Section 6: Double integrals & applications.

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Images from: "Thomas' calculus" by Wier, Hass & Giordano, 2008, Pearson Education Inc.

S1: Motivation.

In this section we learn integration of functions of two variables $f(x, y)$, known as “double integrals”.

We apply the ideas to calculate quantities that vary in two and three dimensions, such as:

mass; moments; centre of mass; volume; area.

An important difference between single and double integrals is that in the latter case, the domain of $f(x, y)$ plays a more prominent role.

S2: What is a double integral?

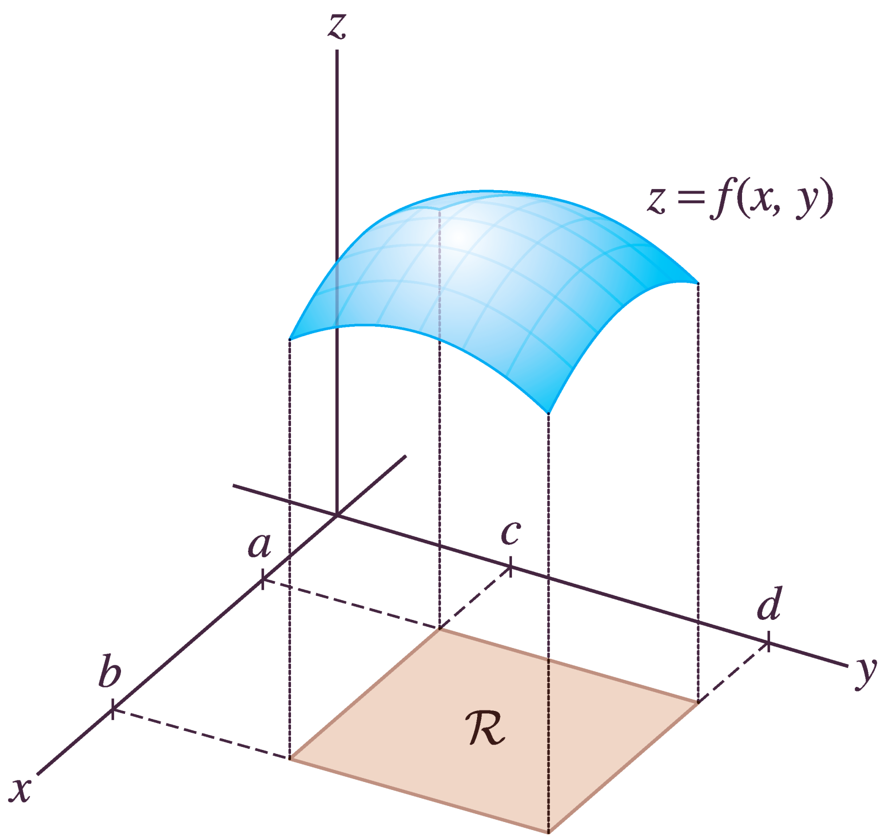
A double integral is just the limit of “Riemann sums”.

Ranger Uranium Mine in Kakadu. The volume of ore removed is one type of quantity that is expressed by a Riemann sum.

Consider a function $f(x, y)$ that is defined on a rectangle $\mathcal{R}$ where

$$\mathcal{R} := \{(x, y) : a \leq x \leq b,\ c \leq y \leq d\}$$

with a, b, c and d being constants.



How to calculate the volume above $\mathcal{R}$ but below the surface?

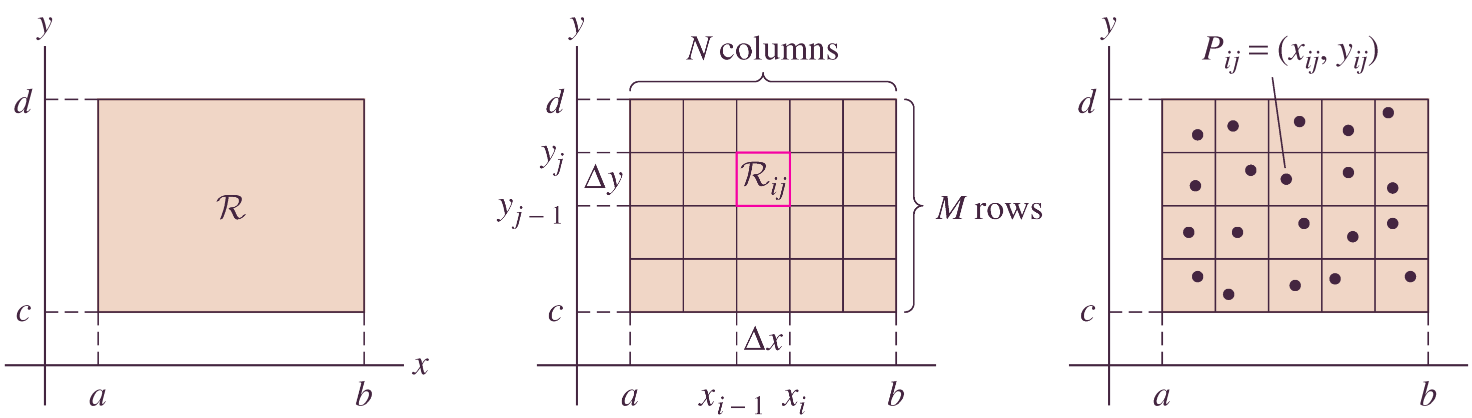
We slice up $\mathcal{R}$ in the following way. Partition $\mathcal{R}$ into NM subrectangles

