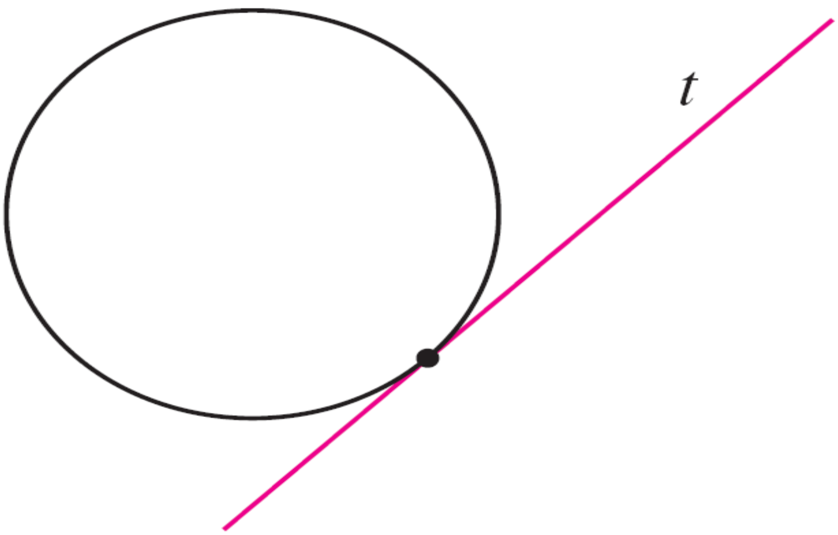
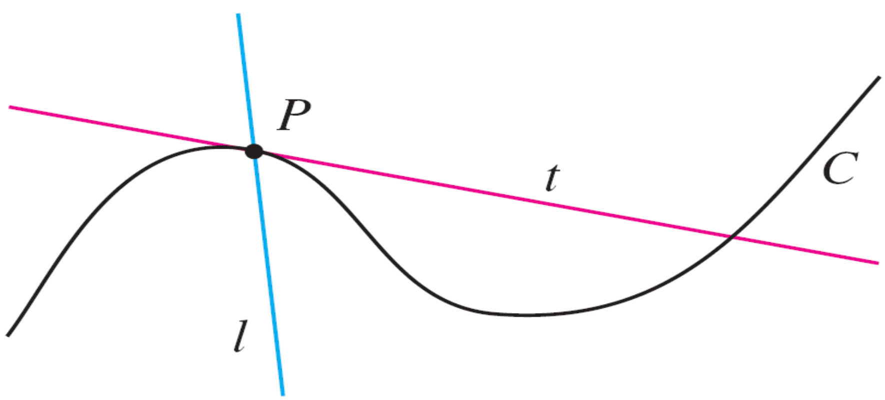
Section 1.4 – Tangents and Velocity

Tangent Lines

A tangent line to a curve is a line that just touches the curve. In terms of a circle, the definition is very simple. A tangent line is a line that intersects the circle at **exactly** one point.



On a curved function, the definition gets a little more complicated.



If we call this curve *C* and look at lines *t* and *l* that pass through point *P*, we see that line *l* intersects curve *C* at exactly one point but does not look like a tangent line. The line *t* looks like a tangent line but intersects *C* at two points. Notice however that "locally" the line *t* only intersects *C* at one point.

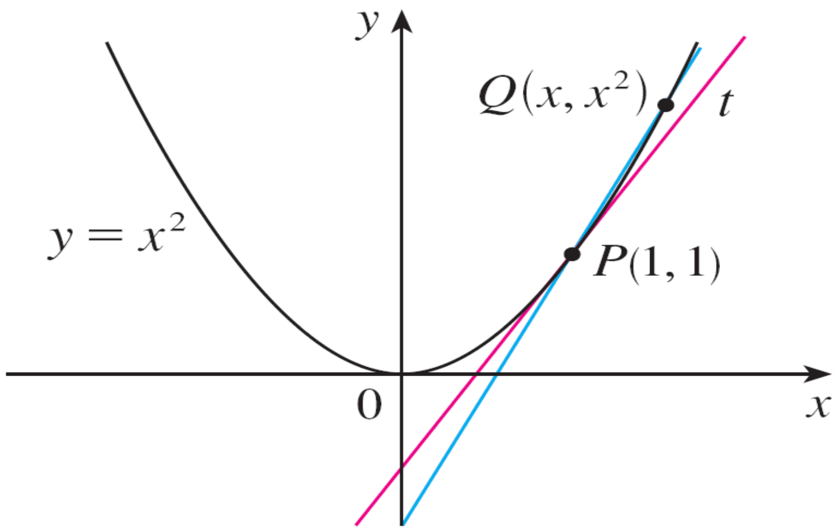
Example 1: Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution: Since we already have a point on the tangent line, we only have to find the slope *m* in order to solve the problem. The difficulty here is that we only have one point, but we need a second in order to compute a slope. Recall that the slope between points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Instead of calculating the slope directly, we will approximate the slope using **secant lines**.

