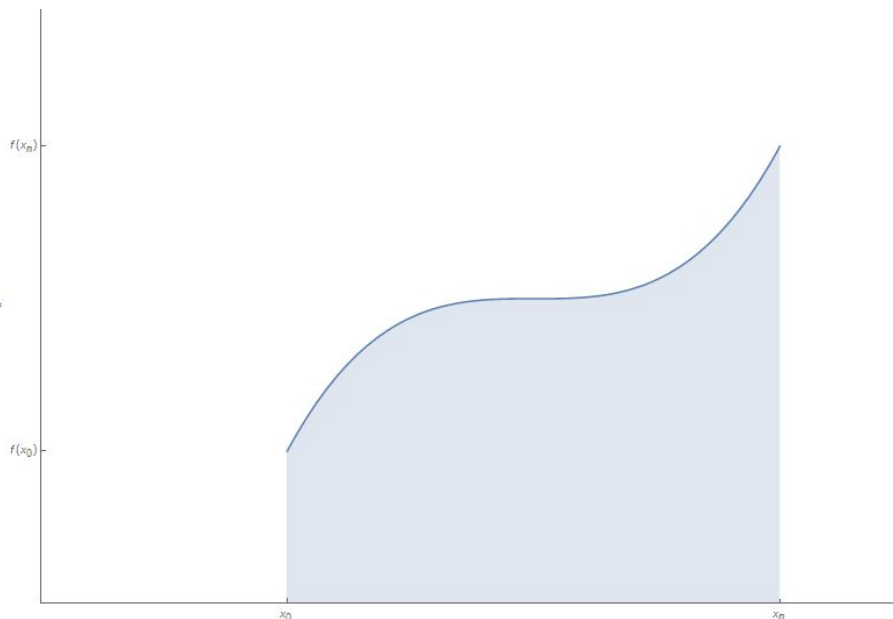### The Fundamental Theorem of Calculus

As we continue to study the area problem, let's think back to what we know about computing areas of regions enclosed by curves. If we want to find the area of the region below the curve $f(x)$, we have seen that we can think of the area of this region as a limit of sums of areas of approximating rectangles, given by the formula

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

In section 4.2, we defined the *definite integral of f from $x = a$ to $x = b$* as

$$\int_{a}^{b} f(x)\ \mathrm{d}x = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x;$$

we saw that if $f(x) \geq 0$ on $[a, b]$, then the area of the region under $f(x)$ from $x = a$ to $x = b$ is precisely

$$A = \int_{a}^{b} f(x)\ \mathrm{d}x.$$

If, however, $f(x)$ takes on values less than 0 on the interval from $x = a$ to $x = b$, then the integral

$$\int_{a}^{b} f(x)\ \mathrm{d}x$$

describes the *difference* between areas of regions above the x-axis and areas of regions below the x-axis.

In 4.2, we also saw an example of the calculation for $\int_{a}^{b} f(x)\ \mathrm{d}x$, which was quite tedious. Our goal for this section is to find a much quicker, less painful way to make the same calculation.