$$\mathcal{R}_{ij} := [x_{i-1}, x_i] \times [y_{i-1}, y_i], \qquad i = 1, \dots, N,\ j = 1, \dots, M$$

whose respective area of each $\mathcal{R}_{ij}$ we denote by ΔA_{ij}. We choose our partition to be "regular", so that the slices are of equal width, ie:

$$\begin{aligned} \Delta x_i &:= x_i - x_{i-1} = (b-a)/N, \qquad i = 1, \dots, N \\ \Delta y_i &:= y_i - y_{i-1} = (d-c)/M, \qquad j = 1, \dots, M \end{aligned}$$

then we can denote: $\Delta x := (b-a)/N$; $\Delta y := (d-c)/M$; $\Delta A := \Delta A_{ij}$.

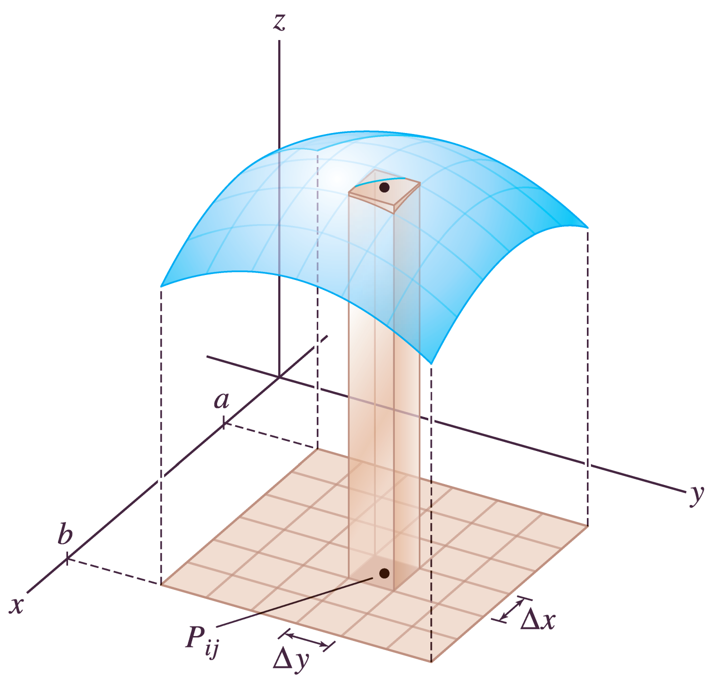


Rectangle $\mathcal{R} = [a, b] \times [c, d]$

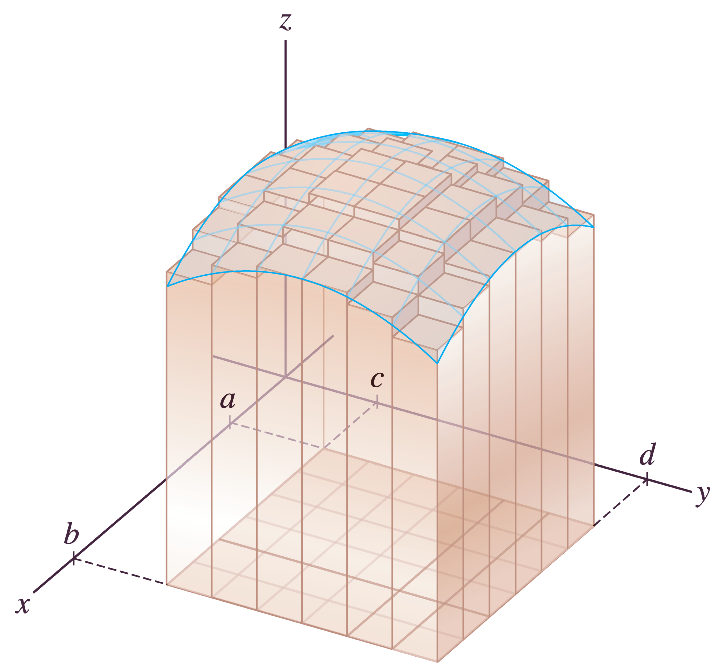
Create $N \times M$ grid of subrectangles

Choose a sample point P_{ij} in each rectangle

We choose a **sample point** $P_{ij} = (x_{ij}, y_{ij})$ in each subrectangle $\mathcal{R}_{ij}$ and construct a box as in Figure 4(B).



