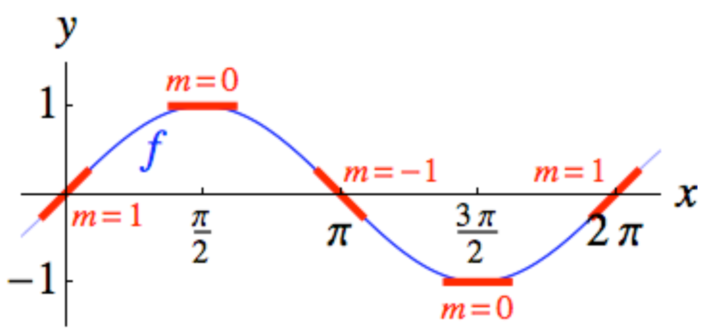
SECTION 3.4: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

LEARNING OBJECTIVES

- Use the Limit Definition of the Derivative to find the derivatives of the basic sine and cosine functions. Then, apply differentiation rules to obtain the derivatives of the other four basic trigonometric functions.
- Memorize the derivatives of the six basic trigonometric functions and be able to apply them in conjunction with other differentiation rules.

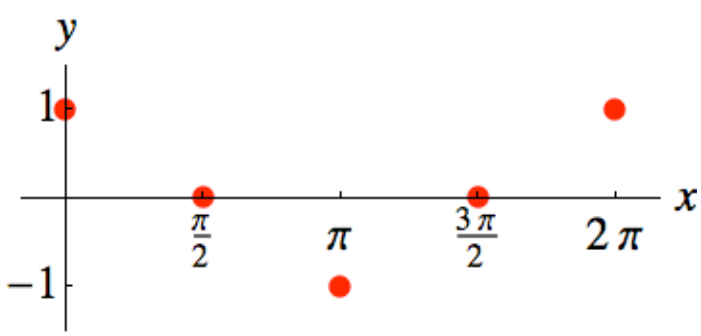
PART A: CONJECTURING THE DERIVATIVE OF THE BASIC SINE FUNCTION

Let $f(x) = \sin x$. The sine function is **periodic** with period 2π. One cycle of its graph is in bold below. Selected [truncated] **tangent lines** and their **slopes** (*m*) are indicated in red. (The leftmost tangent line and slope will be discussed in Part C.)

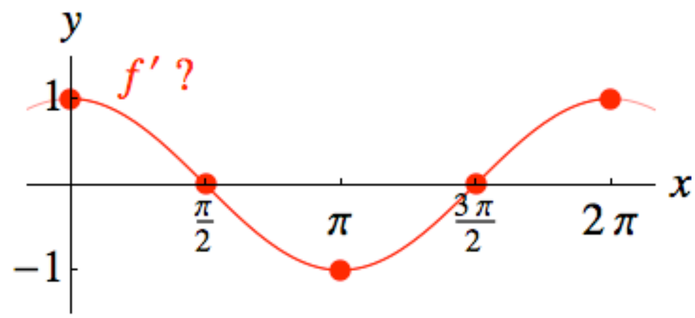


Remember that **slopes** of tangent lines correspond to **derivative** values (that is, values of f').

The graph of f' must then contain the five indicated points below, since their *y*-coordinates correspond to values of f'.



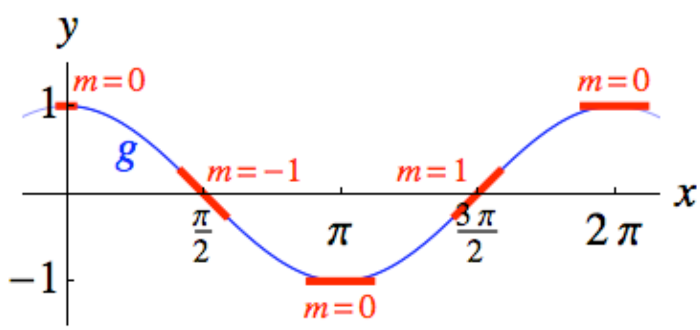
Do you know of a basic periodic function whose graph contains these points?