Secant lines
A secant line is a line that intersects a curve at more one point.



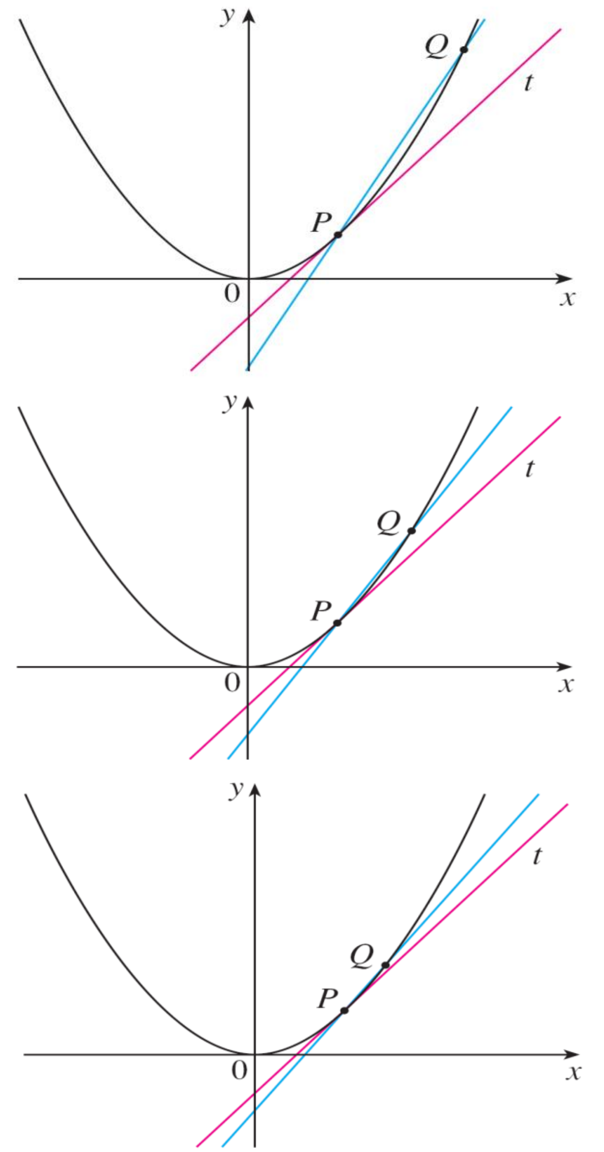
Take the point $Q(x, x^2)$ on the parabola $y = x^2$ near point $P(1, 1)$ and calculate the slope between points *P* and *Q* for different values of *x*.

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

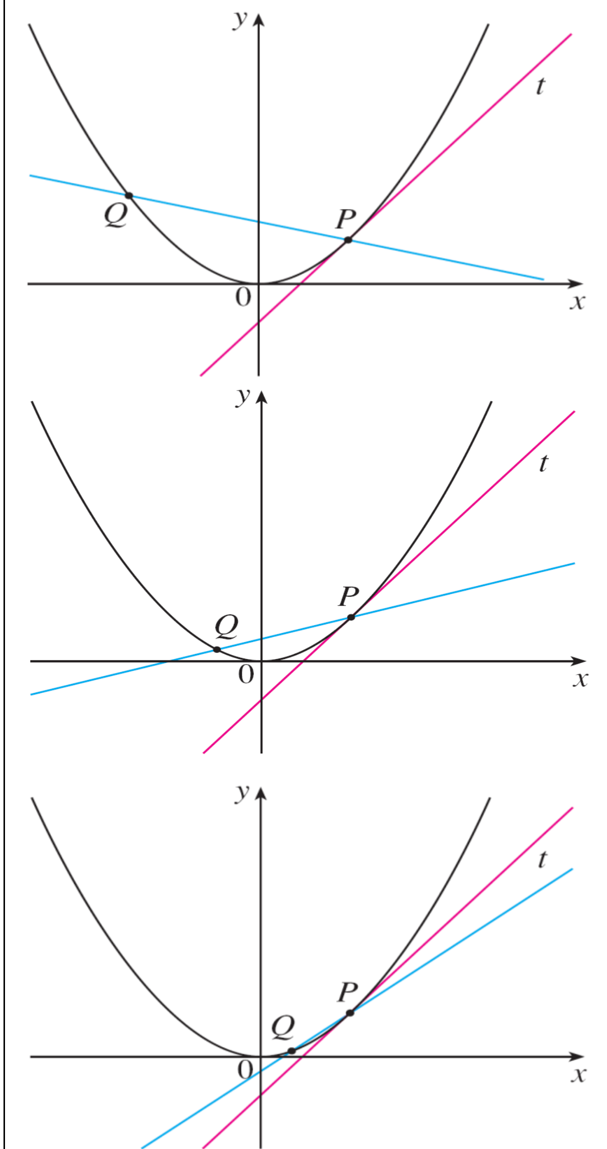
x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

