SECTION 10: FREQUENCY-RESPONSE DESIGN

ESE 499 – Feedback Control Systems

Introduction

- We have seen how to design feedback control systems using the ***root locus***
- In this section of the course, we'll learn how to do the same using the open-loop ***frequency response***
- **Objectives**:
  - Determine static error constants from the open-loop frequency response
  - Determine closed-loop stability from the open-loop frequency response
  - Use the open-loop frequency response for compensator design to:
    - Improve steady-state error
    - Improve transient response

Steady-State Error from Bode Plots

Static Error Constants

- For unity-feedback systems, open-loop transfer function gives ***static error constants***
  - Use static error constants to calculate ***steady-state error***

$$K_p = \lim_{s \to 0} G(s)$$

$$K_v = \lim_{s \to 0} sG(s)$$

$$K_a = \lim_{s \to 0} s^2 G(s)$$

- We can also determine static error constants from a system's ***open-loop Bode plot***

Static Error Constant – Type 0

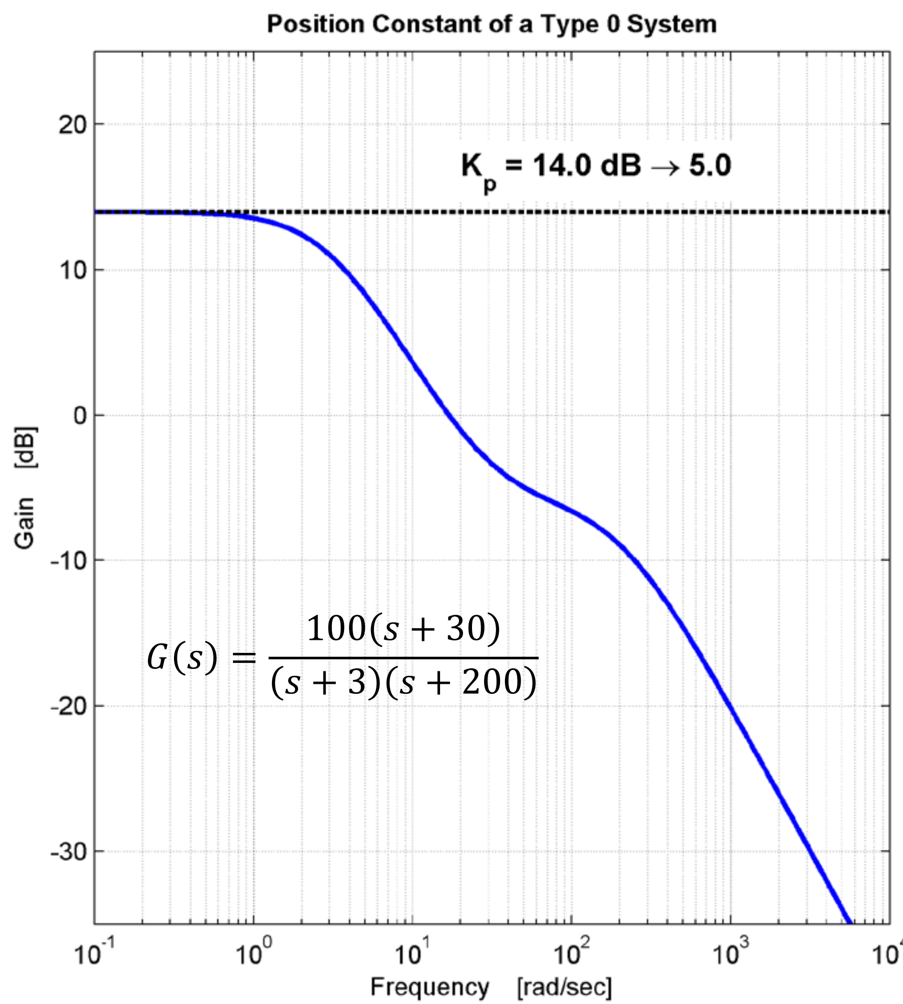
- For a type 0 system

$$K_p = \lim_{s \to 0} G(s)$$

- At low frequency, i.e. below any open-loop poles or zeros

$$G(s) \approx K_p$$

- Read K_p directly from the open-loop Bode plot
  - Low-frequency gain



Static Error Constant – Type 1

- For a type 1 system

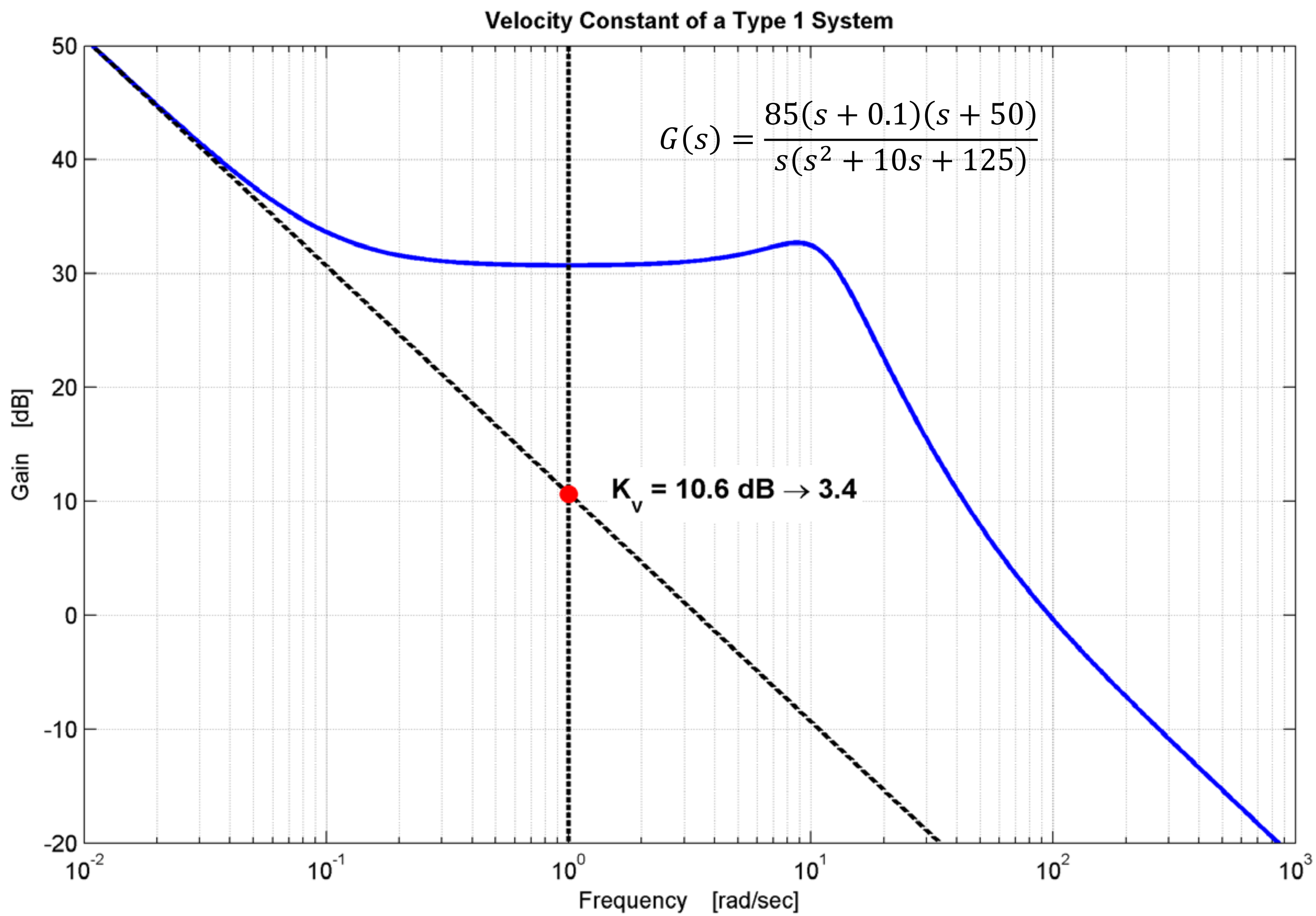
$$K_v = \lim_{s \to 0} sG(s)$$

- At low frequencies, i.e. below any other open-loop poles or zeros

$$G(s) \approx \frac{K_v}{s} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_v}{\omega}$$

- A straight line with a slope of $-20\ dB/dec$
- Evaluating this low-frequency asymptote at $\omega = 1$ yields the velocity constant, K_v
- On the Bode plot, extend the low-frequency asymptote to $\omega = 1$
  - Gain of this line at $\omega = 1$ is K_v

Static Error Constant – Type 1



Static Error Constant – Type 2

- For a type 2 system

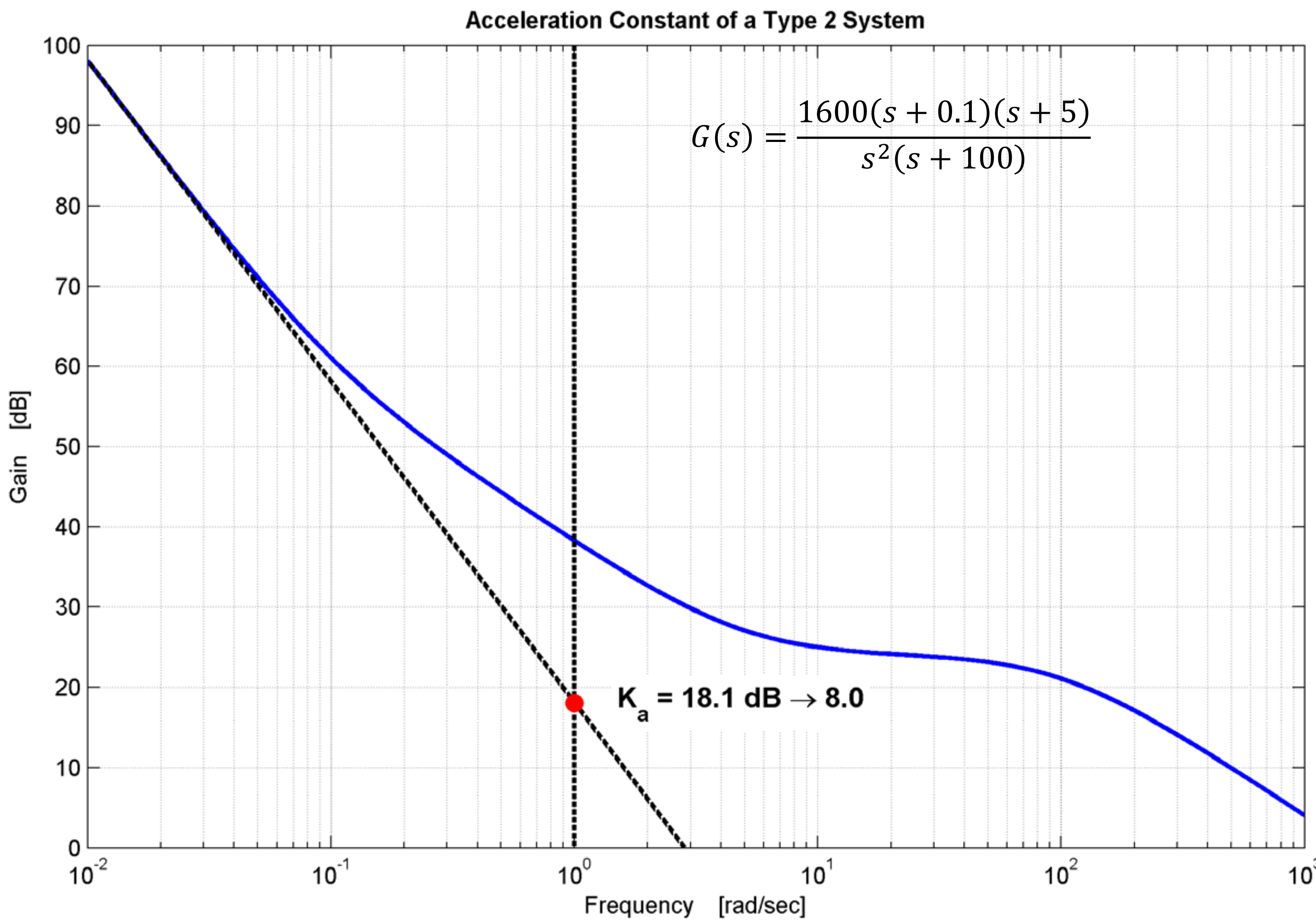
$$K_a = \lim_{s \to 0} s^2 G(s)$$

- At low frequencies, i.e. below any other open-loop poles or zeros

$$G(s) \approx \frac{K_a}{s^2} \quad \text{and} \quad |G(j\omega)| \approx \frac{K_a}{\omega^2}$$

- A straight line with a slope of $-40\ dB/dec$
- Evaluating this low-frequency asymptote at $\omega = 1$ yields the acceleration constant, K_a
- On the Bode plot, extend the low-frequency asymptote to $\omega = 1$
  - Gain of this line at $\omega = 1$ is K_a

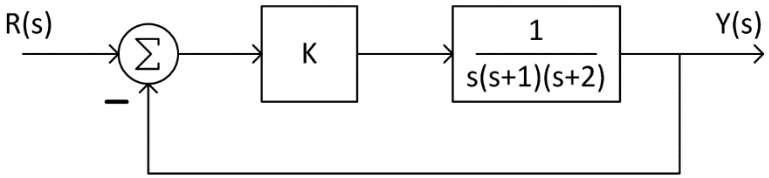
Static Error Constant – Type 2



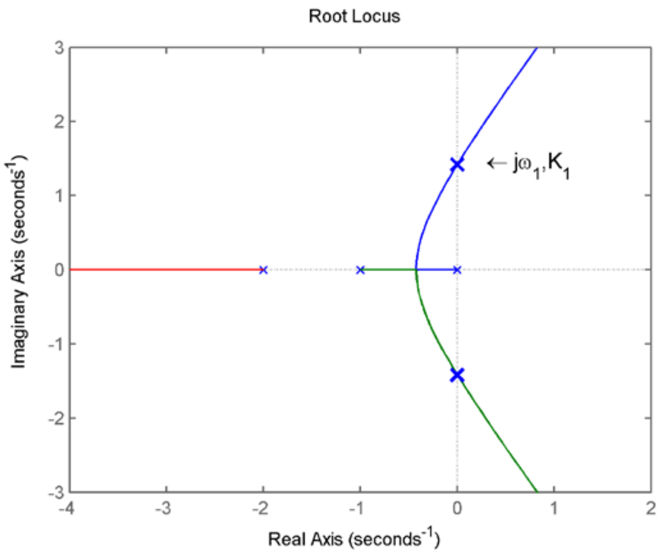
Stability from Open-Loop Bode Plots

Stability

- Consider the following system

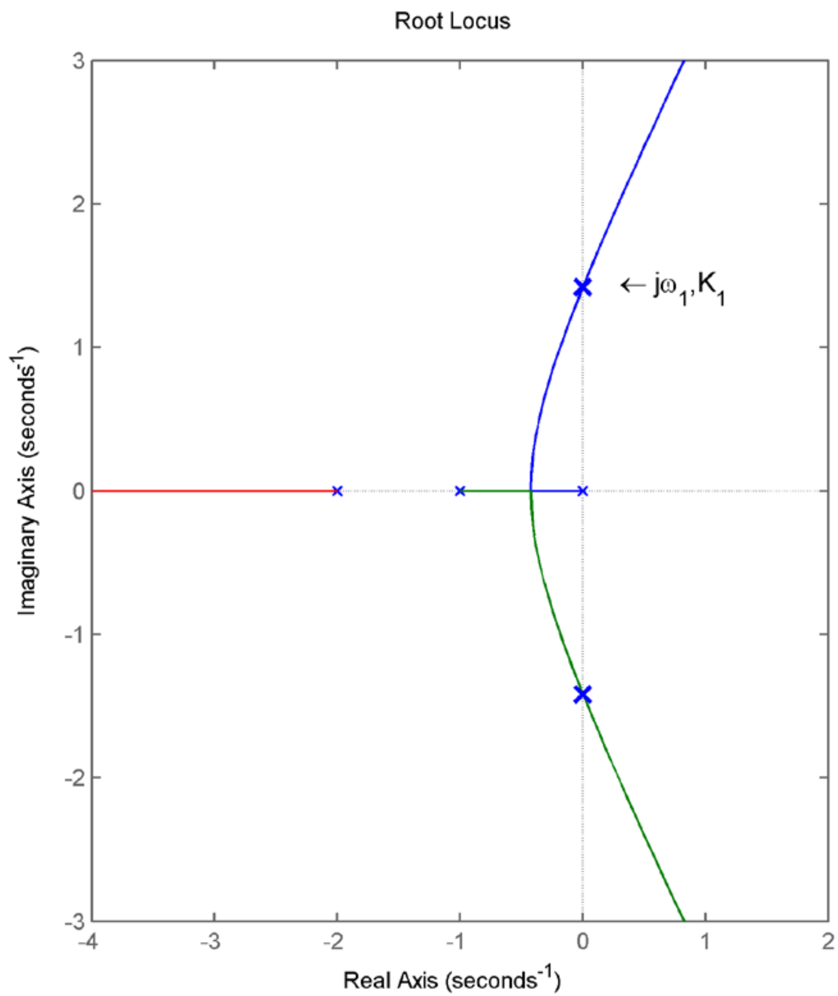


- We already have a couple of tools for assessing stability as a function of loop gain, K
  - Routh Hurwitz
  - Root locus
- Root locus:
  - Stable for some values of K
  - Unstable for others



Stability

- In this case gain is stable ***below*** some value
- Other systems may be stable for gain ***above*** some value
- Marginal stability point:
  - Closed-loop poles on the imaginary axis at $\pm j\omega_1$
  - For gain $K = K_1$