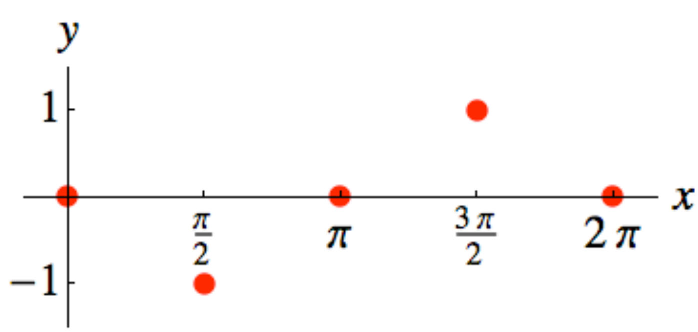
We conjecture that $f'(x) = \cos x$. We will prove this in Parts D and E.

PART B: CONJECTURING THE DERIVATIVE OF THE BASIC COSINE FUNCTION

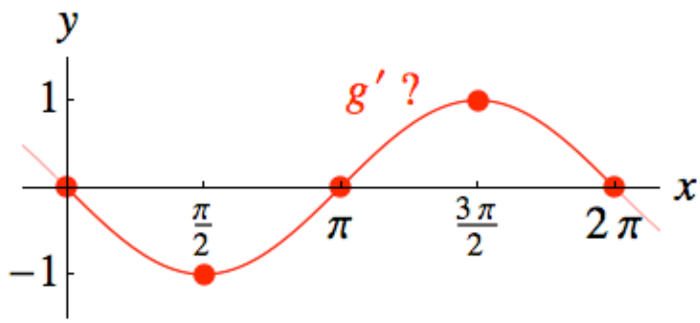
Let $g(x) = \cos x$. The cosine function is also periodic with period 2π.



The graph of g' must then contain the five indicated points below.



Do you know of a (fairly) basic periodic function whose graph contains these points?



We conjecture that $g'(x) = -\sin x$. If f is the sine function from Part A, then we also believe that $f''(x) = g'(x) = -\sin x$. We will prove these in Parts D and E.

PART C: TWO HELPFUL LIMIT STATEMENTS

Helpful Limit Statement #1

$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$

Helpful Limit Statement #2

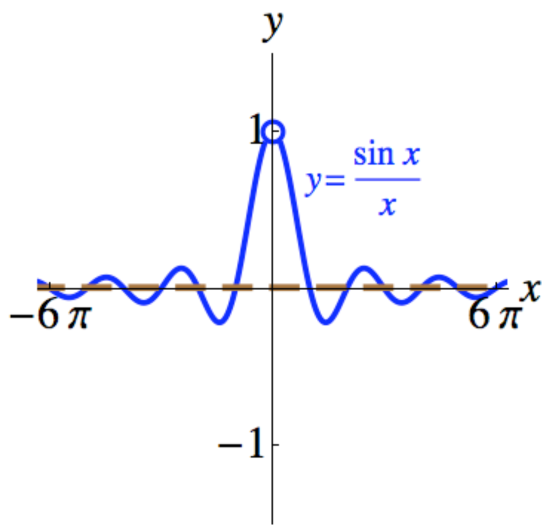
$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \quad \left(\text{or, equivalently, } \lim_{h\to 0} \frac{1-\cos h}{h} = 0 \right)$$

These limit statements, which are proven in Footnotes 1 and 2, will help us **prove** our conjectures from Parts A and B. In fact, only the **first** statement is needed for the proofs in Part E.

Statement #1 helps us graph $y = \frac{\sin x}{x}$.

- In Section 2.6, we proved that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ by the Sandwich (Squeeze) Theorem. Also, $\lim_{x\to-\infty} \frac{\sin x}{x} = 0$.

- Now, Statement #1 implies that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, where we replace h with x. Because $\frac{\sin x}{x}$ is undefined at $x = 0$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$, the graph has a **hole** at the point $(0,1)$.

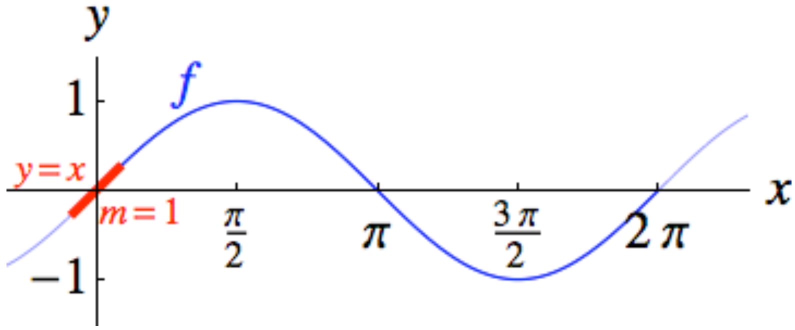


(Axes are scaled differently.)