(B) The volume of the box is $f(P_{ij})\Delta A$ where $\Delta A = \Delta x \Delta y$.



(C) The Riemann sum $S_{N,M}$ is the sum of the volumes of the boxes.

FIGURE 4

The box has volume = height × area of base

$$= f(P_{ij})\Delta x_i \Delta y_j = f(P_{ij})\Delta x \Delta y = f(P_{ij})\Delta A.$$

We form Riemann sums by adding up the volumes of each box

$$\begin{aligned} S_{N,M} &:= \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta A \\ &= \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij}) \Delta x \Delta y. \end{aligned}$$

Now, if we let the size of the rectangles $\mathcal{R}_{ij}$ go to zero (by, say, letting the length of each diagonal line segment of $\mathcal{R}_{ij}$ go to zero) then N and M must approach infinity (why?).

Thus the total volume above $\mathcal{R}$ and below the surface will be given by

$$\lim_{N,M \to \infty} S_{N,M}.$$

Now, if this limit exists (independently of how we chose our partition), then we say that f is integrable on $\mathcal{R}$ and denote this number by the "double integral"

$$\int\int_{\mathcal{R}} f(x,y)\, dA.$$

THEOREM **Fubini's Theorem (First Form)**

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

The integrals in Fubini's theorem are known as *iterated* integrals. Let us see how they naturally arise.

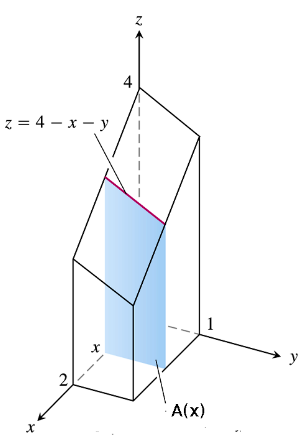
Suppose we wish to calculate the volume of the solid lying above the rectangle

$$R := \{(x, y) : 0 \leq x \leq 2,\ 0 \leq y \leq 1\}$$

and below the plane $z = 4 - x - y$ using integration. We can use slicing techniques from first–year to calculate the volume (and produce an iterated integral).

If we slice up the solid with cuts parallel to the YZ–plane forming cross–sectional areas $A(x)$ then the volume will be given by

$$V = \int_{x=0}^{x=2} A(x)\ dx$$



Now, for each x, the cross–sectional area $A(x)$ may be obtained from integration

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\, dy.$$

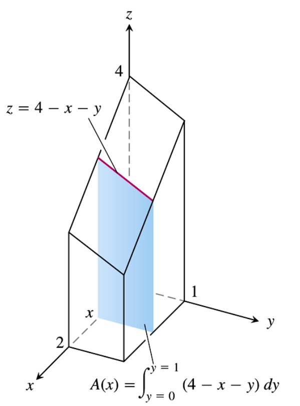


FIGURE To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

Combining the above integrals we obtain the volume V

$$V = \int_{x=0}^{x=2} \left[\int_{y=0}^{y=1} (4 - x - y)\, dy \right] dx$$

or just

$$V = \int_0^2 \int_0^1 (4 - x - y)\, dy\, dx.$$

EXAMPLE Evaluating a Double Integral

Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 1 - 6x^2y \qquad \text{and} \qquad R: \quad 0 \le x \le 2, \quad -1 \le y \le 1.$$

Solution By Fubini's Theorem,

$$\iint_R f(x, y)\, dA = \int_{-1}^{1}\int_{0}^{2}(1 - 6x^2y)\, dx\, dy = \int_{-1}^{1}\left[x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1}(2 - 16y)\, dy = \left[2y - 8y^2\right]_{-1}^{1} = 4.$$

Reversing the order of integration gives the same answer:

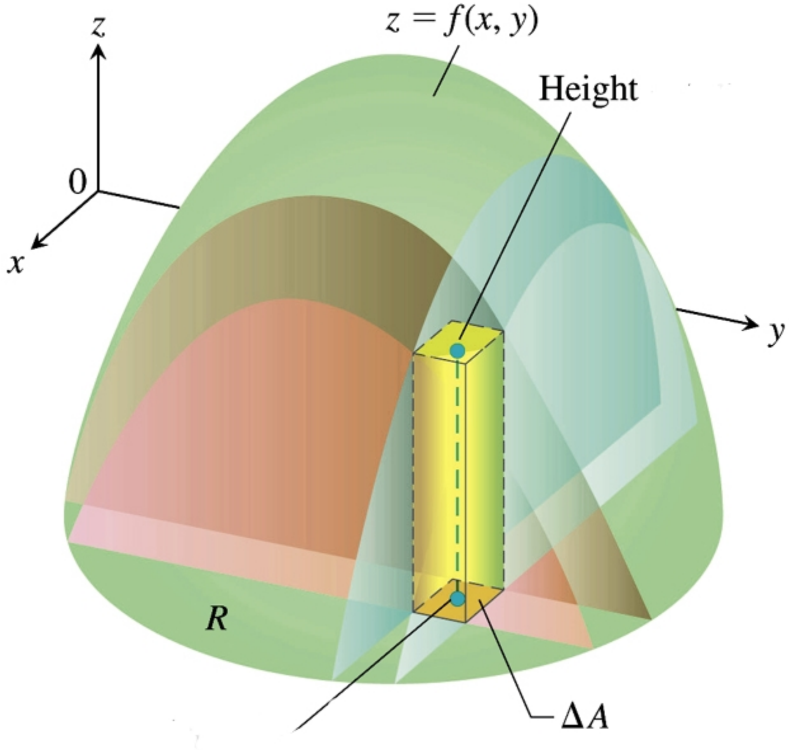
$$\int_{0}^{2}\int_{-1}^{1}(1 - 6x^2y)\, dy\, dx = \int_{0}^{2}\left[y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_{0}^{2}[(1 - 3x^2) - (-1 - 3x^2)]\, dx$$