Open-Loop Frequency Response & Stability

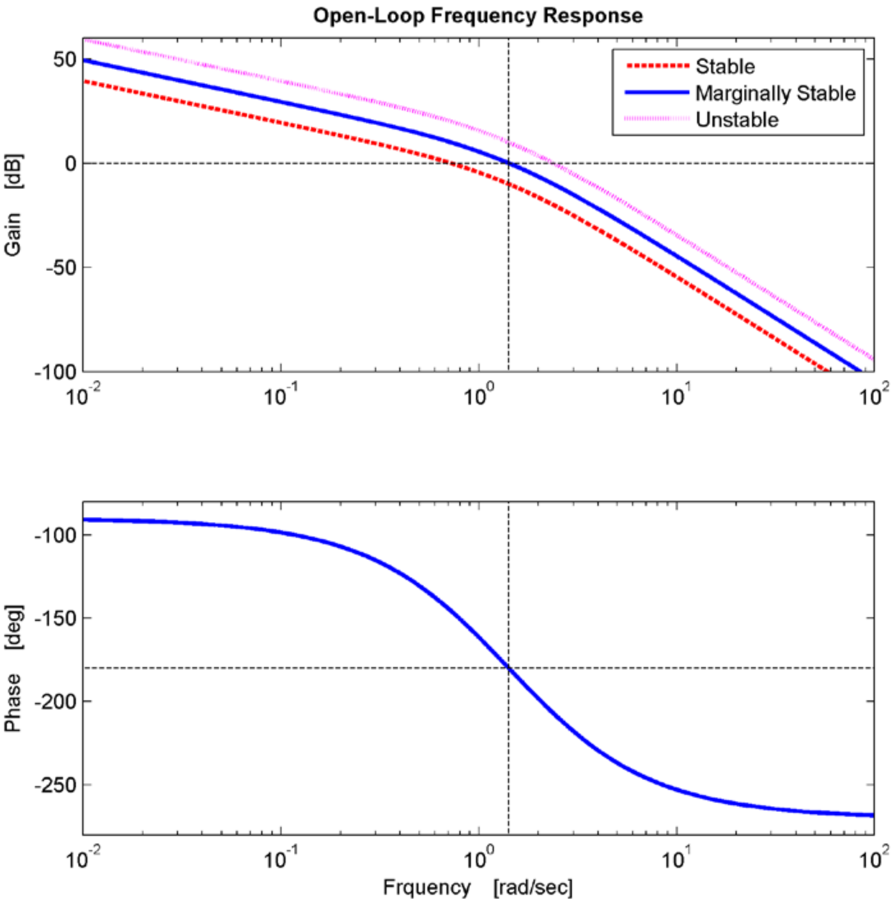
- Marginal stability point occurs when closed-loop poles are on the imaginary axis
  - Angle criterion satisfied at $\pm j\omega_1$

$$|KG(j\omega_1)| = 1 \quad \text{and} \quad \angle KG(j\omega_1) = -180°$$

  - Note that $-180° = 180°$
- $KG(j\omega)$ is the ***open-loop frequency response***
- ***Marginal stability*** occurs when:
  - Open-loop gain is: $|KG(j\omega)| = 0\ dB$
  - Open-loop phase is: $\angle KG(j\omega) = -180°$

Stability from Open-Loop Bode Plots

- Varying K simply shifts gain response up or down
- Here, stable for smaller gain values
  - $|KG(j\omega)| < 0\,dB$ when $\angle KG(j\omega) = -180°$
- Often, stable for larger gain values
  - $|KG(j\omega)| > 0\,dB$ when $\angle KG(j\omega) = -180°$
- Root locus provides this information
  - Bode plot does not



Open-Loop Frequency Response & Stability

- ***Open-loop Bode plot*** can be used to assess stability
  - But, we need to know if system is closed-loop stable for low gain or high gain
- Here, we'll assume ***open-loop-stable systems***
  - Closed-loop stable for low gain
- Open-loop Bode plot can tell us:
  - Is a system closed-loop stable?
  - If so, how stable?
    - I.e. how close to marginal stability
- Two ***stability metrics***:
  - ***Gain margin***
  - ***Phase margin***

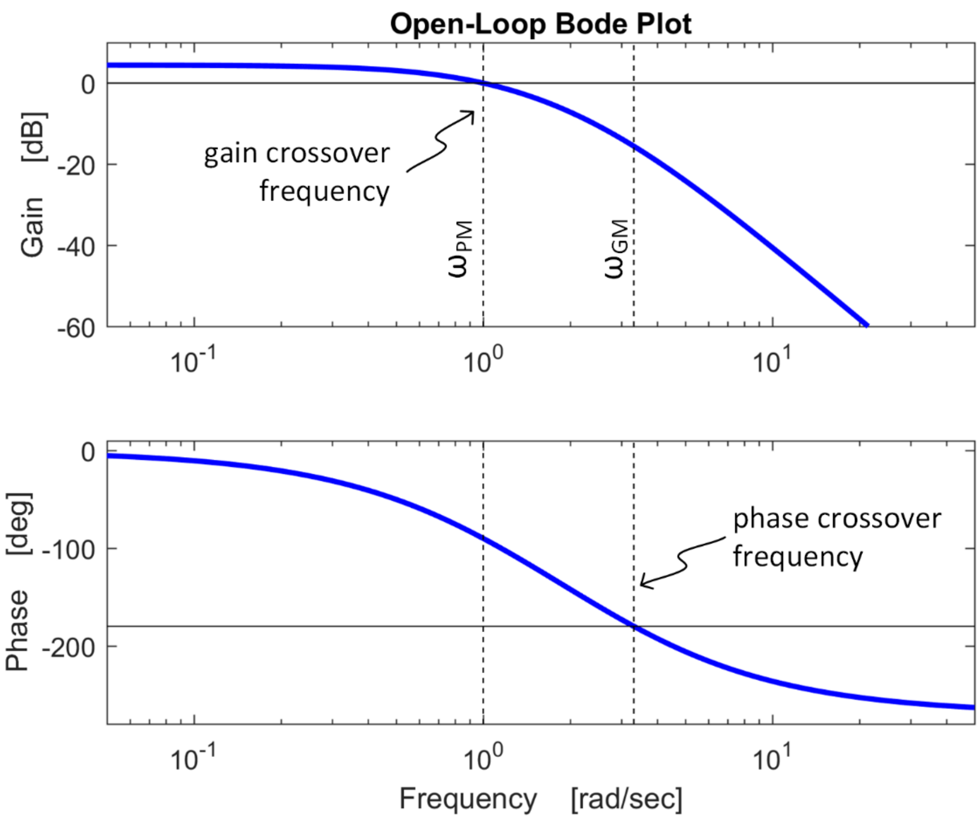
Stability Margins

Crossover Frequencies

- Two important frequencies when assessing stability:

- ***Gain crossover frequency***, ω_{PM}
  - The frequency at which the open-loop gain crosses $0\ dB$

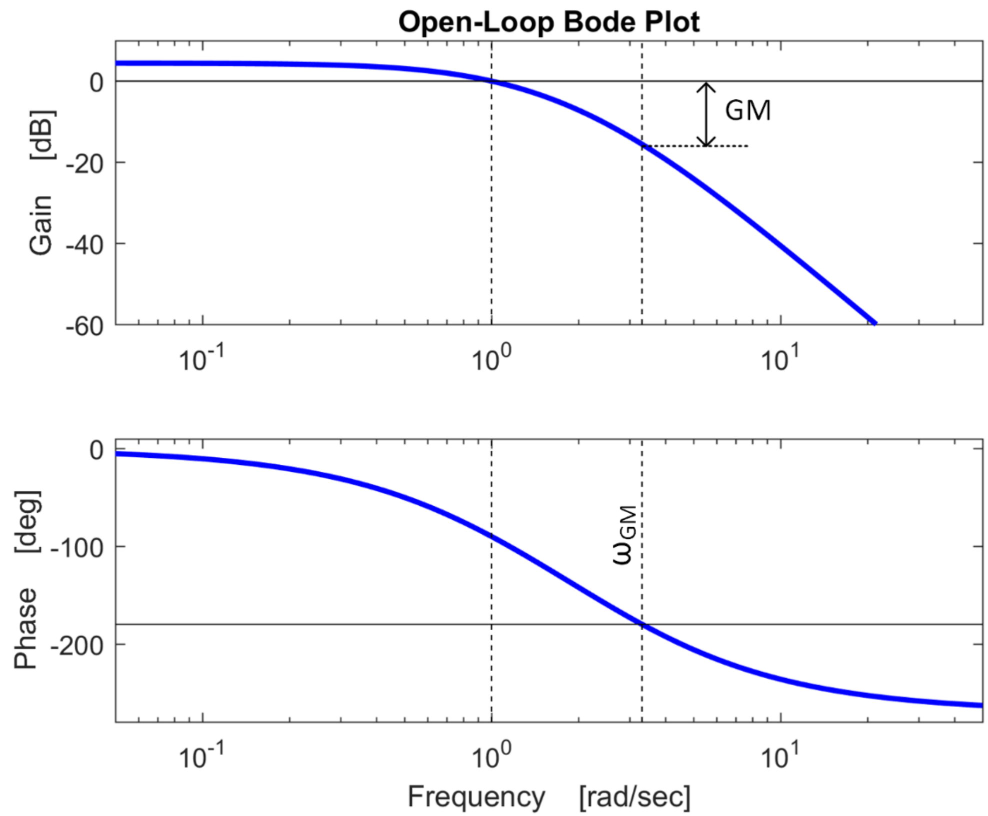
- ***Phase crossover frequency***, ω_{GM}
  - The frequency at which the open-loop phase crosses $-180°$



Gain Margin

- An open-loop-stable system will be closed-loop stable as long as its gain is less than unity at the phase crossover frequency

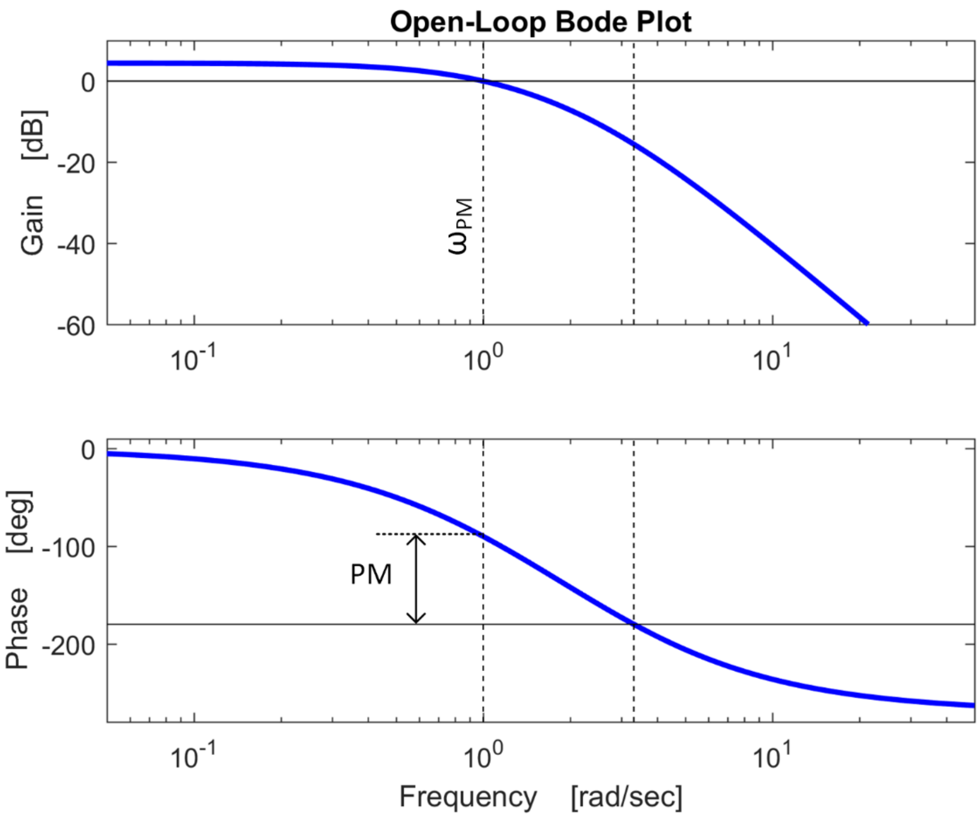
- **<u>Gain margin, GM</u>**
  - The change in open-loop gain at the phase crossover frequency required to make the closed-loop system unstable



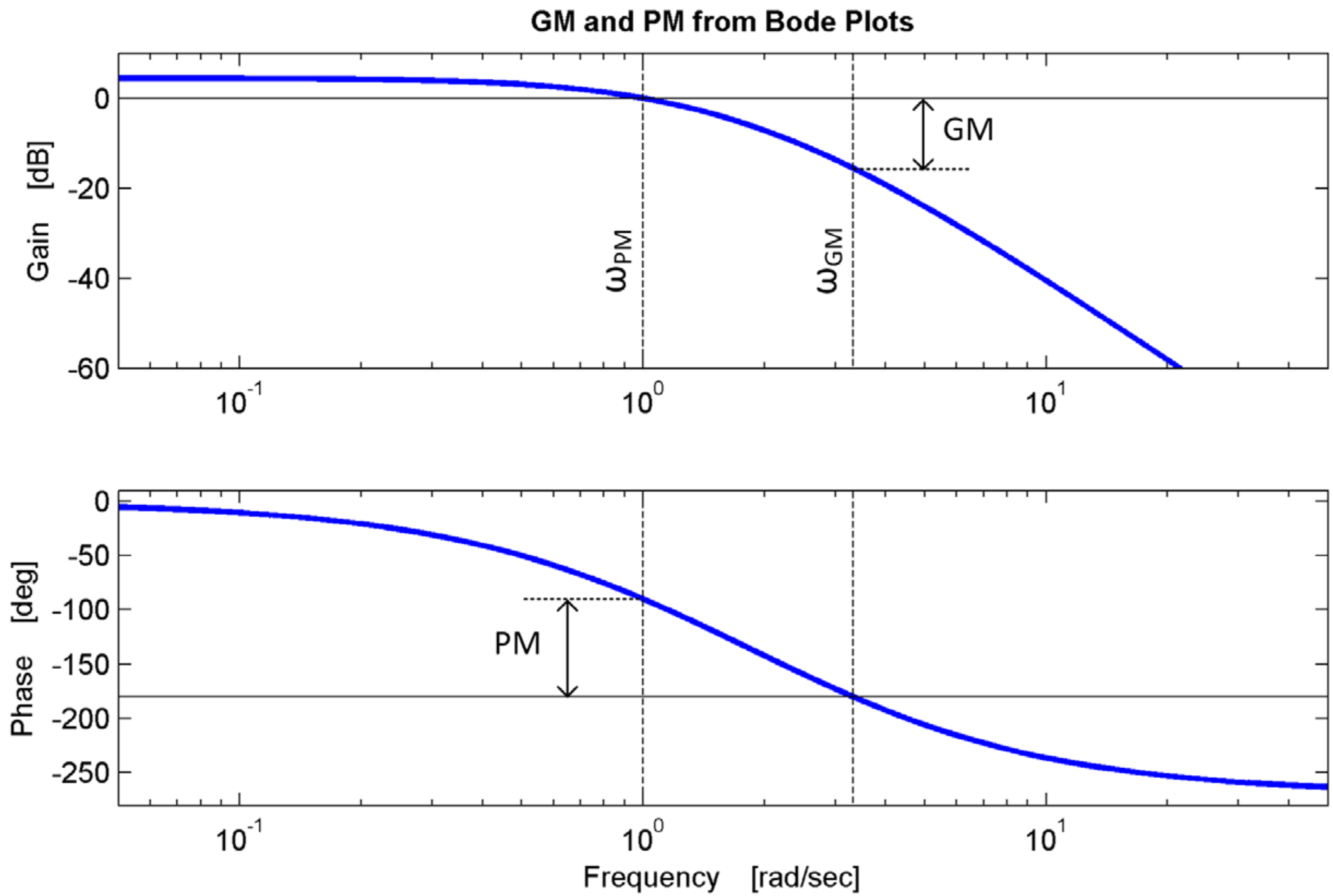
Phase Margin

- An open-loop-stable system will be closed-loop stable as long as its phase has not fallen below $-180°$ at the gain crossover frequency

- **Phase margin, PM**
  - The change in open-loop phase at the gain crossover frequency required to make the closed-loop system unstable



Gain and Phase Margins from Bode Plots



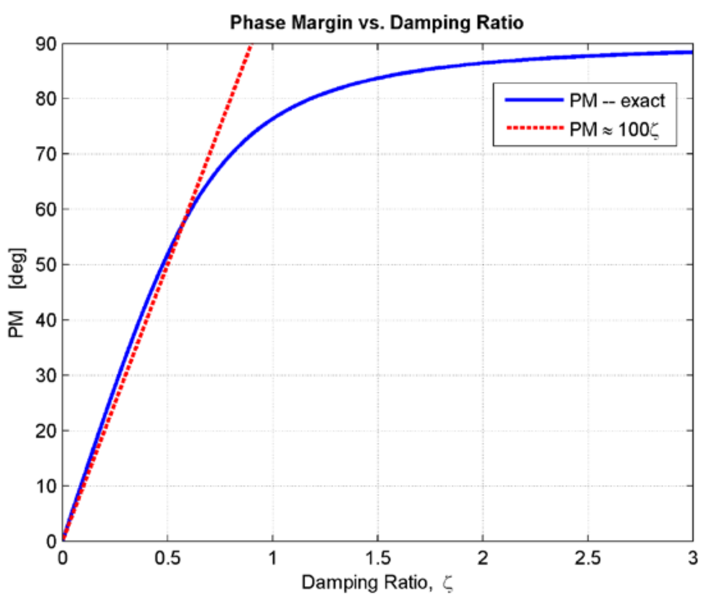
Phase Margin and Damping Ratio, ζ

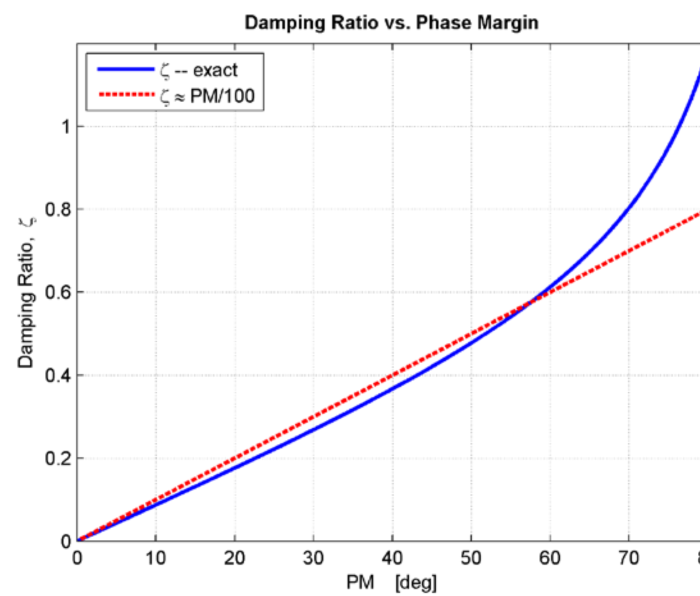
- PM can be expressed as a function of damping ratio, ζ, as

$$PM = \tan^{-1}\left(\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}\right)$$

- For $PM \leq 65°$ or so, we can approximate:

$$PM \approx 100\zeta \quad \text{or} \quad \zeta \approx \frac{PM}{100}$$





Frequency Response Analysis in MATLAB

bode.m


```
[mag,phase] = bode(sys,w)
```

- `sys`: system model – state-space, transfer function, or other
- `w`: *optional* frequency vector – in rad/sec
- `mag`: system gain response vector
- `phase`: system phase response vector – in degrees

- If no outputs are specified, bode response is automatically plotted – preferable to plot yourself
- Frequency vector input is optional
  - If not specified, MATLAB will generate automatically

- May need to do: `squeeze(mag)` and `squeeze(phase)` to eliminate singleton dimensions of output matrices

margin.m


```
[GM,PM,wgm,wpm] = margin(sys)
```

- `sys`: system model – state-space, transfer function, or other
- `GM`: gain margin
- `PM`: phase margin – in degrees
- `wgm`: frequency at which GM is measured, the phase crossover frequency – in rad/sec
- `wpm`: frequency at which PM is measured, the gain crossover frequency

- If no outputs are specified, a Bode plot with GM and PM indicated is automatically generated

