## Section 1.5 - Linear Models

Some real-life problems can be modeled using linear equations. Now that we know how to find the slope of a line, the equation of a line, and the point of intersection of two lines, we will apply these concepts to different types of linear applications.

## Linear Depreciation

An asset is an item owned that has value.
Linear Depreciation refers to the amount of decrease in the book value of an asset, and is frequently used for accounting and tax purposes.

The purchase price, or original cost of an asset is the price paid for the asset when purchased.
The scrap value of an asset is the remaining value after it is no longer considered to be usable.
Over time, items such as cars, boats, planes, machinery, equipment, computers, etc. lose value. When the value of an item depends on time, value is the dependent variable and time is the independent variable. We can then say that value is a function of time.

Let $V(t)$ represent the value of an item after time $t$. Assuming linear depreciation, the value of the item can be modeled by

$$
V(t)=m t+b
$$

Notice that the above equation is in slope-intercept form, and is equivalent to $y=m x+b$, where the variable $t$ is being used instead of $x$, and $V(t)$ is being used instead of $y$.

In linear depreciation problems, the slope, $m$, is always negative since the value of an item is going down over time. Since the term rate of depreciation already assumes that the value is declining, the rate at which something declines will be given as a positive real number. For example, we might say that an item's rate of depreciation is $\$ 2000$ per year, which means that $m=-2000$.

The value $b$ is the purchase price. This makes sense if we think about the graph; $b$ represents the value of the item at $t=0$ (equivalent to the $y$-intercept in $y=m x+b$, which occurs when $x=0$ ).

Example 1: An insurance company purchases an SUV for its employees. The original cost is $\$ 30,500$. The SUV will depreciate linearly over 5 years, and will then have a scrap value of $\$ 10,300$. Answer the following:
A. What is the rate of depreciation?
B. Give a linear equation that describes the SUV's book value at the end of $t$ years of use, where $0 \leq t \leq 5$.
C. What will be the SUV's book value at the end of the third year?

## Solution:

A. Let $V$ represent the value of the SUV in dollars, and let $t$ represent the time in years. We are given two points on the line, $(0,30500)$ and $(5,10300)$. To find the rate of depreciation, we find the slope of the line that passes through the given points.
$m=\frac{V(5)-V(0)}{5-0}=\frac{10,300-30,500}{5-0}=-4,040$
The rate of depreciation is $\$ 4,040$ per year.
B. The linear equation that models this situation can be represented by $V(t)=m t+b$. We have just found that $m=-4,040$. We are given the purchase price, $b$, which is $\$ 30,500$. Therefore, $V(t)=-4,040 t+30,500$.
(We do not include the dollar signs in the formula, since $V$ represents the dollar value of the SUV.)
C. The end of the third year occurs when $t=3$. Substitute $t=3$ into the model found in part B and calculate the value, $V(3)$.
$V(t)=-4,040 t+30,500$
$V(3)=-4,040(3)+30,500=18,380$
The value of the SUV at the end of the third year will be $\$ 18,380$.

Example 2: A rock band has gained much popularity across the country. They buy a bus to travel to their destinations. The purchase price is $\$ 185,000$. The bus will be depreciated linearly over 10 years, and will then have a scrap value of $\$ 75,000$. Answer the following:
A. What is the rate of depreciation?
B. Give a linear equation that describes the book value of the bus at the end of the $t^{\text {th }}$ year of use, where $0 \leq t \leq 10$.
C. What will be the book value of the bus at the end of the seventh year?
D. When will the bus be worth $\$ 130,000$ ?

## Solution:

A. Let $V$ represent the value of the bus in dollars, and let $t$ represent the time in years. We are given two points on the line: $(0,185000)$ and $(10,75000)$. To find the rate of depreciation, we find the slope of the line that passes through the given points.
$m=\frac{V(10)-V(0)}{10-0}=\frac{75,000-185,000}{10-0}=-11,000$
The rate of depreciation is $\$ 11,000$.
B. The linear equation that models this situation can be represented by $V(t)=m t+b$. We have just found that $m=-11,000$. We are given the purchase price, $b$, which is $\$ 185,000$.
Therefore, $V(t)=-11,000 t+185,000$.
(We do not include the dollar signs in the formula, since $V$ represents the dollar amount of the bus.)
C. The end of the seventh year occurs when $t=7$. Substitute $t=7$ into the model found in part B and calculate the value, $V(7)$.
$V(t)=-11,000 t+185,000$
$V(7)=-11,000(7)+185,000=108,000$
The value of the bus at the end of the seventh year will be $\$ 108,000$.
D. We want to know the time, $t$, when $V=\$ 130,000$. We use the model from part B, set it equal to 130,000 , and solve for $t$.

$$
\begin{aligned}
& -11,000 t+185,000=130,000 \\
& -11,000 t=-55,000 \\
& t=5
\end{aligned}
$$

The value of the bus will be $\$ 130,000$ at the end of five years.

Example 3: A recent accounting graduate opened up a new business and installed a computer system that cost $\$ 45,200$. The computer system will be depreciated linearly over 3 years, and will have a scrap value of $\$ 0$.
A. What is the rate of depreciation?
B. Give a linear equation that describes the computer system's book value at the end of the $t^{\text {th }}$ year, where $0 \leq t \leq 3$.
C. What will the computer system's book value be at the end of the first year and a half?

## Solution:

A. Let $V$ represent value the value of the computer system in dollars, and $t$ represent the time in years. We are given two points on the line, $(0,45200)$ and $(3,0)$. To find the rate of depreciation, we find the slope of the line that passes through the given points.
$m=\frac{V(3)-V(0)}{3-0}=\frac{0-45,200}{3-0}=-15,066.67$
The rate of depreciation is $\$ 15,066.67$ per year. Note that the rate of depreciation is given in U.S. dollars, so we rounded to two decimal places.
B. The linear equation that models this situation can be represented by $V(t)=m t+b$. We know $m=-15,066.67$ and $b=45,200$, so
$V(t)=-15,066.67 t+45,200$.
C. The first year and a half is $t=1.5$. Substitute $t=1.5$ into the model in part B and calculate the value, $V$.

$$
\begin{aligned}
V(1.5) & =-15,066.67(1.5)+45,200 \\
& =22,599.995
\end{aligned}
$$

Since the value is given in U.S. dollars, we will round this number to 22,600. (In fact, if we use the exact value for the slope, $m=-15,066 . \overline{6}$, and substitute that into the equation, we obtain exactly $\$ 22,600$.)

The value of the system at the end of the first year and a half will be $\$ 22,600$.

## Cost, Revenue, and Profit Functions

In business, there are cost, revenue, and profit functions. The cost is the total amount of money that a company spends to produce goods or services.

The cost function involves fixed costs and variable costs. Fixed Costs are costs that are independent of the company's production and sales level (e.g. rent or insurance). Variable Costs are costs that are dependent and proportional to the company's production and sales level (e.g. utilities or material costs).

Revenue is the total amount of money received from the sale of goods or services. Profit is the total amount of money earned after all costs have been covered.

Formulas for the linear cost, revenue, and profit functions are shown below.

## Cost, Revenue and Profit Functions

Linear Cost Function: $C(x)=c x+F$,
where $c$ is the production cost per unit, $x$ is the number of units produced, and $F$ represents the fixed costs. The linear cost function is the sum of the variable costs and the fixed costs.

Linear Revenue Function: $R(x)=s x$,
where $s$ is the selling price per unit and $x$ is the number of units sold.
Linear Profit Function: $P(x)=R(x)-C(x)$

$$
\begin{aligned}
& =s x-(c x+F) \\
& =(s-c) x-F
\end{aligned}
$$

Example 4: A company has a cost function of $C(x)=5 x+50,000$ and a revenue function of $R(x)=13 x$.
A. If the company produces 4,850 units, what is the total cost incurred?
B. If the company sells 5,750 units, what is the revenue received?
C. Find the profit function.
D. If the company produces and sells 7,275 units, what is the profit earned or the loss sustained?

Solution: The cost and revenue functions are given to us. Remember that $x$ represents the number of units produced.
A. To find the cost of producing 4,850 units, we substitute $x=4,850$ into the cost function.

$$
\begin{aligned}
& C(x)=5 x+50,000 \\
& C(4,850)=5(4,850)+50,000=74,250
\end{aligned} .
$$

The total cost incurred is $\$ 74,250$.
B. To find the revenue from selling 5,750 units, we substitute $x=5,750$ into the revenue function.

$$
\begin{aligned}
& R(x)=13 x \\
& R(x)=13(5,750)=74,750
\end{aligned}
$$

The revenue received is $\$ 74,750$.
C. To find the profit function, we substitute the given cost and revenue functions into

$$
\begin{aligned}
P(x) & =R(x)-C(x) \\
P(x) & =R(x)-C(x) \\
& =13 x-(5 x+50,000) \\
& =13 x-5 x-50,000 \\
& =8 x-50,000
\end{aligned}
$$

The profit function is $P(x)=8 x-50,000$.
D. To find the profit or loss from selling 7,275 units, we substitute $x=7,275$ into the profit function found in part C .

$$
\begin{aligned}
& P(x)=8 x-50,000 \\
& P(7,275)=8(7,275)-50,000=8,200
\end{aligned}
$$

Since 8,200 is positive, the company earned a profit of $\$ 8,200$.

Example 5: A company has a production cost of $\$ 24$ per unit, a selling price of $\$ 52$ per unit, and fixed monthly costs of $\$ 150,000$.
A. Find the cost function.
B. If the company produces 10,000 units, what is the total cost incurred?
C. Find the revenue function.
D. If the company sells 10,000 units, what is the revenue received?
E. Find the profit function.
F. If the company produces and sells 10,000 units, what is the profit earned or the loss sustained?

## Solution:

A. The production cost, $c$, is $\$ 24$ and the fixed cost, $F$, is $\$ 150,000$. We substitute these into the cost function.

$$
\begin{aligned}
C(x) & =c x+F \\
& =24 x+150,000
\end{aligned}
$$

The cost function is $C(x)=24 x+150,000$.
B. To find the cost of producing 10,000 units, we substitute $x=10,000$ into the cost function from part A.
$C(x)=24 x+150,000$
$C(10,000)=24(10,000)+150,000=390,000$

The total cost incurred is $\$ 390,000$.
C. To find the revenue function, we substitute the selling price of $\$ 52$ into the formula for the revenue function.

$$
\begin{aligned}
R(x) & =s x \\
& =52 x
\end{aligned}
$$

The revenue function is $R(x)=52 x$.
D. To find the revenue from selling 10,000 units, we substitute $x=10,000$ into the revenue function from part C.

$$
\begin{aligned}
& R(x)=52 x \\
& R(10,000)=52(10,000)=520,000
\end{aligned}
$$

The revenue received is $\$ 520,000$.
E. To find the profit function, we substitute the cost and revenue functions from parts A and C into $P(x)=R(x)-C(x)$.

$$
\begin{aligned}
P(x) & =R(x)-C(x) \\
& =52 x-(24 x+150,000) \\
& =28 x-150,000
\end{aligned}
$$

F. To find the amount of profit or loss, we substitute $x=10,000$ into the profit function found in part E .

$$
\begin{aligned}
& P(x)=28 x-150,000 \\
& P(10,000)=28(10,000)-150,000=130,000
\end{aligned}
$$

Since 130,000 is positive, then the company earns a profit of $\$ 130,000$.

Example 6: Living Active, a gym accessory production company, produces foam rollers for $\$ 10$ per unit. They sell each foam roller for $\$ 25$. Their monthly fixed costs are $\$ 135,000$.
A. Find the profit function.
B. Does the company earn a profit or sustain a loss when 15,300 foam rollers are produced and sold?
C. Does the company earn a profit or sustain a loss when 8,500 foam rollers are produced and sold?
D. Does the company earn a profit or sustain a loss when 9,000 foam rollers are produced and sold?