$$= \int_{0}^{2} 2\, dx = 4.$$

S4: Iterated integrals over other regions

What about applying integration to calculate the volume of solids that have non–rectangular bases?



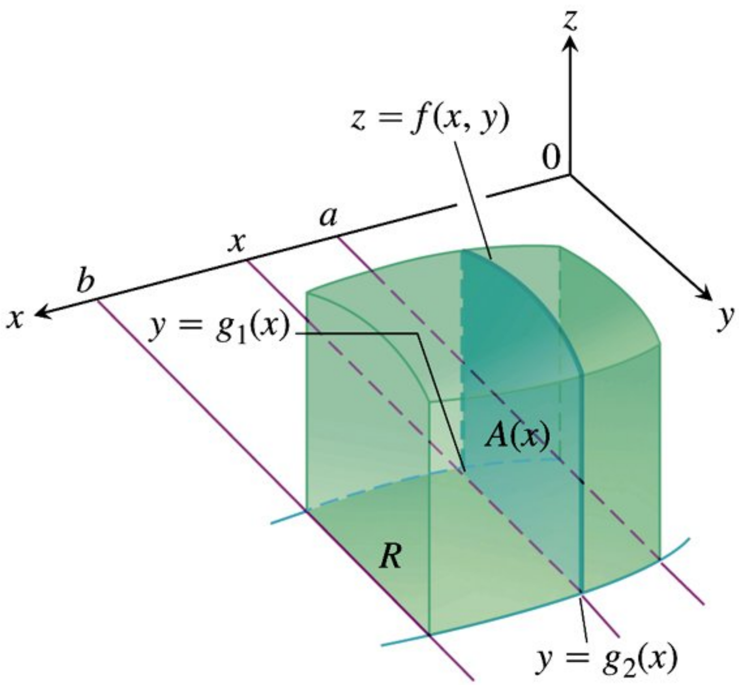


FIGURE The area of the vertical slice shown here is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$.

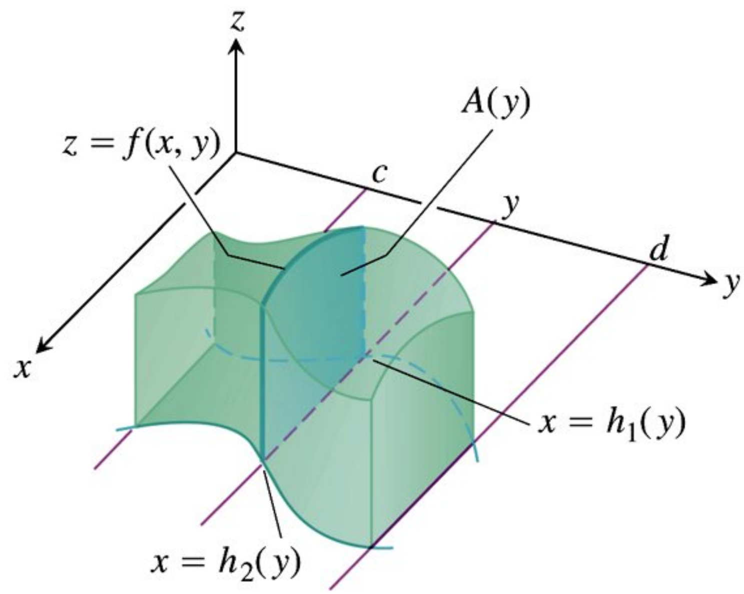


FIGURE The volume of the solid shown here is

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

THEOREM **Fubini's Theorem (Stronger Form)**

Let $f(x, y)$ be continuous on a region R.

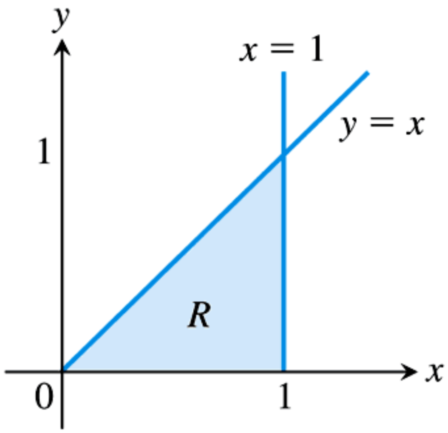
1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

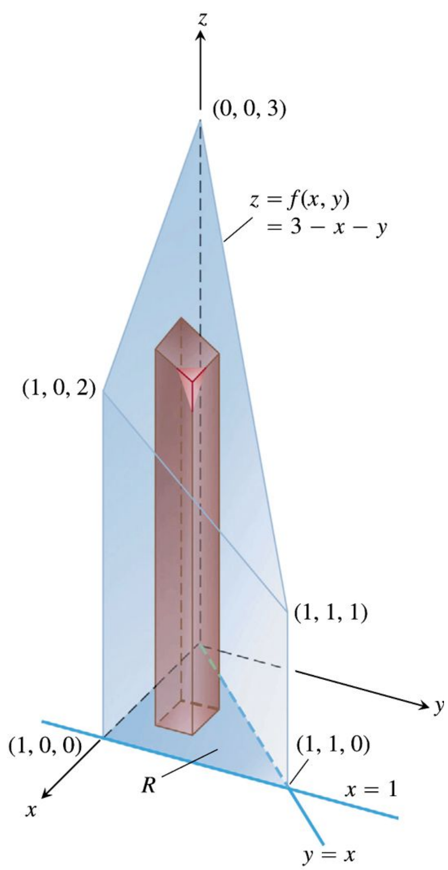
$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Ex: Mathematically describe the following region R in two ways.

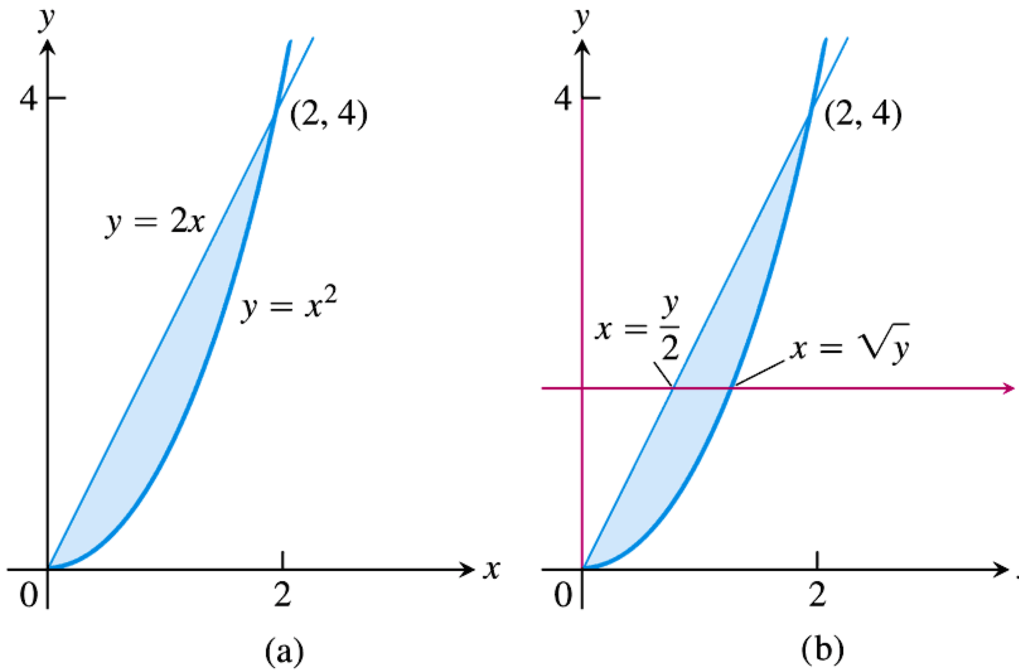


Ex: Apply your ideas from the previous example to calculate

$$\int\int_R 3 - x - y \, dA.$$



Ex: Mathematically describe the following region R in two ways.



Ex: Sketch the region of integration R for

$$\int_0^2 \int_y^2 \sin(x^2)\ dx\ dy.$$

Redescribe R and hence evaluate the double integral.

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c\iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

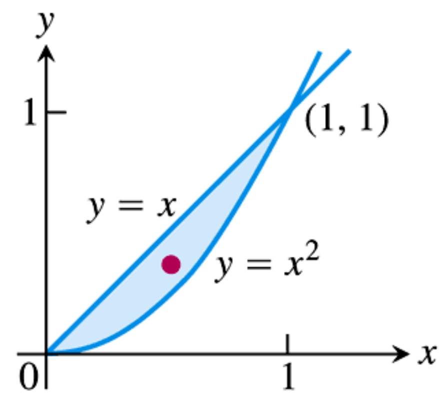
(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

S5: Area of a region.



