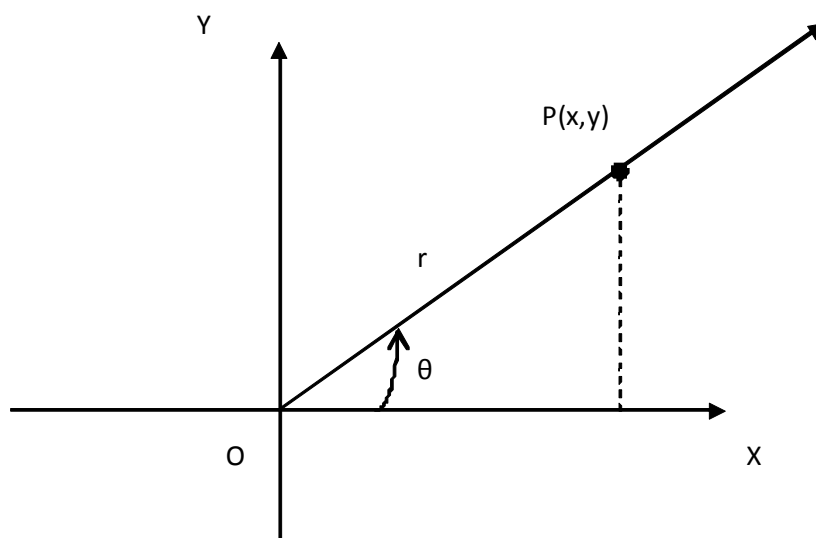# Section 2.6 Cylindrical and Spherical Coordinates

A) Review on the Polar Coordinates

The polar coordinate system consists of the origin O, the rotating ray or half line from O with unit tick. A point P in the plane can be uniquely described by its distance to the origin $r = dist\,(P,O)$ and the angle θ, $0 \leq \theta < 2\pi$:

We call (r, θ) the polar coordinate of P. Suppose that P has Cartesian (standard rectangular) coordinate (x, y). Then the relation between two coordinate systems is displayed through the following conversion formula:

$$\text{Polar Coord. to Cartesian Coord.:} \quad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\text{Cartesian Coord. to Polar Coord.:} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \tan\theta = \dfrac{y}{x} \end{cases}$$

$0 \leq \theta < \pi$ if $y > 0$, $\quad 2\pi \leq \theta < \pi$ if $y \leq 0$.

Note that function $\tan\theta$ has period π, and the principal value for inverse tangent function is

$$-\frac{\pi}{2} < \arctan\frac{y}{x} < \frac{\pi}{2}.$$

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So the angle should be determined by

$$\theta = \begin{cases} \arctan \dfrac{y}{x}, & \text{if } x > 0 \\ \arctan \dfrac{y}{x} + \pi, & \text{if } x < 0 \\ \dfrac{\pi}{2}, & \text{if } x = 0, \ y > 0 \\ -\dfrac{\pi}{2}, & \text{if } x = 0, \ y < 0 \end{cases}$$

Example 6.1. Find (a) Cartesian Coord. of P whose Polar Coord. is $\left(2, \dfrac{\pi}{3}\right)$, and (b) Polar Coord. of Q whose Cartesian Coord. is $(-1, -1)$.
Sol. (a)

