy 3 shirts and 2 pairs of pants, explain what 3s + 2p represents?

12) The equation 5B = 4R describes the relationship between B, the number of blue cars produced and R, the number of red cars produced by a car company. Next to each of the following statements place a T if the statement follows from the equation, an F if the statement contradicts the equation, and a U if there is no certain connection.

a) There are 5 blue cars produced for every 4 red cars

b) The ratio of red to blue cars is five to four.

c) More blue cars are produced than red cars.

13) Mr. Robertson shared his stamp collection with his two sons and the daughter: Javier, Raul and Teresa. Teresa received 5 times the number of stamps than Javier did, and 4 less stamps than those received by Raul. The whole quantity received by Javier and Raul is 22 stamps. How many stamps did Mr. Robertson give to each child?

14) Solve for x.

$4x + 25 = 73$

15) Consider solving the linear system: $a + b = 5$

$a - b = 7$

a) To eliminate a, do you add or subtract the two equations?

b) To eliminate b, do you add or subtract the two equations?

c) Will you obtain the same solution if you add or subtract the two equations? Explain.

16) Solve the following linear system of equations.

$2m + n = 2$

$3m - 2n = 3$

17) Starting with some number, if you multiply it by 4 and then add 25, you get 73. What number did you start with?

18) Kevin has a phone plan. He pays $10.00 each month plus $0.10 each minute of long distance calls.

One month, Kevin made 100 minutes of long distance calls and his bill was $20.00.

In the next month, he made 300 minutes of long distance calls and his bill was $40.00.

Kevin said, “If I talk 3 times as long it only costs me 2 times as much!”

Will Kevin’s rule always work? Explain your reasoning?

19) Jonathan decided to buy a basketball with his four friends. Each friend agreed to pay the same amount and Jonathan paid the balance of $25. The total cost of the basketball was $73. How much did each friend pay?

Appendix 3

Test Instrument - Main Study

Student Name:

This is a non-evaluative assessment. Your performance in this assessment will have no bearing on your grades or evaluations in the course. The assessment is designed to help you with algebra, by helping your teacher understand the mistakes you make, as well as why you make them.

Instructions:

1. Answer all questions.
2. Use algebraic methods to solve all the problems.
3. Time: one hour

1) Antonio sells y donuts. Maria sells three times as many donuts as Antonio. A donut costs 25 cents.
a) Name a variable in this problem.
b) Name another variable in the problem.
c) Name something in the problem that is not a variable.

2) There are n girl scouts in a parade. There are 8 girls in each row. Write an algebraic expression to find out how many rows of girl scouts are marching in the parade.

3) What does yz mean? Write your answer in words.

4) Simplify: a) $A\left(\frac{1}{A}\right)$ b) $0(A)$ c) Expand: $(A+B)^2$

5) Simplify: a) $\frac{r}{4}-\frac{(6-s)}{2}$ b) $x\left(\frac{a}{b}\right)$ c) $\frac{xa+xb}{x+xd}$

d) $\frac{A}{B}+\frac{A}{C}$

6) Subtract $2b$ from 7.

7) Multiply $e+2$ by 3.

8) The letter n represents a natural number. What is more, $\frac{1}{n}$ or $\frac{1}{n+1}$? How do you know?

9) Which is larger y or t in $y = 2t+3$. Explain.

10) Sachin is 14 years old now and his father is 40 years old. How many years will it be until Sachin's father is twice as old as Sachin?

11) Shirts cost s dollars each and pants cost p dollars a pair. If I buy 3 shirts and 2 pairs of pants, explain what 3s + 2p represents?

12) Write an equation using the variables B and R to represent the following statement: "At Tonota car manufacturing company, for every four blue cars produced, there are five red cars produced". Let B represent the number of blue cars and R represent the number of red cars.

13) Mr. Robertson shared his stamp collection with his two sons and the daughter: Javier, Raul and Teresa. Teresa received 5 times the number of stamps than Javier did, and 4 less stamps than those received by Raul. The whole quantity received by Javier and Raul is 22 stamps. How many stamps did Mr. Robertson give to each child?

14) Solve for x: $4x + 25 = 73$

15) Consider solving the linear system:

$$\begin{aligned} a + b &= 5 \\ a - b &= 7 \end{aligned}$$

a) To eliminate a from both equations, do you add or subtract the two equations?
b) To eliminate b from both equations, do you add or subtract the two equations?
c) Will you obtain the same solution if you add or subtract the two equations? Explain.

16) Solve the following linear system of equations.

$$\begin{aligned} 2m + n &= 2 \\ 3m - 2n &= 3 \end{aligned}$$

17) Starting with some number, if you multiply it by 4 and then add 25, you get 73. What number did you start with?