DEFINITION **Area**

The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

EXAMPLE Finding Area

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure below), noting where the two curves intersect, and calculate the area as

$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Notice that the single integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves

S6: Mass, first moments & centre of mass in 2D.

Many structures and mechanical systems behave as if their masses were concentrated at at single point, called the *centre of mass*. It is important locate this point so that we can better understand the behaviour of our structure or system.

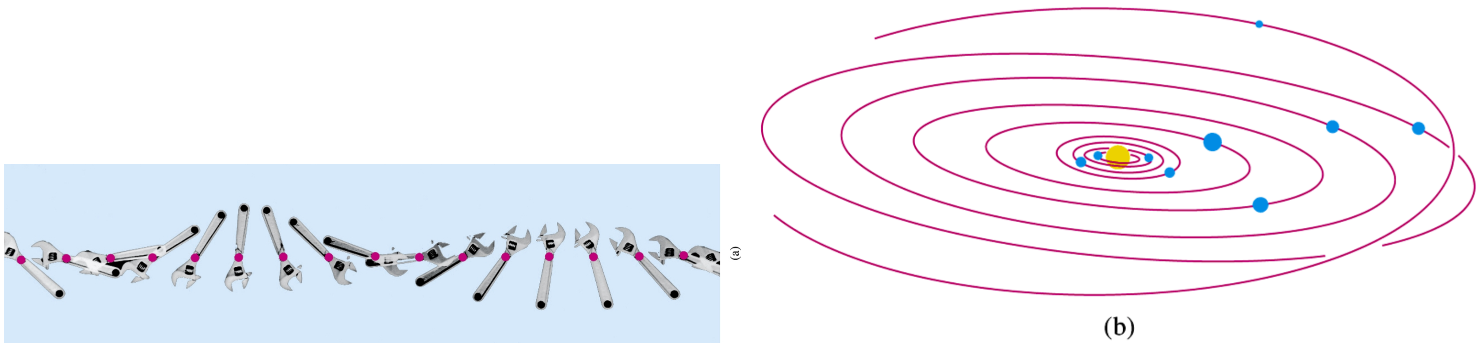


FIGURE (a) The motion of this wrench gliding on ice seems haphazard until we notice that the wrench is simply turning about its center of mass as the center glides in a straight line. (b) The planets, asteroids, and comets of our solar system revolve about their collective center of mass. (It lies inside the sun.)

We now consider the problem of determining the centre of mass of a thin, flat plate: eg, a disc of aluminium; or a triangular sheet of metal.

A material's **density** is its mass per unit volume. In practice, however, we tend to use units that we can conveniently measure. For thin plates (or laminae) we use *mass per unit area*. A body's first moments tells us about balance and about the torque the body exerts about different axes in a gravitational field.

TABLE Mass and first moment formulas for thin plates covering a region R in the xy-plane

Mass: $M = \iint_R \delta(x, y)\, dA$ $\quad$ $\delta(x, y)$ is the density at (x, y)

First moments: $M_x = \iint_R y\delta(x, y)\, dA, \qquad M_y = \iint_R x\delta(x, y)\, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Above, M_x denotes the first moment of the plate about the X–axis; and M_y denotes the first moment of the plate about the Y–axis.

Justification of the mass formula for rectangles $\mathcal{R}$:

Partition $\mathcal{R}$ into subrectangles $\mathcal{R}_{ij} := [x_{i-1}, x_i] \times [y_{i-1}, y_i]$, whose area we denote by ΔA_{ij} and mass by ΔM_{ij}. We make our partition regular so that the slices are of equal width, ie: $\Delta x_i = \Delta x$ and $\Delta y_i = \Delta y$,

We choose a **sample point** $P_{ij} = (x_{ij}, y_{ij})$ in each subrectangle $\mathcal{R}_{ij}$ and and note that the mass ΔM_{ij} of the rectangle $\mathcal{R}_{ij}$ satisfies

$$\delta(P_{ij}) \approx \frac{\Delta M_{ij}}{\Delta A_{ij}}.$$

Rearranging, we sum the ΔM_{ij} to approximately form the mass of the entire plate. Then we take the limit as the size of the rectangles go to zero to obtain our double integral formula for the mass M of the entire plate.

Independent learning ex: Can you justify the other formulae for moments and centre of mass?

Ex: Consider a thin metal plate that covers the triangular region shown below. If the density in the plate is measured by $\delta(x, y) = 6x + 6y + 1$ then calculate the plate's: mass; first moments; centre of mass.

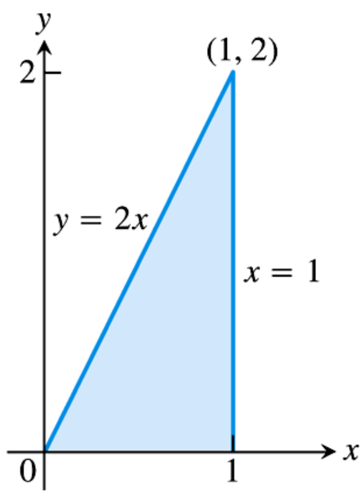


FIGURE The triangular region covered by the plate in Example .

Ex: (continued)

S7: Moments of intertia (2nd moments)

If the body under consideration if a rotating shaft then we are much more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity. This is where the 2nd moment of inertia comes into play.

The moment of inertia of a shaft resembles in some ways the inertia of a locomotive. What makes the locomotive hard to stop or start moving is its mass. What makes the shaft hard to stop of start moving is its moment of inertia.

TABLE Second moment formulas for thin plates in the *xy*-plane

Moments of inertia (second moments):

About the x-axis:	$I_x = \iint y^2 \delta(x, y)\, dA$
About the y-axis:	$I_y = \iint x^2 \delta(x, y)\, dA$
About a line L:	$I_L = \iint r^2(x, y) \delta(x, y)\, dA,$ where $r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2) \delta(x, y)\, dA = I_x + I_y$

Radii of gyration:

About the x-axis:	$R_x = \sqrt{I_x / M}$
About the y-axis:	$R_y = \sqrt{I_y / M}$
About the origin:	$R_0 = \sqrt{I_0 / M}$

Ex: Consider a thin metal plate that covers the triangular region shown below. If the density in the plate is measured by $\delta(x, y) = 6x+6y+1$ then calculate I_x, I_y and I_0.

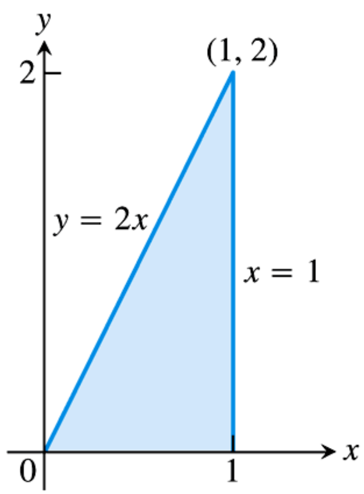


FIGURE The triangular region covered by the plate in Example .

Applications matter! The moment of inertia plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times the moment of inertia of a typical cross–section of the beam about the beam's longitudinal axis.

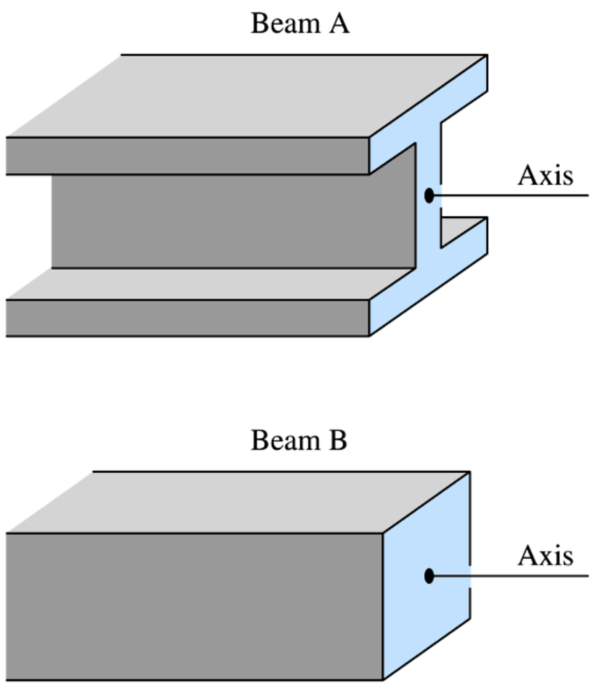
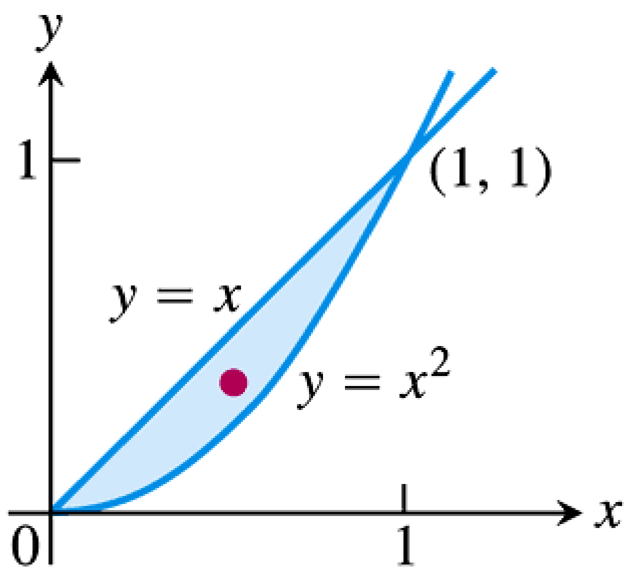


FIGURE The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

The greater the moment of inertia, the stiffer the beam and the less it will bend under a given load. The I–beam holds most of its mass away from the axis to maximize the value of its moment of inertia.