As a precursor to reaching this goal, we begin by building a new type of function–one based on integrals. Below is the graph of a function $f(t)$:
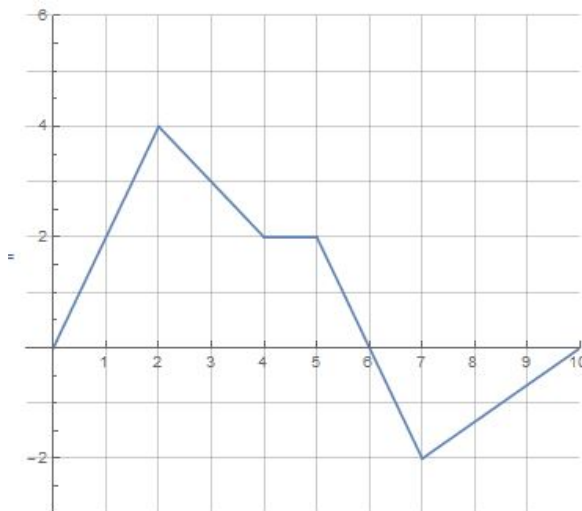
We are going to define the function $g(x)$ *based on* $f(t)$ using the formula
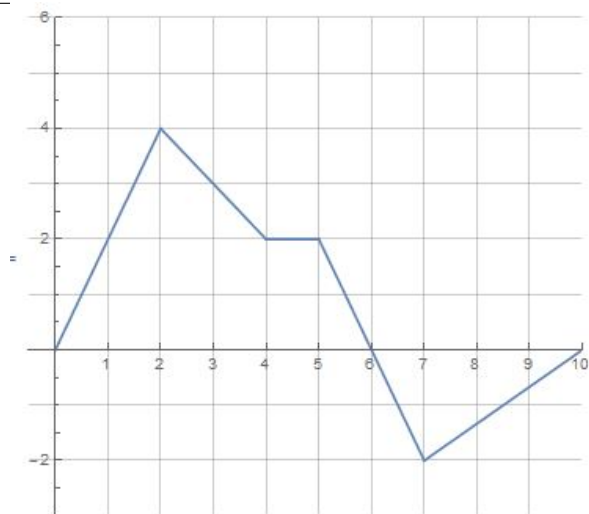
$$g(x) = \int_0^x f(t) \, \mathrm{d}t.$$

Notice that choosing a value for x is essentially the same as evaluating the integral $\int_0^x f(t) \, \mathrm{d}t$ with that choice of x as the upper bound of integration; in other words, g is a function so that the number $g(x)$ describes the *area of the region below $f(t)$ from 0 to x* (or the difference between areas of regions above and below the x-axis).

Example. Given the graph of $f(t)$ below and the function

$$g(x) = \int_0^x f(t) \, \mathrm{d}t,$$

find

1. $g(2)$

2. $g(4)$

3. $g(5)$
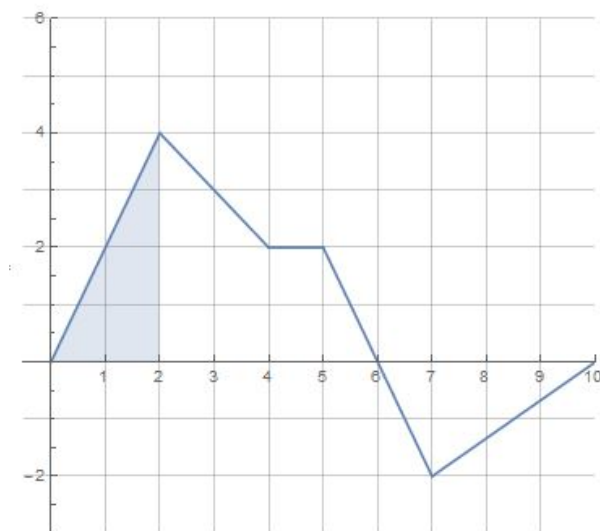
4. $g(6)$

5. $g(7)$

6. $g(10)$

1. Since
$$g(x) = \int_0^x f(t)\ \mathrm{d}t,$$

   we know that
$$g(2) = \int_0^2 f(t)\ \mathrm{d}t,$$

   which is precisely the area of the region below $f(t)$ from $t = 0$ to $t = 2$, shaded below:

   Since the shaded region is a triangle, its area is
$$A = \frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 2 \cdot 4 = 4,$$
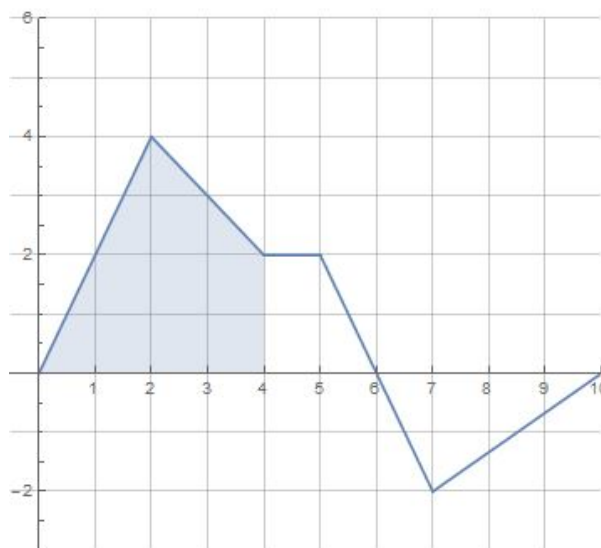
   so we know that
$$g(2) = 4.$$

2. Since

$$g(4) = \int_0^4 f(t) \, \mathrm{d}t,$$

we know that $g(4)$ is the area of the region shaded below:

We've already found the area from $t = 0$ to $t = 2$, so we need to focus on the area of the "new" region from $t = 2$ to $t = 4$; clearly, this is
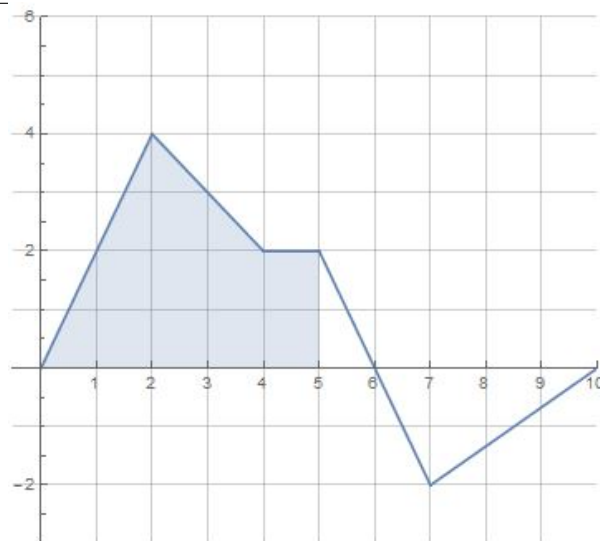
$$\frac{1}{2} + \frac{1}{2} + 1 + 4 = 6,$$

so

$$g(4) = 4 + 6 = 10.$$

3. Again, we can calculate

$$g(5) = \int_0^5 f(t) \, \mathrm{d}t$$

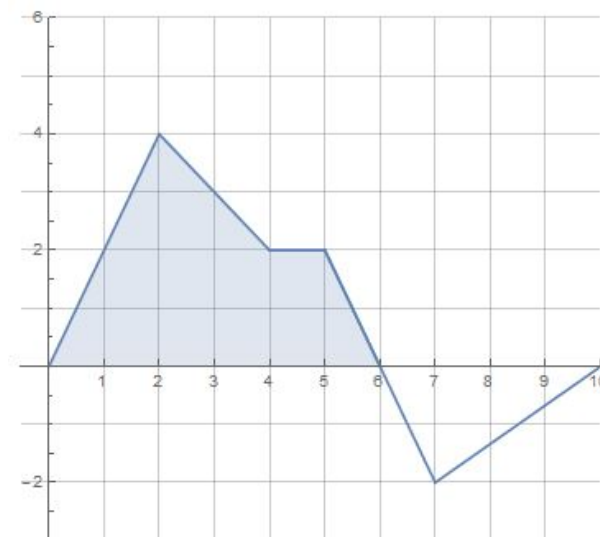by finding the area of the shaded region:

The only new region is the one from $t = 4$ to $t = 5$, whose area is 2, so

$$g(5) = 10 + 2 = 12.$$

4. To get

$$g(6) = \int_0^6 f(t) \, dt,$$

we need to find the area of the shaded region from $t = 0$ to $t = 6$:

Again, the only new region is the one from $t = 5$ to $t = 6$; it's a triangle, so its area is

