Section 3.4 Exponential Growth and Decay

Many natural systems grow or decay over time. For example, population, radioactivity, cooling, heating, chemical reactions, and money.

Let $y = f(t)$ ←some function that represents the number of something with respect to time.

If we want to think about how something changes – as in a "rate of change" what do we look at?

A derivative: $y' = f'(t)$

But wouldn't you agree that it is reasonable that this rate of change (of say, population) must be somehow related to the original $f(t)$?

In fact, most of the time, in growth & decay problems:

$f'(t) = k\ f(t)$ where k is called a "proportionality constant"

Normally, this is written as:

$\frac{dy}{dt} = k\,y$ which is a differential equation

If $k > 0 \Rightarrow$ called "law of natural growth"

If $k < 0 \Rightarrow$ called "law of natural decay"

This particular differential equation is quite easy to solve because you are looking for a function y whose derivative is a constant multiple of itself. Do we know any such functions? Yes and only one: e^t

Theorem: The only solutions of $\frac{dy}{dt} = k\,y$ are the exponential functions $y(t) = y(0) \cdot e^{kt}$.

How can we use this information? To solve practical problems related to growth and decay! You (the class) should review the exercises given as examples on pg. 168 – 173. I will do different ones in the notes so you can see as many as possible.

Examples

1. A common inhabitant of human intestines is the bacterium E. coli. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 60 cells.

   a. Find the relative growth rate. (This means, what is *k*?)

   $$\frac{dy}{dt} = k\,y \quad \text{where} \quad y(t) = y(0)\cdot e^{kt} \quad \text{(note } k \text{ is the same in both equations)}$$

   Let $y(t)$ = population of bacteria
   t = time, in hours (choose your own scale)
   $y(0)$ = initial population

   What do we already know?
   $y(0) = 60$
   at $t = 20 \text{ min} = \frac{1}{3}$ hr, $y(\frac{1}{3}) = 2(60) = 120$
   Use the formula:

   $$y(t) = y(0)\cdot e^{kt}$$
   $$y(\tfrac{1}{3}) = 60\,e^{k\left(\frac{1}{3}\right)}$$
   $$120 = 60\,e^{k\left(\frac{1}{3}\right)}$$
   $$2 = e^{k\left(\frac{1}{3}\right)} \qquad \text{solve for k using logarithms}$$
   $$\ln(2) = \frac{1}{3}k$$
   $$3\ln(2) = k$$
   $$\ln\left(2^3\right) = k$$
   $$k = \ln(8)$$

   b. Find an expression for the number of cells after *t* hours. (That means put *k* into your form and keep general *t*)

   So:
   $$y(t) = y(0)\cdot e^{kt}$$
   $$y(t) = 60\cdot e^{t\ln 8}$$
   $$y(t) = 60\cdot e^{\ln 8^t}$$
   $$y(t) = 60\cdot 8^{t}$$

   c. Find the number of cells after 8 hours. (This means use your formula when $t = 8$)

   $$y(8) = 60\cdot 8^8 \approx 1{,}006{,}632{,}960 \quad \text{(wow!)}$$

   d. Find the rate of growth after 8 hours.

Just like when we did rate of change problems, rate of growth is $\frac{dy}{dt}$.

We know $\frac{dy}{dt} = ky$

So, substitute in what we know when $t = 8$.

$$\frac{dy}{dt}(8) = \underbrace{\ln(8)}_{\text{k}} \cdot \underbrace{60 \cdot 8^8}_{\text{y(8)}} \approx 2.09 \text{ billion}$$

e. When will the population reach 20,000 cells? (This means, find t when $y(t) = 20{,}000$)

So, use your formula:

$$y(t) = y(0) \cdot e^{kt}$$

$$20{,}000 = 60 \cdot 8^{t}$$

$$\frac{20{,}000}{60} = 8^{t}$$

$$\frac{1{,}000}{3} = 8^{t}$$

$$\log_8\left(\frac{1{,}000}{3}\right) = \log_8 8^{t}$$

$$t = \log_8\left(\frac{1{,}000}{3}\right)$$

$$t = \frac{\ln\left(\frac{1{,}000}{3}\right)}{\ln 8} \quad \text{(leave in this form)}$$

$$t \approx 2.8 \text{ hrs}$$

2. The table gives the population of the United States, from census figures in millions, for the years 1900 to 2000.

Year	Population
1900	76
1910	92
1920	106
1930	123
1940	131
1950	150
1960	179
1970	203
1980	227
1990	250
2000	275

a. Use an exponential model and the census figures for 1900 to 1910 to predict the population in 2000. Compare to the actual figure and try to explain the discrepancy.

$$\frac{dy}{dt} = ky \quad \text{where} \quad y(t) = y(0) \cdot e^{kt}$$

Let *y (t)* = population of bacteria
t = time, in hours
y(0) = initial population

What do we already know?
In 1900 $y = 76$ Let 1900 be $t = 0$
In 1910 $y = 92$ So, 1910 is $t = 10$

y(0) = 76
at $t = 10$ years, *y(10)* = 92
Use the formula to find k (the relative growth rate)

$$y(t) = y(0) \cdot e^{kt}$$

$$92 = 76 \cdot e^{k(10)}$$

$$\frac{92}{76} = e^{10k}$$

$$\ln\left(\frac{23}{19}\right) = 10k$$

$$k = \underbrace{\frac{1}{10}\ln\left(\frac{23}{19}\right)}_{\text{Leave like this}} \approx 0.0191$$

Put back into our formula

$$y(t) = 76\,e^{\frac{1}{10}\ln\left(\frac{23}{19}\right)t}$$

What we really want is population estimate when it's the year 2000.
What is *t* then? *t* = 100 yrs

Use your formula:

$$y(100) = 76\,e^{\frac{1}{10}\ln\left(\frac{23}{19}\right)100}$$

$$y(100) = \underbrace{76\,e^{10\ln\left(\frac{23}{19}\right)}}_{\text{Leave like this}}$$

$$y(100) \approx 513.5 \text{ million}$$

Our census said 275 million. Why are we so far off? The formula is based on what has happened in 1900 to 1910 – it doesn't account for outside circumstances. Perhaps declining birth rate, less immigration, etc.

b. Use an exponential model and the census figures for 1980 to 1990 to predict population in 2000.

In 1980	$y = 227$	Let 1980 be $t = 0$
In 1990	y = 250	So, 1990 is t = 10

$y(0) = 227$
at t = 10 years, y(10) = 250

Follow the same process: Find k

$$y(t) = y(0)\cdot e^{kt}$$

$$250 = 227\cdot e^{k(10)}$$

$$\frac{250}{227} = e^{10k}$$

$$\ln\left(\frac{250}{227}\right) = 10k$$

$$k = \underbrace{\frac{1}{10}\ln\left(\frac{250}{227}\right)}_{\text{Leave like this}} \approx 0.00965$$

Put back into our formula

$$y(t) = 227\,e^{\frac{1}{10}\ln\left(\frac{250}{227}\right)t}$$

Population estimate when it's the year 2000. *t* = 20 yrs
Use your formula:

$$y(20) = 227\,e^{\frac{1}{10}\ln\left(\frac{250}{227}\right)200}$$

$$y(20) = 227\,e^{2\ln\left(\frac{250}{227}\right)}$$

$$y(20) = 227\,e^{\ln\left(\frac{250}{227}\right)^2}$$

$$y(20) = \underbrace{227\left(\frac{250}{227}\right)^2}_{\text{Leave like this}} \approx 275.3 \text{ million}$$

As compared to 275 million, not bad at all.

3. Bismuth-210 has a half-life of 5.0 days.
   a. A sample originally has a mass of 800 mg. Find a formula for the mass remaining after t-days.

$$\frac{dy}{dt} = ky \quad \text{where} \quad y(t) = y(0)\cdot e^{kt}$$

Let y(t) = mass of Bismuth-210, in mg
t = time, in days
y(0) = initial mass, in mg

What do we already know?
y(0) = 800
when t = 5 days, y(5) = 400
Note, **half-life** is the amount of time for ½ of the material to decay (or be removed)

Use formula to find k.

$$y(t) = y(0)\cdot e^{kt}$$

$$400 = 800\cdot e^{k(5)}$$

$$\frac{400}{800} = e^{5k}$$

$$\ln\left(\frac{1}{2}\right) = \ln\left(e^{5k}\right)$$

$$\ln\left(\frac{1}{2}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{1}{2}\right) = \frac{1}{5}\left(-\ln(2)\right) = -\frac{1}{5}\ln(2)$$

Put back into formula

$$y(t) = 800\,e^{-\frac{1}{5}\ln(2)t}$$ More common way to write

$$y(t) = 800\,e^{-\frac{t}{5}\ln(2)}$$ Re-arrange

$$y(t) = 800\,e^{\ln 2^{-\frac{t}{5}}}$$ Power Law

$$y(t) = 800\cdot 2^{-\frac{t}{5}}$$ Simplify

b. Find the mass after 30 days (Find y(t) when t = 30)

$$y(30) = 800\cdot 2^{-\frac{30}{5}}$$
$$= 800\cdot 2^{-6}$$
$$= 800\cdot\frac{1}{64}$$
$$= \frac{100}{8} = \frac{25}{2} = 12.5 \text{ mg}$$

c. When is the mass reduced to 1 mg? (Find t when y(t) = 1)

$$1 = 800\cdot 2^{-\frac{t}{5}}$$
$$\frac{1}{800} = 2^{-\frac{t}{5}}$$
$$\log_2\left(\frac{1}{800}\right) = \log_2\left(2^{-\frac{t}{5}}\right)$$
$$\frac{\ln\left(\frac{1}{800}\right)}{\ln 2} = -\frac{t}{5}$$
$$t = \underbrace{\frac{-5\ln\left(\frac{1}{800}\right)}{\ln 2}}_{\text{Leave like this}} \approx 48\ \textit{days}$$

Additional Selected Homework Problems

Note: These are scans of hand-work. You may have difficulty viewing them on some web browsers.

HW Questions from 3.4

1. A population of protozoa develops with a constant relative growth rate of 0.7944 per member per day. On day zero the population consists of two members. Find the population size after six days.

$y(t) = y(0)e^{kt}$ Known: $y(0) = 2$, $k = 0.7944$

$y(t) = 2e^{0.7944t}$ Find: $y(6)$

$y(6) = 2e^{0.7944(6)} \approx 2e^{4.7664} \approx 234.99 \approx 235$ members

3. A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 420.

(a) Find an expression for the number of bacteria after t hours. (i.e. find k)

$y(t) = y(0)e^{kt}$ Known: $y(0) = 100$, $y(1) = 420$ time in hours

$y(1) = y(0)e^{k(1)}$

$420 = 100e^{k}$

$\frac{420}{100} = e^{k} \Rightarrow \ln\left(\frac{42}{10}\right) = k \Rightarrow \ln\left(\frac{21}{5}\right) = k$

$\therefore y(t) = 100e^{t\ln\left(\frac{21}{5}\right)} = 100e^{\ln\left(\frac{21}{5}\right)^t}$ power law

$y(t) = 100\left(\frac{21}{5}\right)^t$

(b) Find the number of bacteria after 3 hours (i.e. find y(3))

$y(3) = 100\left(\frac{21}{5}\right)^3 \approx 7409$

(c) Find the rate of growth after 3 hours. (i.e. find $\frac{dy}{dt}(3)$)

$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{dt} = \ln\left(\frac{21}{5}\right)y$

$\frac{dy}{dt}(3) = \ln\left(\frac{21}{5}\right)y(3) = \ln\left(\frac{21}{5}\right)\cdot 100\cdot\left(\frac{21}{5}\right)^3 \approx 10632$

(d) When will the population reach 10,000? (i.e find t when $y(t) = 10000$)

$10000 = 100\left(\frac{21}{5}\right)^t$

$100 = \left(\frac{21}{5}\right)^t$

$\log_{\frac{21}{5}} 100 = \log_{\frac{21}{5}}\left(\frac{21}{5}\right)^t$

change base: $\frac{\ln(100)}{\ln\left(\frac{21}{5}\right)} = t$ $t \approx 3.2$ hrs

5. The table gives estimates of the world population, in millions, from 1750 to 2000:

YEAR	POP	YEAR	POP
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

(a) Use the exponential model and the population figures for 1750 and 1800 to predict the world population in 1900 and 1950. Compare with actual figures.

$y(t) = y(0)e^{Kt}$ Let $y(0) = 790$ pop in 1750

nd K: $y(50) = y(0)e^{50K}$ $y(50) = 980$ pop in 1800

$980 = 790e^{50K}$

$\frac{980}{790} = e^{50K} \Rightarrow \ln\left(\frac{98}{79}\right) = 50K \Rightarrow K = \frac{1}{50}\ln\left(\frac{98}{79}\right)$

$\therefore\ y(t) = 790e^{\frac{t\ln(98/79)}{50}} = 790e^{\ln(98/79)^{t/50}} = 790\left(\frac{98}{79}\right)^{t/50}$

when 1900, $t = 150 \Rightarrow y(150) = 790\left(\frac{98}{79}\right)^{150/50} = 790\left(\frac{98}{79}\right)^{3} \approx 1508$

when 1950, $t = 200 \Rightarrow y(200) = 790\left(\frac{98}{79}\right)^{200/50} = 790\left(\frac{98}{79}\right)^{4} \approx 1870$

These numbers are small compared to actual values. A bad prediction

(b) Use the exponential model and the population figures for 1850 and 1900 to predict the world population in 1950. Compare with actual figures.

$y(t) = y(0)e^{Kt}$ Let $y(0) = 1260$ pop in 1850

l new K: $y(50) = y(0)e^{50K}$ $y(50) = 1650$ pop in 1900

$1650 = 1260e^{50K}$

$\frac{1650}{1260} = e^{50K} \Rightarrow \ln\left(\frac{165}{126}\right) = 50K \Rightarrow K = \frac{1}{50}\ln\left(\frac{55}{42}\right)$

$\therefore\ y(t) = 1260e^{\frac{t\ln(55/42)}{50}} = 1260e^{\ln(55/42)^{t/50}} = 1260\left(\frac{55}{42}\right)^{t/50}$

when 1950, $t = 100 \Rightarrow y(100) = 1260\left(\frac{55}{42}\right)^{100/50} = 1260\left(\frac{55}{42}\right)^{2} \approx 2160$

A better prediction, but still too low. (the baby boomers)

(c) Use the exponential model and the population figures for 1900 and 1950 to predict the world population in 2000. Compare with the actual figures.

$y(t) = y(0)e^{Kt}$ Let $y(0) = 1650$ pop in 1900

d new K: $y(50) = y(0)e^{K(50)}$ $y(50) = 2560$ pop in 1950

$2560 = 1650e^{50K}$

$\frac{2560}{1650} = e^{50K} \Rightarrow \ln\left(\frac{256}{165}\right) = 50K \Rightarrow K = \frac{1}{50}\ln\left(\frac{256}{165}\right)$

$\therefore\ y(t) = 1650\, e^{t \cdot \frac{1}{50} \ln\left(\frac{256}{165}\right)} = 1650\, e^{\ln\left(\frac{256}{165}\right)^{t/50}} = 1650 \left(\frac{256}{165}\right)^{t/50}$

when 2000, $t = 100 \Rightarrow y(100) = 1650\left(\frac{256}{165}\right)^{100/40} = 1650\left(\frac{256}{165}\right)^2 \approx 3972$

Too high compared to real number. I suspect the data is skewed b/c 50 yr intervals covers more than 1 generation. Weird cycles, therefore, like the baby boomers, gen X'rs and so forth get "lost". World war losses too + regional conflicts surely impacts this number.