Ex: Determine the centroid of the shaded region in the following diagram.



S8: Double integrals in polar co–ordinates.

Sometimes we can reduce a very difficult double integral to a simple one via a substitution. You will have seen this general technique for single integrals. However, for double integrals, we can make a transformation that simplifies the description of the region of integration.

So–called polar co–ordinates are useful when the domain of integration is an angular sector or "polar rectangle".

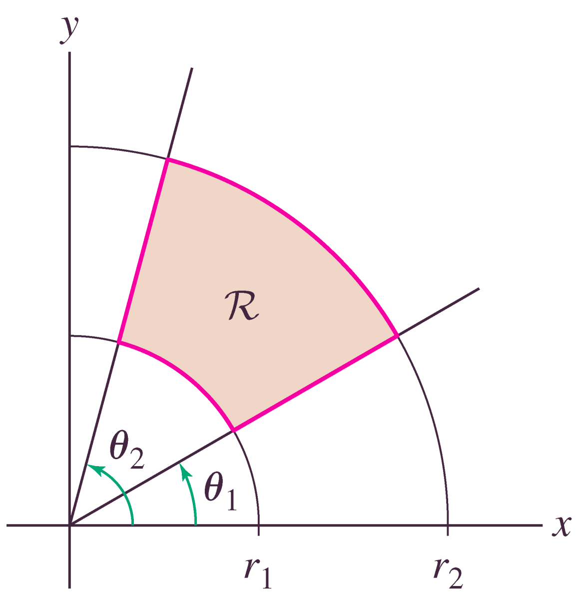


FIGURE Polar rectangle.

Recall that rectangular and polar co–ords are related by:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Thus we write a function $f(x, y)$ in polar co–ords as

$$f(r\cos\theta, r\sin\theta).$$

The change of variables formula for the polar rectangle $\mathcal{R}$ is

$$\iint_{\mathcal{R}} f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Note the extra factor of r in the right–hand side integrand.

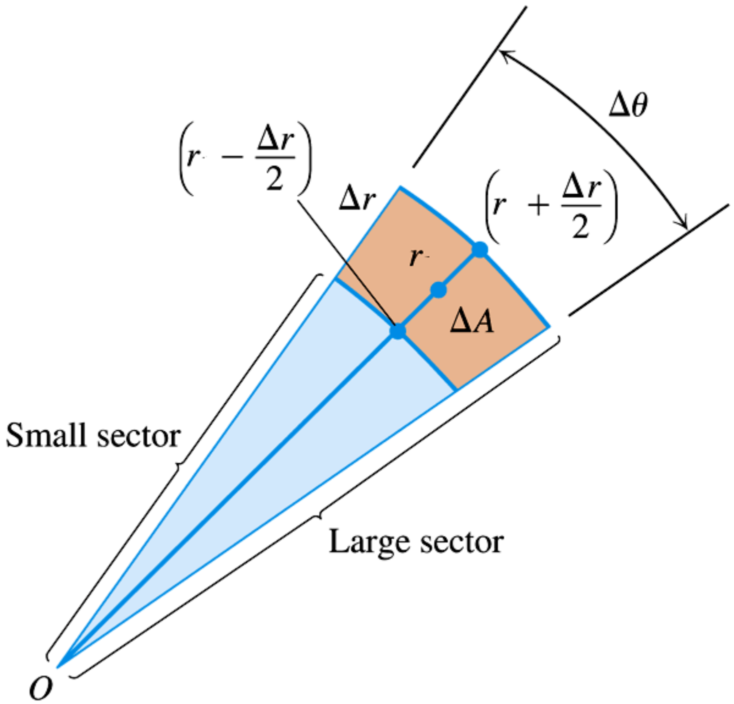


FIGURE The observation that

$$\Delta A = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$$

leads to the formula $\Delta A = r\ \Delta r\,\Delta\theta$.

Ex: Describe, in polar co–ordinates, the set of points of the unit disc Ω that has centre $(0, 0)$ and that lie in the first quadrant ($x \geq 0$, $y \geq 0$).

Hence, evaluate

$$I := \int\int_{\Omega} \sqrt{x^2 + y^2}\, dA.$$

THEOREM **Double Integral in Polar Coordinates** Let $\mathcal{D}$ be a domain of the form $\theta_1 \le \theta \le \theta_2$, $\alpha(\theta) \le r \le \beta(\theta)$, and assume that $f(x, y)$ is continuous on $\mathcal{D}$. Then

$$\iint_{\mathcal{D}} f(x, y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r=\alpha(\theta)}^{\beta(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: Let $\mathcal{R}$ be the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Mathematically describe $\mathcal{R}$ and thus write down the explicit double integral for the area of $\mathcal{R}$.

Ex: Compute the polar moment of inertia about the origin of the thin plate with constant density $\delta \equiv 1$ that covers the region: $x^2+y^2 \leq 1$; $x \geq 0$; $y \geq 0$.

Relationship with the Jacobian.