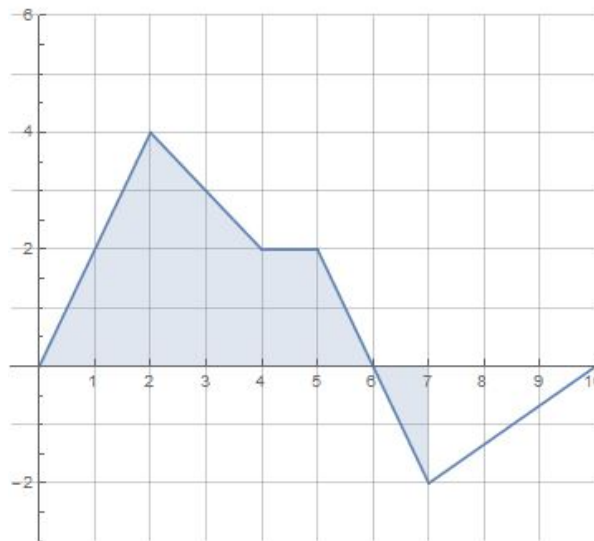$$A = \frac{1}{2} \cdot 1 \cdot 2 = 1.$$

Thus

$$g(6) = 12 + 1 = 13.$$

5

5. Unfortunately,

$$g(7) = \int_0^7 f(t)\ \mathrm{d}t$$

is a bit harder to calculate since $f(t)$ goes below the x-axis:

We will need to calculate the area of the shaded region below the x-axis, then subtract it from the area of the region above the x-axis. Again, this new region is a triangle, so it area is

$$A = \frac{1}{2} \cdot 1 \cdot 2 = 1;$$

but since the integral treats this area as a negative, we know that
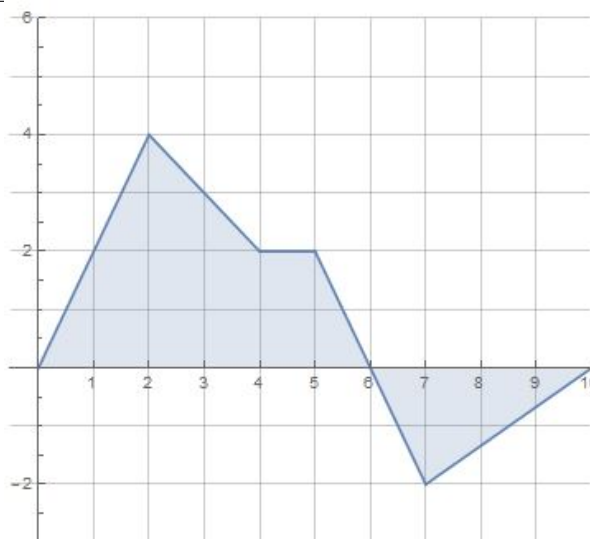
$$g(7) = 13 - 1 = 12.$$

6. Finally, since

$$g(10) = \int_0^{10} f(t)\ \mathrm{d}t,$$

we can use the graph one more time to see that the area of the only new region is

$$A = \frac{1}{2} \cdot 2 \cdot 3 = 3 :$$

This region is below the x-axis, so again the integral treats it as a negative; we have

$$g(10) = 12 - 3 = 9.$$

Since this new object $g(x)$ is actually a function, it has a derivative $g'(x)$ which we might wish to calculate–but as of yet we have no way to do this. Fortunately, the theorem below tells us exactly what $g'(x)$ is:

The Fundamental Theorem of Calculus, Part 1. If $f(t)$ is continuous on $[a, b]$ then the function

$$g(x) = \int_a^x f(t) \, dt$$

is as well, and is differentiable on (a, b); the derivative of $g(x)$ is given by

$$g'(x) = f(x).$$

In a sense, the theorem tells us that *the derivative of the integral of $f(x)$ is $f(x)$*.

