Section 4.7 Scientific Notation

INTRODUCTION

Scientific notation means what it says: it is the notation used in many areas of science. It is used so that scientist and mathematicians can work relatively easily with very large or very small numbers and their related computations. Here is a sample of some very large and very small numbers and to what they refer:

The earth is 93,000,000 (93 million) miles from the sun.

It takes 588,000,000,000,000,000,000,000 (588 billion trillion) atoms of hydrogen to make 1 gram of hydrogen.

Light travels at a rate of about 300,000,000 (300 million) meters per second. (A meter is about 39.6 inches.)

The mass of an atom of hydrogen is 0.00000000000000000000000167. (This number is so small it doesn't even have a name.)

Grass grows at a rate of 0.00000002 (2 hundred-millionths) meters per second

Obviously, these numbers are either much too big or much too small to do easy calculations with, but we can use scientific notation to make them appear less terrifying. Before we work with those numbers specifically, though, we need to do some preparation.

BASE-10 AND NEGATIVE EXPONENTS

Our numbering system is called a **base-10 system** because we have ten digits:

0, 1, 2, 3, 4, 5, 6, 7, 8 and 9.

The tenth counting number is 10 and........... ten 10's make 100: $10 \cdot 10 = 100$;

ten 100's make 1,000: $10 \cdot 100 = 1{,}000$;

and so on.

We know that, for example, in 10^5, there are *five* 0's after the 1: 100,000

and, in 10^3 there are *three* 0's after the 1: 1,000

and, in 10^1 there is *one* 0 after the 1: 10

Likewise, in 10^0 there are *zero* 0's after the 1: 1 (but no zeros)

But what about 10^{-1}? Just as $3^{-1} = \frac{1}{3}$ and $2^{-1} = \frac{1}{2}$, it stands to reason that $10^{-1} = \frac{1}{10}$.

In fact, $10^{-1} = \frac{1}{10^1} = \frac{1}{10} = 0.1$ (one decimal place)

Similarly, $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$ (two decimal places)

$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = 0.001$ (three decimal places)

and $10^{-4} = \frac{1}{10^4} = \frac{1}{10{,}000} = 0.0001$ (four decimal places)

We get two major points out of these:

(a) in base 10,

(i) positive exponents mean large (sometimes very large) numbers

(ii) negative exponents mean small (sometimes very small) numbers

(b) the value of the exponent indicates a position

(i) for positive exponents, the value indicates the number of 0's after the 1

(ii) for negative exponents, the value indicates the number of decimal places

Exercise 1 Rewrite each power of ten as either a large number or as a small number (a decimal). Do not leave any exponents.

a) 10^1

b) 10^{-1}

c) 10^4

d) 10^{-4}

e) 10^{-2}

f) 10^7

g) 10^{12}

h) 10^9

i) 10^{-6}

j) 10^{-8}

WORKING WITH POWERS OF 10

You probably know that multiplying a number, 7, by powers of 10—such as 10 or 100 or 1,000—is the same as placing the same number of zeros at the end of the number:

7 x 1**0** = 7**0**   7 x 1**00** = 7**00**   7 x 1,**000** = 7,**000**

Likewise, a number that ends in one or more zeros can be written as the product of a number and a power of 10:

600 = 6 x 100	9,000 = 9 x 1,000	800,000 = 8 x 100,000
= 6×10^2	= 9×10^3	= 8×10^5

Example 1: Rewrite each expression as a product of a whole number and a power of 10. Use exponents to represent the power of 10.

a) 900   b) 5,000   c) 800,000   d) 40,000,000

Answer: Be sure to count the number of zeros.

a) $900 = 9 \times 100 = 9 \times 10^2$ b) $5{,}000 = 5 \times 1{,}000 = 5 \times 10^3$

c) $800{,}000 = 8 \times 100{,}000 = 8 \times 10^5$ d) $40{,}000{,}000 = 4 \times 10{,}000{,}000 = 4 \times 10^7$

Exercise 2 Rewrite each expression as a product of a whole number and a power of 10. Use exponents to represent the power of 10.

a) 6,000 (6 thousand)   b) 2,000,000 (2 million)

c) 700,000,000,000 (7 hundred billion)   d) 3,000,000,000,000 (3 trillion)

We can represent very small numbers (decimals) in the same way. The difference, of course, is that the powers of 10 will have negative exponents.

