Section 3.2
Solving Systems of Linear Equations Using Matrices

In Section 1.3 we solved 2X2 systems of linear equations using either the substitution or elimination method. If the system is larger than a 2X2, using these methods becomes tedious. In this section we'll learn how matrices can be used to represent system of linear equations and how to solve them, no matter the size.

In order to solve systems of linear equation using matrices, we'll only need the augmented matrix. In a later section we'll need the coefficient and constant matrices.

The following row operations, that are a result of the elimination method in Section 1.3, will allow us to write a linear system in a simplified and equivalent form. Equivalent systems have the same solution sets.

Row Operations

If any of the following row operations are performed on an augmented matrix, the resulting matrix is an equivalent matrix.

• Swap two rows.

Notation: $R_1 \leftrightarrow R_2$ means Row 1 was swapped with Row 2.

• A row is multiplied by a nonzero constant.

Notation: $-5R_1$ means -5 is multiplied to Row 1.

• A row is multiplied by a nonzero constant then added to another row.

Notation: $2R_1 + R_2$ means 2 is multiplied to Row 1 then added to Row 2

We'll use row operations to write the augmented matrix in a specific form called the **row reduced form**, which will allow us to read off the solution to the system quite easily.

Row Reduced Form

A matrix is in row reduced form if the following conditions are satisfied.

1. If a row contains all zeros, it must lie at the bottom of the matrix.

2. The first nonzero element in each row must be a one, called a leading one. Applying any row operations to obtain a leading one is called **pivoting the matrix** about that element that becomes a one.

3. All other elements in each column containing a leading one are zeros. This defines a **unit column**.

4. In any two successive rows, the leading one in the row below lies to the right of the leading one in the row above.

Example 1: Determine which of the following matrices are in row-reduced form. If a matrix is not in row-reduced form, state which condition(s) is/are violated.

a. $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

b. $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 11 \\ 0 & 0 & 8 & 24 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

c. $\left(\begin{array}{ccc|c} 1 & 0 & 3 & -3 \\ 0 & 1 & 5 & 1 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

d. $\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

e. $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

f. $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 2 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

g. $\left(\begin{array}{ccc|c} 1 & -9 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

h. $\left(\begin{array}{cc|c} 1 & 0 & 9 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

i. $\left(\begin{array}{cccc|c} 0 & -5 & 0 & 4 & 30 \\ 0 & 0 & 1 & 11 & 20 \\ 1 & 0 & 0 & 7 & 21 \end{array}\right)$

RRF? Yes or No
Condition(s): 1, 2, 3, 4

j. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & -15 & 30 \\ 0 & 1 & 0 & -3 & -13 \\ 0 & 0 & 0 & 1 & 25 \end{array}\right)$ *RRF? Yes or No* *Condition(s): 1, 2, 3, 4*

Now that we know the row reduced form, let's show how easily the solution can be read from the row reduced augmented matrix. Recall that a linear system of equation can have one solution, no solution or infinitely many solutions.

A Unique Solution

Example 2: The following augmented matrix is in row reduced form

$$\left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -5 \end{array}\right)$$

$\boxed{\begin{array}{l} x = 10 \\ y = -5 \end{array}}$

Give the solution set for the associated linear system.

No Solution

Example 3: The following augmented matrix is in row reduced form

$$\begin{array}{cccc} x & y & z & \\ \multicolumn{4}{c}{\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right)} \end{array}$$

Give the solution set for the associated linear system.

Row 3 Reads $0 \cdot x + 0 \cdot y + 0 \cdot z = 3$

$0 \neq 3$ No solution

The system is inconsistent!

Infinitely Many Solutions

Example 4: The following augmented matrix is in row reduced form

$$\begin{array}{cccc} x & y & z & \\ \multicolumn{4}{c}{\left(\begin{array}{ccc|c} 1 & 0 & 3 & -4 \\ 0 & 1 & 1 & 9 \end{array}\right)} \end{array}$$

Give the solution set for the associated linear system.

$x + 3z = -4 \rightarrow x = -3z - 4$ z = any real #

$y + z = 9 \rightarrow y = -z + 9$

$(-4, 9, 0)$ $(-7, 8, 1)$

Our objective for the rest of this section will be to write augmented matrices in row reduced form. We will use the Gauss-Jordan Elimination Method to do this.

Gauss-Jordan Elimination Method

Basically, you will apply row operations to write the augmented matrix in row reduced form and read off the solution(s) easily.

1. Write the augmented matrix associated with the given system.

2. Use row operations to write the augmented matrix in row reduced form. If at any point a row in the matrix contains zeros to the left of the vertical line and a nonzero number to its right, stop the process the problem has no solution.

3. Read off the solution(s).

The row operations used in Step 2 are not unique; however, the final answer(s) will be equivalent.

Example 5: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$x + 2y = 1$$
$$2x + 3y = -1$$

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 2 & 3 & -1 \end{array}\right] \quad -2R1 + R2 \to R2$$

$$\begin{array}{rrrr} & -2 & -4 & -2 \\ + & 2 & 3 & -1 \\ \hline & 0 & -1 & -3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & -1 & -3 \end{array}\right] \quad -1(R_2) \to R2$$

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 1 \\ 0 & \boxed{1} & 3 \end{array}\right] \quad -2R2 + R1 \to R1$$

$$\begin{array}{rrrr} & 0 & -2 & -6 \\ + & 1 & 2 & 1 \\ \hline & 1 & 0 & -5 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array}\right]$$

$$\boxed{\begin{array}{l} x = -5 \\ y = 3 \end{array}}$$

Example 6: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$y - 8z = 9$$
$$x - 2y + 3z = -3$$
$$7y - 5z = 12$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -8 & 9 \\ 1 & -2 & 3 & -3 \\ 0 & 7 & -5 & 12 \end{array}\right] \xrightarrow{R1 \leftrightarrow R2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 3 & -3 \\ 0 & \boxed{1} & -8 & 9 \\ 0 & 7 & -5 & 12 \end{array}\right]$$

$\to 0$

$2R2 + R1 \to R1$

	0	2	-16	18
+	1	-2	3	-3
	1	0	-13	15

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -13 & 15 \\ 0 & \boxed{1} & -8 & 9 \\ 0 & 7 & -5 & 12 \end{array}\right]$$

$\to 0$

$-7R2 + R3 \to R3$

	0	-7	56	-63
+	0	7	-5	12
	0	0	51	-51

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -13 & 15 \\ 0 & \boxed{1} & -8 & 9 \\ 0 & 0 & \boxed{51} & -51 \end{array}\right]$$

$\frac{R3}{51} \to R3$ or $\frac{1}{51} R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & 0 & -13 & 15 \\ 0 & 1 & -8 & 9 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

$\to 0$ $\to 0$

$8R3 + R2 \to R2$

	0	0	8	-8
+	0	1	-8	9
	0	1	0	1

$13R3 + R1 \to R1$

	0	0	13	-13
+	1	0	-13	15
	1	0	0	2

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\boxed{\begin{array}{l} x = 2 \\ y = 1 \\ z = -1 \end{array}}$$

Example 7: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$x - 2y = 2$$
$$7x - 14y = 14$$
$$3x - 6y = 6$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 7 & -14 & 14 \\ 3 & -6 & 6 \end{array}\right] \quad \begin{array}{l} \frac{1}{7}R2 \to R2 \\ \frac{1}{3}R3 \to R3 \end{array} \quad \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 1 & -2 & 2 \\ 1 & -2 & 2 \end{array}\right]$$

$$x - 2y = 2$$

$$x = 2y + 2 \qquad y = \text{any real \#}$$

Inf. many solutions.

Example 8: Solve the system of linear equations using the Gauss-Jordan elimination method.

$3x + 3y = 9 \rightarrow x + y = 3$

$2x + 3y + z = 6$

$x - z = 1$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 3 \\ 2 & 3 & 1 & 6 \\ 1 & 0 & -1 & 1 \end{array}\right]$$

$-2R1 + R2 \rightarrow R2$

$$\begin{array}{rrrrr} + & -2 & -2 & 0 & -6 \\ & 2 & 3 & 1 & 6 \\ \hline & 0 & 1 & 1 & 0 \end{array}$$

$-R1 + R3 \rightarrow R3$

$$\begin{array}{rrrrr} + & -1 & -1 & 0 & -3 \\ & 1 & 0 & -1 & 1 \\ \hline & 0 & -1 & -1 & -2 \end{array}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 3 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & -1 & -1 & -2 \end{array}\right]$$

$R2 + R3 \rightarrow R3$

$0 \quad 0 \quad 0 \mid -2$

No solution!

Example 9: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$x + 2y - 3z = -2$$
$$3x - y - 2z = 1$$
$$2x + 3y - 5z = -3$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & -2 \\ 3 & -1 & -2 & 1 \\ 2 & 3 & -5 & -3 \end{array}\right]$$

$-3R1 + R2 \to R2$

$$\begin{array}{rrrrr} & -3 & -6 & 9 & 6 \\ + & 3 & -1 & -2 & 1 \\ \hline & 0 & -7 & 7 & 7 \end{array}$$

$\frac{R2}{-7} \to R2$

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & -2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & -1 & 1 & 1 \end{array}\right]$$

$-2R1 + R3 \to R3$

$$\begin{array}{rrrrr} & -2 & -4 & 6 & 4 \\ + & 2 & 3 & -5 & -3 \\ \hline & 0 & -1 & 1 & 1 \end{array}$$

$R2 + R3 \to R3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & -2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2 \to 0$

$-2R2 + R1 \to R1$

$$\begin{array}{rrrrr} & 0 & -2 & 2 & 2 \\ + & 1 & 2 & -3 & -2 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - z = 0 \to x = z$

$y - z = -1 \to y = z - 1$

$z =$ any real #

Inf. many solutions.

Example 10: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$2x + 3y = 2$$
$$x + 3y = -2$$
$$x - y = 3$$

$$\left[\begin{array}{cc|c} 2 & 3 & 2 \\ 1 & 3 & -2 \\ 1 & -1 & 3 \end{array}\right] \quad R_1 - R_2 \to R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 4 \\ 1 & 3 & -2 \\ 1 & -1 & 3 \end{array}\right]$$

$-R1 + R2 \to R2$

$$\begin{array}{rrr} -1 & 0 & -4 \\ +\ 1 & 3 & -2 \\ \hline 0 & 3 & -6 \end{array}$$

$-R1 + R3 \to R3$

$$\begin{array}{rrr} -1 & 0 & -4 \\ +\ 1 & -1 & 3 \\ \hline 0 & -1 & -1 \end{array}$$

$\frac{1}{3}R2 \to R2$

$-1(R3) \to R3$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

$-R2 + R3 \to R3$

$$\begin{array}{rrr} 0 & -1 & 2 \\ +\ 0 & 1 & 1 \\ \hline 0 & 0 & 3 \end{array}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \end{array}\right]$$

$\boxed{\text{No solution!}}$

Try this one: Solve the system of linear equations using the Gauss-Jordan elimination method.

$$2x + y - 2z = 4$$
$$x + 3y - z = -3$$
$$3x + 4y - z = 7$$