Example. Find $g'(x)$ given that

$$g(x) = \int_2^x \frac{4 \sin t}{t^2} \, dt.$$

Since $g(x)$ is defined as an integral, we can find its derivative using the Fundamental Theorem of Calculus:

$$g'(x) = \frac{4 \sin x}{x^2}.$$

7

Example. Given

$$g(x) = \int_0^{\sin x} t^3 \; \mathrm{d}t,$$

find $g'(x)$.

Unfortunately, we can't directly apply the Fundamental Theorem of Calculus here; $g(x)$ is actually a composition of

$$\text{(inside) } h(x) = \sin x \text{ and (outside) } j(x) = \int_0^x t^3 \; \mathrm{d}t.$$

Thus we'll need to use the chain rule to integrate; fortunately, we know the derivatives of both the inside and outside functions:

$$h'(x) = \cos x \text{ and } j'(x) = x^3.$$

Using the chain rule, we have

$$
\begin{aligned}
g'(x) &= h'(x) \cdot j'(h(x)) \\
&= (\cos x) \cdot (j'(\sin x)) \\
&= (\cos x) \cdot ((\sin x)^3) \\
&= \cos x \sin^3 x.
\end{aligned}
$$

Thus

$$g'(x) = \cos x \sin^3 x.$$

Finally, we get to return to our original question: How do we quickly evaluate definite integrals such as

$$\int_a^b f(x) \; \mathrm{d}x?$$

The question is answered by

The Fundamental Theorem of Calculus, Part 2. If $f(x)$ is continuous on $[a, b]$, and if $F(x)$ is any antiderivative of $f(x)$ (so that $F'(x) = f(x)$), then

$$\int_a^b f(x) \; \mathrm{d}x = F(b) - F(a).$$

FTC 2 says that, in order to evaluate a definite integral of $f(x)$, we merely need to evaluate an antiderivative $F(x)$ of f at the bounds b and a of integration!

Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are processes that "reverse" each other.

Example. Find the area A of the region below $f(x) = \sqrt{x}$ from $x = 0$ to $x = 1$.

In the previous section, we expressed this area as the definite integral

$$A = \int_0^1 \sqrt{x} \ \mathrm{d}x.$$

Now we have the tools to actually calculate the integral! FTC 2 tells us that we should find any antiderivative of $f(x) = \sqrt{x} = x^{1/2}$;

$$F(x) = \frac{1}{\frac{3}{2}} x^{3/2} = \frac{2}{3} x^{3/2}$$

will do. Since $F(x) = 2/3x^{3/2}$ is an antiderivative of $f(x) = \sqrt{x}$, FTC 2 says that

$$
\begin{aligned}
A &= \int_0^1 \sqrt{x} \ \mathrm{d}x \\
&= F(1) - F(0) \\
&= \frac{2}{3}(1)^{3/2} - \frac{2}{3}(0)^{3/2} \\
&= \frac{2}{3}(1)^{3/2} \\
&= \frac{2}{3}.
\end{aligned}
$$

Thus the area of the region below $f(x) = \sqrt{x}$ from $x = 0$ to $x = 1$ is $A = 2/3$.

Example. Evaluate

$$\int_0^\pi \cos x \ \mathrm{d}x$$

and interpret your result.

Again, FTC 2 tells us that evaluating this integral really comes down to finding an antiderivative $F(x)$ of the integrand $f(x) = \cos x$; $F(x) = \sin x$ will work. Thus

$$
\begin{aligned}
\int_0^\pi \cos x \ \mathrm{d}x &= F(\pi) - F(0) \\
&= \sin \pi - \sin 0 \\
&= 0.
\end{aligned}
$$

Since the value for the integral is 0, we conclude that the curve $\cos x$ encloses the same area above the x-axis as it does below the x-axis from $x = 0$ to $x = \pi$; since the integral counts area below the axis as negative, the areas "cancel" each other, to yield an integral of 0. This interpretation is confirmed by the graph of $f(x) = \cos x$: