

Section 6.2 Law of Cosines

- If ABC is any oblique triangle with sides a , b , and c , then the Law of Cosines says:

$$(a) \quad a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$(b) \quad b^2 = a^2 + c^2 - 2ac \cos B \quad \text{or} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$(c) \quad c^2 = a^2 + b^2 - 2ab \cos C \quad \text{or} \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

- You should be able to use the Law of Cosines to solve an oblique triangle for the remaining three parts, given:

(a) Three sides (SSS)

(b) Two sides and their included angle (SAS)

- Given any triangle with sides of lengths a , b , and c , the area of the triangle is

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{a+b+c}{2}. \quad (\text{Heron's Formula})$$

Solutions to Odd-Numbered Exercises

1. Given: $a = 6$, $b = 8$, $c = 12$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{64 + 144 - 36}{2(8)(12)} \approx 0.8958 \Rightarrow A \approx 26.4^\circ$$

$$\sin B = \frac{b \sin A}{a} \approx \frac{8 \sin 26.4^\circ}{6} \approx 0.5928 \Rightarrow B \approx 36.4^\circ$$

$$C \approx 180^\circ - 26.4^\circ - 36.4^\circ = 117.2^\circ$$

3. Given: $A = 50^\circ$, $b = 15$, $c = 30$

$$a^2 = b^2 + c^2 - 2bc \cos A = 225 + 900 - 2(15)(30) \cos 50^\circ \approx 546.49 \Rightarrow a \approx 23.4$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} \approx \frac{547.56 + 900 - 225}{2(23.4)(30)} \approx 0.8708 \Rightarrow B \approx 29.5^\circ$$

$$C = 180^\circ - A - B \approx 180^\circ - 50^\circ - 29.5^\circ = 100.5^\circ$$

5. Given: $a = 9$, $b = 12$, $c = 15$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{81 + 144 - 225}{2(9)(12)} = 0 \Rightarrow C = 90^\circ$$

$$\sin A = \frac{9}{15} = \frac{3}{5} \Rightarrow A \approx 36.9^\circ$$

$$B \approx 180^\circ - 90^\circ - 36.9^\circ = 53.1^\circ$$

7. Given: $a = 75.4, b = 48, c = 48$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{48^2 + 48^2 - 75.4^2}{2(48)(48)} = -0.2338 \Rightarrow A \approx 103.5^\circ$$

$$\sin B = \frac{b \sin A}{a} = \frac{48 \sin(103.5^\circ)}{75.4} = 0.6190 \Rightarrow B \approx 38.2$$

$C = B \approx 38.2$ (Because of roundoff error, $A + B + C \neq 360^\circ$)

9. Given: $B = 8^\circ 15' = 8.25^\circ, a = 26, c = 18$

$$b^2 = a^2 + c^2 - 2ac \cos B = 26^2 + 18^2 - 2(26)(18) \cos(8.25) \approx 73.6863 \Rightarrow b \approx 8.6$$

$$\sin C = \frac{c \sin B}{b} = \frac{18 \sin(8.25)}{8.6} \approx 0.3 \Rightarrow C \approx 17.5^\circ$$

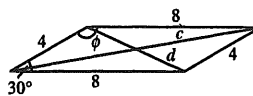
$$A = 180^\circ - B - C = 180^\circ - 8.25^\circ - 17.5^\circ = 154.25$$

11. $d^2 = 4^2 + 8^2 - 2(4)(8) \cos 30^\circ \approx 24.57 \Rightarrow d \approx 4.96$

$$2\phi = 360^\circ - 2\theta \Rightarrow \phi = 150^\circ$$

$$c^2 = 4^2 + 8^2 - 2(4)(8) \cos 150^\circ \approx 135.43$$

$$c \approx 11.64$$



13. $\cos \phi = \frac{10^2 + 14^2 - 20^2}{2(10)(14)}$

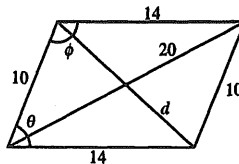
$$\phi \approx 111.8$$

$$2\theta \approx 360^\circ - 2(111.80^\circ)$$

$$\theta \approx 68.2^\circ$$

$$d^2 = 10^2 + 14^2 - 2(10)(14) \cos 68.2^\circ$$

$$d \approx 13.86$$



15. $\cos \alpha = \frac{(9)^2 + (10)^2 - (6)^2}{2(9)(10)}$

$$\alpha \approx 36.3^\circ$$

$$\cos \beta = \frac{6^2 + 10^2 - 9^2}{2(6)(10)}$$

$$\beta \approx 62.7^\circ$$

$$z = 180^\circ - \alpha - \beta \approx 80.9$$

$$\mu = 180^\circ - z \approx 99.1$$

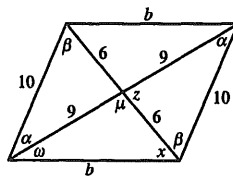
$$b^2 = 9^2 + 6^2 - 2(9)(6)(\cos 99.0^\circ)$$