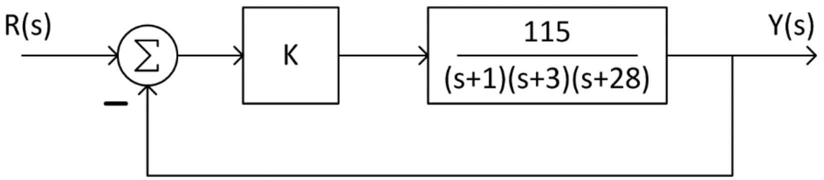
Frequency-Response Design

Frequency-Response Design

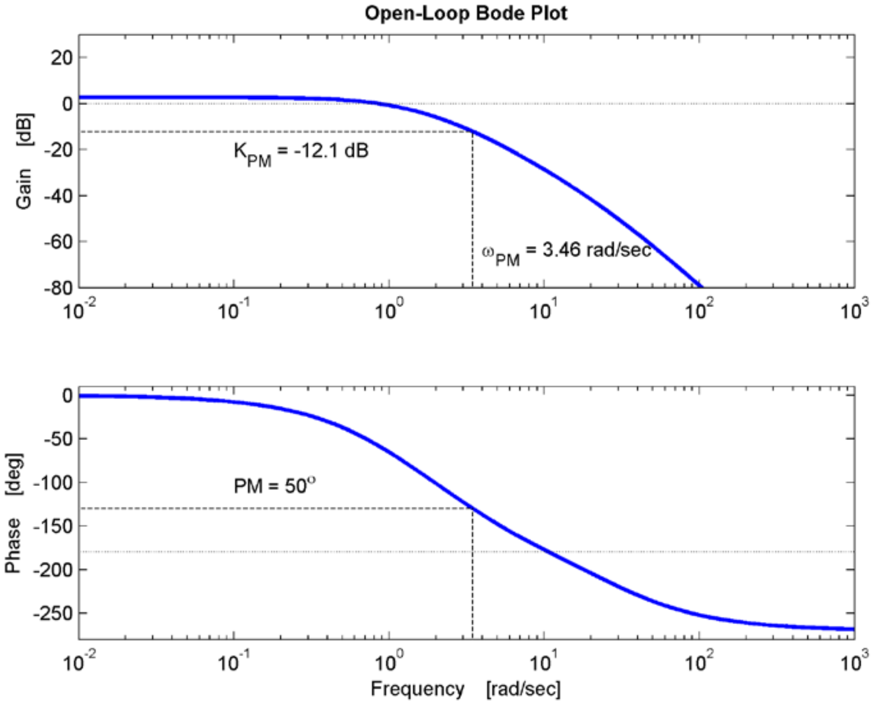
- In a previous section of notes, we saw how we can use root-locus techniques to design compensators
- Two primary objectives of compensation
  - Improve steady-state error
    - Proportional-integral (PI) compensation
    - Lag compensation
  - Improve dynamic response
    - Proportional-derivative (PD) compensation
    - Lead compensation
- Now, we'll learn to design compensators using a system's ***open-loop frequency response***
  - We'll focus on lag and lead compensation

Improving Steady-State Error

Improving Steady-State Error

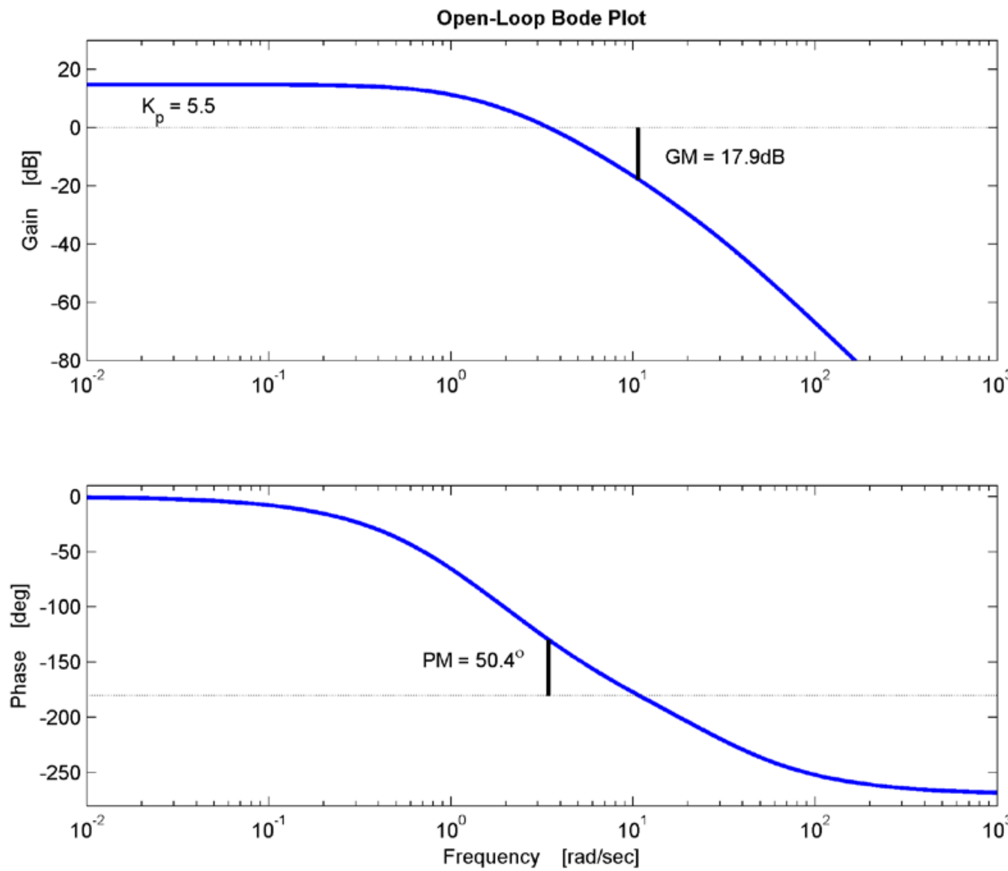


- Consider the system above with a desired phase margin of $PM \approx 50°$
- According to the Bode plot:
  - $\phi = -130°$ at $\omega_{PM} = 3.46\ rad/sec$
  - Gain is $K_{PM} = -12.1\ dB$ at ω_{PM}
  - Set $K = -K_{PM} = 12.1 dB = 4$ for desired phase margin



Improving Steady-State Error

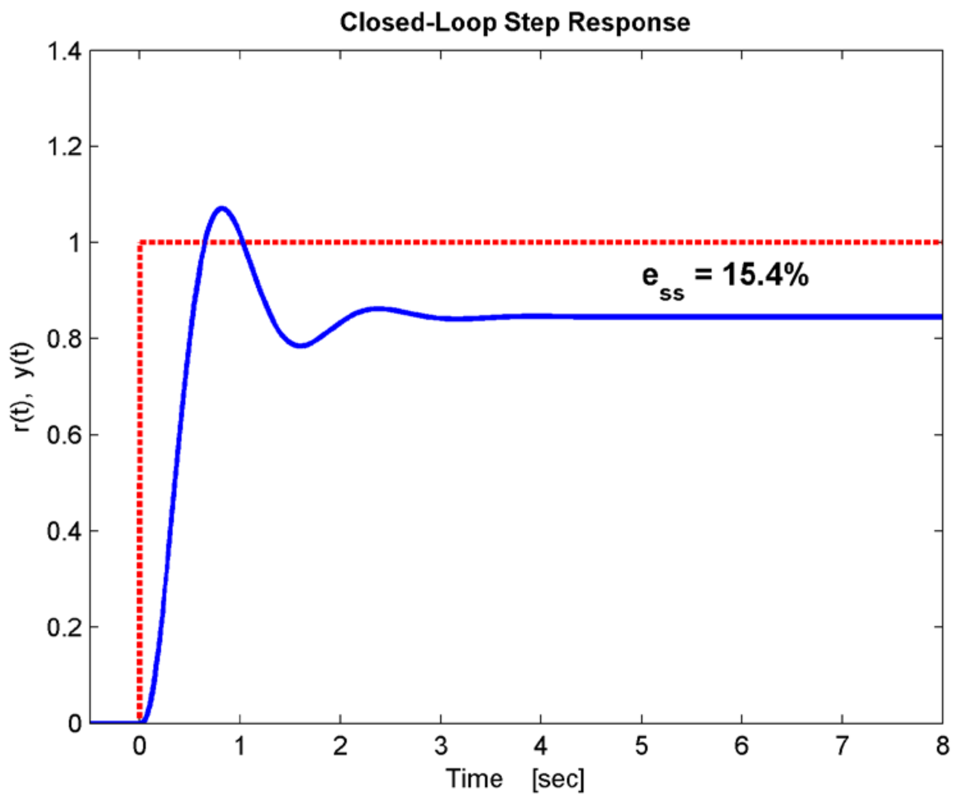
- Can read the position constant directly from the Bode plot: $K_p = 14.8\,dB \;\; \rightarrow \;\; 5.5$
- Note that $PM \approx 50°$, as desired
- Gain margin is $GM = 17.9\,dB$



Improving Steady-State Error

- Steady-state error to a constant reference is

$$e_{ss} = \frac{1}{1 + K_p} = 0.154 \;\rightarrow\; 15.4\%$$

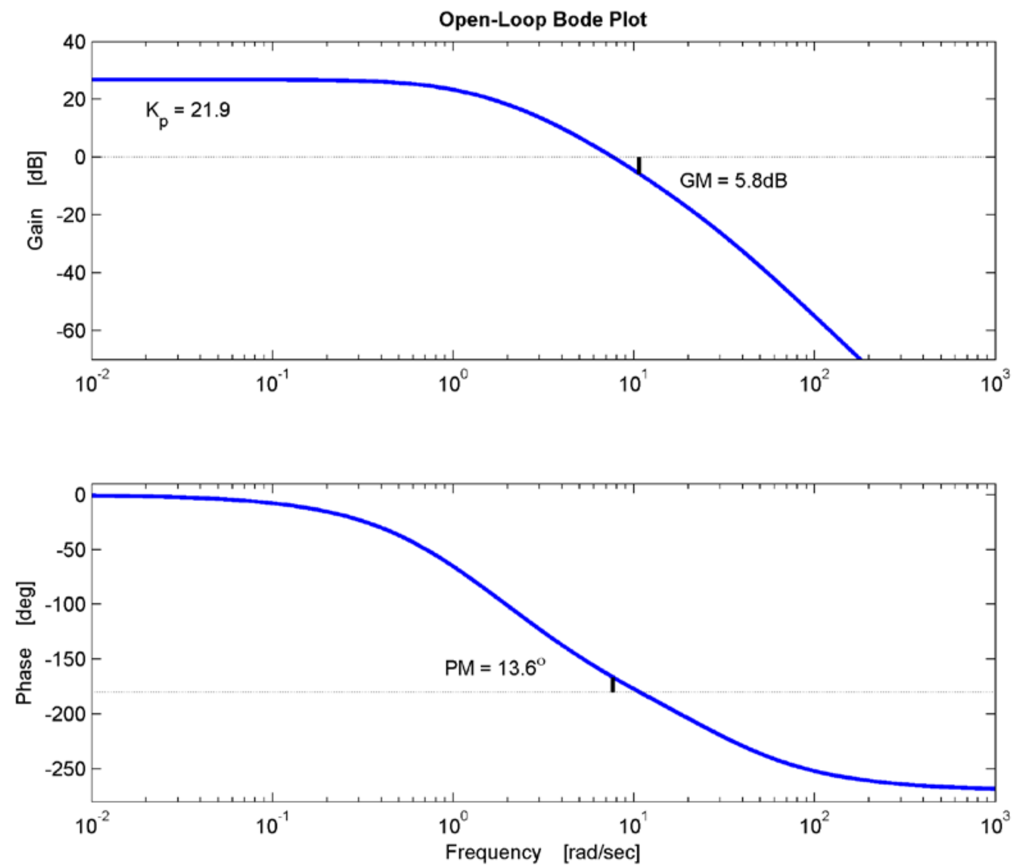


Improving Steady-State Error

- Let's say we want to reduce steady-state error to $e_{ss} < 5\%$
- Required position constant

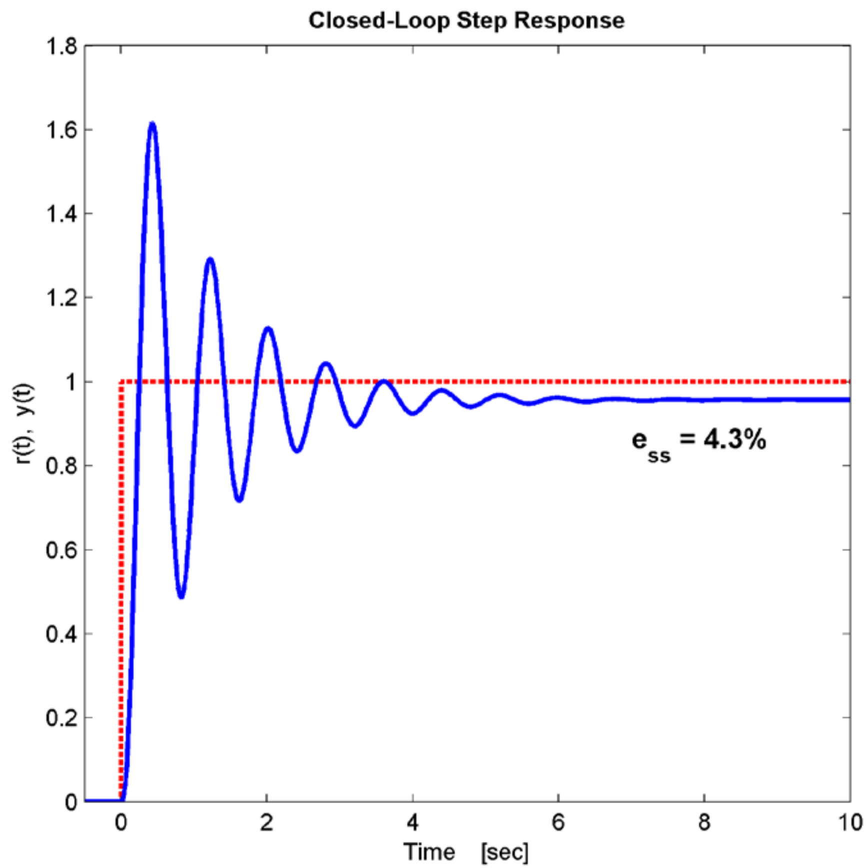
$$K_p > \frac{1}{0.05} - 1 = 19$$

- Increase gain by 4x
  - Bode plot shows desired position constant
  - But, phase margin has been degraded significantly



Improving Steady-State Error

- Step response shows that error goal has been met
  - But, reduced phase margin results in significant overshoot and ringing
- Error improvement came at the cost of degraded phase margin
- Would like to be able to improve steady-state error without affecting phase margin
  - Integral compensation
  - Lag compensation

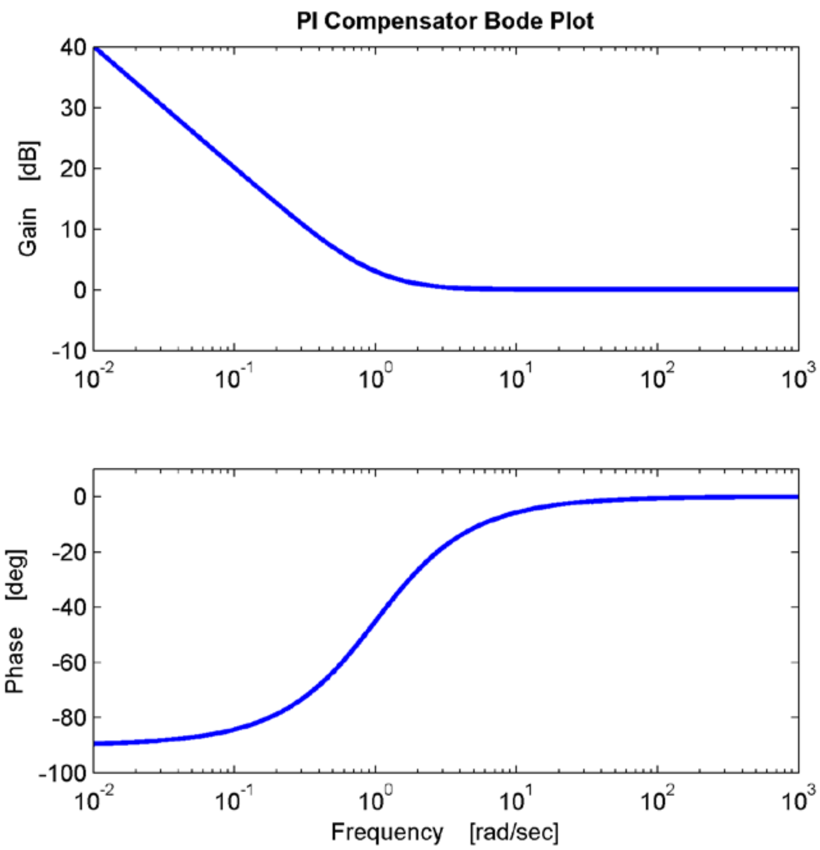


Integral Compensation

PI Compensation

- Proportional-integral (PI) compensator:

$$D(s) = \frac{1}{T_D}\frac{(T_D s + 1)}{s}$$



- Low-frequency gain increase
  - Infinite at DC
  - System type increase
- For $\omega \gg 1/T_D$
  - Gain unaffected
  - Phase affected little
  - PM unaffected
- Susceptible to integrator overflow
  - Lag compensation is often preferable