Statement #1 also implies that, if $f(x) = \sin x$, then $f'(0) = 1$.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin(0+h) - \sin(0)}{h} \\ &= \lim_{h\to 0} \frac{\sin h - 0}{h} \\ &= \lim_{h\to 0} \frac{\sin h}{h} \\ &= 1 \end{aligned}$$

This verifies that the **tangent line** to the graph of $y = \sin x$ at the **origin** does, in fact, have **slope** 1. Therefore, the tangent line is given by the **equation** $y = x$. By the **Principle of Local Linearity** from Section 3.1, we can say that $\sin x \approx x$ when $x \approx 0$. That is, the tangent line closely **approximates** the sine graph close to the origin.



PART D: "STANDARD" PROOFS OF OUR CONJECTURES

Derivatives of the Basic Sine and Cosine Functions

1) $D_x(\sin x) = \cos x$
2) $D_x(\cos x) = -\sin x$

§ Proof of 1)

Let $f(x) = \sin x$. Prove that $f'(x) = \cos x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\overbrace{\sin x \cos h + \cos x \sin h}^{\text{by Sum Identity for sine}} - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\overbrace{(\sin x \cos h - \sin x)}^{\text{Group terms with } \sin x.} + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{(\sin x)(\cos h - 1) + \cos x \sin h}{h}
\end{aligned}$$

(Now, group expressions containing h.)

$$\begin{aligned}
&= \lim_{h\to 0} \left[(\sin x) \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} + (\cos x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= \cos x
\end{aligned}$$

Q.E.D. §

§ Proof of 2)

Let $g(x) = \cos x$. Prove that $g'(x) = -\sin x$.
(This proof parallels the previous proof.)

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{\cos x \cos h - \sin x \sin h}^{\text{by Sum Identity for cosine}} - \cos x}{h} \\
&= \lim_{h \to 0} \frac{\overbrace{(\cos x \cos h - \cos x)}^{\text{Group terms with } \cos x.} - \sin x \sin h}{h} \\
&= \lim_{h \to 0} \frac{(\cos x)(\cos h - 1) - \sin x \sin h}{h}
\end{aligned}$$

(Now, group expressions containing h.)

$$\begin{aligned}
&= \lim_{h \to 0} \left[(\cos x) \underbrace{\left(\frac{\cos h - 1}{h} \right)}_{\to 0} - (\sin x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= -\sin x
\end{aligned}$$

Q.E.D.

• Do you see where the "$-$" sign in $-\sin x$ arose in this proof? §

PART E: MORE ELEGANT PROOFS OF OUR CONJECTURES

Derivatives of the Basic Sine and Cosine Functions

1) $D_x(\sin x) = \cos x$
2) $D_x(\cos x) = -\sin x$

Version 2 of the Limit Definition of the Derivative Function in Section 3.2, Part A, provides us with more elegant proofs. In fact, they do not even use Limit Statement #2 in Part C.

§ Proof of 1)

Let $f(x) = \sin x$. Prove that $f'(x) = \cos x$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\overbrace{(\sin x\cos h + \cos x \sin h)}^{\text{by Sum Identity for sine}} - \overbrace{(\sin x\cos h - \cos x \sin h)}^{\text{by Difference Identity for sine}}}{2h} \\
&= \lim_{h\to 0} \frac{\cancel{2}\cos x \sin h}{\cancel{2}h} \\
&= \lim_{h\to 0} \left[(\cos x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= \cos x
\end{aligned}$$

Q.E.D. §

§ Proof of 2)