Notice that as x gets close to 1, the slope gets very close to 2. Also see that as Q gets close to P, the secant line comes very close to the tangent line.

Q gets close to P from the right



Q gets close to P from the left



(This is called a limit, but we'll talk more about that later)

Concluding that the slope of the tangent line must be 2, we the equation of the tangent line
$y - 1 = 2(x - 1)$

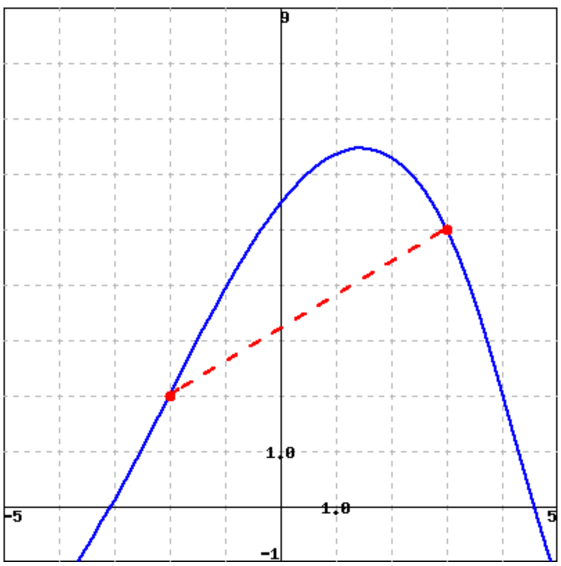
Example 2: Suppose you drop a ball off a tall building and that ball hits the ground after 3 seconds. The ball starts off slow but gains speed as it falls so that it is moving fast right before it hits the ground. The ball's average velocity is given by

$$average\ velocity = \frac{change\ in\ position}{time\ elapsed}$$

If that building is 144 feet tall, then we have the points $(0, 144)$ and $(3, 0)$ in the form (t, d) where *t* is time in seconds and *d* is distance in feet. The average velocity is

$$\frac{d_2 - d_1}{t_2 - t_1} = \frac{0 - 144}{3 - 0} = -48\ ft/s$$

Example 3: Determine the average rate of change of the function between the *x* coordinates of the two points on the graph of the function.



The average rate of change of a function $f(x)$ on the interval $[a, b]$ is

$$average\ rate\ of\ change = \frac{f(b) - f(a)}{b - a}$$

Example 4: If a ball is thrown straight up into the air with an initial velocity of 48 ft/s, its height in feet after t second is given by $s = 48t - 16t^2$.