7. Experiments show that if the chemical reaction $N_2O_5 \rightarrow 2NO_2 + \frac{1}{2} O_2$ takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows: $-\frac{d[N_2O_5]}{dt} = 0.0005\,[N_2O_5]$

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.

Notice that $-\dfrac{d[N_2O_5]}{dt} = 0.0005\,[N_2O_5]$

looks just like: $-\dfrac{dy}{dt} = ky$ So, we know $k = -0.0005$ (just move ⊖ sign from other side)

So: $y(t) = y(0)e^{kt}$

Or: $[N_2O_5](t) = [N_2O_5](0)\, e^{-0.0005t}$

We are told initial concentration is $C \Rightarrow [N_2O_5](0) = C$

So: $[N_2O_5](t) = C e^{-0.0005t}$

(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value: i.e. What is t when $[N_2O_5](t) = 0.9C$?

$$0.9\cancel{C} = \cancel{C} e^{-0.0005t}$$

$$0.9 = e^{-0.0005t}$$

$$\ln(0.9) = -0.0005t \Rightarrow t = \frac{\ln(0.9)}{-0.0005}$$

$$t \approx 210.7 \text{ seconds}$$

9. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after t years

$y(t) = y(0)e^{Kt}$ Known: $y(0) = 100$, $y(30) = 50$, t in years

Find K: $y(30) = y(0)\, e^{K(30)}$

$50 = 100\, e^{30K} \Rightarrow \frac{50}{100} = e^{30K} \Rightarrow \ln\left(\frac{1}{2}\right) = 30K \Rightarrow K = \frac{1}{30}\ln\left(\frac{1}{2}\right)$

$\therefore\ y(t) = 100\, e^{t/30 \ln(\frac{1}{2})} = 100\, e^{\ln(\frac{1}{2})^{t/30}} = 100\, e^{-\ln(2)^{t/30}} = 100\, e^{\ln(2)^{-t/30}}$

$y(t) = 100 \cdot 2^{-t/30}$

(b) How much of the sample remains after 100 years?

$y(100) = 100 \cdot 2^{-100/30} = 100\,(2)^{-10/3} \approx 9.9$ mg

(c) After how long will only 1mg remain?

$1 = 100 \cdot 2^{-t/30}$

$\frac{1}{100} = 2^{-t/30}$

$\log_2\left(\frac{1}{100}\right) = -t/30 \Rightarrow \frac{\ln\left(\frac{1}{100}\right)}{\ln(2)} = -\frac{t}{30}$

So, $t = -30\,\frac{\ln\left(\frac{1}{100}\right)}{\ln(2)} \approx +199.3$ years

11. Scientists can determine the age of ancient objects by a method called radiocarbon dating. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore, the level of radiactivity must also decay exponentially. A parchmant fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on Earth today. Estimate the age of the parchment.

$y(t) = y(0)e^{Kt}$ Known : $y(0) = 1$; $y(5730) = 0.5$, t in years

Find K: $y(5730) = y(0)\, e^{K(5730)}$

$0.5 = 1\, e^{5730K} \Rightarrow \frac{1}{2} = e^{5730K} \Rightarrow -\ln(2) = 5730k \Rightarrow k = \frac{-\ln(2)}{5730}$

$\therefore\ y(t) = 1\, e^{-t\frac{\ln(2)}{5730}} = e^{-\ln(2)^{t/5730}} = (2)^{-t/5730}$

So, our problem is to find t when $y(t) = 0.74$

$0.74 = 2^{-t/5730}$

$$\log_2(0.74) = \log_2 2^{-t/5730}$$

$$\frac{\ln(0.74)}{\ln(2)} = \frac{-t}{5730} \Rightarrow t = \frac{-5730\ln(0.74)}{\ln(2)} \approx 2489 \text{ years.}$$

13. A roast turkey is taken from an oven when its temperature has reached 185°F, and is placed on a table in a room where the temperature is 75°F.

(a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 minutes?

p. 171

Newton's Law of Cooling: $\frac{dT}{dt} = k(T - T_s)$ (ambient T)

Let $y(t) = T - T_s$ ← Keys

Convert $\frac{dy}{dt} = ky \Rightarrow y(t) = y(0)e^{kt}$

Known: $y(0) = T(0) - T_s$

$y(0) = 185 - 75 = 110$

Likewise $y(0.5) = 150 - 75 = 75$ when $t = 0.5$ (half an hour)

$y(t) = y(0)e^{kt}$

Find k: $y(0.5) = y(0)e^{k(0.5)}$

$75 = 110e^{k(0.5)}$

$\frac{75}{110} = e^{0.5k} \Rightarrow \ln\left(\frac{15}{22}\right) = 0.5k \Rightarrow k = 2\ln\left(\frac{15}{22}\right) \approx -0.7659$

$y(t) = 110e^{2\ln\left(\frac{15}{22}\right)t} = 110e^{\ln\left(\frac{15}{22}\right)^{2t}} = 110\left(\frac{15}{22}\right)^{2t}$

Now, 45 min = 0.75 hr = t

$y(0.75) = 110\left(\frac{15}{22}\right)^{2(0.75)} = 110\left(\frac{15}{22}\right)^{1.5} = 62\text{ F}$ but be careful!

Remember that $y(t) = T - T_s$ and what we want is T.

$62\text{F} = T - 75°\text{F} \Rightarrow T = 137°\text{F}.$

(b) When will the turkey have cooled to 100°F? That's the T.

(*) So $y(t) = T - T_s = 100 - 75 = 25$

$y(t) = 110\left(\frac{15}{22}\right)^{2t}$

$25 = 110\left(\frac{15}{22}\right)^{2t}$

$\left(\frac{25}{110}\right) = \left(\frac{15}{22}\right)^{2t}$

$\log_{\frac{15}{22}}\left(\frac{5}{22}\right) = \log_{\frac{15}{22}}\left(\frac{15}{22}\right)^{2t}$

$\frac{\ln\left(\frac{5}{22}\right)}{\ln\left(\frac{15}{22}\right)} = 2t$

$\Rightarrow t = \frac{1}{2}\frac{\ln\left(\frac{5}{22}\right)}{\ln\left(\frac{15}{22}\right)} \approx 1.93\text{ hr} \approx 116\text{ min}$

15. When a cold drink is taken from a refrigerator, its temperature is 5°C. After 25 minutes in a 20°C room its temperature has increased to 10°C.

(a) What is the temperature of the drink after 50 minutes?

$\frac{dT}{dt} = k(T - T_s)$ Let $y(t) = T - T_s$

onvert $\frac{dy}{dt} = ky \Rightarrow y(t) = y(0)e^{kt}$ Known: $y(0) = 5°C - 20°C = -15°C$

$t = 25\,\text{min}$ $y(25) = 10°C - 20°C = -10°C$

d K: $y(t) = y(0)e^{kt}$

$-10 = -15e^{k \cdot 25}$

$\frac{10}{15} = e^{25k} \Rightarrow \ln\left(\frac{10}{15}\right) = 25k \Rightarrow k = \frac{1}{25}\ln\left(\frac{2}{3}\right) \approx -0.0162$

$y(t) = -15e^{t \cdot \frac{1}{25}\ln\left(\frac{2}{3}\right)} = -15e^{\ln\left(\frac{2}{3}\right)^{t/25}} = -15\left(\frac{2}{3}\right)^{t/25}$