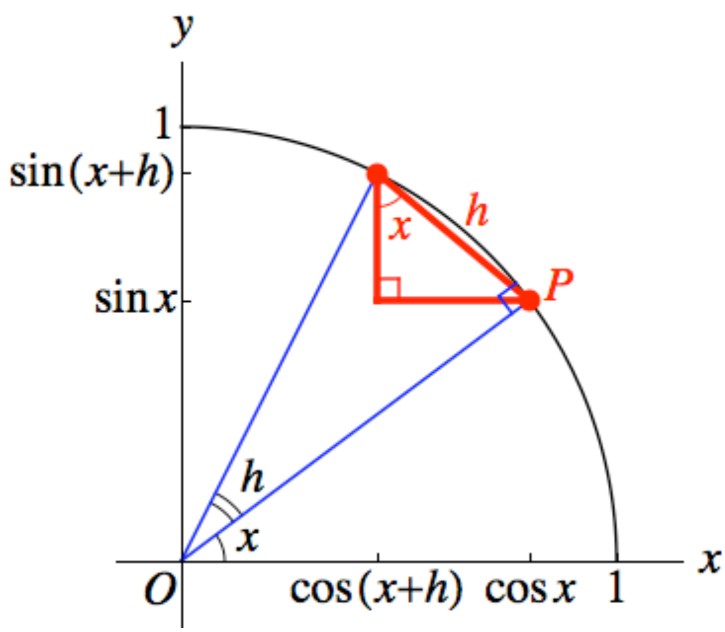
Let $g(x) = \cos x$. Prove that $g'(x) = -\sin x$.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x-h)}{2h} \\
&= \lim_{h\to 0} \frac{\overbrace{(\cos x \cos h - \sin x \sin h)}^{\text{from Sum Identity for cosine}} - \overbrace{(\cos x \cos h + \sin x \sin h)}^{\text{from Difference Identity for cosine}}}{2h} \\
&= \lim_{h\to 0} \frac{-\not{2} \sin x \sin h}{\not{2} h} \\
&= \lim_{h\to 0} \left[(-\sin x) \underbrace{\left(\frac{\sin h}{h} \right)}_{\to 1} \right] \\
&= -\sin x
\end{aligned}$$

Q.E.D. §

§ A Geometric Approach

Jon Rogawski has recommended a more geometric approach, one that stresses the concept of the derivative. Examine the figure below.



• Observe that: $\sin(x+h)-\sin x \approx h\cos x$, which demonstrates that the **change** in a differentiable function on a small interval *h* is related to its derivative. (We will exploit this idea when we discuss **differentials** in Section 3.5.)

• Consequently, $\dfrac{\sin(x+h)-\sin x}{h} \approx \cos x$.

• In fact, $D_x(\sin x) = \lim_{h\to 0} \dfrac{\sin(x+h)-\sin(x)}{h} = \cos x$.

• A similar argument shows: $D_x(\cos x) = \lim_{h\to 0} \dfrac{\cos(x+h)-\cos(x)}{h} = -\sin x$.

• Some angle and length measures in the figure are **approximate**, though they become more accurate as $h \to 0$. (For clarity, the figure does not employ a small value of *h*.)

• Exercises in Sections 3.6 and 3.7 will show that the **tangent line** to any point *P* on a **circle** with center *O* is **perpendicular** to the line segment $\overline{OP}$.

§

PART F: DERIVATIVES OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

Basic Trigonometric Rules of Differentiation

1) $D_x(\sin x) = \cos x$	2) $D_x(\cos x) = -\sin x$
3) $D_x(\tan x) = \sec^2 x$	4) $D_x(\cot x) = -\csc^2 x$
5) $D_x(\sec x) = \sec x \tan x$	6) $D_x(\csc x) = -\csc x \cot x$

WARNING 1: Radians. We assume that *x*, *h*, etc. are measured in **radians** (corresponding to **real numbers**). If they are measured in degrees, the rules of this section and beyond would have to be modified. (Footnote 3 in Section 3.6 will discuss this.)

TIP 1: Memorizing.

- The sine and cosine functions are a pair of **cofunctions**, as are the tangent and cotangent functions and the secant and cosecant functions.
- Let's say you know Rule 5) on the derivative of the **secant** function. You can quickly modify that rule to find Rule 6) on the derivative of the **cosecant** function.
- You take $\sec x \tan x$, multiply it by -1 (that is, do a "**sign flip**"), and take the **cofunction** of each factor. We then obtain: $-\csc x \cot x$, which is $D_x(\csc x)$.
- This method also applies to Rules 1) and 2) and to Rules 3) and 4).
- The Exercises in Section 3.6 will demonstrate why this works.

TIP 2: Domains. In Rule 3), observe that $\tan x$ and $\sec^2 x$ share the same domain. In fact, all six rules exhibit the same property.

Rules 1) and 2) can be used to **prove** Rules 3) through 6). The proofs for Rules 4) and 6) are left to the reader in the Exercises for Sections 3.4 and 3.6 (where the **Cofunction Identities** will be applied).