18) Kevin has a phone plan. He pays $10.00 each month plus $0.10 each minute of long distance calls.
One month, Kevin made 100 minutes of long distance calls and his bill was $20.00.
In the next month, he made 300 minutes of long distance calls and his bill was $40.00.
Kevin said, "If I talk 3 times as long it only costs me 2 times as much!"
Will Kevin's rule always work? Explain your reasoning.

19) Jonathan decided to buy a basketball with his four friends. Each friend agreed to pay the same amount and Jonathan paid the balance of $25. The total cost of the basketball was $73. How much did each friend pay?

Appendix 4

Student Interview Format

Process	Interview question
1. Reading	Please read the question
2. Comprehension/Interpretation	What does the question mean?
3. Strategy selection/skills selection	How will you do the question?
4. Process	Work out the question. Tell me what you do as you proceed
5. Encoding	Write down the answer
6. Consolidation	What does the answer mean?
7. Verification	Is there any way you can check to make sure your answer is right?
8. Conflict	Is there any conflict? (Here the interviewer will ask some conflicting questions to verify whether the student has a conflict in the solving process)

Appendix 5
Letter to school principals

Dear ________,

I am a third year Ph. D. student in the University of Toronto. My thesis supervisor is Dr. Douglas McDougall. I am also an instructor of mathematics programs for adult students at George Brown College and the Ryerson University. For the final thesis in my Ph. D. program, I am hoping to conduct a research study which examines grade 11 students' difficulties in algebra. I have selected your school as one of the two schools to collect data for this study.

The purpose of this study is to identify student difficulties in solving algebraic problems and to suggest some remedial measures to overcome these difficulties. In order to examine student errors and misconceptions, I wish to administer a test instrument to 60 students in two grade 11 classrooms. Later, six students will be selected for interviews based on their answers to the test. The test paper will take approximately one hour to answer and each interview will last within 20 to 30 minutes. Further, I hope to interview two mathematics teachers of those students to get their views on student errors and misconceptions. Each interview will be tape-recorded for later transcription.

I would like to request the participation of your school in this study by allowing me to conduct the test and the interviews. The teachers will be given a summary of their interviews later. You will also be given an opportunity to receive a summary of the findings. I will not use teachers' or students' names or anything else that might identify them in the written work, oral presentations, or publications. The information remains confidential. They are free to change their minds at any time, and to withdraw even after they have consented to participate. They may decline to answer any specific questions. I will destroy the tape recording after the research has been presented and/or published which may take up to five years after the data has been collected. There are no known risks to you for assisting in this study.

This study has been reviewed by OISE/UT, by University of Toronto's Ethical Review Office, and by the TDSB's External Research Review Committee. Please find a copy of the letter of approval from the TDSB ERRC. If you would like more information, please contact me by phone at 416-413-0280 or by e-mail at egunawardena@oise.utoronto.ca. Please contact me at your earliest convenience to discuss the work or to provide your consent to participate.

Thank you for your consideration.
Yours sincerely,

Gunawardena Egodawatte

Appendix 6

Parent/Guardian consent letter

Dear Parent or Guardian,

I am a third year Ph. D. student in the University of Toronto. My thesis supervisor is Dr. Douglas McDougall. I am also an instructor of mathematics programs for adult students at George Brown College and the Ryerson University. For the final thesis in my Ph. D. program, I am hoping to conduct a research study which examines grade 11 students' difficulties in algebra. I have selected your child's school as one of the two schools to collect data for this study.

The purpose of this study is to identify student difficulties in solving algebraic problems and to suggest some remedial measures to overcome these difficulties. In order to examine student errors and misconceptions, I wish to administer a test instrument to 60 students in two grade 11 mathematics classrooms. Your child will be asked to participate in a written test during the Fall semester of 2009. This test will take approximately one hour. The test contains about 30 short answer items. Based on the results, your child may be asked to participate in an interview to identify his or her difficulties in algebraic problem solving. This interview will take not more than 30 minutes.

I would like to request the participation of your child in this study. Participation in this study is voluntary and will not affect your child's attendance in class or his/her evaluation by the school. All information collected will be anonymous. In a way, the results of this study may help the school as well to identify students' difficulties in algebra and propose remedial work.

Please indicate on the attached form whether you permit your child to take part in this study. Your cooperation will be very much appreciated. If you have any questions or would like more information, please contact me by phone at 416-413-0280 or by e-mail at egunawardena@oise.utoronto.ca. Or, if you have any questions about your child's rights as a participant in this study, please contact The Office of Research Ethics of University of Toronto at 416-946-3273 or by email at ethics.review@utoronto.ca.

Thank you for your consideration.
Yours sincerely,

Gunawardena Egodawatte

Parent/Guardian Consent Form

I agree to allow my child ______________________________________ to participate
(son/daughter's name)

In the test ☐

In the interview ☐

Parent's/Guardian's signature: __________________________ Date:_________________

Appendix 7

Mean percentage of incorrect responses for "variables"

Question number	Number of incorrect responses	Percentage	Mean percentage
1(a)	5	17	37
1(b)	16	53	
1(c)	10	33	
3	18	60	
9	10	33	
12	8	27	

Appendix 8
Students' response categories for variables

Student responses for question 1(a)

Student answers	No. of students
y (Expected answer)	24
donuts	1
Let d represent the number of donuts*	1
$3y + y = 25$	1
No answer	3
Total	**30**

* Considered as a correct answer

Student responses for question 1(b)

Student answers	No. of students
$3y$ (Expected answer)	5
x*	3
c*	3
3 times	2
3	2
Cost 25cents	2
Amount of donuts Maria sells*	1
cents	1
25	1
There is only one	1
cost*	1
$y \times 3y = 25$	1
$x =$ no. of sells*	1
No answer	6
Total	**30**

* Considered as a correct answer

Student responses for question 1(c)

Student answers	No. of students
25 cents, 25 (Expected answer)	16
Donuts	3
The cost of donuts*	3
Antonio	2
Three times*	1
cents	1
Maria or Antonio	1
No answer	3
Total	**30**

* Considered as a correct answer

Student responses for question 3

Student answers	No. of students
y multiplied by z, $y \times z$ (Expected answer)	10
yz means a variable	2
Two variables each letter representing an unknown number*	1
yz are two different variables that are combined together*	1
yz means the variable represents part of the question	1
The number of people per row	1
A variable to represent something and another variable to change the value of the first variable	1
yz both are variables	1
No answer	12
Total	**30**

* Considered as a correct answer

Student responses for question 9

Student answers	No. of students
y is larger because $2t$ is part of y and what's part of it is always smaller than the whole*	10
y is larger because it is the sum of the equation*	9
t has a bigger value beside it	3
You cannot know because they are 2 different variables either one could be bigger depending on the number (Expected answer)	1
t is larger because you can multiply by 2	1
y, because you are finding out the value of y *	1
I don't know, they are both variables	1
t, because it's not negative and if it were y, it will be negative	1
None is larger	1
No answer	2
Total	**30**

* Considered as a correct answer

Student responses for question 11

Student answers	No. of students
$3s + 2p$ represents the cost of 3 shirts and 2 pairs of pants (Expected answer)	17
$3s$ represents 3 shirts and $2p$ represents 2 pants	3
The total price for all the clothing being bought*	2
It means you bought 3 shirts for s dollars and 2 pairs of	1

pants for p dollars	
$3s+2p$ represent that 3 shirts will cost an unknown amount and $2p$ will represent 2 pants will be an unknown amount. You add both together and you get a total amount*	1
$3s$ is the cost of 3 shirts $2p$ is the cost of 2 pairs of pants*	1
3 shirts plus 2 pairs of pants as well as 3 dollars plus 2 dollars equals 5 dollars	1
It represents buying 3 shirts and 2 pants	1
It represents the equation to find the price*	1
$3s+2p$ represents 3 shirts and 2 pants being bought for the price of 1	1
No answer	1
Total	**30**

* Considered as a correct answer

Appendix 9

Students' response categories for algebraic expressions

Question 4(a)	Question 4(b)	Question 4(c)	Question 5(a)	Question 5(b)	Question 5(c)	Question 5(d)
$\frac{1A}{A}$ (2) $\frac{A}{A}$ (4) $1A$ (1) $\frac{1}{A^2}$ (2) 1 (3)* $\frac{1A}{A^2}$ (1) $A\times A\times 1 = A$ (1) $\frac{A1}{1A}$ (1) $A\times A\times 1$ (1) $\frac{1}{2A}$ (1) $A\times A+1$ (1) No answer (12)	0 (20)* A (1) $0A$ (1) A^0 (1) $A=0$ (1) No answer (6)	$A^2+AB+BA+B^2$ (2)* A^2+B^2 (5) A^2B^2 (4) $A^2+A^2B^2+B^2$ (1) $A^2+B^2+AB^2$ (1) A^2+B^2+2AB (7) * A^2+B^2+AB (1) $(AB)^2$ (1) $(A+B)(A+B)$ (2) AB^2 (1) No answer (5)	$\frac{r}{4}-\frac{6s}{2}$ (1) $\frac{r-6-s}{2}$ (1) $\frac{r-12-2s}{4}$ (1) $\frac{r-(6-s)}{2}$ (1) $r-\frac{4(6-s)}{2}$ (1) $\frac{r}{4}(6-s)$ (1) $sr-12$ (1) $\frac{r-12+2s}{4}$ (1)* $0.25r-3-s$ (1) $\frac{r}{4}-\frac{(6-s)}{4}$ (1) No answer (20)	$\frac{ax}{bx}$ (9) $\frac{ax}{b}$ (4)* $xa+xb$ (2) $xa=b$ (1) $\frac{a}{xb}$ (1) $x(a\div b)$ (1) No answer (12)	$\frac{x^2ab}{x^2d}$ (7) $\frac{x^2ab}{x^2d}=\frac{ab}{d}$ (1) $\frac{a+b}{d}$ (2) $2x^2abd$ (1) $\frac{xab}{xd}$ (1) $\frac{2xab}{2xd}$ (1) No answer (17)	$\frac{A^2}{BC}$ (4) $\frac{A}{BC}$ (3) $\frac{2A}{BC}$ (3) $\frac{A^2}{B+C}$ (1) $\frac{AC+AB}{BC}$ (2)* $\frac{2A}{B+C}$ (1) $\frac{A}{C}+\frac{A}{C}$ (1) No answer (15)