Find the average velocity for the time period beginning when $t = 1$ and lasting

i) 0.1 seconds

ii) 0.01 seconds

iii) 0.001 seconds

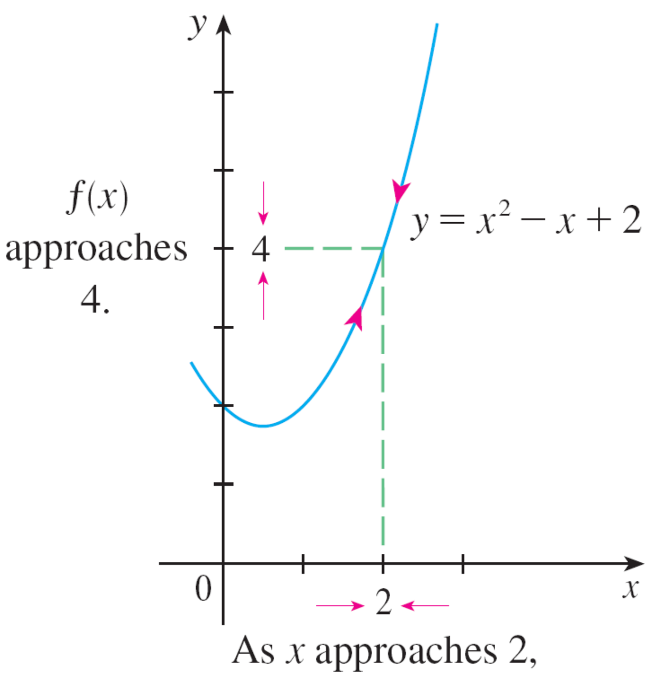
Based on the above results, guess what the instantaneous velocity of the ball is when $t = 1$

Section 1.5 – Limits

Example 1: Let's look at the function $f(x) = x^2 - x + 2$ near $x = 2$.

x	$f(x)$	x	$f(x)$
1.0	2.000000	3.0	8.000000
1.5	2.750000	2.5	5.750000
1.8	3.440000	2.2	4.640000
1.9	3.710000	2.1	4.310000
1.95	3.852500	2.05	4.152500
1.99	3.970100	2.01	4.030100
1.995	3.985025	2.005	4.015025
1.999	3.997001	2.001	4.003001

Looking at the values of the function, it seems as though we could get value as close to 4 as we want by making x get closer to 2. Graphically, we can see this is true below.



We say *the limit of the function* $f(x) = x^2 - x + 2$ *as* x *approaches* 2 *is equal to* 4. The notation for which is

$$\lim_{x \to 2}(x^2 - x + 2) = 4$$

Which seems pretty clear since $f(2) = 4$

1. **Definition** We write

$$\lim_{x \to a} f(x) = L$$

and say *the limit of $f(x)$ as x apprachess a is equal to L*

if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, but not equal to a.

Another notation sometimes used is the following

$$f(x) \to L \quad as \quad x \to a$$

Which reads $f(x)$ *approaches L as x approaches a*

So why do we need limits? In **Example 1** we found that $\lim_{x \to 2}(x^2 - x + 2) = 4$, but we could have gotten that information by just plugging in 2. Limits are useful because they allow us to talk about function at point where the function is not defined.

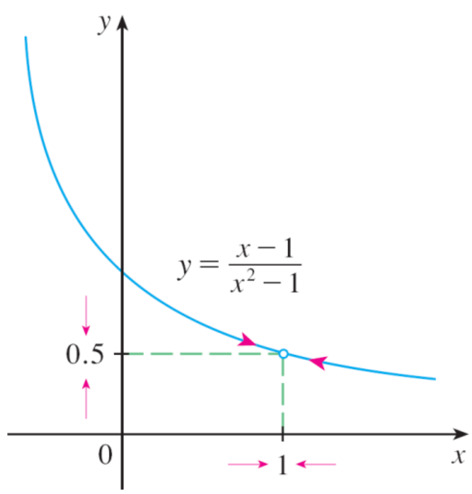
Example 2: Guess the following limit and notice that the function is not defined at $x = 1$.

$$\lim_{x \to 1} \frac{x-1}{x^2-1}$$

$x < 1$	$f(x)$
0.5	0.666667
0.9	0.526316
0.99	0.502513
0.999	0.500250
0.9999	0.500025

$x > 1$	$f(x)$
1.5	0.400000
1.1	0.476190
1.01	0.497512
1.001	0.499750
1.0001	0.499975

We can see from the tables that the limit is probably 0.5 since the function values get close to that number.



Example 3: Consider the function $f(x) = \begin{cases} 1 & if\ x \geq 0 \\ 0 & if\ x < 0 \end{cases}$ and try to find $\lim_{x \to 0} f(x)$

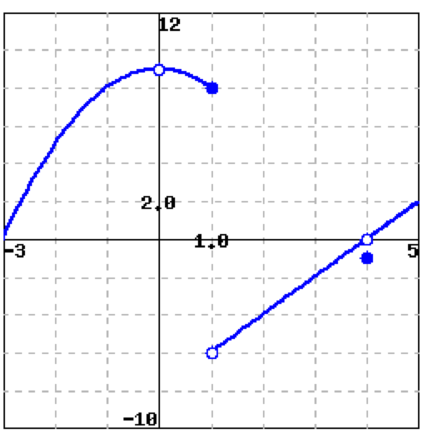
Notice that this is somewhat difficult since the function approaches 1 on the right side, but 0 on the left side.

2. **Definition** We write $$\lim_{x \to a^-} f(x) = L$$ to mean the **left-handed limit** of $f(x)$. We say $as\ x\ approaches\ a\ from\ the\ left\ldots$
Similarly $$\lim_{x \to a^+} f(x) = L$$ is the **right-handed limit** of $f(x)$ and we say $as\ x\ approaches\ a\ from\ the\ left\ldots$

Now we can come back to **Example 3** and find the left and right handed limits at 0. But as it turns out, $\lim_{x \to 0} f(x)$ does not exist.

3. $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Example 4: Find the following limits



a) $\lim_{x \to 0} f(x)$

b) $\lim_{x \to 1^+} f(x)$

c) $\lim_{x \to 1} f(x)$

d) $\lim_{x \to 4} f(x)$

e) Find $f(0)$, $f(1)$, and $f(4)$

Example 5: Let $f(x) = \begin{cases} x + 42 & if\ x \neq 11 \\ 42 & if\ x = 11 \end{cases}$

Find the following limits

$$\lim_{x \to 11^-} f(x)$$

$$\lim_{x \to 11^+} f(x)$$

$$\lim_{x \to 11} f(x)$$

4. **Definition** Let f be a function defined on both sides of a, but not necessarily at a.

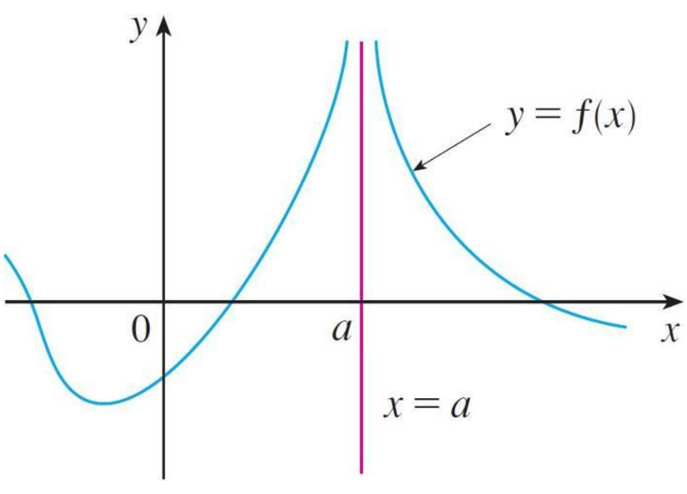
$$\lim_{x \to a} f(x) = \infty$$

Means that we can make the values of $f(x)$ arbitrarily large by taking x sufficiently close to a, but not equal to a.

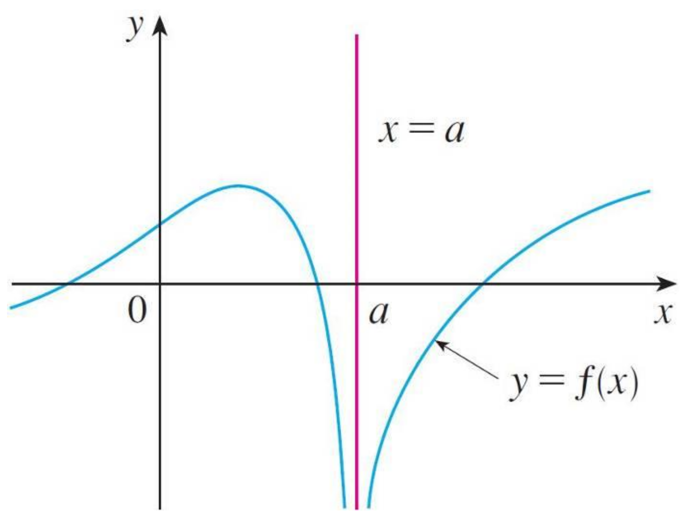
5. **Definition** Let f be a function defined on both sides of a, but not necessarily at a.

$$\lim_{x \to a} f(x) = -\infty$$

Means that we can make the values of $f(x)$ arbitrarily large negative by taking x sufficiently close to a, but not equal to a.



$$\lim_{x \to a} f(x) = \infty$$



$$\lim_{x \to a} f(x) = -\infty$$

Example 6: Find the following limits

$$\lim_{x\to 4^-}\frac{7}{x-4}$$

$$\lim_{x\to 4^-}\frac{7}{x-4}$$

6. **Definition** The line $x=a$ is a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following is true

$$\lim_{x\to a} f(x)=\infty \qquad \lim_{x\to a^-} f(x)=\infty \qquad \lim_{x\to a^+} f(x)=\infty$$

$$\lim_{x\to a} f(x)=-\infty \qquad \lim_{x\to a^-} f(x)=-\infty \qquad \lim_{x\to a^+} f(x)=-\infty$$

Example 7: Find the following limits

$$\lim_{x\to 5}\frac{8}{(x-5)^2}$$

$$\lim_{x\to 3^+}\frac{7}{(x-3)^3}$$

$$\lim_{x\to 6^-}\frac{1}{(x-6)^7}$$

$$\lim_{x\to 4}\frac{5}{(4-x)^8}$$

$$\lim_{x\to 4^+}\frac{5}{(4-x)^9}$$

$$\lim_{x\to -3}\frac{1}{x^3(x+3)^2}$$

Section 1.6 – Limit Laws

Suppose that c is a constant number and the limits

$$\lim_{x\to a} f(x) \qquad and \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

3. $\lim_{x\to a}[c\cdot f(x)] = c\cdot\lim_{x\to a} f(x)$

4. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

5. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Example 1: Given that $\lim_{x\to -2} f(x) = 7$, $\lim_{x\to -2} g(x) = 4$, and $\lim_{x\to -2} h(x) = 2$, find

a) $\lim_{x\to -2}[f(x)+h(x)] =$ ____________________

b) $\lim_{x\to -2}[2f(x)-3g(x)] =$ __________________

c) $\lim_{x\to -2}[g(x)\cdot 5h(x)] =$ ___________________

d) $\lim_{x\to -2}\left[\frac{f(x)}{g(x)-2h(x)}\right] =$ ______________________

A few more limit laws

6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$

7. $\lim_{x\to a} c = c$

8. $\lim_{x\to a} x = a$

9. $\lim_{x \to a} x^n = a^n$

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ (if *n* is even, assume *a*>*0*)

11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where *n* is a positive integer. If *n* is even, assume $\lim_{x \to a} f(x) > 0$.

Example 2: Evaluate the following limits.

a) $\lim_{x \to 4}(3x^2 - x + 4)$

b) $\lim_{x \to 2} \frac{3x^2 - x + 4}{5 - 3x}$

Direct Substitution Property: If *f* is a polynomial or a rational function and *a* is in the **domain** of *f*, then $\lim_{x \to a} f(x) = f(a)$.

12. If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ provided the limit exists.

Example 3: Find $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

Example 4: Find $\lim_{x \to 1} g(x)$ where

$$g(x) = \begin{cases} x + 2 & if\ x \neq 1 \\ \pi & if\ x = 1 \end{cases}$$

Example 5: Evaluate $\lim_{h\to 0} \frac{(4+h)^2-16}{h}$

Example 6: Find $\lim_{h\to 0} g(h)$ where

$$g(h) = \frac{\frac{4}{5+h} - \frac{4}{5}}{h}$$

Example 7: Show that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist

Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possible at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

Then

$$\lim_{x \to a} g(x) = L$$

Example 8: Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

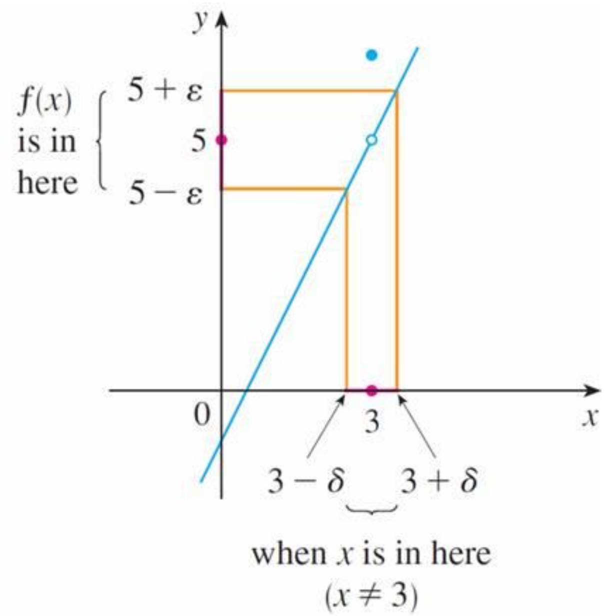
Section 1.7 – Limit Definition

Definition: Let *f* be a function defined on some open interval that contains the number a, except possibly a itself. Then we say **the limit of *f(x)* as *x* approaches *a* is *L***, and we write

$$\lim_{x \to a} f(x) = L$$

If for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$\text{If } 0 < |x - a| < \delta \qquad \text{then} \qquad |f(x) - L| < \epsilon$$



Example 1: Given that $f(x) = x - 2, a = 12, L = 10, and\ \epsilon = .04$, find the largest $\delta > 0$ in the formal definition of a limit so that $|f(x) - L| < \epsilon$.

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Example 2: Given that $f(x) = \sqrt{38 - x}, a = 13, L = 5, and\ \epsilon = 1$, find the largest $\delta > 0$ in the formal definition of a limit so that $|f(x) - L| < \epsilon$.

Example 3: Prove $\lim_{x \to 10}(x - 9) = 1$ using the formal definition of a limit.

Example 4: Prove $\lim_{x \to 8}(7x - 3) = 53$ using the formal definition of a limit.

Example 5: Prove $\lim_{x \to 5}(x^2 - 10x + 42) = 17$ using the formal definition of a limit.

Definition: Let *f* be a function defined on some open interval that contains the number *a*, except possibly *a* itself. Then

$$\lim_{x \to a} f(x) = \infty$$

Means that for every positive number *M* there is a positive number $\delta > 0$ such that

If $0 < |x - a| < \delta$ then $f(x) > M$

Example 6: Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Section 1.8 – Continuity

Definition: A function *f* is **continuous at a number *a*** if

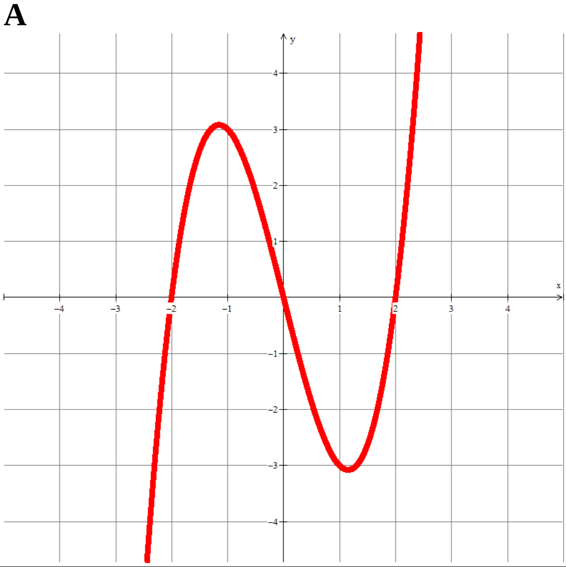
$$\lim_{x \to a} f(x) = f(a)$$

Definition: A function *f* is **continuous on an interval** if it is continuous at every number in that interval.

A function is a **continuous function** if it is continuous at all points in it's **domain**.

Example 1: Which of the following are continuous?

A



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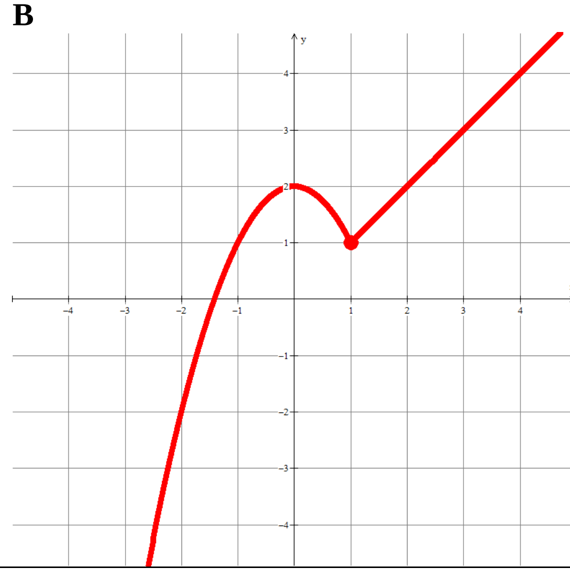

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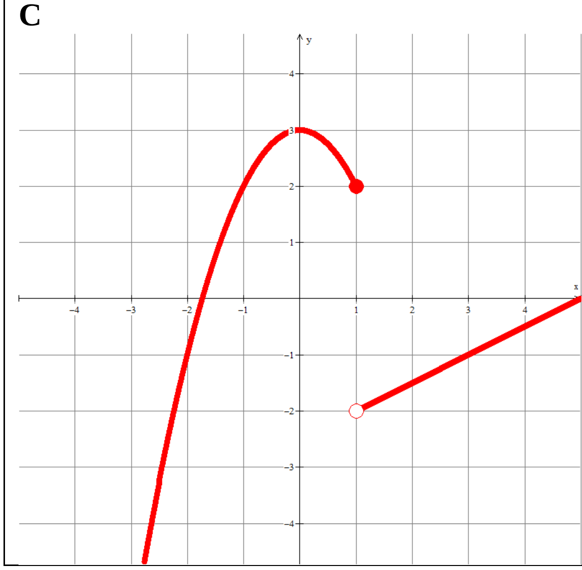

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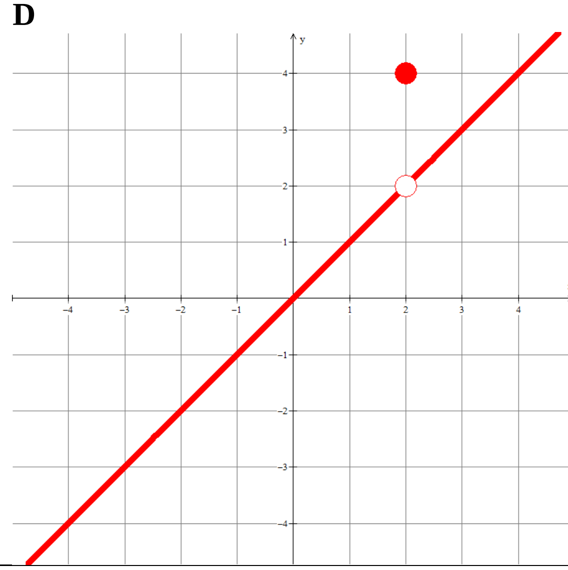

A function *f* is discontinuous at a point *a* if it is not continuous.

Example 2. Where are each of the following discontinuous?

a) $f(x) = \frac{x^2-x-2}{x-2}$

b) $f(x) = \begin{cases} \frac{1}{x^2} & if\ x \neq 0 \\ 1 & if\ x = 0 \end{cases}$

c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & if\ x \neq 2 \\ 3 & if\ x = 2 \end{cases}$

If *f* and *g* are continuous at *a* and *c* is a constant, then the following combinations are also continuous at *a*:

1. $f + g$ **2.** $f - g$ **3.** cf **4.** fg **5.** $\frac{f}{g}$ $if\ g(a) \neq 0$

There are some functions that we know are continuous at every number in their domains:

Polynomials **Rational functions**
Root functions **Trig functions**

Example 3: Give the interval(s) on which each of the following functions are continuous.

a) $f(x) = \frac{x^2-x-2}{x-2}$

b) $f(x) = \sin x$

c) $f(x) = x^5 + 3x^2 - 4$

d) $f(x) = \sqrt{4x-7}$

e) $f(x) = \tan x$

Example 4: Find the value of *a* that makes the following function continuous at 6

$$f(x) = \begin{cases} x^2 - 3x - 28 & if\ x < 6 \\ 2x + a & if\ x \geq 6 \end{cases}$$

Example 5: Find the value of *c* that makes the following function continuous everywhere

$$f(x) = \begin{cases} x^2 - 4 & if\ x < c \\ 8x - 20 & if\ x \geq c \end{cases}$$

If *f* is continuous at *b* and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. Or in other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Furthermore, if *g* is continuous at *a* and *f* is continuous at *g(a)*, then the composite function $f \circ g = f(g(x))$ is continuous at *a*.

Example 6: Find $\lim_{x \to \pi} \sin(3x - \sin(5x))$

The Intermediate Value Theorem (IVT): Suppose f is continuous on the closed interval $[a, b]$ and let N be a number between $f(a)$ and $f(b)$ and $f(a) \neq f(b)$, then there exists some number c in (a, b) such that $f(c) = N$.

Think: If Steve is on the North side of Shaw lane at 11:00 and on the South side at 11:05, then he must have crossed the street sometime between 11:00 and 11:05.

Example 7: Show that there is a root of function $2x^3 - 4x^2 + 3x - 5 = 0$ between 1 and 2.

Example 8: Use the IVT to show that the following function has a solution in the interval $(41, 54)$.

$$\frac{1}{x-41} + \frac{1}{x-54} = 0$$