§ Proof of 3)

$$D_x(\tan x) = D_x\left(\frac{\sin x}{\cos x}\right) \qquad (\text{Quotient Identities})$$

$$= \frac{\text{Lo}\cdot \text{D}(\text{Hi}) - \text{Hi}\cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \qquad (\text{Quotient Rule of Differentiation})$$

$$= \frac{[\cos x]\cdot[D_x(\sin x)] - [\sin x]\cdot[D_x(\cos x)]}{(\cos x)^2}$$

$$= \frac{[\cos x]\cdot[\cos x] - [\sin x]\cdot[-\sin x]}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad \left(\text{Can: } = \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x = \sec^2 x\right)$$

$$= \frac{1}{\cos^2 x} \qquad (\text{Pythagorean Identities})$$

$$= \sec^2 x \qquad (\text{Reciprocal Identities})$$

Q.E.D.

• Footnote 3 gives a proof using the Limit Definition of the Derivative. §

§ Proof of 5)

$$D_x(\sec x) = D_x\left(\frac{1}{\cos x}\right)$$

Quotient Rule of Differentiation — Reciprocal Rule

$$= \frac{\text{Lo} \cdot \text{D}(\text{Hi}) - \text{Hi} \cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \quad \text{or} \quad -\frac{\text{D}(\text{Lo})}{(\text{Lo})^2}$$

$$= \frac{[\cos x] \cdot [D_x(1)] - [1] \cdot [D_x(\cos x)]}{(\cos x)^2} \quad \text{or} \quad -\frac{D_x(\cos x)}{(\cos x)^2}$$

$$= \frac{\cancel{[\cos x] \cdot [0]} - [1] \cdot [-\sin x]}{(\cos x)^2} \quad \text{or} \quad -\frac{-\sin x}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \quad (\text{Factoring or "Peeling"})$$

$$= \sec x \tan x \quad (\text{Reciprocal and Quotient Identities})$$

Q.E.D. §

Example 1 (Finding a Derivative Using Several Rules)

Find $D_x(x^2 \sec x + 3\cos x)$.

§ Solution

We apply the **Product Rule of Differentiation** to the first term and the **Constant Multiple Rule** to the second term. (The Product Rule can be used for the second term, but it is inefficient.)

$$D_x(x^2 \sec x + 3\cos x) = D_x(x^2 \sec x) + D_x(3\cos x) \quad (\text{Sum Rule of Diff'n})$$

$$= \left([D_x(x^2)][\sec x] + [x^2][D_x(\sec x)]\right) + 3 \cdot [D_x(\cos x)]$$

$$= \left([2x][\sec x] + [x^2][\sec x \tan x]\right) + 3 \cdot [-\sin x]$$

$$= 2x \sec x + x^2 \sec x \tan x - 3\sin x$$

§

Example 2 (Finding and Simplifying a Derivative)

Let $g(\theta) = \dfrac{\cos\theta}{1-\sin\theta}$. Find $g'(\theta)$.

§ Solution

Note: If $g(\theta)$ were $\dfrac{\cos\theta}{1-\sin^2\theta}$, we would be able to simplify considerably before we differentiate. Alas, we cannot here. Observe that we cannot "split" the fraction through its denominator.

$$g'(\theta) = \frac{\text{Lo}\cdot \text{D}(\text{Hi}) - \text{Hi}\cdot \text{D}(\text{Lo})}{(\text{Lo})^2} \quad \text{(Quotient Rule of Diff'n)}$$

$$= \frac{[1-\sin\theta]\cdot\left[D_\theta(\cos\theta)\right] - [\cos\theta]\cdot\left[D_\theta(1-\sin\theta)\right]}{(1-\sin\theta)^2}$$

Note: $(1-\sin\theta)^2$ is **not** equivalent to $1-\sin^2\theta$.

$$= \frac{[1-\sin\theta]\cdot[-\sin\theta] - [\cos\theta]\cdot[-\cos\theta]}{(1-\sin\theta)^2}$$

TIP 3: Signs. Many students don't see why $D_\theta(-\sin\theta) = -\cos\theta$. Remember that differentiating the basic **sine** function does **not** lead to a "**sign flip**," while differentiating the basic **cosine** function **does**.

$$= \frac{-\sin\theta + \sin^2\theta + \cos^2\theta}{(1-\sin\theta)^2}$$

WARNING 2: Simplify.

$$= \frac{-\sin\theta + 1}{(1-\sin\theta)^2} \quad \text{(Pythagorean Identities)}$$

$$= \frac{1-\sin\theta}{(1-\sin\theta)^2} \quad \text{(Rewriting)}$$

$$= \frac{1}{1-\sin\theta}$$

§

Example 3 (Simplifying Before Differentiating)

Let $f(x) = \sin x \csc x$. Find $f'(x)$.

§ Solution

Simplifying $f(x)$ first is preferable to applying the Product Rule directly.