Now, $t = 50$ min

$y(50) = -15\left(\frac{2}{3}\right)^{50/25} = -15\left(\frac{2}{3}\right)^2 = -15\left(\frac{4}{9}\right) = -\frac{60}{9} = -\frac{20}{3} \sim -6.7$

but $y(50) = T - 20$

$-6.7 = T - 20 \Rightarrow T = 13.3°C$

(b) When will its temperature be 15°C? $y(t) = 15 - 20 = -5$

$-5 = -15\left(\frac{2}{3}\right)^{t/25} \Rightarrow \frac{5}{15} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow \frac{1}{3} = \left(\frac{2}{3}\right)^{t/25}$

$\log_{\frac{2}{3}}\left(\frac{1}{3}\right) = \log_{\frac{2}{3}}\left(\frac{2}{3}\right)^{t/25} \Rightarrow \frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{2}{3}\right)} = \frac{t}{25}$

$t = 25\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{2}{3}\right)} \approx 67.7\,\text{min}$

17. The rate of change of atmospheric pressure P with respect to altitude h is proportional to P, provided that the temperature is constant. At 15°C the pressure is 101.3 kPa at sea level and 87.14 kPa at $h = 1000$m.

(a) What is the pressure at an altitude of 3000 m?

$\frac{dP}{dh} = kP \Rightarrow P(h) = P(0)e^{kh}$ Known: $P(0) = 101.3$ kPa (sea level)

$P(1000) = 87.14$ kPa

Find k: $P(h) = P(0)e^{kh}$

$P(1000) = P(0)e^{k \cdot 1000}$

$87.14 = 101.3e^{k \cdot 1000}$ $\qquad k \approx -0.0001505$

$\frac{87.14}{101.3} = e^{k \cdot 1000} \Rightarrow \ln\left(\frac{87.14}{101.3}\right) = 1000 \cdot k \Rightarrow k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right)$

$P(h) = 101.3\, e^{\frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right)t}$

$P(h) = 101.3\, e^{\ln\left(\frac{87.14}{101.3}\right)^{t/1000}} \Rightarrow P(h) = 101.3\left(\frac{87.14}{101.3}\right)^{t/1000}$

Now, $h = 3000$

$P(3000) = 101.3\left(\frac{87.14}{101.3}\right)^{\frac{3000}{1000}} = 101.3\left(\frac{87.14}{101.3}\right)^{3} \approx 64.48$ kPa

(b) What is the pressure at the top of Mount McKinley at an altitude of 6187m?

$h = 6187$

$P(6187) = 101.3\left(\frac{87.14}{101.3}\right)^{\frac{6187}{1000}} \approx 39.90$ kPa

19. If \$3000 is invested at 5% interest, find the value of the investment at the end of 5 years if the interest is compounded ; gen form: $A(t) = A(0)\left(1 + \frac{r}{n}\right)^{nt}$

(a) annually: r = interest rate ; n = # times/year t = total time

Known: $A(0) = 3000 \quad r = 0.05 \quad n = 1 \quad t = 5$

$A(5) = 3000\left(1 + \frac{.05}{1}\right)^{(1)(5)} = 3000(1.05)^5 = \$3828.84$

(b) semi-annually: change $n = 2$

$A(5) = 3000\left(1 + \frac{.05}{2}\right)^{(2)(5)} = 3000(1.025)^{10} = \$3840.25$

(c) monthly: change $n = 12$

$A(5) = 3000\left(1 + \frac{.05}{12}\right)^{(12)(5)} = 3000(1.0041)^{60} = \$3850.08$

(d) weekly: change $n = 52$

$A(5) = 3000\left(1 + \frac{.05}{52}\right)^{(52)(5)} = 3000(1.00096)^{260} = \$3851.61$

(e) daily: change $n = 365$

$A(5) = 3000\left(1 + \frac{.05}{365}\right)^{(365)(5)} = 3000(1.00013)^{1825} = \$3852.01$

(f) continuously: Let $n \to \infty$; $A(t) = A(0)e^{rt}$

$A(5) = 3000e^{.05(5)} = 3000e^{.25} = 3000(1.2840) = \$3852.08$