$$x = 2\cos\frac{\pi}{3} = 1,$$
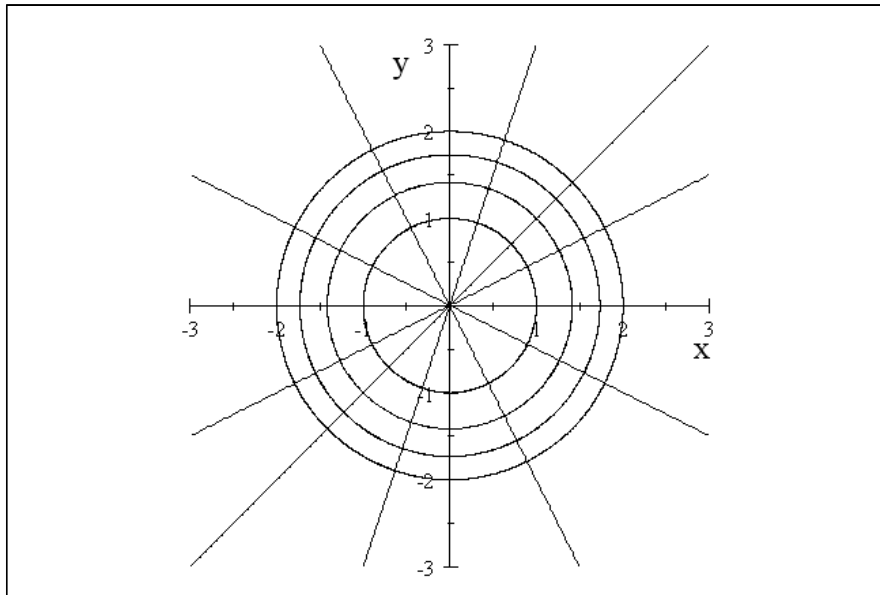$$y = 2\sin\frac{\pi}{3} = \sqrt{3}.$$

(b)

$$r = \sqrt{1+1} = \sqrt{2}$$
$$\tan\theta = \frac{-1}{-1} = 1 \implies \theta = \frac{\pi}{4} \text{ or } \theta = \frac{\pi}{4} + \pi = \frac{5\pi}{4}.$$

Since $(-1, -1)$ is in the third quadrant, we choose $\theta = \dfrac{5\pi}{4}$ so

$$\left(\sqrt{2}, \frac{5\pi}{4}\right) \text{ is Polar Coord.}$$

Under Polar Coordinate system, the graph of any equation of two variables r and θ is a curve. In particular, there are two families of coordinate curves that form Polar grid:

$$r = \text{constant} \quad - \quad \text{circle centered at } O \text{ with radius } r$$
$$\theta = \text{constant} \quad - \quad \text{ray with angle } \theta.$$

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B) Cylindrical Coordinate System

The cylindrical coordinate system basically is a combination of the polar coordinate system $xy - plane$ with an additional $z - coordinate$ vertically. In the cylindrical coordinate system, a point $P(x, y, z)$, whose Cartesian coordinate is (x, y, z), is assigned by the ordered triple (r, θ, z), where (r, θ) is the polar coordinate of (x, y), the vertical projection along $z - axis$ of P onto $xy - plane$.
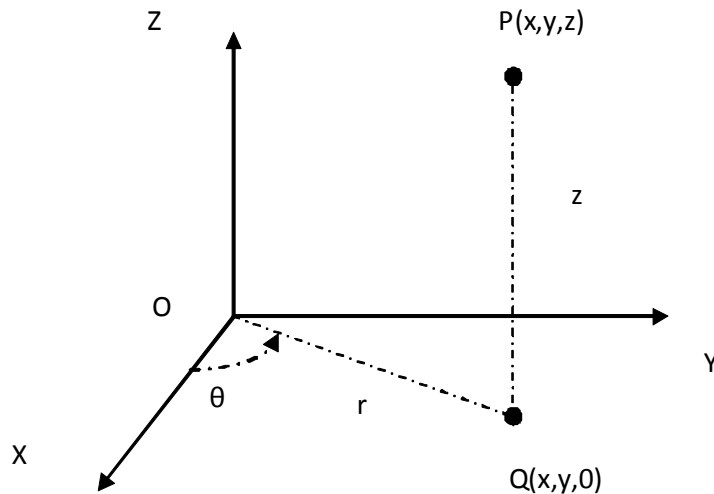
Thus, we readily have the conversion formula:

$$x = r \cos \theta$$
$$y = r \sin \theta$$
$$z = z.$$

The reserve formula from Cartesian coordinates to cylindrical coordinates follows from the conversion formula from $2D$ Cartesian to $2D$ polar coordinates:

$$r^2 = x^2 + y^2$$
$$\theta = \arctan \frac{y}{x} \ \ or \ \ \arctan \frac{y}{x} + \pi.$$

Example 6.2. (a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$, and find its rectangular coordinates. (b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Sol. (a) We first plot the point with 2D polar coordinate $(2, 2\pi/3)$ in $xy-plane : r = 2, \theta = 2\pi/3 = 120^o$. Then we raise it up vertically 1 unit. To find its rectangular coordinates, we use the formula

$$x = r \cos \theta = 2 \cos \frac{2\pi}{3} = 2 \left(-\frac{1}{2} \right) = -1$$
$$y = r \sin \theta = 2 \sin \frac{2\pi}{3} = 2 \left(\frac{\sqrt{3}}{2} \right) = \sqrt{3}$$


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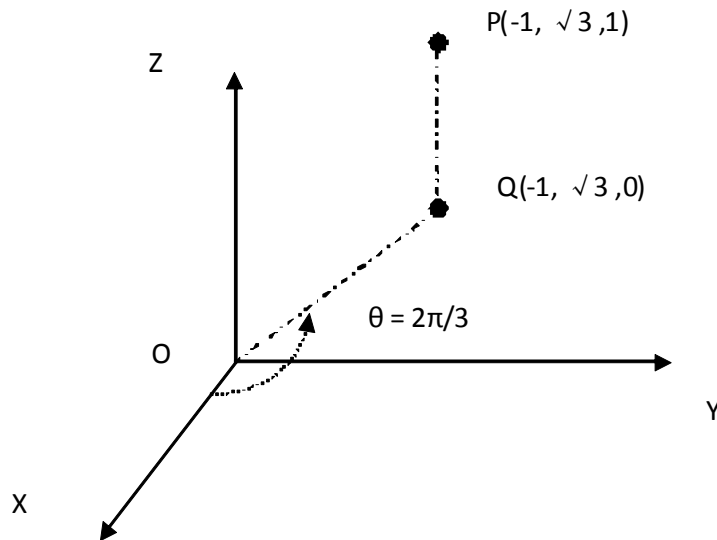
(b)

$$r = \sqrt{x^2 + y^2} = \sqrt{9 + 9} = \sqrt{18}$$
$$\tan \theta = \frac{y}{x} = \frac{-3}{3} = -1, \quad \arctan(-1) = -\frac{\pi}{4}.$$

There are two choices for $\theta : -\pi/4$ or $3\pi/4$. Since $(3, -3)$ is in the 4th quadrant, we find that the cylindrical coordinates for the point with rectangular coord $(3, -3, -7)$ is

$$\left(\sqrt{18}, -\pi/4, -7 \right).$$

As with the polar coordinate system, one finds it very convenient and simple to represent many surface using cylindrical coordinates instead of the rectangular coordinate system. For instance, the coordinate planes, under the cylindrical coordinate system, consists of three families surfaces described as follows:

- Equation $r =$ constant represents a circular cylinder with $z - axis$ as its symmetric axis.
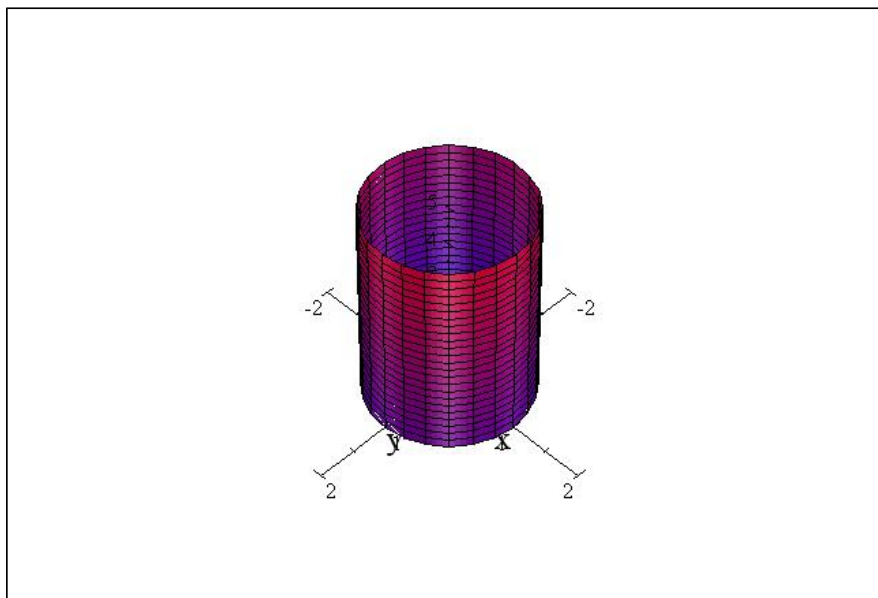
$$r = 1$$

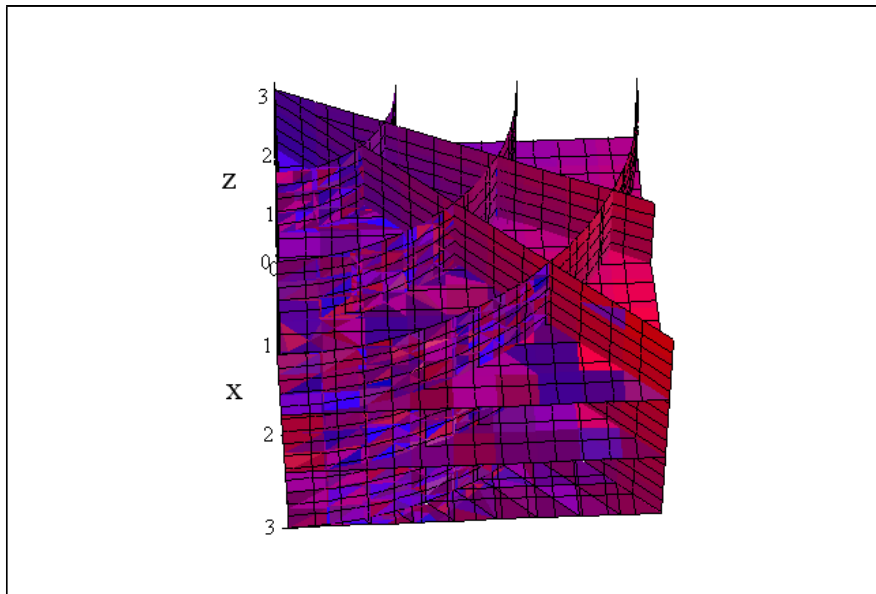- Equation $\theta =$ constant represents a half-plane originating from $z - axis$ with the constant angle to $zx - plane \ y = x$

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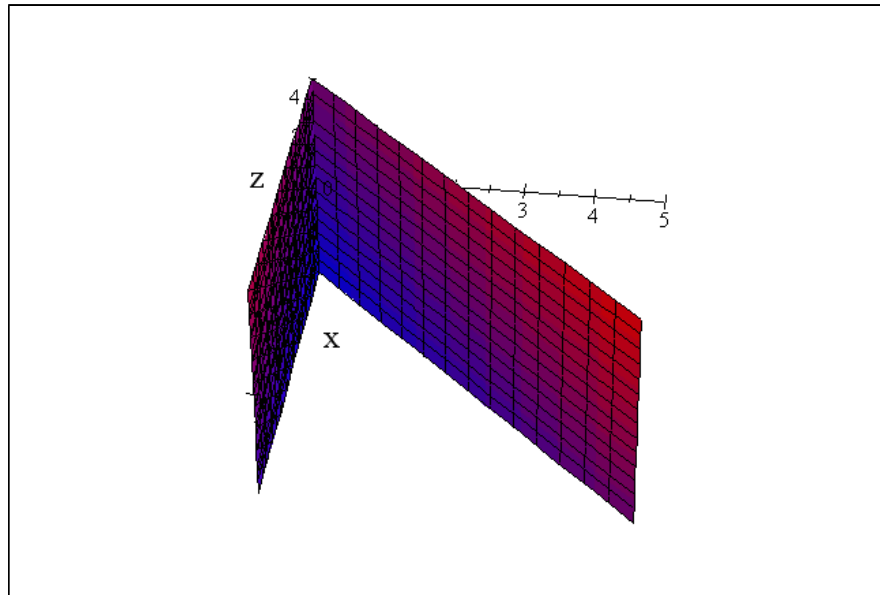
$$\theta = 0, \quad \theta = \frac{\pi}{4}$$

- $z = $ constant represents a plane parallel to $xy-plane$, the same as in the rectangular system.

Put all three families together, we have the cylindrical grids:

Cylindrical Grids

Therefore, equations for cylinder-like surfaces may be much easier using the cylindrical coordinate system.

Example 6.3. (a) Describe he surface whose cylindrical equation is $z = r$.

(b) Find the cylindrical equation for the ellipsoid $4x^2 + 4y^2 + z^2 = 1$.

(c) Find the cylindrical equation for the ellipsoid $x^2 + 4y^2 + z^2 = 1$.

Solution: (a)

$$z = r \implies z^2 = r^2 \implies$$
$$z^2 = x^2 + y^2$$

This a cone with its axis on $z - axis$.

(b)

$$4x^2 + 4y^2 + z^2 = 1 \implies$$
$$4r^2 + z^2 = 1$$

(c) If we use the cylindrical coordinate as we introduced above, we would get

$$x^2 + 4y^2 + z^2 = 1$$
$$r^2 \cos^2\theta + 4r^2 \sin^2\theta + z^2 = 1$$

or

$$r^2 + 3r^2 \sin^2\theta + z^2 = 1.$$

However, if we change the axis of the cylindrical coordinate system to $y-axis$, i.e.,

$$z = r \cos\varphi$$
$$x = r \sin\varphi$$
$$y = y,$$

where (r, φ) is the polar coordinate of the projection onto $zx - plane$, then the equation between

$$x^2 + 4y^2 + z^2 = 1 \implies$$
$$4y^2 + r^2 = 1.$$

C) Spherical Coordinate System.

In the spherical coordinate system, a point $P(x, y, z)$, whose Cartesian coordinates are (x, y, z), is described by an ordered triple (ρ, θ, ϕ), where

$$\rho > 0, \ 0 \leq \theta \leq 2\pi, \ \ 0 \leq \phi \leq \pi$$

are defined as follows.

- $\rho = dist(P, O)$

- θ is defined in the same way in the cylindrical coordinate system: Angle from $zx - plane$, counter-clockwise, to the half-plane originating from $z - axis$ and containing P

- $\phi =$ angle from positive $z - axis$ to vector $\overrightarrow{OP}$.
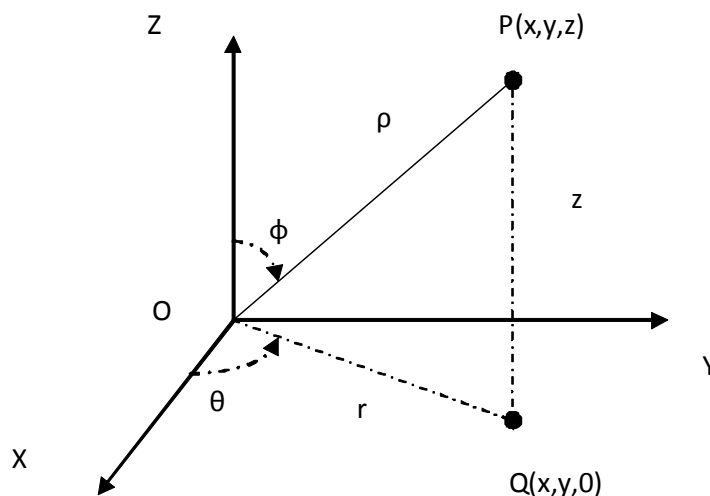
Note that when P is on $z - axis$, $\phi = 0$, and

ϕ increases from 0 to $\dfrac{\pi}{2}$ as P moves closer to $xy - plane$,

and ϕ keeps increasing as P moves below $xy - plane$,

and ϕ reaches the maximum value π when P is on the negative $z - axis$.

Conversion formula (rectangular $\leftrightarrow$ cylindrical $\leftrightarrow$ spherical)

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$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = r\cot\phi = \rho\cos\phi$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$
$$\tan\theta = \frac{y}{x}$$
$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Note that when we determine θ, we also need to consider which quadrant the point is in. More precisely,

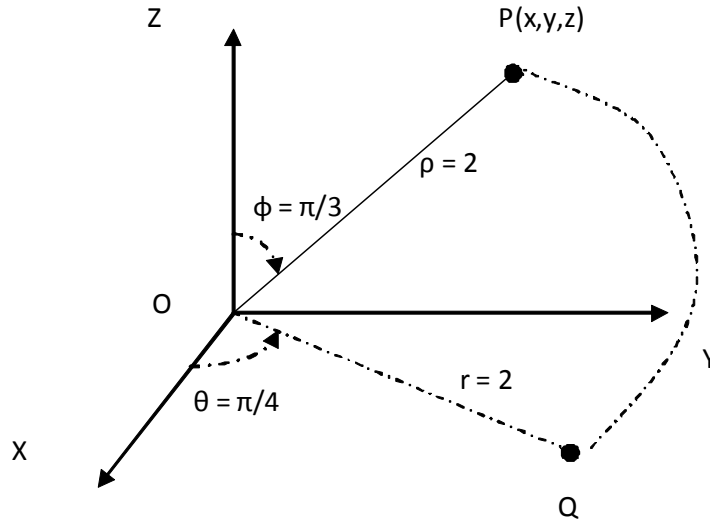$$\theta = \begin{cases} \arctan\dfrac{y}{x}, & \text{if } x > 0 \\ \arctan\dfrac{y}{x} + \pi, & \text{if } x < 0 \\ \dfrac{\pi}{2}, & \text{if } x = 0,\ y > 0 \\ -\dfrac{\pi}{2}, & \text{if } x = 0,\ y < 0 \end{cases}.$$

However, ϕ can be determined uniquely as

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

Example 6.4. (a) Plot the point with spherical coordinates $(2, \pi/4, \pi/3)$, and find its rectangular coordinates. (b) Find the spherical coordinates for the point with rectangular coordinates $(0, 2\sqrt{3}, -2)$.

Sol: (a) We first plot the point Q on $xy - plane$ with polar coordinate $(2, \pi/4)$. We then rotate $\overrightarrow{OQ}$ in the vertical direction (i.e., the rotation remains in the plane spanned by z-axis and line OQ) till its angle with $z - axiz$ reaches $\pi/3$.

$$x = \rho \sin \phi \cos \theta = 2 \sin \left(\frac{\pi}{3} \right) \cos \left(\frac{\pi}{4} \right) = \frac{\sqrt{6}}{2}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \left(\frac{\pi}{3} \right) \sin \left(\frac{\pi}{4} \right) = \frac{\sqrt{6}}{2}$$

$$z = \rho \cos \phi = 2 \cos \left(\frac{\pi}{3} \right) = 1.$$

(b)

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(2\sqrt{3} \right)^2 + 4} = \sqrt{16} = 4$$

$$\theta = \frac{\pi}{2}, \quad \text{since} \quad \left(0, 2\sqrt{3}, -2 \right) \quad \text{is on positive } y - axis$$

$$\cos \phi = \frac{z}{\rho} = \frac{-2}{4} = -\frac{1}{2} \implies \phi = \frac{2\pi}{3}.$$

Ans: Spherical coord. $= \left(4, \dfrac{\pi}{2}, \dfrac{2\pi}{3} \right).$

The coordinate surfaces are:

- $\rho = \rho_0$ (constant) is a sphere centered at the origin with radius ρ_0

- $\theta = \theta_0$ (constant) is a half-plane originated from $z - axis$ with angle θ_0 to $zx - plane$ (the same as in the cylindrical coordinate system

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- $\phi = \phi_0$ (constant) is a circular cone with $z - axis$ as its symmetric axis and the opening angle ϕ_0

In fact, if we convert $\phi = \phi_0$ into the rectangular coordinate system, we have

$$x = \rho \sin \phi_0 \cos \theta$$
$$y = \rho \sin \phi_0 \sin \theta$$
$$z = \rho \cos \phi_0.$$

We can eliminate θ by

$$x^2 + y^2 = \rho^2 \sin^2 \phi_0$$
$$z^2 = \rho^2 \cos^2 \phi_0.$$

We then eliminate ρ by dividing the first equation by the second

$$\frac{x^2 + y^2}{z^2} = \tan^2 \phi_0 \quad \text{or} \quad \frac{x^2}{\tan^2 \phi_0} + \frac{y^2}{\tan^2 \phi_0} = \frac{z^2}{1}$$

which is a cone.

Again, these three families of coordinate surfaces form the spherical grids:

Spherical Grids

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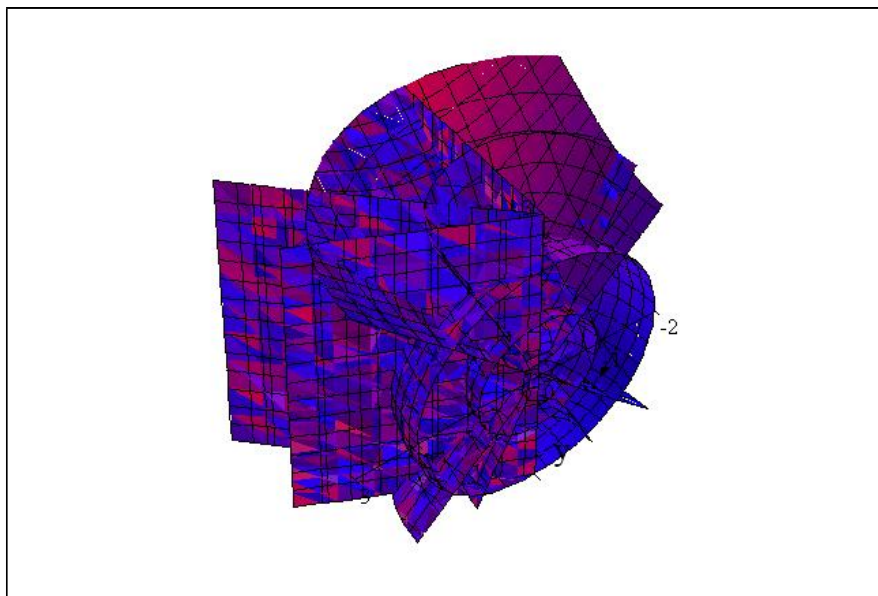
Example 6.5. Find the spherical equation for the hyperboloid of two sheets $x^2 - y^2 - z^2 = 1$.

Solution: By direct substitution, we obtain, under the standard spherical coordinate system

$$(\rho \sin \phi \cos \theta)^2 - (\rho \sin \phi \sin \theta)^2 - (\rho \cos \phi)^2 = 1$$

or

$$\rho^2 \left(\sin^2 \phi \cos^2 \theta - \sin^2 \phi \sin^2 \theta - \cos^2 \phi \right) = 1.$$

Example 6.6. Find a rectangular equation for the surface whose spherical equation is $\rho = 2 \sin \phi \sin \theta$.

Solution: We want to eliminate the spherical variables ρ, θ, ϕ and replace them with x, y, z. To this end, we multiply the equation by ρ to obtain

$$\rho^2 = 2\rho \sin \phi \sin \theta.$$

From the conversion formula, we have

$$x^2 + y^2 + z^2 = 2y$$

or

$$x^2 + (y - 1)^2 + z^2 = 1.$$

This is the sphere centered at $(0, 1, 0)$ with radius $R = 1$.

Homework:

1. Change from rectangular to (i) cylindrical coordinates and (ii) to spherical coordinates.

   (a) $\left(1, \sqrt{3}, -2\sqrt{3} \right)$
   (b) $(0, 1, -1)$
   (c) $\left(-1, -\sqrt{3}, 2 \right)$

2. Describe in words and then sketch the surface.

   (a) $r = 2$
   (b) $\rho = 2$

(c) $\phi = \dfrac{\pi}{6}$

(d) $\theta = \dfrac{\pi}{6}$

3. Convert into (i) cylindrical equation and (ii) spherical equation. You must simply your solutions.

   (a) $z = 2x^2 + 2y^2$

   (b) $x^2 + y^2 - 2z^2 = 3$

   (c) $x^2 + 2y^2 - z^2 = 3$

4. (optional) Identify the surface by converting into rectangular equation.

   (a) $r = 3\cos\theta$

   (b) $\rho = 3\cos\phi$

   (c) $r^2 - 2z^2 = 1$.