For example, 0.005 = 5×0.001 (which has *three* decimal places)

= 5×10^{-3} (notice the exponent of negative *three*)

Example 2: Rewrite each expression as a product of a whole number and a power of 10.

a) 0.04 b) 0.00009

Answer: Be sure to count the number of decimal places, not the number of zeros.

a) $0.04 = 4 \times 0.01 = 4 \times 10^{-2}$ b) $0.00009 = 9 \times 0.00001 = 9 \times 10^{-5}$

Exercise 3 Rewrite each expression as a product of a whole number and a power of 10.

a) 0.006 (6 thousandths) b) 0.00002 (2 hundred-thousandths)

c) 0.000007 (7 millionths) d) 0.00000003 (3 hundred-millionths)

SCIENTIFIC NOTATION

When a number is written as the product of a number—called a **coefficient**—and a power of 10, we say that the number is in **scientific notation**.

So, the scientific notation form of 300 is $\underbrace{3}_{\text{The coefficient}} \times \underbrace{10^2}_{\text{The power of 10}}$ and of 0.004 is $\underbrace{4}_{\text{The coefficient}} \times \underbrace{10^{-3}}_{\text{The power of 10}}$

Scientific Notation is a *product* of two factors:

a **numerical coefficient** and a **power of 10**.

In **proper scientific notation** (or **proper form**),
the coefficient is a number between 1.00000 and 9.99999
(in other words, it has a *whole* number 1 through 9).

The coefficient is typically a single-digit whole number followed by one or two decimal places. Here is a table of some numbers that are proper coefficients and some that are not:

proper coefficients	Not proper coefficients (with reasons)	
3.11	23.11	(the whole number has two digits; it can have only one.)
6	0.6	(the number is less than 1; the whole number can't be 0.)
1.308		
9.96		

Example 3: Identify which of the following numbers could be the coefficient in proper scientific notation. If it cannot be a proper coefficient, state why.

	Number	**Could it be a coefficient?**
a)	9.75	yes (the whole number, 9, is only one digit)
b)	15.4	No, the whole number, 15, has more than one digit
c)	4	yes, the whole number doesn't need to be followed by a decimal
d)	0.56	no, the whole number can't be zero.

Exercise 4 Identify which of the following numbers could be the coefficient in proper scientific notation. If it cannot be a proper coefficient, state why.

	Number	**Could it be a coefficient?**
a)	3	______________________________
b)	2.09	______________________________
c)	31.4	______________________________
d)	0.91	______________________________
e)	10	______________________________

WRITING LARGE NUMBERS IN SCIENTIFIC NOTATION:

In order to write a large number in *proper* scientific notation we need to think about three things:

(a) The coefficient must have a single-digit whole number, 1 through 9;

(b) we'll need to count the number of spaces needed to move a decimal point;

(c) large numbers have powers of 10 with positive exponents

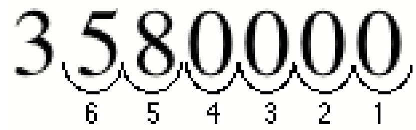
A number like 3,000,000 is easy to write in proper scientific notation because the coefficient is just the whole number 3 with no decimals; in this case, we need only count the number of zeros, and there are six:

$$3{,}000{,}000 = 3 \times 10^6.$$

However, a number like 3,580,000 is a little more challenging because we can't just count the number of zeros. Instead, we need to recognize that

(a) The coefficient is 3.58

(b) we'll need to count the number of spaces needed to move a decimal point to between the 3 and 5; it must move *6* places.

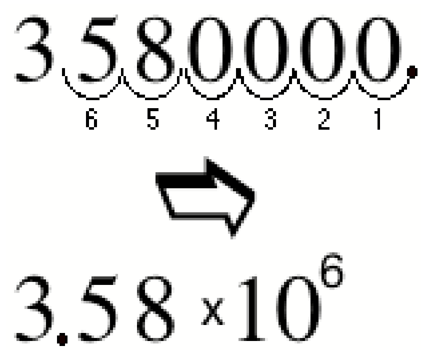


(c) Of course, 3,580,000 is a large number so it will have a power of 10 with a positive exponent.

So, in proper scientific notation, $3{,}580{,}000 = 3.58 \times 10^6$.

You might ask, "Where is the decimal point in the first place? If there's none there, how can we move it?"

Well, of course, whole numbers don't need decimal points, but we can always place one at the end of it. For example, 6 can be thought of as "6**.**" and 4,500 can be thought of as "4,500**.**" With this in mind, the diagram becomes:



Example 4: Rewrite each number into proper scientific notation.

a) 960,000 b) 745,000,000

Answer: Decide on the coefficient and count the number of places the decimal point would move. Also, these are both large numbers.

a) The coefficient is 9.6; the decimal will need to move *five* places: $960{,}000 = 9.6 \times 10^5$

b) The coefficient is 7.45; the decimal will need to move *eight* places: $745{,}000{,}000 = 7.45 \times 10^8$

Exercise 5 Rewrite each number into proper scientific notation.

a) 28,000 =

b) 413,000 =

c) 9,070,000 =

d) 62,150,000,000 =

e) 580 =

f) 73 =

WRITING SMALL NUMBERS IN SCIENTIFIC NOTATION:

Writing small numbers (positive numbers less than 1) in proper scientific notation is similar to writing large numbers. We still have the three things to think about:

(a) The coefficient must have a single-digit whole number, 1 through 9;

(b) we'll need to count the number of spaces needed to move a decimal point;

(c) small numbers have powers of 10 with negative exponents.

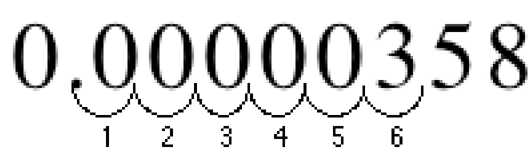
A number like 0.000003 is easy to write in proper scientific notation because the coefficient is just the whole number 3 with no decimals; in this case, we need only count the number of decimal places, and there are six. Also, because this is a small number, the exponent will be negative:

$$0.000003 = 3 \times 10^{-6}.$$

However, a number like 0.00000358 is a little more challenging because we can't just count the number of decimal places. Instead, we need to recognize that

(a) The coefficient is 3.58

(b) we'll need to count the number of spaces needed to *move* a decimal point to between the 3 and 5; it must move *6* places.



(c) Of course, 0.00000358 is a small number so it will have a power of 10 with a negative exponent.

So, in proper scientific notation, $0.00000358 = 3.58 \times 10^{-6}$.

Here is a diagram of the change:

$$0.\underbrace{0}_{1}\underbrace{0}_{2}\underbrace{0}_{3}\underbrace{0}_{4}\underbrace{0}_{5}\underbrace{3}_{6}58$$

$$\Rightarrow$$

$$3.58 \times 10^{-6}$$

Example 5:	Rewrite each number into proper scientific notation.	
	a) 0.00096	b) 0.00000000745
Answer:	Decide on the coefficient and count the number of places the decimal point would move. Also, these are both small numbers.	
a)	The coefficient is 9.6; the decimal needs to move *four* places:	$0.00096 = 9.6 \times 10^{-4}$
b)	The coefficient is 7.45; the decimal needs to move *nine* places:	$0.00000000745 = 7.45 \times 10^{-9}$

Exercise 6 Rewrite each number into proper scientific notation.

a) 0.0028 =

b) 0.0000413 =

c) 0.000000907 =

d) 0.00000006215 =

e) 0.034 =

f) 0.92 =

EXPANDING FROM SCIENTIFIC NOTATION

Sometimes, when a number is written in scientific notation, it's easier to work with but it isn't always easy to know what the number actually is. Of course, if you work long enough with any system it becomes second nature on how to interpret it.

For us, though, since we are new to scientific notation, it's good to be able to write numbers in their more familiar form. In other words, whereas scientific notation abbreviates very large or very small numbers, expanding them to their natural form "*un*-abbreviates" them.

Example 6: Expand each number to its natural form.

a) 4.6×10^4 b) 8.07×10^{-5}

Answer: First, decide if the number is going to be a large number (positive exponent on 10) or a small number (negative exponent on 10).

Second, place a number of zeros to the beginning or end of the coefficient so that the exponent can be moved easily.

Third, check your answer by *thinking* about how it would be if you were to abbreviate it back into scientific notation.

a)	4.6×10^4	This is going to be a *large* number, so place some zeros *after* the coefficient.
	4.6000000×10^4	Now move the decimal four places to create a large number.
	46000.000	Eliminate the unnecessary zeros (after the decimal point) and use a comma.
	46,000	Check this to see if it is right (large number, move the decimal 4 places...yep)
b)	8.07×10^{-5}	This is going to be a *small* number; place some zeros *before* the coefficient.
	$000000008.07 \times 10^{-5}$	Now move the decimal five places to create a small number.
	0000.0000807	Eliminate the unnecessary zeros (before the decimal point).
	0.0000807	Check this to see if it is right (small number, move the decimal 5 places...yep)

Exercise 7 Expand each number to its natural form.

a) $6.1 \times 10^3 =$

b) $9.2 \times 10^{-2} =$

c) $4.33 \times 10^5 =$

d) $3.06 \times 10^{-4} =$

e) $2.084 \times 10^8 =$

f) $4.138 \times 10^{-7} =$

ADJUSTING THE COEFFICIENT

If a number is written in scientific notation, but it's coefficient is not in *proper form*, then we need to adjust the coefficient to put it into proper form.

For example, the coefficient of 12×10^5 is not in proper form (it is too large). If we expand it, we can better see what it will be in proper form:

12×10^5 is 12 followed by five zeros = 1,200,000. This, however, can be written as 1.2×10^6.

In other words, $12 \times 10^5 = 1.2 \times 10^6$. (Notice the changes in the coefficients and the powers of 10.)

We call this—going from an improper form to a proper one—***adjusting the coefficient***. When making this adjustment, if the coefficient is too large, then we usually divide it by 10 and add 1 to the power of 10. (This has the overall effect of both dividing and multiplying by 10, which won't change the value, just the way it looks.)

For example, $12 \times 10^5 = \underbrace{\frac{12}{10}}_{\text{Dividing by 10}} \times \underbrace{10^{5+1}}_{\text{Multiplying by 10 (adding 1 to the power of 10)}} = 1.2 \times 10^6$

Dividing by 10 has the effect of moving the decimal point one place to the left.

For example, $95.3 \times 10^7 = \underbrace{\frac{95.3}{10}}_{\text{Dividing by 10}} \times \underbrace{10^{7+1}}_{\text{Multiplying by 10 (adding 1 to the power of 10)}} = 9.53 \times 10^8$

In some instances, we might even need to divide by 100 and add *2* to the power of 10.

For example, $264 \times 10^3 = \underbrace{\frac{264}{100}}_{\text{Dividing by 100}} \times \underbrace{10^{3+2}}_{\text{Multiplying by 100 (adding 2 to the power of 10)}} = 2.64 \times 10^5$

Example 7: Adjust each so that the coefficient is in proper form.

a) 56×10^4 b) 30.7×10^8

Answer: Each of these has a coefficient with a two-digit whole number. Divide the coefficient by 10 (or 100, whichever is appropriate) and add 1 (or 2) to the power of 10.

a) $56 \times 10^4 = \frac{56}{10} \times 10^{4+1} = 5.6 \times 10^5$

b) $30.7 \times 10^8 = \frac{30.7}{10} \times 10^{8+1} = 3.07 \times 10^9$

Example 8: Adjust each so that the coefficient is in proper form.

a) 29.3×10^{-5} b) 158×10^{-9}

Answer: Each of these has a coefficient with a two-digit whole number. Divide the coefficient by 10 (or 100, whichever is appropriate) and add 1 (or 2) to the power of 10.

a) $29.3 \times 10^{-5} = \frac{29.3}{10} \times 10^{-5+1} = 2.93 \times 10^{-4}$

b) $158 \times 10^{-9} = \frac{158}{100} \times 10^{-9+2} = 1.58 \times 10^{-7}$

Exercise 8 Adjust each so that the coefficient is in proper form.

a) $61 \times 10^{3} =$ b) $49.2 \times 10^{12} =$

c) $38 \times 10^{-2} =$ d) $73.5 \times 10^{-6} =$

e) $506 \times 10^{9} =$ f) $241 \times 10^{-8} =$

If the coefficient is too small, such as 0.87 then we need to *multiply* it by 10 and *subtract* 1 from the power of 10.

For example, $0.12 \times 10^{5} = \underbrace{(0.12 \times 10)}_{\text{Multiplying by 10}} \times \underbrace{10^{5-1}}_{\text{Dividing by 10 (subtracting 1 from the power of 10)}} = 1.2 \times 10^{4}$

Example 9: Adjust each so that the coefficient is in proper form.

a) 0.5×10^{4} b) 0.37×10^{8} c) 0.206×10^{-5}

Answer: Each of these has a coefficient less than 1. Multiply the coefficient by 10 and subtract 1 from the power of 10.

a) $0.5 \times 10^{4} = (0.5 \times 10) \times 10^{4-1} = 5 \times 10^{3}$

b) $0.37 \times 10^{8} = (0.37 \times 10) \times 10^{8-1} = 3.7 \times 10^{7}$

c) $0.206 \times 10^{-5} = (0.206 \times 10) \times 10^{-5-1} = 2.06 \times 10^{-6}$

Exercise 9 Adjust each so that the coefficient is in proper form.

a) $0.6 \times 10^3 =$

b) $0.492 \times 10^{10} =$

c) $0.38 \times 10^{-4} =$

d) $0.735 \times 10^{-2} =$

MULTIPLYING AND DIVIDING WITH SCIENTIFIC NOTATION

Multiplying $300 \times 4{,}000$ is the same as multiplying $3 \times 4 = 12$ and placing the total number of zeros after the 12:

$300 \times 4{,}000 = 12$ followed by a total of five zeros: 1,200,000

If each of these numbers was first written in scientific notation, the product would look like this:

	$300 \times 4{,}000$	
=	$(3 \times 10^2) \times (4 \times 10^3)$	written in scientific notation
=	$(3 \times 4) \times (10^2 \times 10^3)$	using associative and commutative properties
=	12×10^5	using the product rule: $10^2 \times 10^3 = 10^{2+3} = 10^5$.
=	$\frac{12}{10} \times 10^{5+1}$	Adjusting the coefficient.
=	1.2×10^6	If we wanted, we could expand this to be 1,200,000 (the same answer as above).

If two numbers are already written in scientific notation, then the process of *multiplying* them together requires that we

(i) multiply the coefficients together

(ii) combine the powers of 10 by adding the exponents (even if one is positive and the other is negative)

(iii) adjust the coefficient, if necessary.

Likewise, if two numbers are already written in scientific notation, then the process of *dividing* them together requires that we

(i) divide the coefficients appropriately; round off to the nearest hundredth

(ii) combine the powers of 10 by subtracting the exponents (as the *quotient rule* requires)

(iii) adjust the coefficient, if necessary.

Example 10: Perform the indicated operation. Write the answer in proper scientific notation.

a) $(1.2 \times 10^{8}) \cdot (2.7 \times 10^{-5})$ b) $\dfrac{3.5 \times 10^{9}}{2.1 \times 10^{3}}$

c) $(3.5 \times 10^{4}) \cdot (6.4 \times 10^{7})$ d) $\dfrac{3.6 \times 10^{6}}{7.2 \times 10^{-2}}$

Procedure: You may do the multiplying and dividing of the coefficients off to the side or on a calculator (if allowed); they are not shown here. *Be sure to adjust the coefficient, if necessary.*

Answer: a) $(1.2 \times 10^{8}) \cdot (2.7 \times 10^{-5})$

$= (1.2 \times 2.7) \times (10^{8 + (-5)})$

$= 3.24 \times 10^{3}$

(we don't need to adjust this coefficient)

b) $\dfrac{3.5 \times 10^{9}}{2.1 \times 10^{3}}$

$= (3.5 \div 2.1) \times 10^{9 - 3}$

$= 1.67 \times 10^{6}$

(1.67 has been rounded to the nearest hundred)

c) $(3.5 \times 10^{4}) \cdot (6.4 \times 10^{7})$

$= (3.5 \times 6.4) \times (10^{4 + 7})$

$= 22.4 \times 10^{11}$

$= \dfrac{22.4}{10} \times 10^{11 + 1}$

$= 2.24 \times 10^{12}$

d) $\dfrac{3.6 \times 10^{6}}{7.2 \times 10^{-2}}$

$= (3.6 \div 7.2) \times 10^{6 - (-2)}$

$= 0.5 \times 10^{8}$

$= 0.5 \times 10 \times 10^{8 - 1}$

$= 5 \times 10^{7}$

Exercise 10 Perform the indicated operation. Write the answer in proper scientific notation.

a) $(1.5 \times 10^{9}) \cdot (4.4 \times 10^{5})$

b) $\dfrac{6.5 \times 10^{8}}{1.3 \times 10^{5}}$

c) $(7.6 \times 10^{10}) \cdot (4.0 \times 10^{6})$

d) $\dfrac{6.3 \times 10^{4}}{1.8 \times 10^{-4}}$

e) $(9.8 \times 10^{3}) \cdot (2.5 \times 10^{-7})$

f) $\dfrac{1.2 \times 10^{-5}}{4.8 \times 10^{4}}$

Answers to each Exercise

Section 4.7

Exercise 1:	a)	10	b)	0.1	c)	10,000		
	d)	.0001	e)	0.01	f)	10,000,000		
	g)	1,000,000,000,000	h)	1,000,000,000				
	i)	0.000001	j)	0.00000001				
Exercise 2:	a)	6×10^{3}	b)	2×10^{6}	c)	7×10^{11}	d)	3×10^{12}
Exercise 3:	a)	6×10^{-3}	b)	2×10^{-5}	c)	7×10^{-6}	d)	3×10^{-8}

Exercise 4:
- a) yes (the whole number is a single digit)
- b) yes (the whole number is a single digit)
- c) no (the whole number has more than one digit)
- d) no (it is a number less than 1)
- e) no (the whole number has more than one digit)

Exercise 5:	a)	2.8×10^{4}	b)	4.13×10^{5}	c)	9.07×10^{6}	d)	6.215×10^{10}
	e)	5.8×10^{2}	f)	7.3×10^{1}				
Exercise 6:	a)	2.8×10^{-3}	b)	4.13×10^{-5}	c)	9.07×10^{-7}	d)	6.215×10^{-8}
	e)	3.4×10^{-2}	f)	9.2×10^{-1}				
Exercise 7:	a)	6,100	b)	0.092	c)	433,000	d)	0.000306
	e)	208,400,000	f)	0.0000004138				
Exercise 8:	a)	6.1×10^{4}	b)	4.92×10^{13}	c)	3.8×10^{-1}	d)	7.35×10^{-5}
	e)	5.06×10^{11}	f)	2.41×10^{-6}				
Exercise 9:	a)	6×10^{2}	b)	4.92×10^{9}	c)	3.8×10^{-5}	d)	7.35×10^{-3}
Exercise 10:	a)	6.6×10^{14}	b)	5×10^{3}	c)	3.04×10^{17}	d)	3.5×10^{8}
	e)	2.45×10^{-3}	f)	2.5×10^{-10}				

Section 4.7 Focus Exercises

1. Rewrite each expression as a product of a whole number and a power of 10. Use exponents to represent the power of 10.

a) 500,000,000,000 b) 4,000,000 c) 900

d) 0.03 e) 0.0002 f) 0.000007

2. Rewrite each number into proper scientific notation.

a) 330,000 = b) 78,000 = c) 5,090,000 =

d) 130 = e) 0.28 = f) 0.042 =

g) 0.00913 = h) 0.0000708 = i) 0.0000002914 =

3. Expand each number to its natural form.

a) 5.6×10^{2} = b) 2.9×10^{9} = c) 7.3×10^{5} =

d) 2.3×10^{-2} = e) 4.01×10^{-4} = f) 1.89×10^{-1} =

4. Adjust each so that the coefficient is in proper form.

a) 85×10^{6} = b) 90.3×10^{3} = c) 768×10^{4} =

d) 71.6×10^{-7} = e) 0.602×10^{-5} = f) 349×10^{-4} =

5. Perform the indicated operation. Write the answer in proper scientific notation.

a) $(1.1 \times 10^{6}) \cdot (3.7 \times 10^{4})$

b) $\dfrac{3.6 \times 10^{7}}{2.4 \times 10^{2}}$

c) $(8.1 \times 10^{7}) \cdot (3.0 \times 10^{-3})$

d) $\dfrac{7.2 \times 10^{4}}{4.5 \times 10^{9}}$

e) $(6.4 \times 10^{-1}) \cdot (5.5 \times 10^{-4})$

f) $\dfrac{1.1 \times 10^{-4}}{8.8 \times 10^{-6}}$