## Solution:

A. The production cost, $c$, is $\$ 10$, the selling price, $s$, is $\$ 25$, and the fixed cost, $F$, is $\$ 135,000$. We first find $R(x)$ and $C(x)$, in order to use them for the profit formula.

$$
\begin{aligned}
& R(x)=s x=25 x \\
& C(x)=c x+F=10 x+135,000
\end{aligned}
$$

The profit function is:

$$
\begin{aligned}
P(x) & =R(x)-C(x) \\
& =25 x-(10 x+135,000) \\
& =15 x-135,000
\end{aligned}
$$

B. Substitute $x=15,300$ into the profit function found in part A.

$$
\begin{aligned}
& P(x)=15 x-135,000 \\
& P(15,300)=15(15,300)-135,000=94,500
\end{aligned}
$$

Since 94,500 is positive, the company earns a profit of $\$ 94,500$.
C. We substitute $x=8,500$ into the profit formula found in part B.

$$
\begin{aligned}
& P(x)=15 x-135000 \\
& P(8,500)=15(8,500)-135,000=-7,500
\end{aligned}
$$

Since $-7,500$ is negative, the company sustains a loss of $\$ 7,500$.
(Notice that we write "a loss of $\$ 7500$ " and not "a loss of $-\$ 7500$ ", since the word "loss" already indicates a negative amount.)
D. To find the amount of profit or loss, we substitute $x=9,000$ into the profit formula found in part B..

$$
\begin{aligned}
& P(x)=15 x-135000 \\
& P(9,000)=15(9,000)-135,000=0
\end{aligned}
$$

Since $P(9,000)=0$, the company neither makes a profit nor sustains a loss. The company "breaks even" when 9,000 units are produced and sold.

## Break-Even Point

When a company "breaks even", this means that the company neither makes a profit nor sustains a loss. In this case, the profit is zero. Since $P(x)=R(x)-C(x)=0$, this implies that $R(x)=C(x)$. Hence, the break-even point refers the point of intersection of the revenue function and the cost function. An illustration of this is shown below.


The $x$-value of the point of intersection is called the break-even quantity. The $y$-value of the point of intersection is called the break-even revenue.

Note:
If an $x$-value is larger than the break-even quantity, this results in a profit.
If an $x$-value is smaller than the break-even quantity, this results in a loss.
Example 7: A company has a cost function of $C(x)=14 x+133,600$ and a revenue function of $R(x)=22 x$.
A. Find the break-even quantity.
B. Find the break-even revenue.
C. Find the break-even point.
D. If the company produces and sells 20,000 units, does this result in a profit or a loss?

## Solution:

A. The break-even quantity is the $x$-value of the point of intersection of the revenue function and the cost function. To find the break-even quantity, we set the revenue function equal to the cost function and solve for $x$.

$$
\begin{aligned}
& R(x)=C(x) \\
& 22 x=14 x+133,600 \\
& 8 x=133,600 \\
& x=16,700
\end{aligned}
$$

The break-even quantity is 16,700 units.
B. The break-even revenue is the $y$-value of the point of intersection of the revenue function and the cost function. In part A, we found the $x$-value of the point of intersection; we now want to find the corresponding $y$-value. We can substitute the break-even quantity from part A into either the revenue function or the cost function. Both substitutions are shown below, though only one is needed.

$$
\begin{aligned}
& R(x)=22 x \\
& R(16,700)=22(16,700) \\
& R(16,700)=367,400
\end{aligned}
$$

or

$$
\begin{aligned}
& C(x)=14 x+133,600 \\
& C(16,700)=14(16,700)+133,600 \\
& C(16,700)=367,400
\end{aligned}
$$

The break-even revenue is $\$ 367,400$.
C. The break-even point is the point on the graph where the intersection occurs, so we write the break-even quantity and the break-even revenue as an ordered pair. The break-even point is $(16700,367400)$.
D. We want to determine if the production and sale of 20,000 units results in a profit or a loss. Since 20,000 is more than the break-even quantity of 16,700 , this results in a profit.

Example 8: Easy Cooking produces small crock pots for $\$ 18$ per unit. They sell each crock pot for $\$ 42$. Their monthly fixed costs are $\$ 264,000$.
A. Find the break-even quantity.
B. Find the break-even revenue.
C. Find the break-even point.
D. If the company produces and sells 10,500 units, does this result in a profit or a loss?
E. If the company produces and sells 25,000 units, does this result in a profit or a loss?

## Solution:

A. The break-even quantity is the $x$-value of the point of intersection of the revenue function and the cost function. To find the break-even quantity, we set the revenue function equal to the cost function and solve for $x$.

$$
\begin{aligned}
& R(x)=C(x) \\
& 42 x=18 x+264,000 \\
& 24 x=264,000 \\
& x=11,000
\end{aligned}
$$

The break-even quantity is 11,000 crock pots.
B. The break-even revenue is the $y$-value of the point of intersection of the revenue function and the cost function. In part A, we found the $x$-value of the point of intersection; we now want to find the corresponding $y$-value. We can substitute the break-even quantity from part A into either the revenue function or the cost function. Both substitutions are shown below, though only one is needed.

$$
\begin{array}{lll}
R(x)=42 x \\
R(11,000)=42(11,000) & \text { or } & C(x)=18 x+264,000 \\
R(11,000)=462,000 & & C(11,000)=18(11,000)+264,000 \\
& & C(11,000)=462,000
\end{array}
$$

The break-even revenue is $\$ 462,000$.
C. The break-even point is the point on the graph where the intersection occurs, so we write the break-even quantity and the break-even revenue as an ordered pair. The break-even point is $(11000,462000)$.
D. We want to determine if the production and sale of 10,500 units results in a profit or a loss. Since 10,500 is fewer than the break-even quantity of 11,000 , this results in a loss.
E. We want to determine if the production and sale of 25,000 units results in a profit or a loss. Since 25,000 is more than the break-even quantity of 11,000 , this results in a profit.

## Supply, Demand and Market Equilibrium

A linear demand function describes the relationship between price and quantity demanded.

A linear demand function is of the form $D(p)=m p+b$, where $p$ is the price per unit, $D(p)$ is the quantity demanded at the price $p, m$ is the slope, and $b$ is the $y$-intercept. An illustration is shown below. The slope of this function is negative since as the price increases, the quantity demanded decreases.


A linear supply function describes the relationship between price and quantity supplied.
A linear supply function is of the form $S(p)=m p+b$, where $p$ is the price per unit, $S(p)$ is the quantity supplied at the price $p, m$ is the slope, and $b$ is the $y$-intercept. An illustration is shown below. The slope of this function is positive since as the price increases, the seller is willing to supply more.


Notice that the functions $D(p)=m p+b$ and $S(p)=m p+b$ are both in slope-intercept form. They are in the same form as $y=m x+b$, where $p$ is in the place of $x$, and the $y$-value represents either the demand or supply, respectively. Also note that although $m$ is used in both formulas, the slope in the demand function is not equal to the slope in the supply function.

Market Equilibrium occurs when the demand for a product or service equals the supply for the product or service.

Graphically, this is represented by the intersection of the demand and supply functions. The point of intersection is called the equilibrium point.


The price value of the point of intersection is called the equilibrium price, and it represents the price per unit for which the quantity demanded is equal to the quantity supplied.

The function value of the point of intersection is called the equilibrium quantity, and it represents the quantity for which the quantity demanded is equal to the quantity supplied.

Example 9: The demand for a product is given by the demand function $D(p)=-32 p+900$.
The supply for the product is given by the supply function $S(p)=8 p+300$. In these functions, $p$ is the price per unit. Find the following:
A. the equilibrium price.
B. the equilibrium quantity
C. the equilibrium point

## Solution:

A. The equilibrium price is the $p$-value of the point of intersection of the demand function and the supply function. To find the equilibrium price, we set the demand function equal to the supply function and solve for $p$.

$$
\begin{aligned}
& D(p)=S(p) \\
& -32 p+900=8 p+300 \\
& -40 p=-600 \\
& p=15
\end{aligned}
$$

The equilibrium price is $\$ 15$.
B. The equilibrium quantity is the function value of the point of intersection of the demand function and the supply function. We can substitute the equilibrium price from part A into either the demand function or the supply function. Both substitutions are shown below, though only one is needed.

$$
\begin{array}{lll}
D(p)=-32 p+900 & & S(p)=8 p+300 \\
D(15)=-32(15)+900 & \text { or } & S(15)=8(15)+300 \\
D(15)=420 & & S(15)=420
\end{array}
$$

The equilibrium quantity is 420 units.
C. The equilibrium point is the equilibrium price and the equilibrium quantity written as an ordered pair. The equilibrium point is $(15,420)$.

Example 10: A local sporting goods store provides the following demand and supply functions for a particular heart monitor. The demand function is given by $D(p)=-65 p+1,940$. The supply function is given by $S(p)=87 p+420$. In these functions, $p$ represents the price per heart monitor. Find the following:
A. the equilibrium price.
B. the equilibrium quantity
C. the equilibrium point

## Solution:

A. The equilibrium price is the $p$-value of the point of intersection of the demand function and the supply function. To find the equilibrium price, we set the demand function equal to the supply function and solve for $p$.

$$
\begin{aligned}
& D(p)=S(p) \\
& -65 p+1,940=87 p+420 \\
& -152 p=-1520 \\
& p=10
\end{aligned}
$$

The equilibrium price is $\$ 10$.
B. The equilibrium quantity is the function value of the point of intersection of the demand function and the supply function. We can substitute the equilibrium price from part A into either the demand function or the supply function. Both substitutions are shown below, though only one is needed.

$$
\begin{aligned}
& D(p)=-65 p+1940 \\
& D(10)=-65(10)+1940 \\
& D(10)=1,290
\end{aligned}
$$

$$
\begin{aligned}
& S(p)=87 p+420 \\
& S(10)=87(10)+420 \\
& S(10)=1,290
\end{aligned}
$$

The equilibrium quantity is 1,290 units.
C. The equilibrium point is the equilibrium price and the equilibrium quantity written as an ordered pair. The equilibrium point is $(10,1290)$.

## Least Squares

Suppose we are given a set of data points such that when they are plotted, they do not all lie on a single straight line. We can find a linear equation that "best fits" the set of data. The procedure for finding this best-fit line is called the Least-Squares Method. The best-fit line is the line that produces the minimum sum of the squares of the deviations of the points to the line. The best-fit line is called the least-squares line or regression line and is of the form $f(x)=m x+b$.

## The Least Squares Method

Given a set of $n$ data points: $\left(x_{1}, y_{1}\right),\left(x_{2}, y_{2}\right), \ldots,\left(x_{n}, y_{n}\right)$

1. Calculate the sum of the $x$-values, $\sum x_{i}$, where $\sum x_{i}=x_{1}+x_{2}+\ldots+x_{n}$.
2. Calculate the sum of the $y$-values, $\sum y_{i}$, where $\sum y_{i}=y_{1}+y_{2}+\ldots+y_{n}$.
3. Square each $x$-value, and then find the sum of these squares, $\sum x_{i}{ }^{2}$, where $\sum x_{i}{ }^{2}=x_{1}{ }^{2}+x_{2}{ }^{2}+\ldots+x_{n}{ }^{2}$.
4. Multiply the corresponding $x$ - and $y$-values, then find the sum of these products, $\sum x_{i} y_{i}$, where $\sum x_{i} y_{i}=x_{1} y_{1}+x_{2} y_{2}+\ldots+x_{n} y_{n}$.
5. Substitute the sums from above into the following equations, known as normal equations. Then use either the substitution method of the elimination method to solve for the variables $m$ and $b$.

$$
\left\{\begin{aligned}
n b+\left(x_{1}+\ldots+x_{n}\right) m & =\left(y_{1}+\ldots+y_{n}\right) \\
\left(x_{1}+\ldots+x_{n}\right) b+\left(x_{1}^{2}+\ldots+x_{n}^{2}\right) m & =\left(x_{1} y_{1}+\ldots+x_{n} y_{n}\right)
\end{aligned}\right.
$$

The same normal equations are shown below with summation notation:
$\left\{\begin{aligned} n b+\left(\sum x_{i}\right) m & =\sum y_{i} \\ \left(\sum x_{i}\right) b+\left(\sum x_{i}^{2}\right) m & =\sum\left(x_{i} y_{i}\right)\end{aligned}\right.$
6. Substitute $m$ and $b$ into $f(x)=m x+b$ to obtain the least-squares line.

Example 11: Determine the least-squares line for the given data set: $(1,3),(2,4),(5,6)$
Solution: A table is shown below. We make a column for the $x$-values (1,2, and 5), and a column for the $y$-values ( 3,4 , and 6 ). We square each of the $x$-values and make a column for those squares, $x^{2}$. We then find the product of each corresponding $x$ - and $y$-value, and make a column for those products, $x y$. Finally, we create one more row at the bottom of the table to find the sum of the numbers in each column.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{x}^{\mathbf{2}}$ | $\boldsymbol{x y}$ |
| :---: | :---: | :---: | :---: |
| 1 | 3 | 1 | 3 |
| 2 | 4 | 4 | 8 |
| 5 | 6 | 25 | 30 |
| SUMS: | $\mathbf{8}$ | $\mathbf{1 3}$ | $\mathbf{3 0}$ |
| $\mathbf{4 1}$ |  |  |  |

Since we are given 3 data points, $n=3$. We now substitute $n=3$ along with the sums from the table into the normal equations, as shown below.
$\left\{\begin{aligned} n b+\left(x_{1}+\ldots+x_{n}\right) m & =\left(y_{1}+\ldots+y_{n}\right) \\ \left(x_{1}+\ldots+x_{n}\right) b+\left(x_{1}^{2}+\ldots+x_{n}^{2}\right) m & =\left(x_{1} y_{1}+\ldots+x_{n} y_{n}\right)\end{aligned}\right.$
$\left\{\begin{array}{l}3 b+8 m=13 \\ 8 b+30 m=41\end{array}\right.$
We need to solve for $m$ and $b$. Let us choose to eliminate the variable $b$. Looking at the $b$-terms, the least common multiple of $3 b$ and $8 b$ is $24 b$. We can multiply the first equation by 8 so that it contains the term $24 b$, and multiply the second equation by -3 so that it contains the term $-24 b$.
$8(3 b+8 m)=8(13)$
$-3(8 b+30 m)=-3(41)$
We obtain the resulting system of equations:
$\left\{\begin{aligned} 24 b+64 m & =104 \\ -24 b-90 m & =-123\end{aligned}\right.$

Adding the resulting equations, we obtain $-26 m=-19$, so $m=\frac{19}{26}$.
Next, we substitute $m=\frac{19}{26}$ into one of the equations, and then solve for $b .3 b+8 m=13$, so $3 b+8\left(\frac{19}{26}\right)=13$. Simplifying further, $3 b+\frac{76}{13}=13$, so $3 b=13-\frac{76}{13}$. Making a common denominator, $3 b=\frac{169}{13}-\frac{76}{13}$, so $3 b=\frac{93}{13}$. Multiplying both sides by $\frac{1}{3}$, $\frac{1}{3} \cdot 3 b=\frac{1}{3} \cdot \frac{93}{13}$, so $b=\frac{31}{13}$.

Finally, we substitute the values of $m$ and $b$ into the equation $f(x)=m x+b$.
The least-squares line is $f(x)=\frac{19}{26} x+\frac{31}{13}$.
***
Example 12: In a certain part of a particular city, there is an increase in new construction of apartment complexes. The city gathered the following set of data. The first column represents the number of years since the data was first noted, and the second column represents the total number of new apartment complexes built up to that time. (The "new" buildings are any buildings built since the data was first noted.)

| Year | Complexes |
| :---: | :---: |
| 1 | 3 |
| 2 | 7 |
| 3 | 12 |

A. Determine the least-squares line for the given data.
B. Assuming the trend continues, use the least-squares line to approximate the total number of new apartment complexes that will be built by the end of the fifth year.

## Solution:

A. Let $x$ represent the number of years since the data was first noted, and let $y$ represent the total number of new apartment complexes built up to that time. A table is shown below.

We make a column for the $x$-values (1,2, and 3), and a column for the $y$-values (3, 7, and 12). We square each of the $x$-values and make a column for those squares, $x^{2}$. We then find the product of each corresponding $x$ - and $y$-value, and make a column for those products, $x y$. Finally, we create one more row at the bottom of the table to find the sum of the numbers in each column.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\boldsymbol{x}^{\mathbf{2}}$ | $\boldsymbol{x y}$ |  |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 3 | 1 | 3 |  |
| 2 | 7 | 4 | 14 |  |
| 3 | 12 | 9 | 36 |  |
| SUMS: | $\mathbf{6}$ | $\mathbf{2 2}$ | $\mathbf{1 4}$ | $\mathbf{5 3}$ |

Since we are given 3 data points, $n=3$. We now substitute $n=3$ along with the sums from the table into the normal equations, as shown below. (The summation notation is shown in the formulas below.)

$$
\begin{aligned}
& \left\{\begin{array}{c}
n b+\left(\sum x_{i}\right) m=\sum y_{i} \\
\left(\sum x_{i}\right) b+\left(\sum x_{i}^{2}\right) m=\sum\left(x_{i} y_{i}\right)
\end{array}\right. \\
& \left\{\begin{array}{l}
3 b+6 m=22 \\
6 b+14 m=53
\end{array}\right.
\end{aligned}
$$

We need to solve for $m$ and $b$. Let us choose to eliminate the variable $b$. Looking at the $b$ terms, the least common multiple of $3 b$ and $6 b$ is $6 b$. We can multiply the first equation by -2 so that it contains the term $-6 b$, and we can keep the second equation the same since it already contains the term $6 b$.
$-2(3 b+6 m)=-2(22)$
$6 b+14 m=53$
We obtain the resulting system of equations:
$\left\{\begin{aligned}-6 b-12 m & =-44 \\ 6 b+14 m & =53\end{aligned}\right.$

Adding the resulting equations, we obtain $2 m=9$, so $m=\frac{9}{2}$.
Next, we substitute $m=\frac{9}{2}$ into one of the equations, and then solve for $b .3 b+6 m=22$,
so $3 b+6\left(\frac{9}{2}\right)=22$. Simplifying further, $3 b+27=22$, so $3 b=-5$. Dividing both sides by 3 , we find that $b=-\frac{5}{3}$.

Finally, we substitute the values of $m$ and $b$ into the equation $f(x)=m x+b$.

The least-squares line is $f(x)=\frac{9}{2} x-\frac{5}{3}$.
B. We need to use the least-squares line to approximate the total number of new apartment complexes that will be built by the end of the fifth year.

To do this, we substitute $x=5$ into $f(x)=\frac{9}{2} x-\frac{5}{3}$ and find the corresponding $y$-value.
$f(5)=\frac{9}{2}(5)-\frac{5}{3}=\frac{125}{6}=205 / 6 \approx 20$
Assuming the trend continues, a total of 20 new apartment complexes will be built by the end of the fifth year. (We are rounding down since according to our model, the $21^{\text {st }}$ complex will not be completed by the end of the fifth year.)