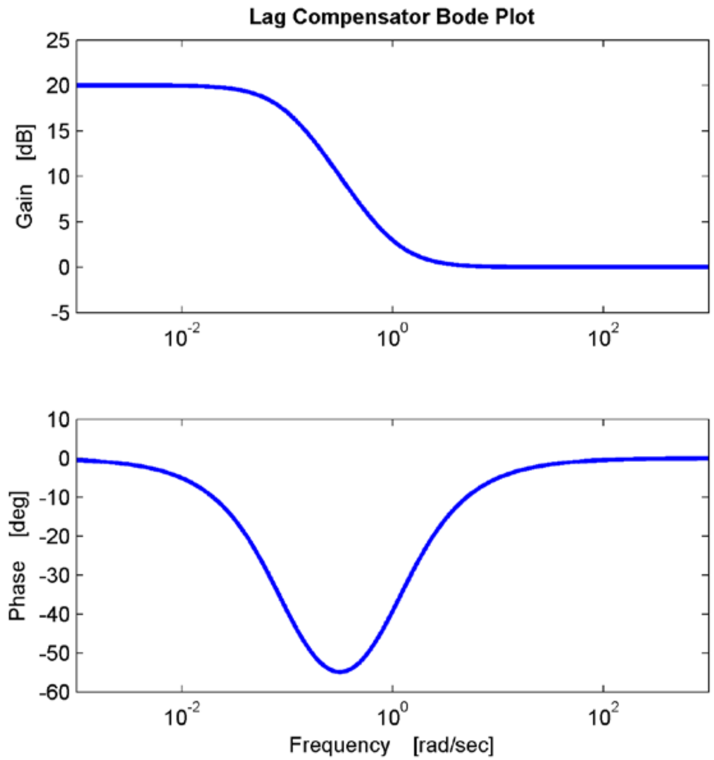
Lag Compensation

Lag Compensation

- Lag compensator

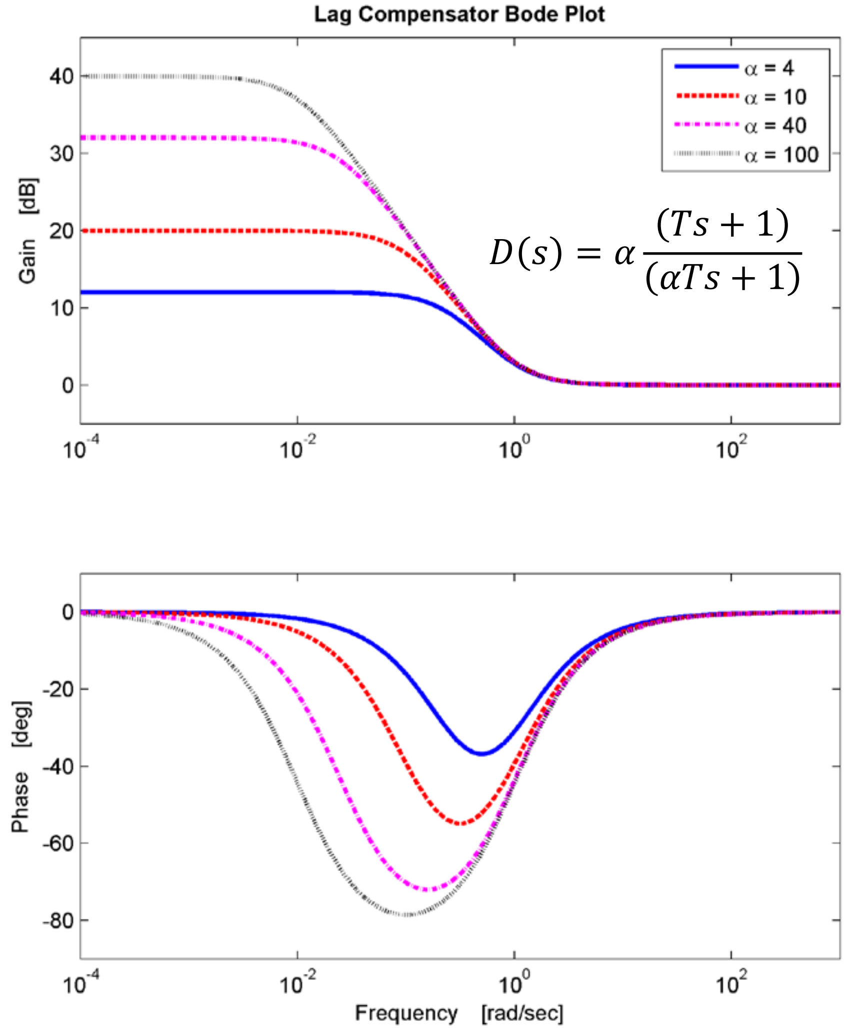
$$D(s) = \alpha \frac{(Ts+1)}{(\alpha Ts+1)}, \qquad \alpha > 1$$

- Objective: add a gain of α at low frequencies without affecting phase margin
- Lower-frequency pole: $s = -1/\alpha T$
- Higher-frequency zero: $s = -1/T$
- Pole/zero spacing determined by α
- For $\omega \ll 1/\alpha T$
  - Gain: $\sim 20\log(\alpha)\ dB$
  - Phase: $\sim 0°$
- For $\omega \gg 1/T$
  - Gain: $\sim 0\ dB$
  - Phase: $\sim 0°$



Lag Compensation vs. α

- Gain increased at low frequency only
  - Dependent on α
  - DC gain: $20\log(\alpha)\ dB$
- Phase lag added between compensator pole and zero
  - $0° \leq \phi_{max} \leq 90°$
  - Dependent on α
- Lag pole/zero well below crossover frequency
  - Phase margin unaffected



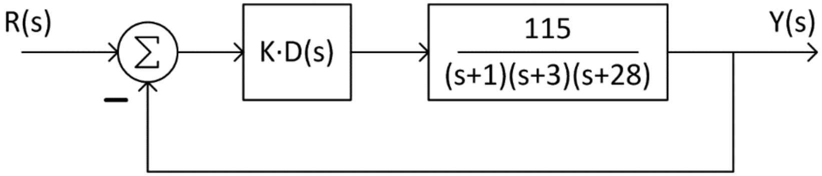
Lag Compensator Design Procedure

- Lag compensator adds gain at low frequencies without affecting phase margin
- ***Basic design procedure***:
  - Adjust gain to achieve the desired phase margin
  - Add compensation, increasing low-frequency gain to achieve desired error performance
- Same as adjusting gain to place poles at the desired damping on the root locus, then adding compensation
  - ***Root locus is not changed***
  - Here, the ***frequency response near the crossover frequency is not changed***

Lag Compensator Design Procedure

1. ***Adjust gain***, K, of the *uncompensated* system to provide the ***desired phase margin*** plus $5° \ldots 10°$ (to account for small phase lag added by compensator)
2. Use the open-loop Bode plot for the uncompensated system with the value of gain set in the previous step to ***determine the static error constant***
3. ***Calculate α*** as the low-frequency gain increase required to provide the desired error performance
4. ***Set the upper corner frequency*** (the zero) to be one decade below the crossover frequency: $1/T = \omega_{PM}/10$
   - Minimizes the added phase lag at the crossover frequency
5. ***Calculate the lag pole***: $1/\alpha T$
6. ***Simulate*** and ***iterate***, if necessary

Lag Example – Step 1



- Design a lag compensator for the above system to satisfy the following requirements
  - $e_{ss} < 2\%$ for a step input
  - $\%OS \approx 12\%$

- First, determine the required phase margin to satisfy the overshoot requirement

$$\zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.559$$

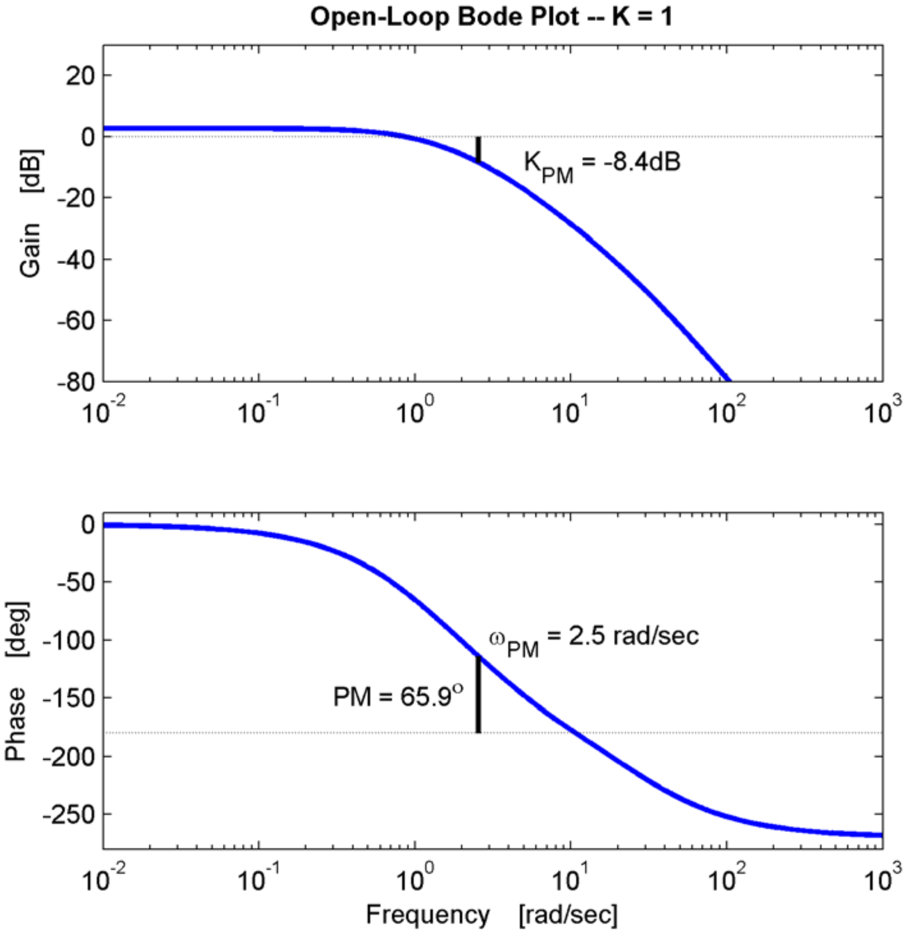
$$PM \approx 100\zeta = 55.9°$$

- Add $\sim 10°$ to account for compensator phase at ω_{PM}

$$PM = 65.9°$$

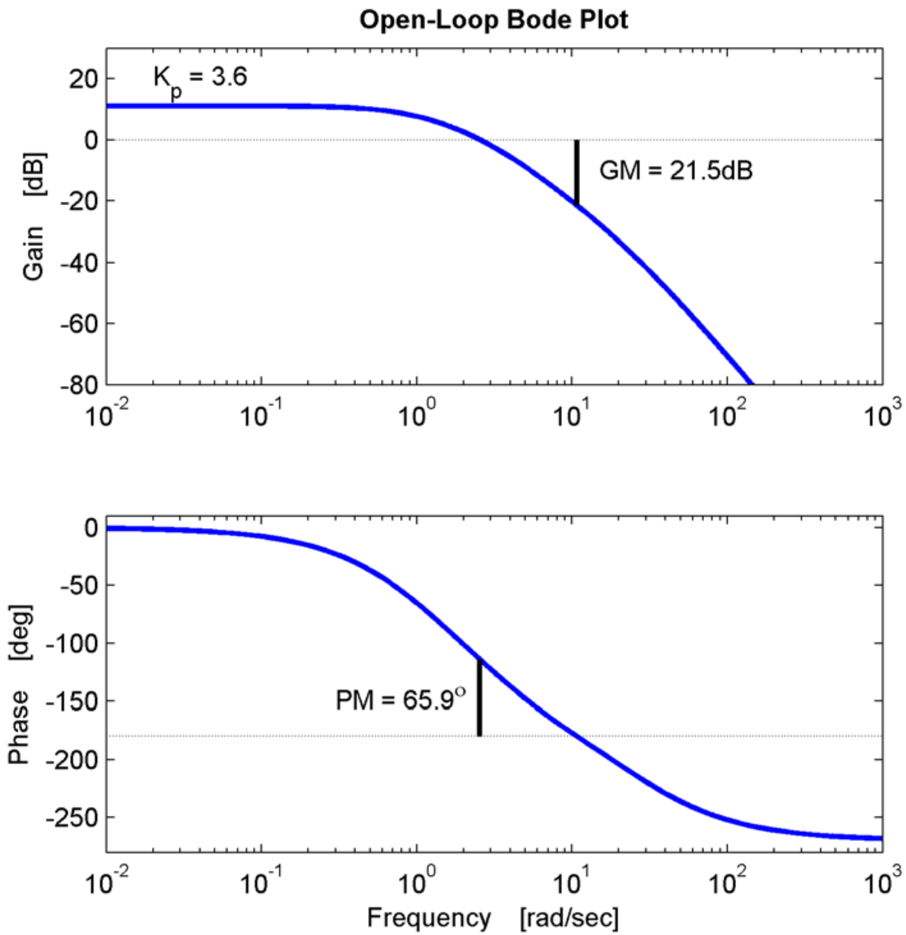
Lag Example – Step 1

- Plot the open-loop Bode plot of the uncompensated system for $K = 1$
- Locate frequency where phase is $-180° + PM = -114.1°$
  - This is ω_{PM}, the desired crossover frequency
  - $\omega_{PM} = 2.5\ rad/sec$
- Gain at ω_{PM} is K_{PM}
  - $K_{PM} = -8.4\ dB\ \rightarrow 0.38$
- Increase the gain by $1/K_{PM}$
  - $K = 8.4\ dB\ \rightarrow 2.63$



Lag Example – Step 2

- Gain has now been set to yield the desired phase margin of $PM = 65.9°$
- Use the new open-loop bode plot to determine the static error constant



- Position constant of the uncompensated system given by the DC gain:

$$K_{pu} = 11.14\, dB \ \rightarrow 3.6$$

Lag Example – Step 3

- Calculate α to yield desired steady-state error improvement
- Steady-state error:

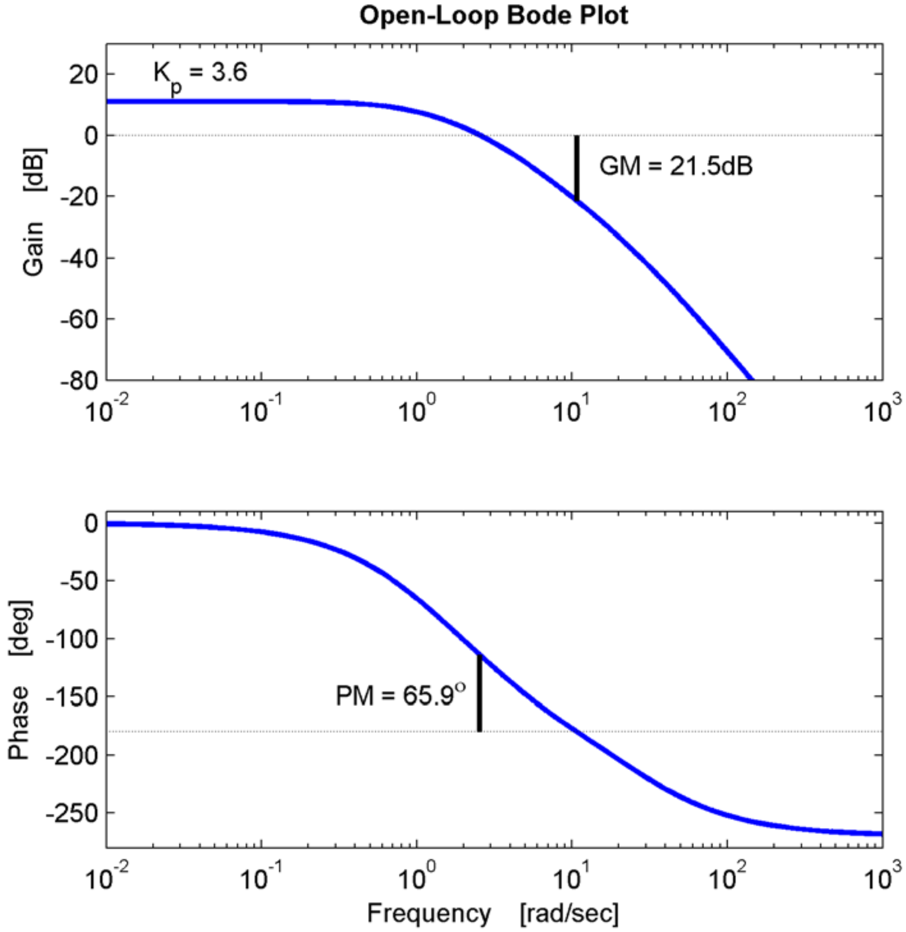
$$e_{ss} = \frac{1}{1+K_p} < 0.02$$

- The required position constant:

$$K_p > \frac{1}{e_{ss}} - 1 = 49 \;\rightarrow K_p = 50$$

- Calculate α as the required position constant improvement

$$\alpha = \frac{K_p}{K_{pu}} = 13.9 \;\rightarrow \alpha = 14$$



Lag Example – Steps 4 & 5

- Place the compensator zero one decade below the crossover frequency, $\omega_{PM} = 2.5\ rad/sec$

$$1/T = 0.25\ rad/sec$$

$$T = 4\ sec$$

- The compensator pole:

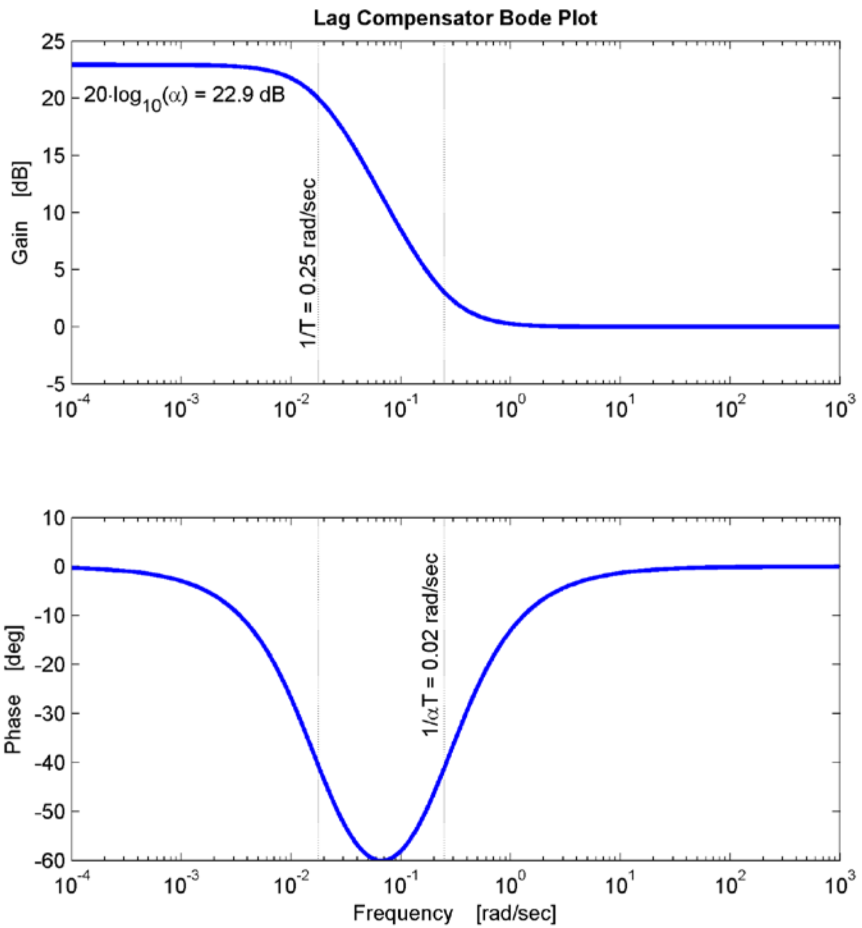
$$1/\alpha T = \frac{0.25}{14}$$

$$1/\alpha T = 0.018\ rad/sec$$

- Lag compensator transfer function

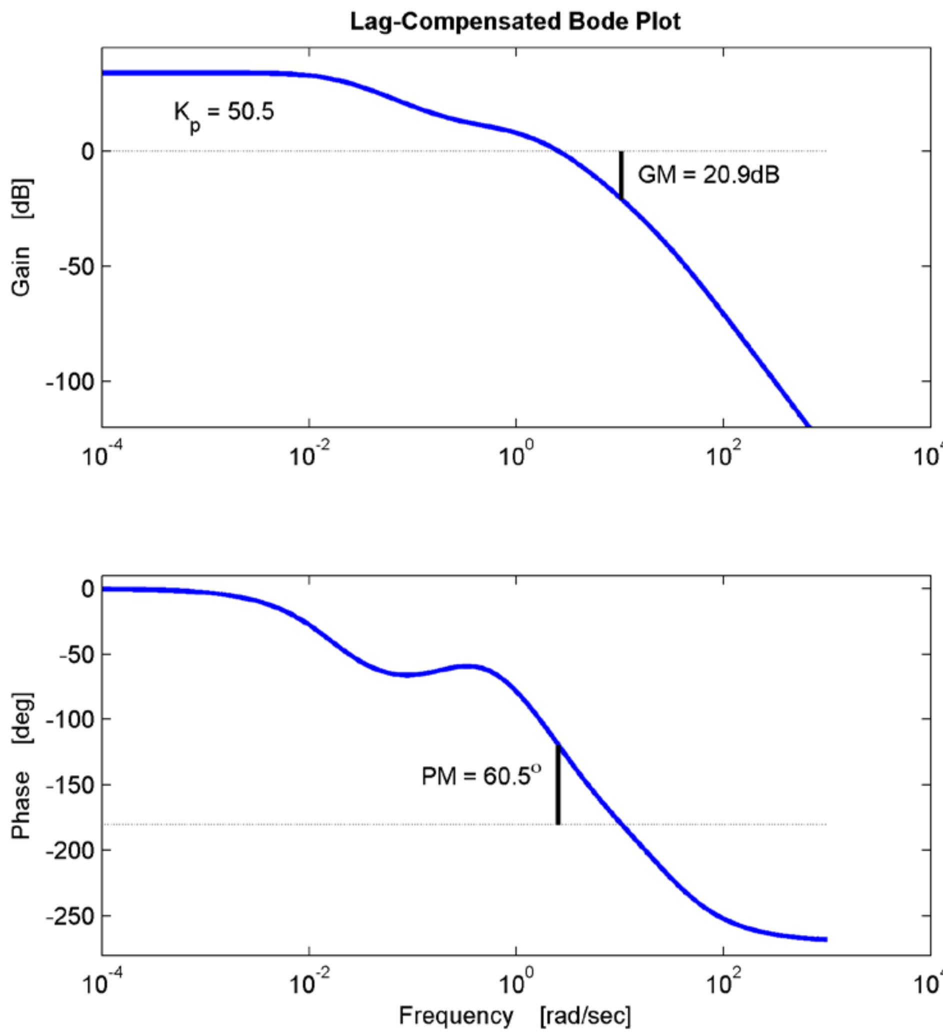
$$D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)}$$