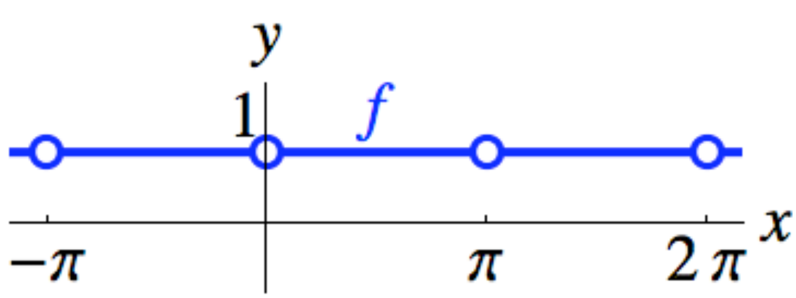
$$\begin{aligned} f(x) &= \sin x \csc x \\ &= (\sin x)\left(\frac{1}{\sin x}\right) \qquad \text{(Reciprocal Identities)} \\ &= 1, \quad (\sin x \neq 0) \quad \Rightarrow \end{aligned}$$

$$f'(x) = 0, \quad (\sin x \neq 0)$$

TIP 4: Domain issues.

$$\text{Dom}(f) = \text{Dom}(f') = \{x \in \mathbb{R} \mid \sin x \neq 0\} = \{x \in \mathbb{R} \mid x \neq \pi n,\ (n \in \mathbb{Z})\}.$$

In routine differentiation exercises, domain issues are often ignored. Restrictions such as $(\sin x \neq 0)$ here are rarely written.



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PART G: TANGENT LINES

Example 4 (Finding Horizontal Tangent Lines to a Trigonometric Graph)

Let $f(x) = 2\sin x - x$. Find the *x*-coordinates of all points on the graph of $y = f(x)$ where the **tangent line** is **horizontal**.

§ Solution

• We must find where the **slope** of the tangent line to the graph is 0.
We must solve the equation:

$$\begin{aligned} f'(x) &= 0 \\ D_x(2\sin x - x) &= 0 \\ 2\cos x - 1 &= 0 \\ \cos x &= \frac{1}{2} \\ x &= \pm\frac{\pi}{3} + 2\pi n,\ (n \in \mathbb{Z}) \end{aligned}$$

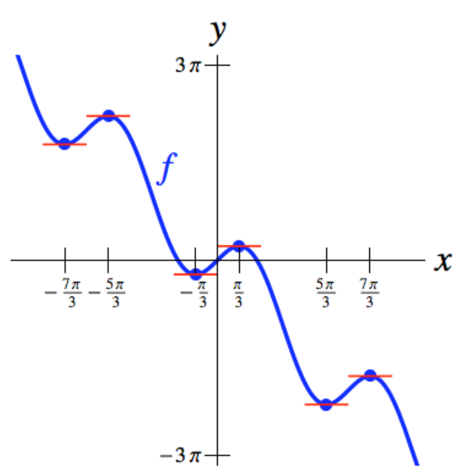
The desired *x*-coordinates are given by:

$$\left\{ x \in \mathbb{R} \,\middle|\, x = \pm\frac{\pi}{3} + 2\pi n,\ (n \in \mathbb{Z}) \right\}.$$

• Observe that there are **infinitely many** points on the graph where the tangent line is horizontal.

• Why does the graph of $y = 2\sin x - x$ below make sense? Observe that f is an **odd** function. Also, the "$-x$" term leads to downward drift; the graph oscillates about the line $y = -x$.

• The **red tangent lines** below are truncated.



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Example 5 (Equation of a Tangent Line; Revisiting Example 4)

Let $f(x) = 2\sin x - x$, as in Example 4. Find an equation of the **tangent line** to the graph of $y = f(x)$ at the point $\left(\frac{\pi}{6},\, f\left(\frac{\pi}{6}\right)\right)$.

§ Solution

• $f\left(\frac{\pi}{6}\right) = 2\sin\left(\frac{\pi}{6}\right) - \frac{\pi}{6} = 2\left(\frac{1}{2}\right) - \frac{\pi}{6} = 1 - \frac{\pi}{6}$, so the **point** is at $\left(\frac{\pi}{6},\, 1 - \frac{\pi}{6}\right)$.

• Find *m*, the **slope** of the tangent line there. This is given by $f'\left(\frac{\pi}{6}\right)$.

$$m = f'\left(\frac{\pi}{6}\right)$$

Now, $f'(x) = 2\cos x - 1$ (see Example 4).

$$= 2\cos\left(\frac{\pi}{6}\right) - 1$$

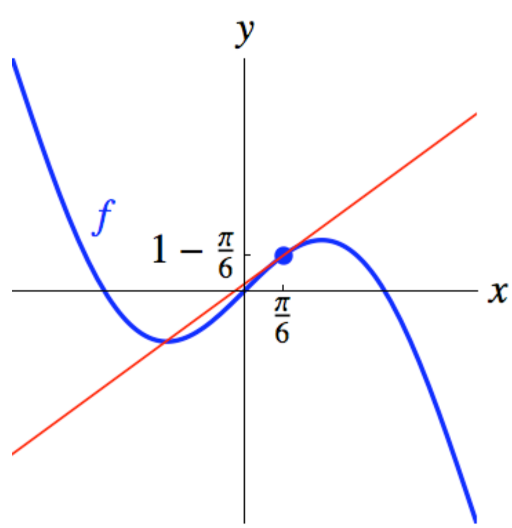
$$= 2\left(\frac{\sqrt{3}}{2}\right) - 1$$

$$= \sqrt{3} - 1$$

• Find a **Point-Slope Form** for the equation of the desired tangent line.

$$y - y_1 = m(x - x_1)$$

$$y - \left(1 - \frac{\pi}{6}\right) = \left(\sqrt{3} - 1\right)\left(x - \frac{\pi}{6}\right), \text{ or } y - \frac{6-\pi}{6} = \left(\sqrt{3} - 1\right)\left(x - \frac{\pi}{6}\right)$$



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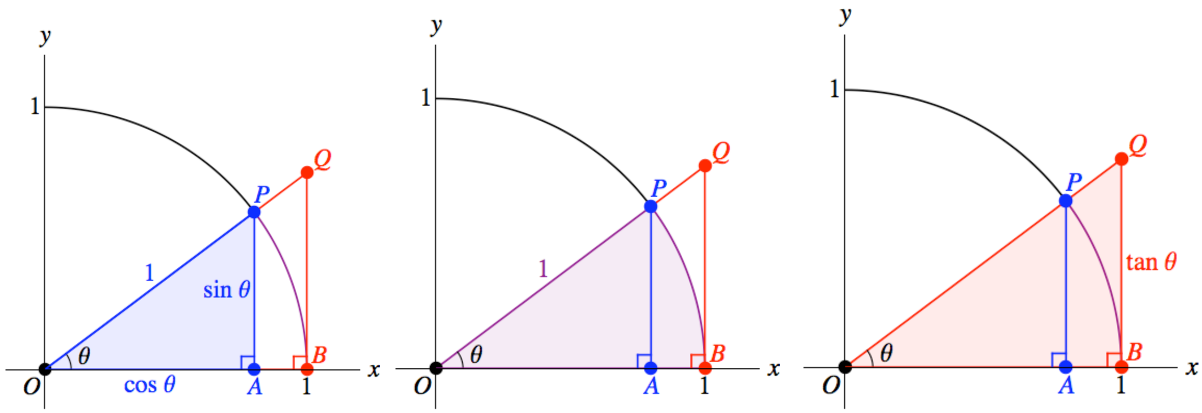
FOOTNOTES

1. **Proof of Limit Statement #1 in Part C.** First prove that $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$, where we use θ to represent angle measures instead of h.

   Side note: The area of a circular sector such as *POB* below is given by:

$$\begin{pmatrix}\text{Ratio of } \theta \text{ to a} \\ \text{full revolution}\end{pmatrix}\begin{pmatrix}\text{Area of} \\ \text{the circle}\end{pmatrix} = \left(\frac{\theta}{2\pi}\right)\left(\pi r^2\right) = \frac{1}{2}\theta r^2 = \frac{1}{2}\theta \;\; (\text{if } r = 1), \text{ where } \theta \in [0, 2\pi].$$

Area of Triangle *POA* ≤ Area of Sector *POB* ≤ Area of Triangle *QOB*



$$\begin{aligned}
\frac{1}{2}\sin\theta\cos\theta \;&\le\; \frac{1}{2}\theta \;\le\; \frac{1}{2}\tan\theta, \quad \forall\theta \in \left(0, \frac{\pi}{2}\right) \\
\sin\theta\cos\theta \;&\le\; \theta \;\le\; \tan\theta, \quad \text{"} \quad \text{"} \\
\cos\theta \;&\le\; \frac{\theta}{\sin\theta} \;\le\; \frac{1}{\cos\theta}, \quad \text{"} \quad \text{"}
\end{aligned}$$

(Now, we take reciprocals and reverse the inequality symbols.)

$$\begin{aligned}
\frac{1}{\cos\theta} \;&\ge\; \frac{\sin\theta}{\theta} \;\ge\; \cos\theta, \quad \text{"} \quad \text{"} \\
\underbrace{\cos\theta}_{\to 1} \;&\le\; \frac{\sin\theta}{\theta} \;\le\; \underbrace{\frac{1}{\cos\theta}}_{\to 1}, \quad \forall\theta \in \left(0, \frac{\pi}{2}\right)
\end{aligned}$$

as $\theta \to 0^+$. Therefore, $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$ by a one-sided variation of the Squeeze (Sandwich) Theorem from Section 2.6.

Now, prove that $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1$. Let $\alpha = -\theta$.

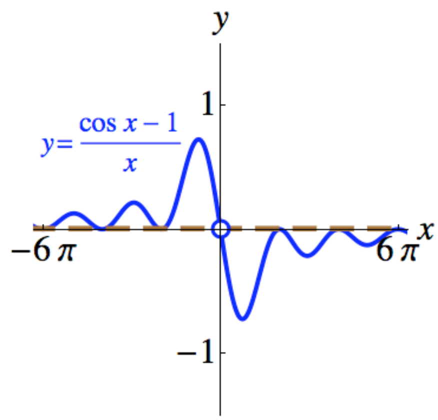
$\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = \lim_{\alpha \to 0^+} \frac{\sin(-\alpha)}{-\alpha} = \lim_{\alpha \to 0^+} \frac{-\sin(\alpha)}{-\alpha} = \lim_{\alpha \to 0^+} \frac{\sin(\alpha)}{\alpha} = 1$. Therefore, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

2. **Proof of Limit Statement #2 in Part C.**

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\left(-\frac{1-\cos h}{h}\right)=\lim_{h\to 0}\left[-\frac{(1-\cos h)}{h}\cdot\frac{(1+\cos h)}{(1+\cos h)}\right]=\lim_{h\to 0}\left[-\frac{1-\cos^2 h}{h(1+\cos h)}\right]$$

$$=\lim_{h\to 0}\left[-\frac{\sin^2 h}{h(1+\cos h)}\right]=\lim_{h\to 0}\left(-\frac{\sin h}{h}\cdot\frac{\sin h}{1+\cos h}\right)=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{\sin h}{1+\cos h}\right)$$

$$=-(1)\cdot(0)=0\,.$$



3. **Proof of Rule 3) $D_x(\tan x)=\sec^2 x$, using the Limit Definition of the Derivative.**

Let $f(x)=\tan x$. $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h}$

$$=\lim_{h\to 0}\frac{\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x}{h}=\lim_{h\to 0}\left[\frac{\left(\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x\right)}{h}\cdot\frac{(1-\tan x\tan h)}{(1-\tan x\tan h)}\right]$$

$$=\lim_{h\to 0}\frac{\tan x+\tan h-(\tan x)(1-\tan x\tan h)}{h(1-\tan x\tan h)}=\lim_{h\to 0}\frac{\tan h+\tan^2 x\tan h}{h(1-\tan x\tan h)}$$

$$=\lim_{h\to 0}\frac{(\tan h)(1+\tan^2 x)}{h(1-\tan x\tan h)}=\lim_{h\to 0}\frac{(\tan h)(\sec^2 x)}{h(1-\tan x\tan h)}$$ (Assume the limits in the next step exist.)

$$=\left(\lim_{h\to 0}\frac{\tan h}{h}\right)\cdot(\sec^2 x)\cdot\left(\lim_{h\to 0}\frac{1}{1-\tan x\tan h}\right)=\left[\lim_{h\to 0}\left(\frac{\sin h}{\cos h}\cdot\frac{1}{h}\right)\right]\cdot(\sec^2 x)\cdot(1)$$

$$=\left[\lim_{h\to 0}\left(\frac{\sin h}{h}\cdot\frac{1}{\cos h}\right)\right]\cdot(\sec^2 x)=\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cdot\left(\lim_{h\to 0}\frac{1}{\cos h}\right)\cdot(\sec^2 x)=(1)(1)(\sec^2 x)$$

$$=\sec^2 x$$

4. **A joke.** The following is a “sin”: $\frac{\sin \not x}{\not x}=\sin$. Of course, this is ridiculous!