$$b \approx 11.58$$

$$\cos \omega = \frac{9^2 + 11.58^2 - 6^2}{2(9)(11.58)}$$

$$\omega \approx 30.8^\circ$$

$$\theta = \alpha + \omega \approx 67.1^\circ$$



$$\cos x = \frac{6^2 + 11.58^2 - 9^2}{2(6)(11.58)}$$

$$x \approx 50.1$$

$$\phi = \beta + x \approx 112.8^\circ$$

17. Given:
- $a = 5, b = 9, c = 10$

$$s = \frac{a + b + c}{2} = 12$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{12(7)(3)(2)} \approx 22.45 \text{ square units}$$

19. Given:
- $a = 3.5, b = 10.2, c = 9$

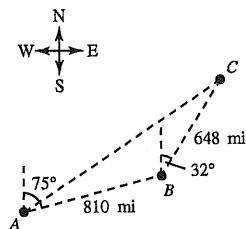
$$s = \frac{a + b + c}{2} = 11.35$$

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{11.35(7.85)(1.15)(2.35)} \approx 15.52 \text{ square units}$$

- 21.
- $a = 20, b = 20, c = 10 \Rightarrow s = \frac{20 + 20 + 10}{2} = 25$

$$\text{Area} = \sqrt{25(5)(5)(15)} \approx 96.82 \text{ square units}$$

- 23.



$$B = 105^\circ + 32^\circ = 137^\circ$$

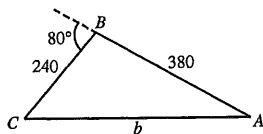
$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac \cdot \cos B \\ &= 648^2 + 810^2 - 2(648)(810) \cos(137^\circ) \\ &= 1,843,749.862 \end{aligned}$$

$$b = 1357.8 \text{ miles}$$

$$\begin{aligned} \text{From the Law of Sines, } \frac{a}{\sin A} &= \frac{b}{\sin B} \Rightarrow \sin A = \frac{a}{b} \sin B = \frac{648}{1357.8} \sin(137^\circ) \approx 0.32548 \\ &\Rightarrow A \approx 19^\circ \Rightarrow \text{Bearing S } 56^\circ \text{ W} \end{aligned}$$

25. Angle at
- $B = 180^\circ - 80^\circ = 100^\circ$

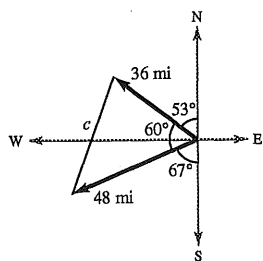
$$b^2 = 240^2 + 380^2 - 2(240)(380) \cos 100^\circ \approx 233,673.4 \Rightarrow b \approx 483.4 \text{ meters}$$



- 27.
- $C = 180^\circ - 53^\circ - 67^\circ = 60^\circ$

$$c^2 = a^2 + b^2 - 2ab \cos C = 36^2 + 48^2 - 2(36)(48)(0.5) = 1872$$

$$c \approx 43.3 \text{ mi}$$



$$29. (a) \cos \theta = \frac{273^2 + 178^2 - 235^2}{2(273)(178)}$$

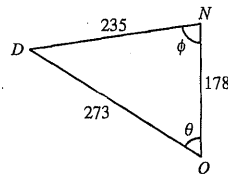
$$\theta \approx 58.4^\circ$$

Bearing: N 58.4° W

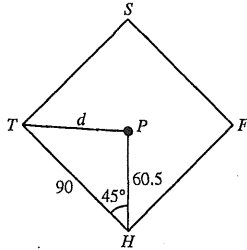
$$(b) \cos \phi = \frac{235^2 + 178^2 - 273^2}{2(235)(178)}$$

$$\phi \approx 81.5^\circ$$

Bearing: S 81.5° W



$$31. d^2 = 60.5^2 + 90^2 - 2(60.5)(90) \cos 45^\circ \approx 4059.9 \Rightarrow d \approx 63.7 \text{ ft}$$



$$33. \overline{RS} = \sqrt{8^2 + 10^2} = \sqrt{164} = 2\sqrt{41} \approx 12.8 \text{ ft}$$

$$\overline{PQ} = \frac{1}{2}\sqrt{16^2 + 10^2} = \frac{1}{2}\sqrt{356} = \sqrt{89} \approx 9.4 \text{ ft}$$

$$\tan P = \frac{10}{16}$$

$$P = \arctan \frac{5}{8} \approx 32.0^\circ$$

$$\overline{QS} = \sqrt{8^2 + 9.4^2 - 2(8)(9.4) \cos 32^\circ} \approx \sqrt{24.81} \approx 5.0 \text{ ft}$$

$$35. (a) 7^2 = 1.5^2 + x^2 - 2(1.5)(x) \cos \theta$$

$$49 = 2.25 + x^2 - 3x \cos \theta$$

$$(b) \quad x^2 - 3x \cos \theta = 46.75$$

$$x^2 - 3x \cos \theta + \left(\frac{3 \cos \theta}{2}\right)^2 = 46.75 + \left(\frac{3 \cos \theta}{2}\right)^2$$

$$\left[x - \frac{3 \cos \theta}{2}\right]^2 = \frac{187}{4} + \frac{9 \cos^2 \theta}{4}$$

$$x - \frac{3 \cos \theta}{2} = \pm \sqrt{\frac{187 + 9 \cos^2 \theta}{4}}$$

Choosing the positive values of x , we have

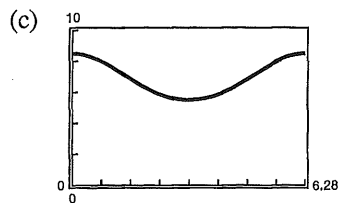
$$x = \frac{1}{2}(3 \cos \theta + \sqrt{9 \cos^2 \theta + 187}).$$

$$37. A = 180^\circ - 40^\circ - 20^\circ = 120^\circ$$

$$\frac{x}{\sin 20^\circ} = \frac{7}{\sin 120^\circ}$$

$$x = \frac{7 \sin 20^\circ}{\sin 120^\circ}$$

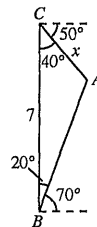
$$x \approx 2.76 \text{ feet}$$



$$(d) x = \frac{1}{2}(3 \cos \pi + \sqrt{9 \cos^2 \pi + 187})$$

$$= 5.5$$

$$\approx 6 \text{ inches}$$



39. False. This is not a triangle! $5 + 10 < 16$

41. (a) Working with $\triangle ODC$, we have $\cos \alpha = \frac{a/2}{R}$.

This implies that $2R = a/\cos \alpha$. Since we know that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

we can complete the proof by showing that $\cos \alpha = \sin A$. The solution of the system

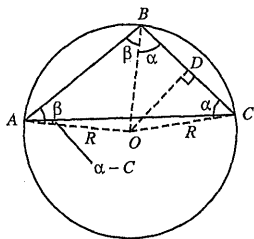
$$A + B + C = 180^\circ$$

$$\alpha - C + A = \beta$$

$$\alpha + \beta = B$$

is $\alpha = 90^\circ - A$. Therefore:

$$2R = \frac{a}{\cos \alpha} = \frac{a}{\cos(90^\circ - A)} = \frac{a}{\sin A}.$$



- (b) By Heron's Formula, the area of the triangle is

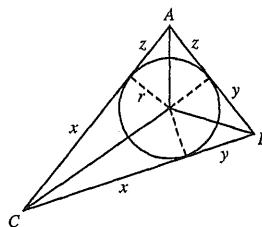
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

We can also find the area by dividing the area into six triangles and using the fact that the area is $1/2$ the base times the height. Using the figure as given, we have

$$\begin{aligned} \text{Area} &= \frac{1}{2}xr + \frac{1}{2}xr + \frac{1}{2}yr + \frac{1}{2}yr + \frac{1}{2}zr + \frac{1}{2}zr \\ &= r(x + y + z) \\ &= rs. \end{aligned}$$

$$\text{Therefore: } rs = \sqrt{s(s-a)(s-b)(s-c)} \Rightarrow$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$



43. Given: $a = 200$ ft, $b = 250$ ft, $c = 325$ ft

$$s = \frac{200 + 250 + 325}{2} = 387.5$$

$$\text{Radius of the inscribed circle: } r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{\frac{(187.5)(137.5)(62.5)}{387.5}} \approx 64.5 \text{ ft}$$

$$\text{Circumference of an inscribed circle: } C = 2\pi r \approx 2\pi(64.5) \approx 405 \text{ ft}$$

45. $3 \sec x + 4 = 10$

$$\sec x = 2$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$47. \frac{3}{\cos x \sin x} = 2\sqrt{3} \csc x = 2\sqrt{3} \left(\frac{1}{\sin x} \right)$$

$$\frac{3}{\cos x} = 2\sqrt{3}$$

$$\cos x = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$\begin{aligned} 49. \cos \frac{5\pi}{6} - \cos \frac{\pi}{3} &= -2 \sin \left(\frac{\frac{5\pi}{6} + \frac{\pi}{3}}{2} \right) \sin \left(\frac{\frac{5\pi}{6} - \frac{\pi}{3}}{2} \right) \\ &= -2 \sin \frac{7\pi}{12} \sin \frac{\pi}{4} \end{aligned}$$