$$D(s) = 14 \frac{(4s + 1)}{(56s + 1)}$$



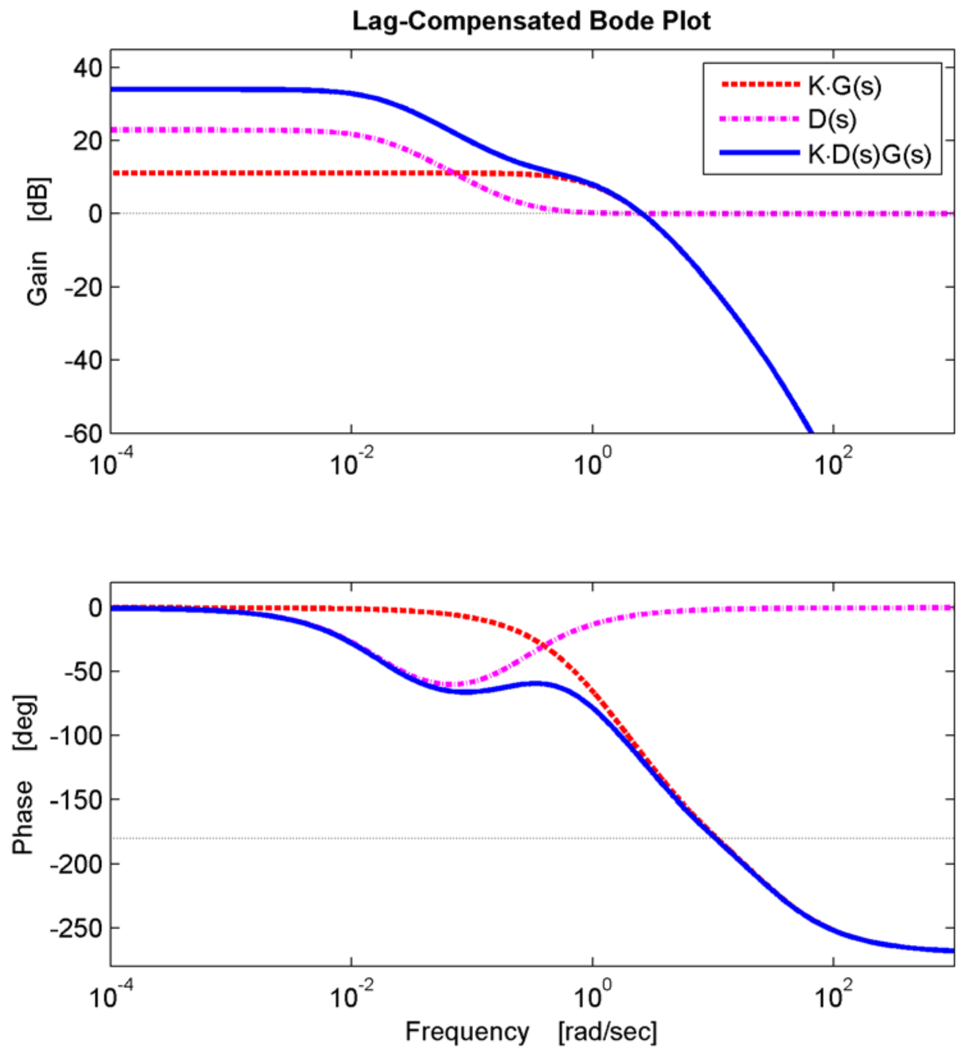
Lag Example – Step 6

- Bode plot of compensated system shows:
  - $PM = 60.5°$
  - $K_p = 50.5$



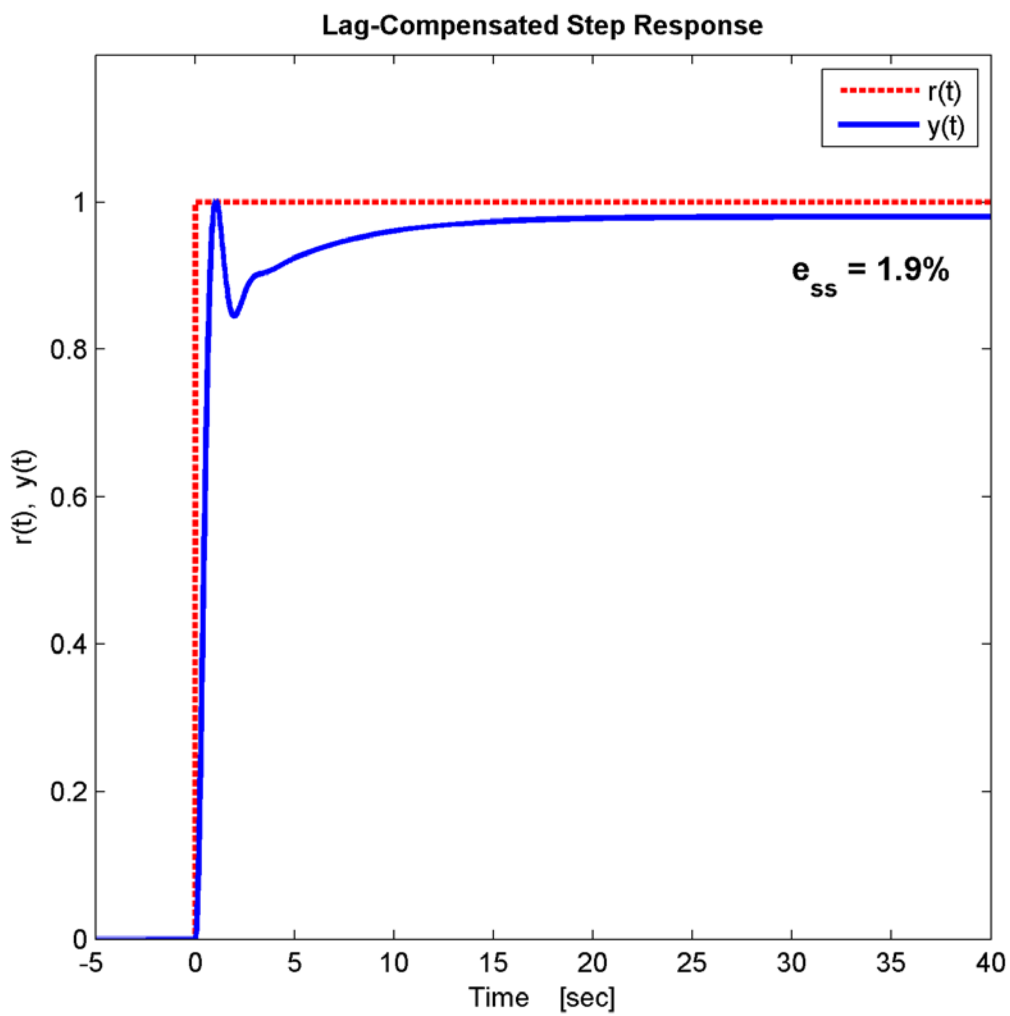
Lag Example – Step 6

- Lag compensator adds gain at low frequencies only
- Phase near the crossover frequency is nearly unchanged



Lag Example – Step 6

- Steady-state error requirement has been satisfied
- Overshoot spec has been met
  - Though slow tail makes overshoot assessment unclear



Lag Compensator – Summary

$$D(s) = \alpha \frac{(Ts + 1)}{(\alpha Ts + 1)}$$

- Higher-frequency zero: $s = -1/T$
  - Place one decade below crossover frequency, ω_{PM}
- Lower-frequency pole: $s = -1/\alpha T$
  - α sets pole/zero spacing
- DC gain: $\alpha \; \rightarrow \; 20\log_{10}(\alpha) \; dB$
- Compensator adds *low-frequency* gain
  - Static error constant improvement
  - Phase margin unchanged

Improving Dynamic Response

Improving Dynamic Response

- We've already seen two types of compensators to ***improve dynamic response***
  - Proportional derivative (PD) compensation
  - Lead compensation
- Unlike with the lag compensator we just looked at, here, the objective is to ***alter the open-loop phase***
- We'll look briefly at PD compensation, but will focus on ***lead compensation***

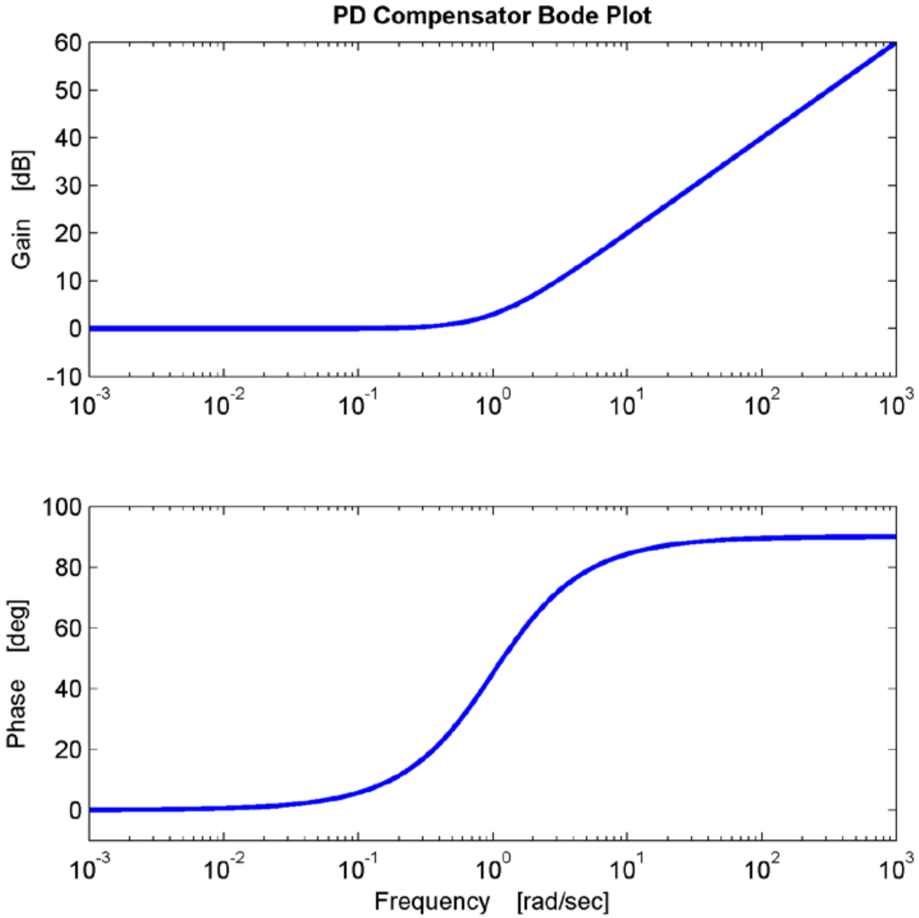
Derivative Compensation

PD Compensation

- Proportional-Derivative (PD) compensator:

$$D(s) = (T_D s + 1)$$

- Phase added near (and above) the crossover frequency
  - Increased phase margin
  - Stabilizing effect
- Gain continues to rise at high frequencies
  - Sensor noise is amplified
  - Lead compensation is usually preferable



Lead Compensation

Lead Compensation

- With lead compensation, we have three design parameters:
  - ***Crossover frequency***, ω_{PM}
    - Determines closed-loop bandwidth, ω_{BW}; risetime, t_r; peak time, t_p; and settling time, t_s
  - ***Phase margin***, PM
    - Determines damping, ζ, and overshoot
  - ***Low-frequency gain***
    - Determines steady-state error performance

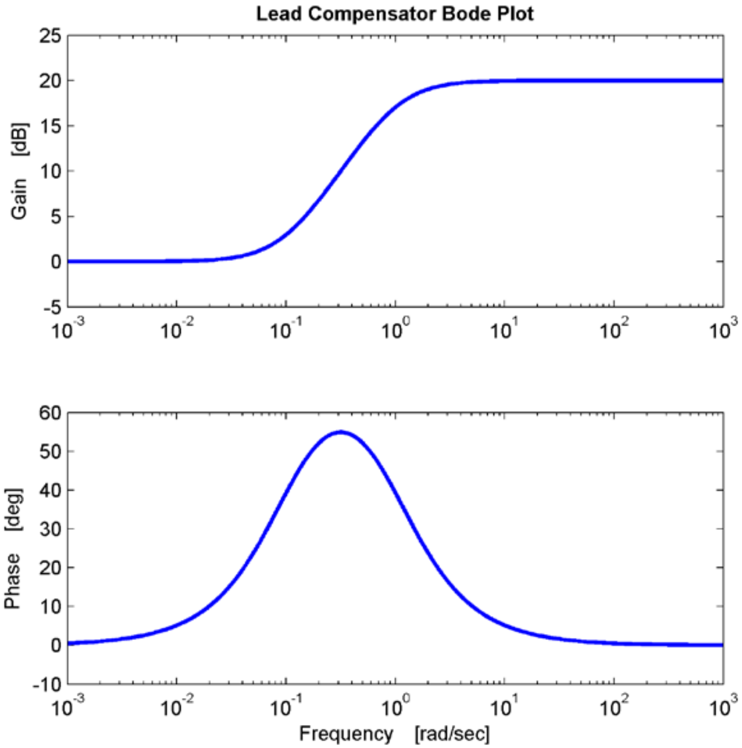
- We'll look at the design of lead compensators for two common scenarios, *either*
  - Designing for ***steady-state error*** and ***phase margin***, *or*
  - Designing for ***closed-loop bandwidth*** and ***phase margin***

Lead Compensation

- Lead compensator

$$D(s) = \frac{(Ts+1)}{(\beta Ts+1)}\ , \qquad \beta < 1$$

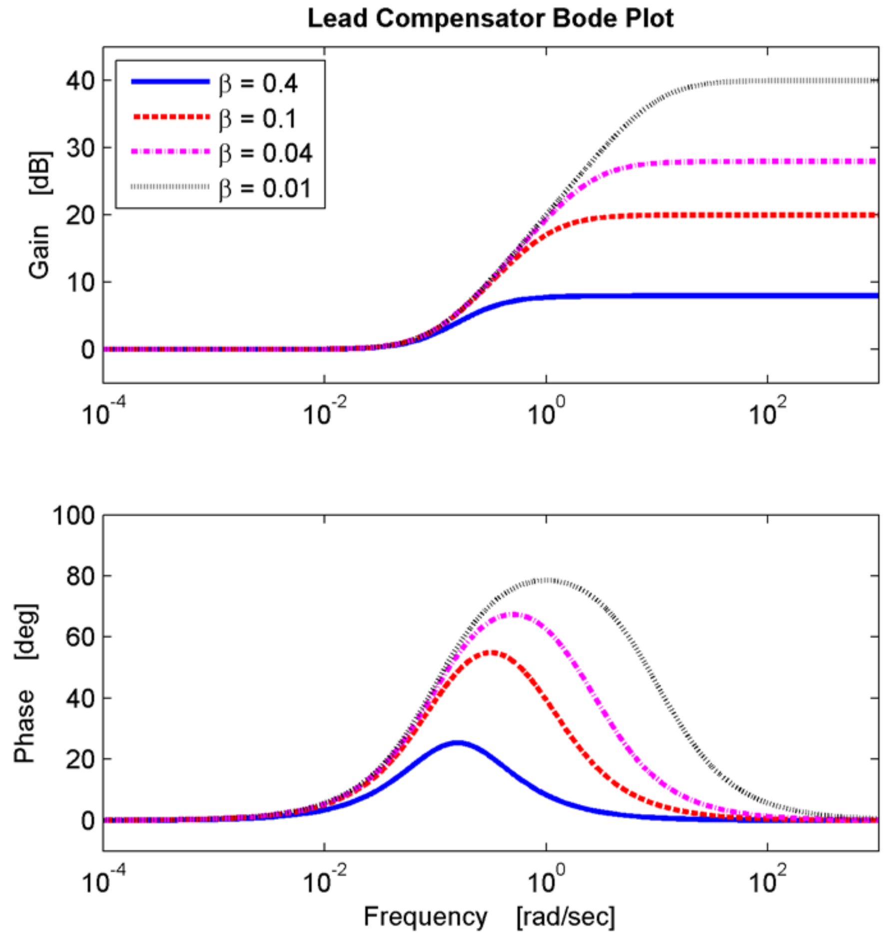
- Objectives: add phase lead near the crossover frequency and/or alter the crossover frequency
- Lower-frequency zero: $s = -1/T$
- Higher-frequency pole: $s = -1/\beta T$
- Zero/pole spacing determined by β
- For $\omega \ll 1/T$
  - Gain: $\sim 0\ dB$
  - Phase: $\sim 0°$
- For $\omega \gg 1/\beta T$
  - Gain: $\sim 20\log(1/\beta)\ \ dB$
  - Phase: $\sim 0°$



Lead Compensation vs. β

$$D(s) = \frac{(Ts+1)}{(\beta Ts+1)} \ , \qquad \beta < 1$$

- β determines:
  - Zero/pole spacing
  - Maximum compensator phase lead, ϕ_{max}
  - High-frequency compensator gain



Lead Compensation – ϕ_{max}

- β, zero/pole spacing, determines maximum phase lead

$$\phi_{max} = \sin^{-1}\left(\frac{1-\beta}{1+\beta}\right)$$

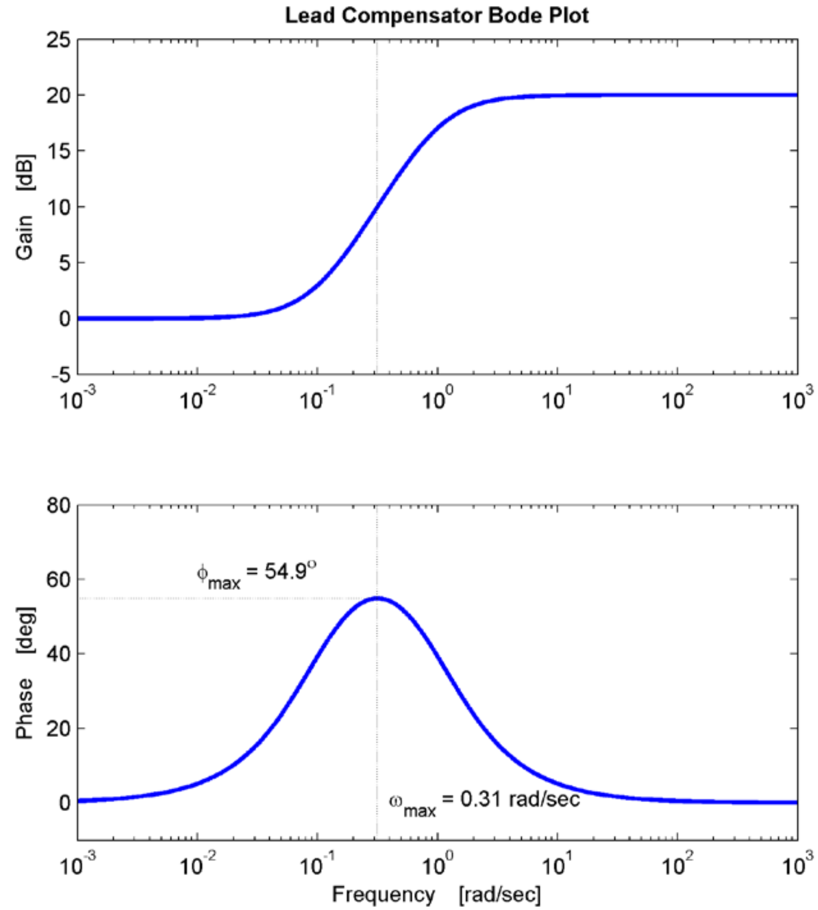
- Can use a desired ϕ_{max} to determine β

$$\beta = \frac{1-\sin(\phi_{max})}{1+\sin(\phi_{max})}$$

- ϕ_{max} occurs at ω_{max}

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}}$$



Lead Compensation – Design Procedure

1. Determine loop gain, K, to satisfy *either* steady-state error requirements *or* bandwidth requirements:
   a) Set K to provide the required static error constant, *or*
   b) Set K to place the crossover frequency an octave below the desired closed-loop bandwidth
2. Evaluate the phase margin of the uncompensated system, using the value of K just determined
3. If necessary, determine the required PM from ζ or overshoot specifications. Evaluate the PM of the uncompensated system and determine the required phase lead at the crossover frequency to achieve this PM. Add $\sim 10°$ additional phase – this is ϕ_{max}
4. Calculate β from ϕ_{max}
5. Set $\omega_{max} = \omega_{PM}$. Calculate T from ω_{max} and β
6. Simulate and iterate, if necessary

Closed-Loop Bandwidth and Transient Response

- ***Closed-loop bandwidth***, ω_{BW}, is one possible design criterion
  - How is it related to transient response?
- For a ***second-order system*** (or approximate second-order system):
  - Closed-loop bandwidth and ***damping ratio*** and ***natural frequency***, ζ and ω_n

$$\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

  - Closed-loop bandwidth and $\pm 1\%$ ***settling time***, t_s

$$\omega_{BW} \approx \frac{4.6}{t_s \zeta} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

  - Closed-loop bandwidth and ***peak time***, t_p

$$\omega_{BW} = \frac{4}{t_p \sqrt{1 - \zeta^2}} \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Double-Lead Compensation

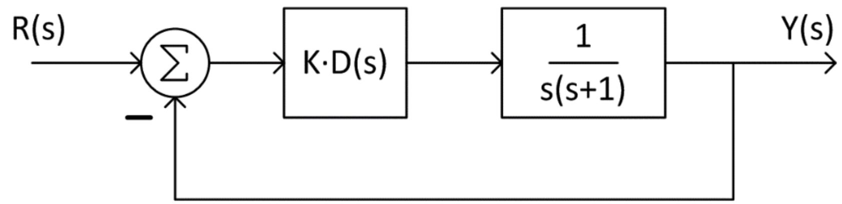
- A lead compensator can add, at most, $90°$ of phase lead
- If more phase is required, use a double-lead compensator

$$D(s) = \left[\frac{(Ts+1)}{(\beta Ts+1)}\right]^2$$

- For phase lead over $\sim 60° \ldots 70°$, $1/\beta$ must be very large, so typically use double-lead compensation

Lead Compensation – Example 1

- Consider the following system



- Design a compensator to satisfy the following
  - $e_{ss} < 0.1$ for a ramp input
  - $\%OS < 15\%$
- Here, we'll design a lead compensator to simultaneously adjust ***low-frequency gain*** and ***phase margin***

Lead Example 1 – Steps 1 & 2

- The velocity constant for the uncompensated system is

$$K_v = \lim_{s \to 0} sKG(s)$$

$$K_v = \lim_{s \to 0} \frac{K}{s+1} = K$$

- Steady-state error is

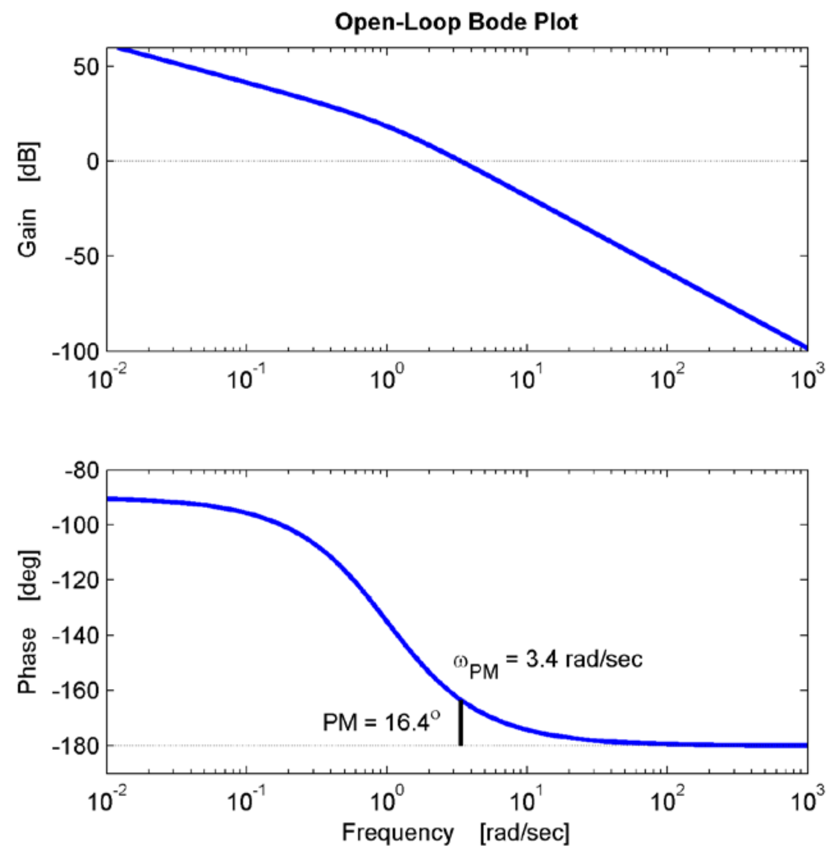
$$e_{ss} = \frac{1}{K_v} < 0.1$$

$$K_v = K > 10$$

- Adding a bit of margin

$$K = 12$$

- Bode plot shows the resulting phase margin is $PM = 16.4°$



Lead Example 1 – Step 3

- Approximate required phase margin for $\%OS < 15\%$
  - Design for 13%
- First calculate the required damping ratio

$$\zeta = -\frac{\ln(OS)}{\sqrt{\pi^2 + \ln^2(OS)}} = 0.545$$

- Approximate corresponding PM, and add 10° correction factor

$$PM \approx 100\zeta + 10^\circ = 64.5^\circ$$

- Calculate the required phase lead

$$\phi_{max} = 64.5^\circ - 16.4^\circ = 48^\circ$$

Lead Example 1 – Steps 4 & 5

- Calculate β from ϕ_{max}

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.147$$

- Set $\omega_{max} = \omega_{PM}$, as determined from Bode plot, and calculate T

$$\omega_{max} = \omega_{PM} = 3.4\,rad/sec$$

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = \frac{1}{3.4\sqrt{0.147}} = 0.766$$

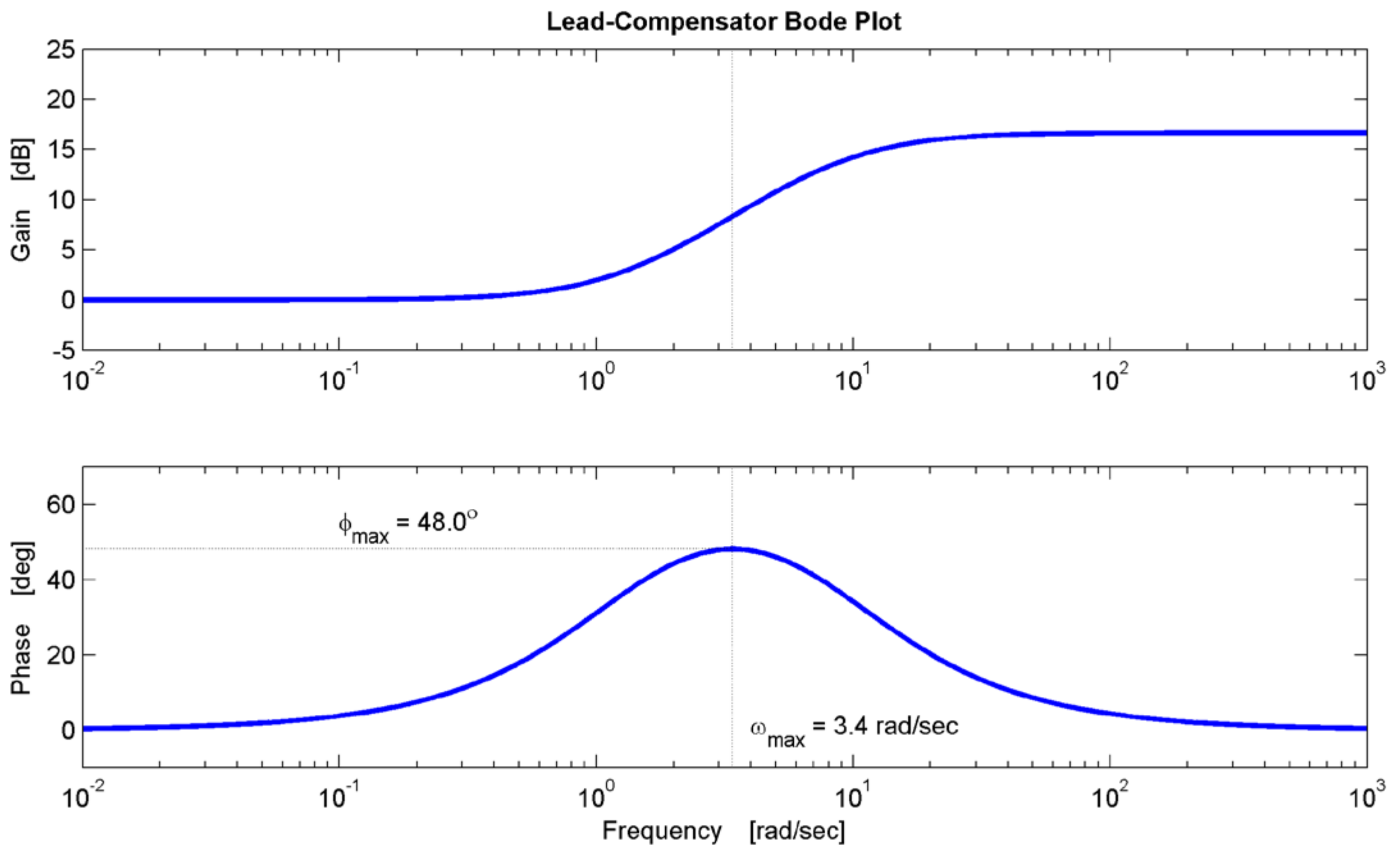
- The resulting lead compensator transfer function is

$$KD(s) = K\frac{(Ts + 1)}{(\beta Ts + 1)} = 12\frac{(0.766s + 1)}{(0.113s + 1)}$$

Lead Example 1 – Step 6

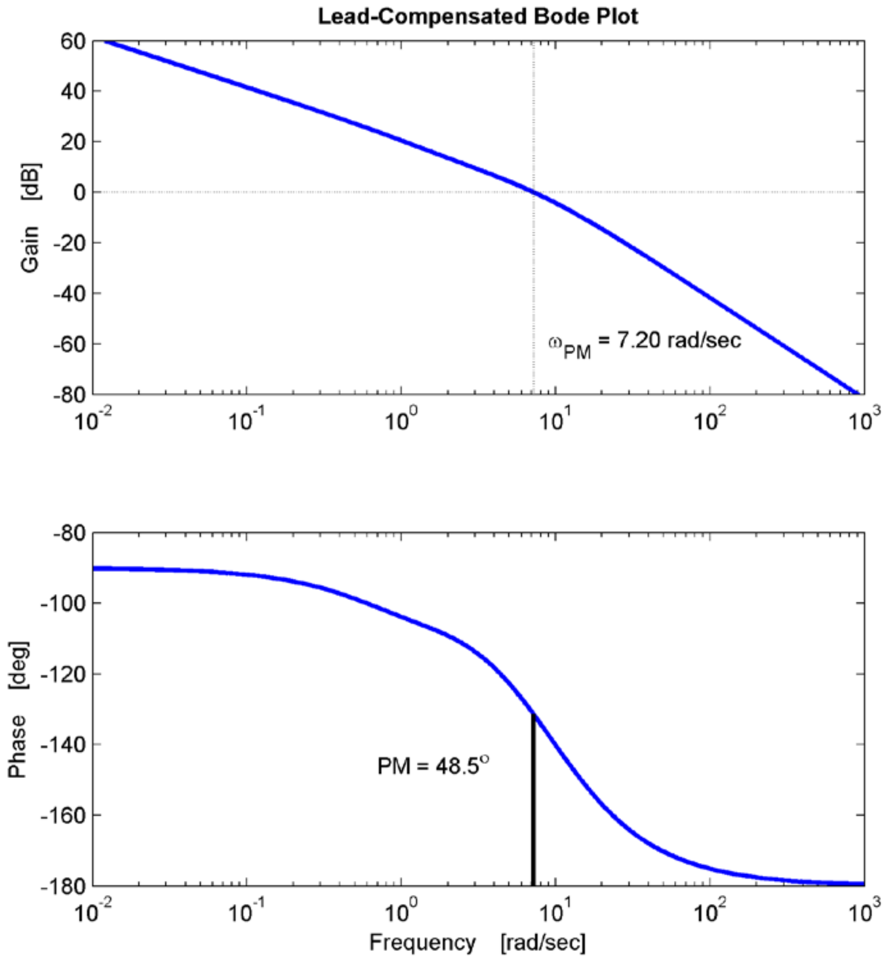
$$KD(s) = 12\frac{(0.766s + 1)}{(0.113s + 1)}$$

- The lead compensator Bode plot



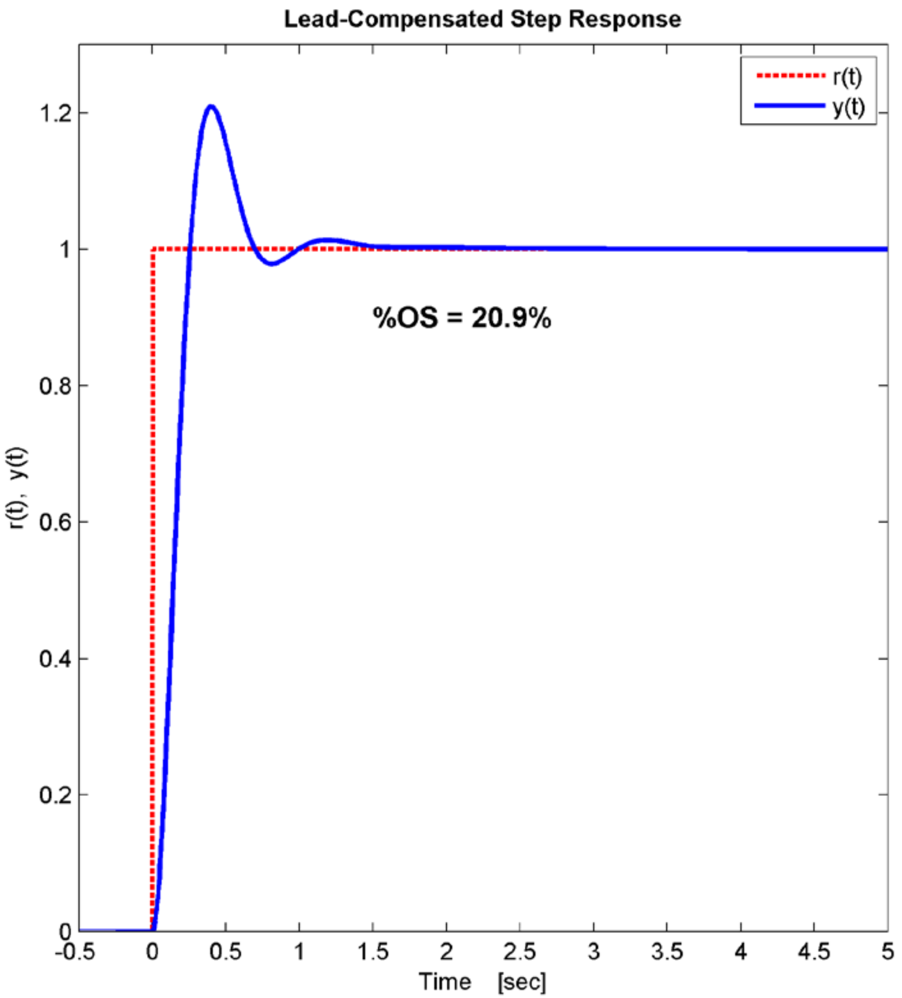
Lead Example 1 – Step 6

- Lead-compensated system:
  - $PM = 48.5°$
  - $\omega_{PM} = 7.2\ rad/sec$
- High-frequency compensator gain increased the crossover frequency
  - Phase was added at the *previous* crossover frequency
  - PM is below target
- Move lead zero/pole to higher frequencies
  - Reduce the crossover frequency increase
  - Improve phase margin



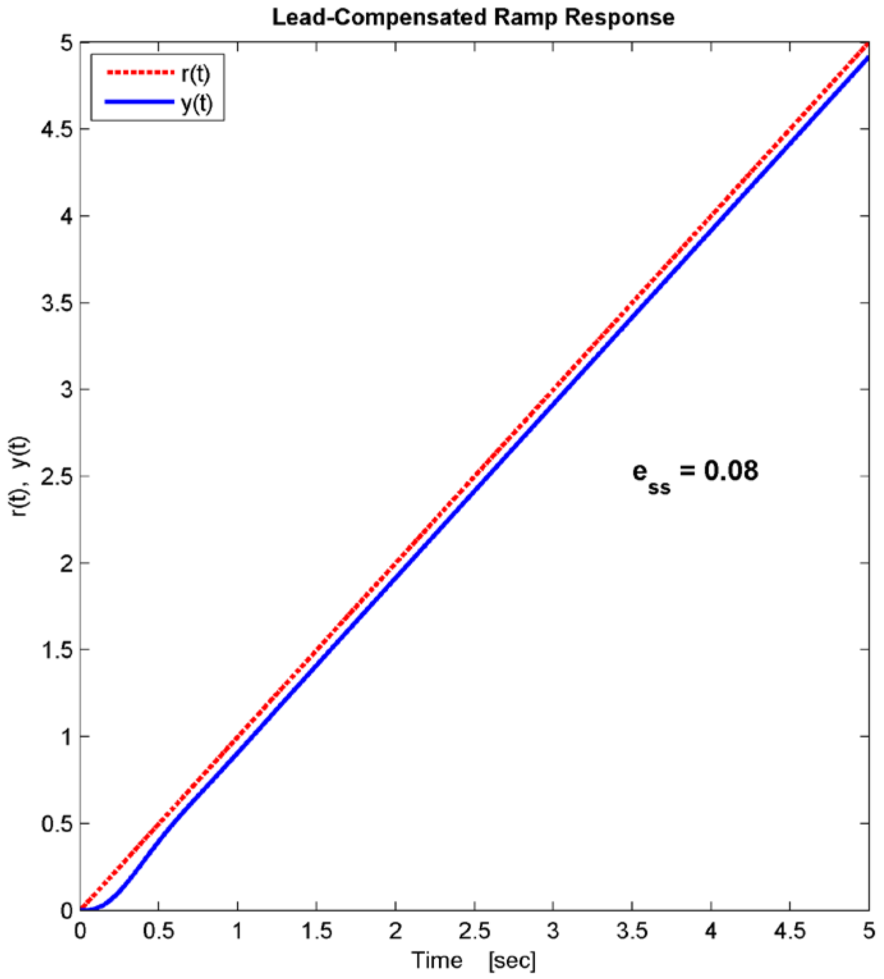
Lead Example 1 – Step 6

- As predicted by the insufficient phase margin, overshoot exceeds the target
  - $\%OS = 20.9\% > 15\%$
- Redesign compensator for higher ω_{max}
  - Improve phase margin
  - Reduce overshoot



Lead Example 1 – Step 6

- The steady-state error requirement has been satisfied
  - $e_{ss} = 0.08 < 0.1$
- Will not change with compensator redesign
  - Low-frequency gain will not be changed



Lead Example 1 – Step 6

- Iteration yields acceptable value for ω_{max}
  - $\omega_{max} = 5.5 \text{ rad/sec}$
  - Maintain same zero/pole spacing, β, and, therefore, same ϕ_{max}
- Recalculate zero/pole time constants:

$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = \frac{1}{5.5\sqrt{0.147}} = 0.4742$$

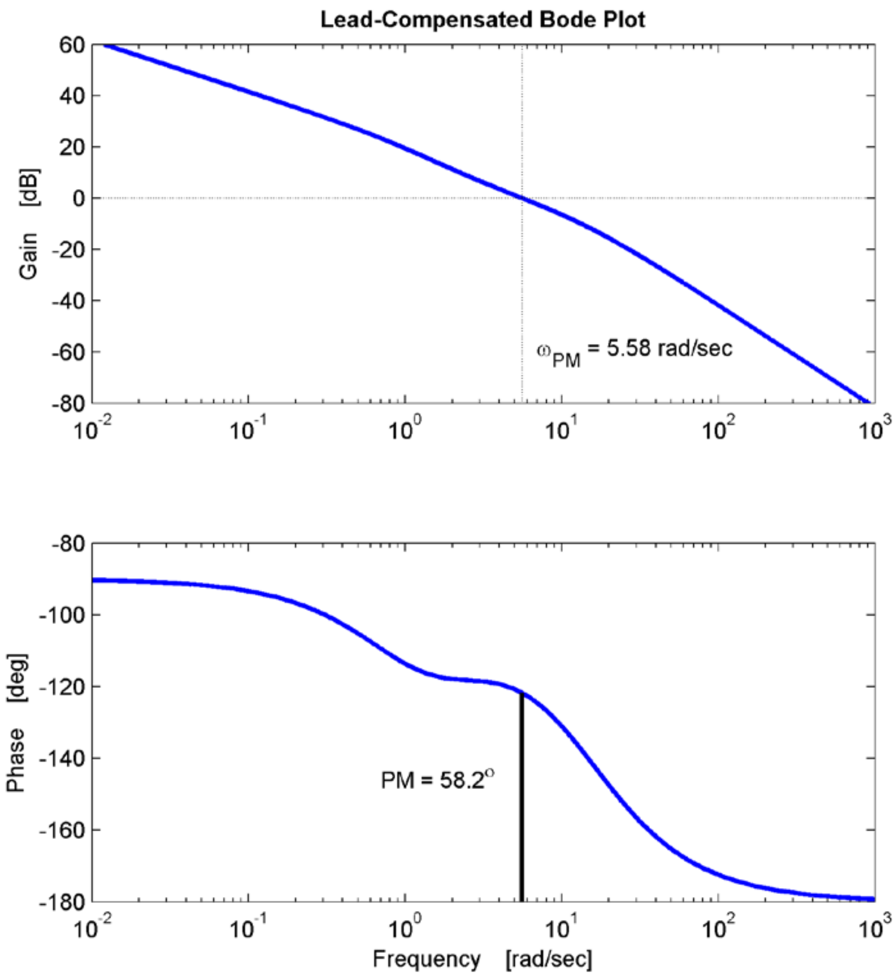
$$\beta T = 0.147 \cdot 0.4742 = 0.0697$$

- The updated lead compensator transfer function:

$$D(s) = 12\frac{(0.4742s + 1)}{(0.0697s + 1)}$$

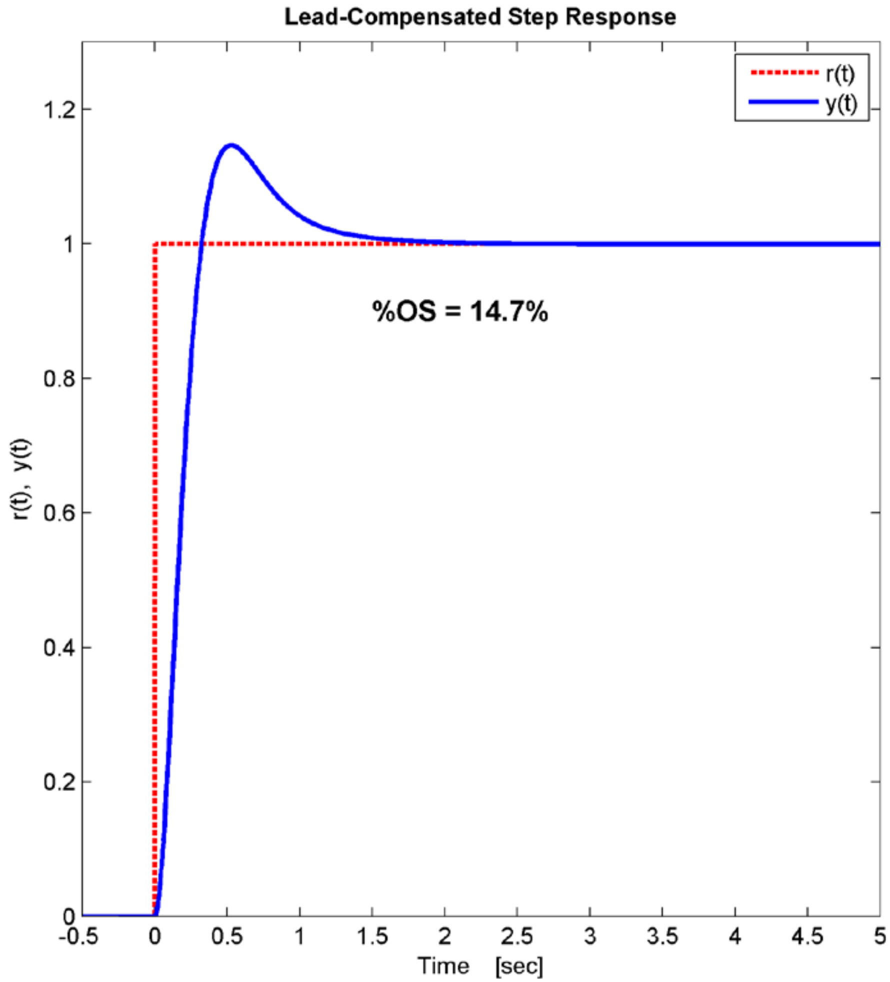
Lead Example 1 – Step 6

- Crossover frequency has been reduced
  - $\omega_{PM} = 5.58\, rad/sec$
- Phase margin is close to the target
  - $PM = 58.2°$
- Dip in phase is apparent, because ω_{max} is now placed at point of lower open-loop phase



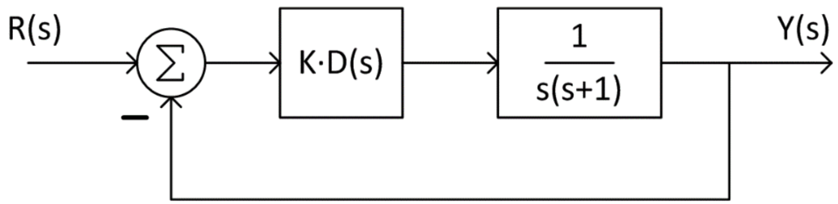
Lead Example 1 – Step 6

- Overshoot requirement now satisfied
  - $\%OS = 14.7\% < 15\%$
- Low-frequency gain has not been changed, so error requirement is still satisfied
- Design is complete



Lead Compensation – Example 2

- Again, consider the same system



- Design a compensator to satisfy the following
  - $t_s \approx 1.2\ sec \quad (\pm 1\%)$
  - $\%OS \approx 10\%$
- Now, we'll design a lead compensator to simultaneously adjust ***closed-loop bandwidth*** and ***phase margin***

Lead Example 2 – Step 1

- The required damping ratio for 10% overshoot is

$$\zeta = -\frac{\ln(OS)}{\sqrt{\pi^2+\ln^2(OS)}} = 0.5912$$

- Given the required damping ratio, calculate the required closed-loop bandwidth to yield the desired settling time

$$\omega_{BW} = \frac{4.6}{t_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$\omega_{BW} = 7.52\ rad/sec$$

- We'll initially set the gain, K, to place the crossover frequency, ω_{PM}, one octave below the desired closed-loop bandwidth

$$\omega_{PM} = \omega_{BW}/2 = 3.8\ rad/sec$$

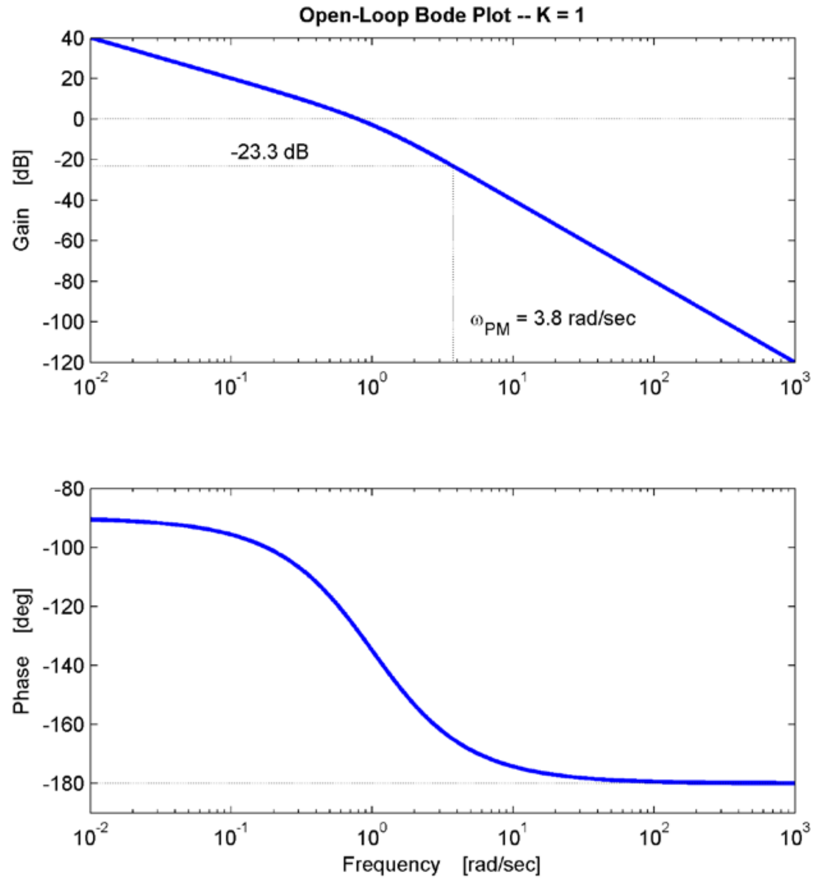
Lead Example 2 – Step 1

- Plot the Bode plot for $K = 1$
  - Determine the loop gain at the desired crossover frequency

$$K_{PM} = -23.3\ db$$

- Adjust K so that the loop gain at the desired crossover frequency is $0\ dB$

$$K = \frac{1}{K_{PM}} = 23.3\ dB = 14.7$$



Lead Example 2 – Steps 2 & 3

- Generate a Bode plot using the gain value just determined
- Phase margin for the uncompensated system:

$$PM_u = 14.9°$$

- Required phase margin to satisfy overshoot requirement:

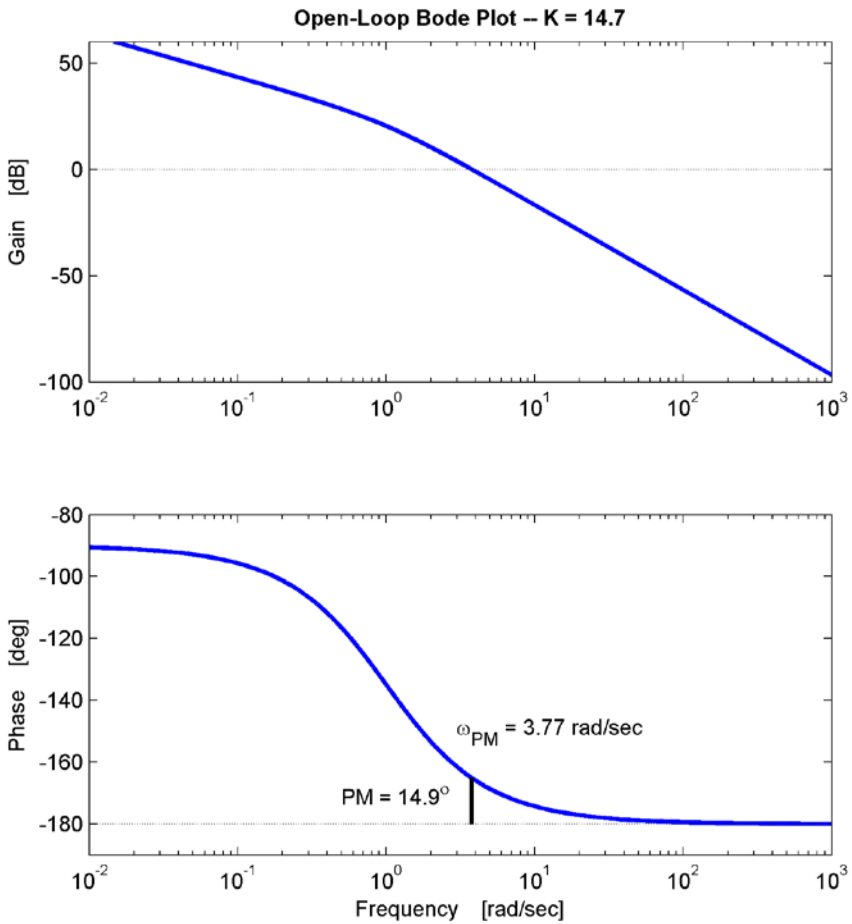
$$PM \approx 100\zeta = 59.1°$$

- Add 10° to account for crossover frequency increase

$$PM = 69.1°$$

- Required phase lead from the compensator

$$\phi_{max} = PM - PM_u = 54.2°$$



Lead Example 2 – Steps 4 & 5

- Calculate zero/pole spacing, β, from required phase lead, ϕ_{max}

$$\beta = \frac{1 - \sin(\phi_{max})}{1 + \sin(\phi_{max})} = 0.1040$$

- Calculate zero and pole time constants

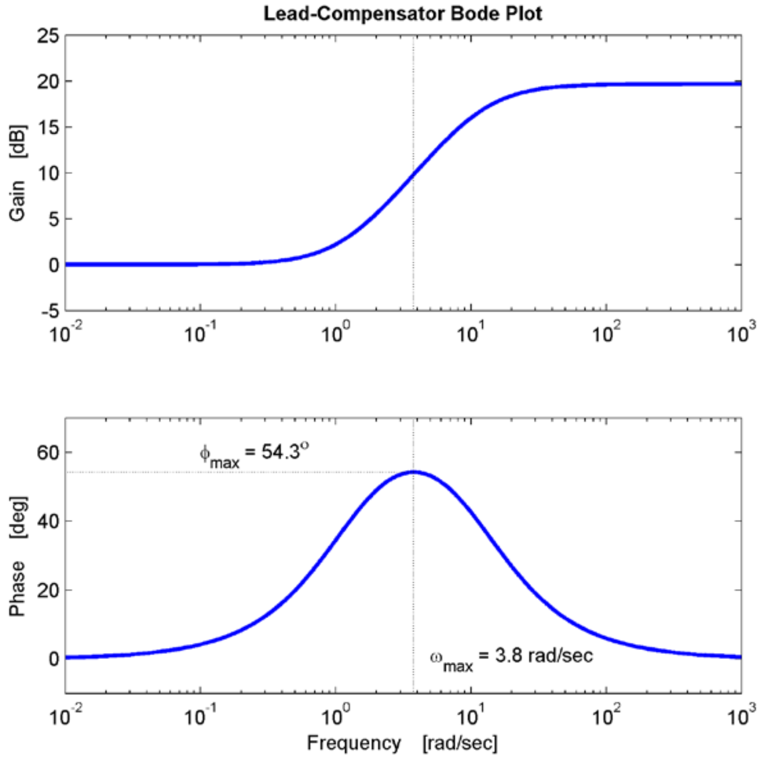
$$T = \frac{1}{\omega_{max}\sqrt{\beta}} = 0.8228\ sec$$

$$\beta T = 0.0855\ sec$$

- The resulting lead compensator transfer function:

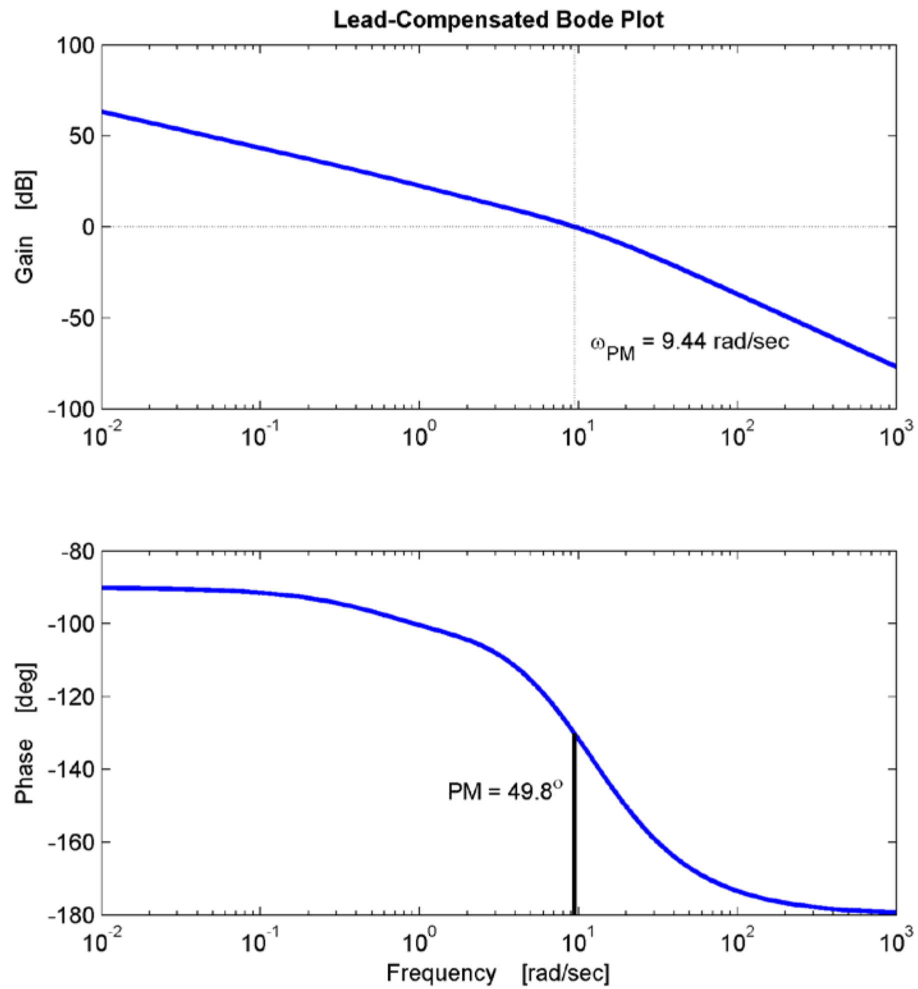
$$KD(s) = K\frac{(Ts + 1)}{(\beta Ts + 1)}$$

$$KD(s) = 14.7\frac{(0.8228s + 1)}{(0.0855s + 1)}$$



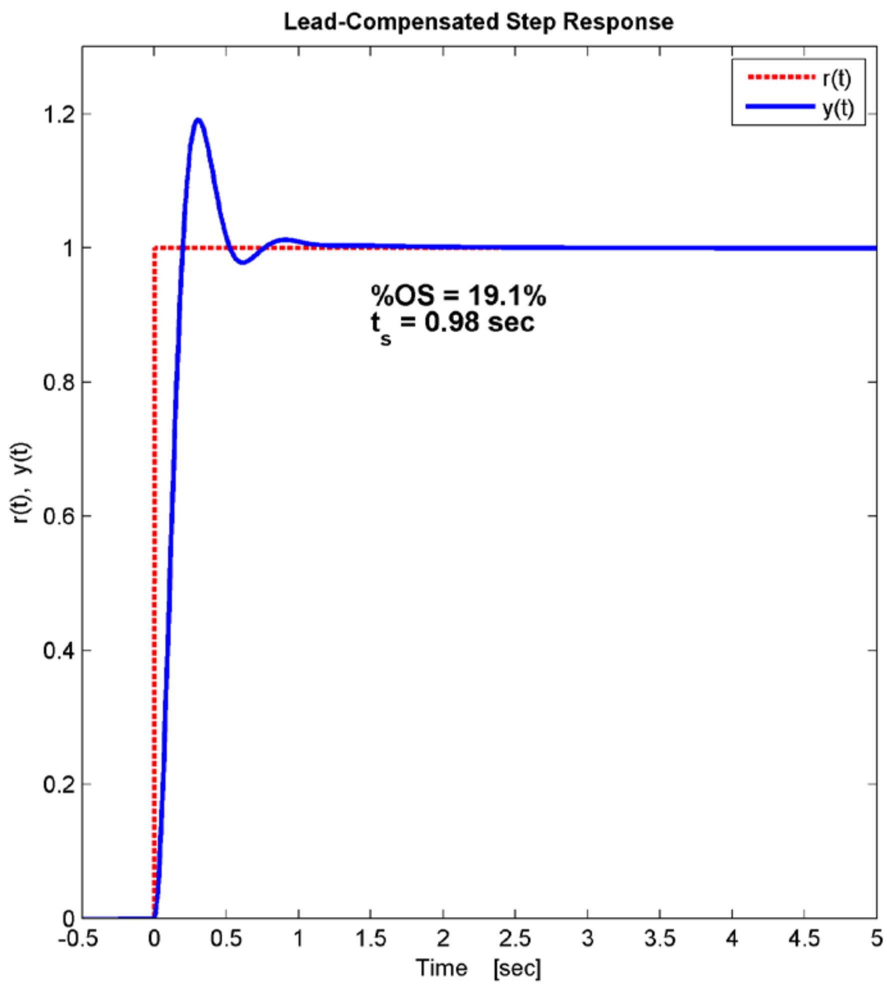
Lead Example 2 – Step 6

- Bode plot of the compensated system
  - $PM = 49.8°$
  - Substantially below target
- Crossover frequency is well above the desired value
  - $\omega_{PM} = 9.44\ rad/sec$
- Iteration will likely be required



Lead Example 2 – Step 6

- Overshoot exceeds the specified limit
  - $\%OS = 19.1° > 10\%$
- Settling time is faster than required
  - $t_s = 0.98\ sec < 1.2\ sec$
- Iteration is required
  - Start by reducing the target ω_{PM}



Lead Example 2 – Step 6

- Must redesign the compensator to meet specifications
  - Must ***increase PM*** to reduce overshoot
  - Can afford to ***reduce crossover***, ω_{PM}, to improve PM
- Try various combinations of the following
  - Reduce crossover frequency, ω_{PM}
  - Increase compensator zero/pole frequencies, ω_{max}
  - Increase added phase lead, ϕ_{max}, by reducing β
- Iteration shows acceptable results for:
  - $\omega_{PM} = 2.4\,rad/sec$
  - $\omega_{max} = 3.4\,rad/sec$
  - $\phi_{max} = 52°$

Lead Example 2 – Step 6

- Redesigned lead compensator:

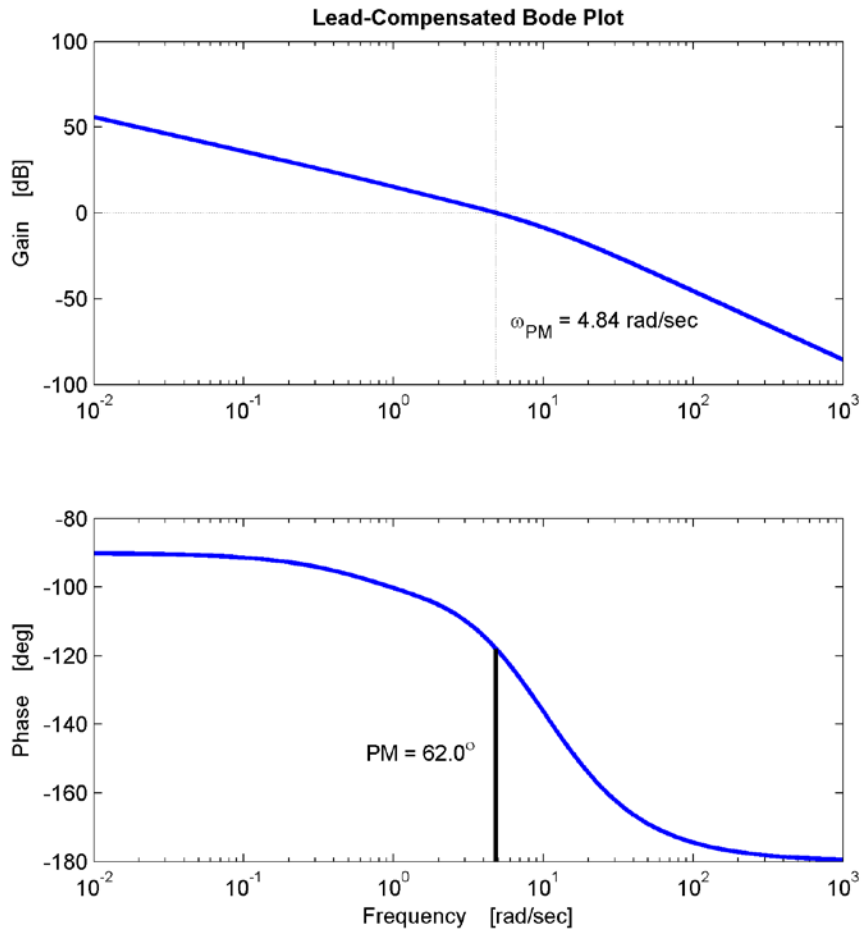
$$KD(s) = 6.27\frac{(0.8542s + 1)}{(0.1013s + 1)}$$

- Phase margin:

$$PM = 62°$$

- Crossover frequency:

$$\omega_{PM} = 4.84\, rad/sec$$



Lead Example 2 – Step 6

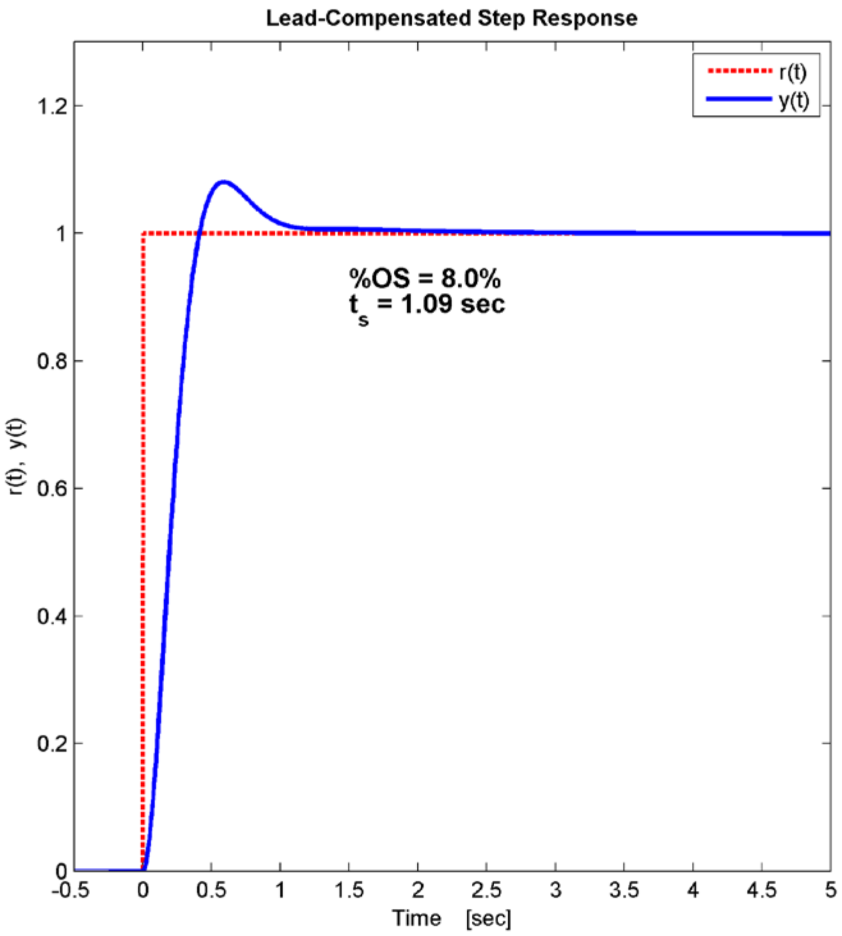
- Dynamic response requirements are now satisfied

- Overshoot:

  $$\%OS = 8\%$$

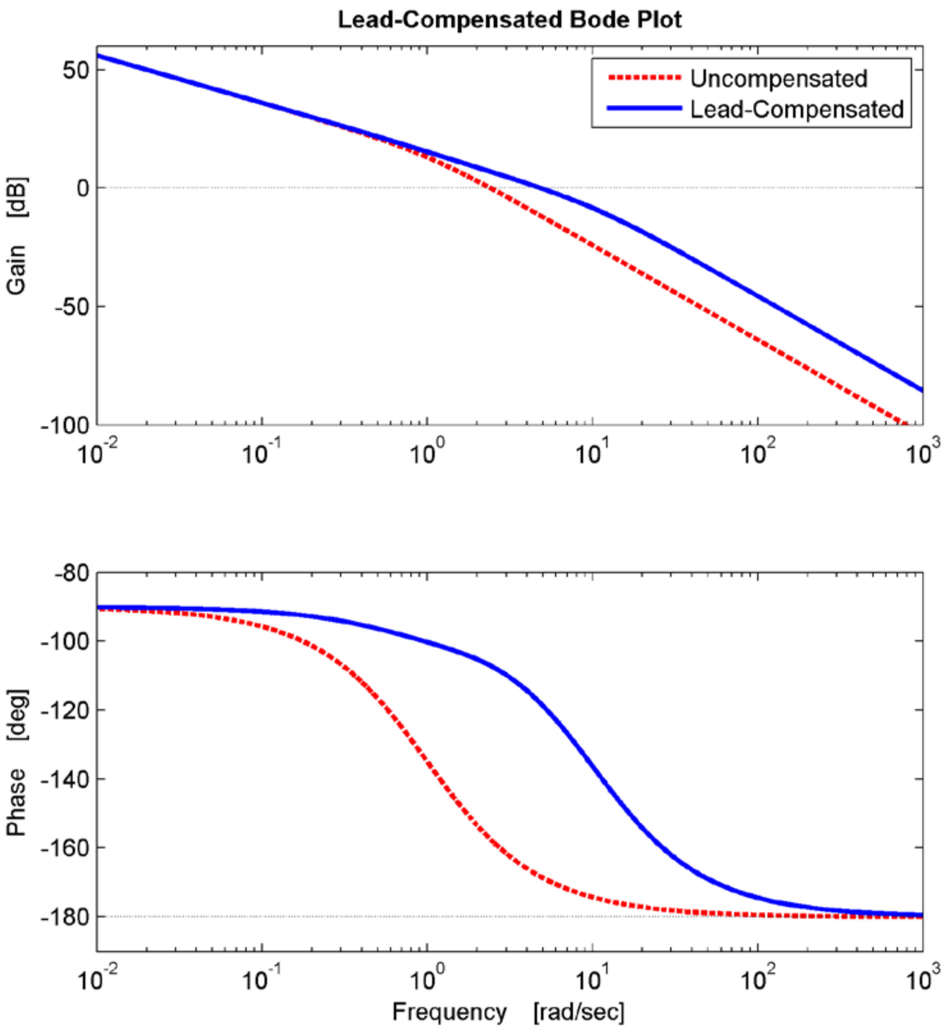
- Settling time:

  $$t_s = 1.09\ sec$$



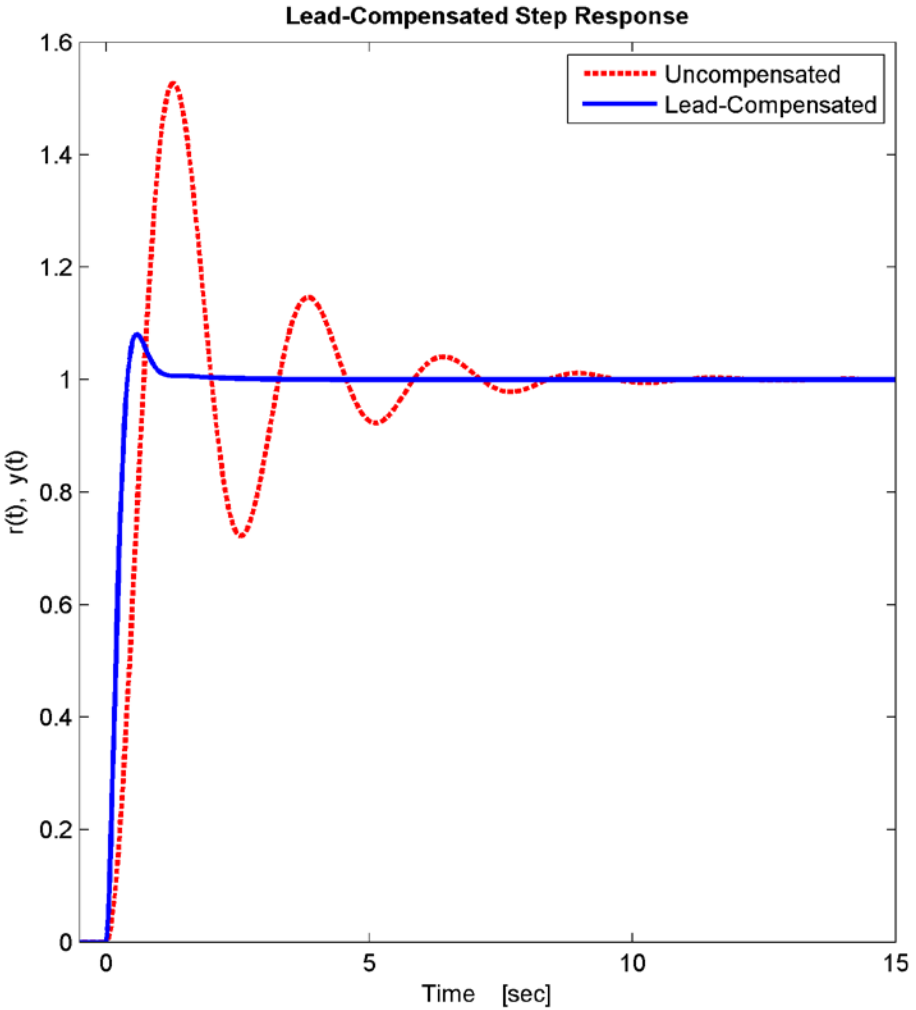
Lead Compensation – Example 2

- Lead compensator adds gain at higher frequencies
  - Increased crossover frequency
  - Faster response time
- Phase added near the crossover frequency
  - Improved phase margin
  - Reduced overshoot



Lead Compensation – Example 2

- Step response improvements:
  - Faster settling time
  - Faster risetime
  - Significantly less overshoot and ringing



Lead-Lag Compensation

- If performance specifications require adjustment of:
  - Bandwidth
  - Phase margin
  - Steady-state error
- Lead-lag compensation may be used

$$D(s) = \alpha \frac{(T_{lag}s + 1)}{(\alpha T_{lag}s + 1)} \frac{(T_{lead}s + 1)}{(\beta T_{lead}s + 1)}$$

- Many possible design procedures – one possibility:
  1. Design lag compensation to satisfy steady-state error and phase margin
  2. Add lead compensation to increase bandwidth, while maintaining phase margin