Question 6	Question 7	Question 8
$-5b$ (2) $7-2b=5b$ (3) $7-2b$ (7)* $7-2b=5$ (1) $2b-7$ (4) $b=3.5$ (2) You cannot subtract(1) 7(1) $5b$ (2) $b=5$ (1) No answer (6)	$6e$ (2) $(e+2)3=3e+2$ (1) $3\times 2(e)$ (2) $e+2(3)=e+6$ (1) $e+2\times 3=2e\times 3=6e$ (1) $3e+6$ (7)* $3(e+2)$ (3) $(e+2)(e+2)(e+2)=e^3+2^3=e^3+8$ (1) $3\times e+2$ (1) $e+6$ (2)	$\frac{1}{n}$ (13)* $\frac{1}{n+1}$ (12) No answer (5)

	$2e \times 3 = 5e$ (1) $e = -6$ (2) $(e+2)^2 = e^2 + 4$ (1) No answer (5)	

* Correct answer

Appendix 10

Student response categories for equations

Question 14	Question 15(a)	Question 15(b)	Question 15(c)	Question 16
$x=12$ (28)* $x=8$ (48+25=73) (1) No answer (1)	Add (1) Subtract (20)* The first you subtract, the second you add (2) No answer (7)	Subtract (4) Add (12)* Both (4) Subtract for first equation and add for the next equation (1) Depends on the equation (1) Add for first equation and subtract for the next equation (1) No answer (7)	Yes (4)* No (13) No answer (13)	$m=1, n=0$ (1)* $2m+n-2=3m-2n-3$ (2) $5m-1n=3$ (1) $6m+3n=6$; $6m-4n=6$; $n=-12$ (1) $-7n=0$, $n=7$ (1) $2m+n=2$; $n=2-2m$; $\frac{n}{0}=\frac{0m}{0}$ (1) $3-2n=3$, $n=3$ (1) $2m+n=2$; $2m+\left(\frac{2}{2}\right)^2-\left(\frac{2}{2}\right)^2+n=2$ (1) No answer (21)

Question 17	Question 19
12 (working backwards method) (10)* $\frac{48}{3}=x$, $x=16$ (2) If you multiply 12 by 4 and then add 25 you will get 73 (Guess and check) (9)* 12 (algebraic method) (6)* No answer (3)	\$12 (working backwards) (16)* $\frac{73}{4}=\$18.25$ (3) $73-25=\frac{48}{3}=\$16$ (4) \$12 (algebraic method) (2)* \$12 (guessing – no method) (1)* No answer (4)

* Correct answer

Appendix 11

Student response categories for word problems

Question 2	Question 10	Question 12	Question 13	Question 18
n^8 (1) $n = \frac{8}{r}$ (1) $8n + n$ (1) $8n$ (6) 8 (1) $\frac{n}{8}$ (4)* $8 \times n = 0$ (1) $y = nx + 8$ (1) $y = \frac{n}{8}$ (2) $x = 8n$ (1) $y = n + 8$ (1) $n + 8$ (1) $x = n8$ (1) $n = 8$ (1) $n \div 8$ (2) $y = 8n$ (1) $r = n \div 8$ (1) No answer (3)	$x = \frac{28 \times 14}{42}$ (1) Arithmetic method (15) 12 years (arithmetic method) (5)* No answer (9)	$4B = 5R$ (12) $Total = 4B + 5R$ (7) $Total = (4B)(5R)$ (2) $B = R + 1$ (2) $R = B + 1$ (1) B - no. of blue cars; R - no. of red cars; Ratio 4:5(1) No answer (5)	$J + R = 22, T = 5 < T < 4$ (1) Arithmetic method –wrong answer (12) $J = 3, T = 15, R = 19$ (algebraic method) (1)* No answer (16)	No, guess reasons (8) No, with reasons (7)* Yes, work with given data (3) Yes, no reasons(1) Yes-wrong reasoning (3) No answer (8)

* Correct answer