Section 6 – Quadratics – Part 1

The following Mathematics Florida Standards will be covered in this section:	
MAFS.912.A-SSE.1.2	Use the structure of an expression to identify ways to rewrite it. *For example, see* $x^4 - y^4$ *as* $(x^2)^2 - (y^2)^2$, *thus recognizing it as a difference of squares that can be factored* $(x^2 - y^2)(x^2 + y^2)$.
MAFS.912.A-SSE.2.3	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. a. Factor a quadratic expression to reveal the zeros of the function it defines.
MAFS.912.F-IF.3.8	Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. a. Use the process of factoring and completing the square in quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.
MAFS.912.A-REI.2.4	Solve quadratic equations in one variable. a. Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x-p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. b. Solve quadratic equations by inspection (*e.g., for* $x^2 = 49$), taking square roots, completing the square, the quadratic formula, and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions.
MAFS.912.F-IF.2.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.*

Videos in this Section

Video 1: Real-World Examples of Quadratic Functions
Video 2: Solving Quadratics Using the Quadratic Formula
Video 3: Factoring Quadratic Expressions
Video 4: Solving Quadratics by Factoring – Part 1
Video 5: Solving Quadratics by Factoring – Part 2
Video 6: Solving Quadratics by Factoring – Special Cases
Video 7: Solving Quadratics by Taking Square Roots
Video 8: Solving Quadratics by Completing the Square
Video 9: Quadratics in Action

Section 6 – Video 1
Real-World Examples of Quadratic Functions

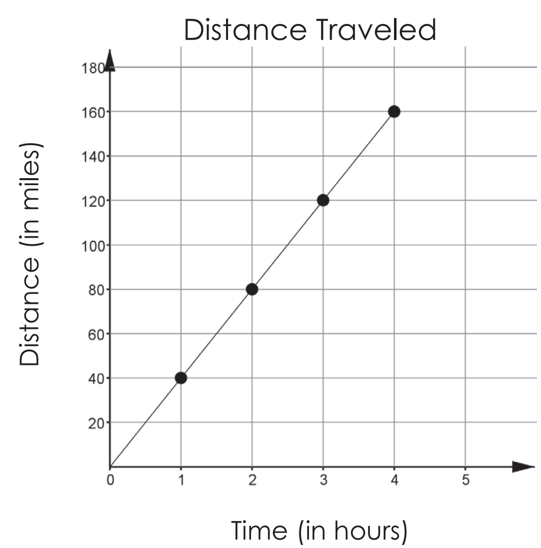
Let's revisit linear functions.

Imagine that you are driving down the road at a constant speed of 40 mph. This is a linear function.

We can represent the distance traveled versus time on a table:

Time (in hours)	Distance Traveled (in miles)
1	40
2	80
3	120
4	160

We can represent the scenario on a graph:



We can represent the distance traveled in terms of time with the equation $d(t) = 40t$.

Linear functions always have a constant rate of change. In this section, we are going to discover a type of non-linear function.

Consider the following situation:

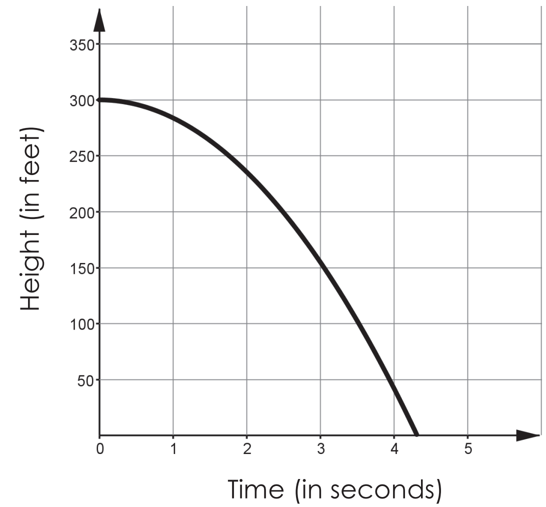
Liam dropped a watermelon from the top of a 300 feet tall building. He wanted to know if the watermelon was falling at a constant rate over time. He filmed the watermelon's fall and then recorded his observations in the following table:

Time (in seconds)	Height (in feet)
0	300
1	283.9
2	235.6
3	155.1
4	42.4

What do you notice about the rate of change?

Why do you think that the rate of change is not constant?

Liam entered the data into his graphing calculator. The graph below displays the first quadrant of the graph.



What is the independent variable?

What is the dependent variable?

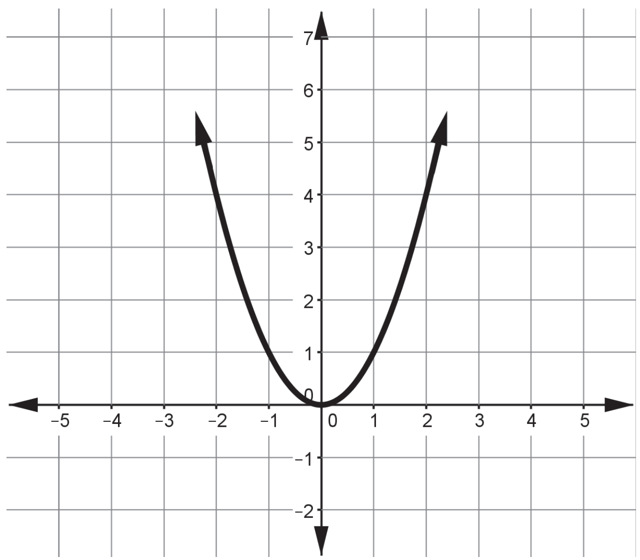
Liam then used his calculator to find the equation of the function:

$$h(t) = -16.1t^2 + 300$$

Important facts:

- We call this non-linear function a ________________________.
- The general form (parent function) of the equation is ________________________.

The graph of $f(x) = x^2$ is shown below:



- This graph is called a ________________________.

Why did we only consider the first quadrant of Liam's graph?

In Liam's graph, what was the watermelon's height when it hit the ground?

The time when the watermelon's height was at zero is called the solution to this quadratic equation. We also call these the ***zeros of the equation***.

There was only one solution to Liam's equation. Describe a situation where there could be two solutions.

What about no solutions?

To solve a quadratic equation using a graph**:**

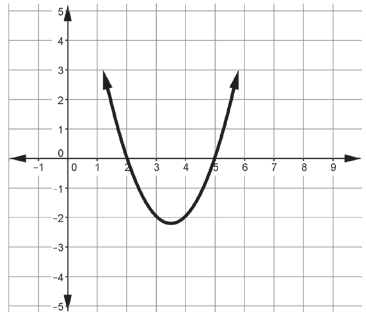
- look for the ________________________ of the graph
- the solution(s) are the values where the graph intercepts the ________________________

STUDY EDGE TIP

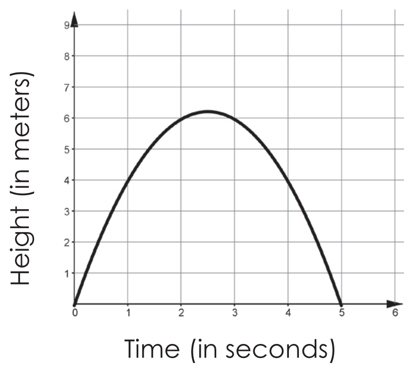
ZEROS = X-INTERCEPTS = SOLUTIONS

Try It!

What are the solutions to the quadratic equation graphed below?



Aaron shoots a water bottle rocket from the ground. A graph of height over time is shown below:



What type of function best models the rocket's motion?

After how many seconds did the rocket hit the ground?

Estimate the maximum height of the rocket.

This point is called the ***vertex***, the maximum or minimum point of a parabola.

BEAT THE TEST!

1. A ball is thrown straight up into the air.

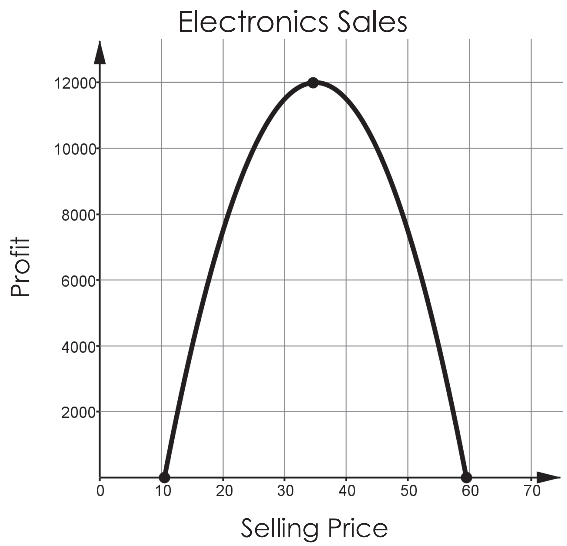
   Part A: What type of function best models the ball's height over time? What would it look like?

   Part B: You want to know how long it will take until the ball hits the ground. What part of the function's graph is needed to answer this question?

   Part C: You want to know when the ball will reach its maximum height. What part of the function's graph is needed to answer this question?

   Part D: What information do you need to graph the ball's height over time?

2. Jordan owns an electronics business. During her first year in the business, she collected data on different prices that yielded different profits. She used the data to create the following graph showing the relationship between the selling price of an item and the profit:



Part A: Circle the solutions to the quadratic function graphed above.

Part B: What do the solutions represent?

Part C: Box the vertex of the graph.

Part D: What does the vertex represent?

Section 6 – Video 2
Solving Quadratics Using the Quadratic Formula

How can we find the solutions when the quadratic is given in the form of an equation?

We can always use the ***quadratic formula***.

For any quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the quadratic formula:

1. Set the quadratic equation equal to zero.

2. Identify a, b, and c.

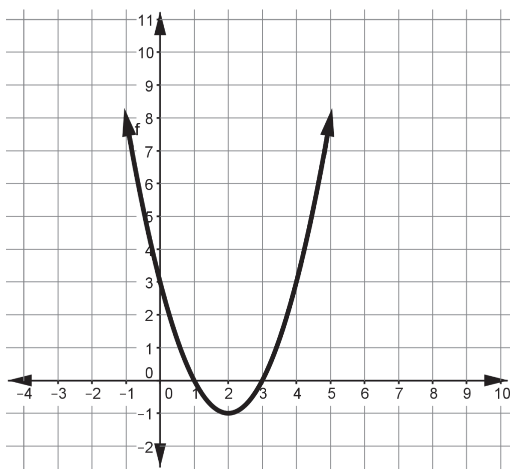
3. Substitute a, b, and c into the quadratic formula and evaluate to find the zeros.

Let's Practice!

Use the quadratic formula to solve $x^2 - 4x + 3 = 0$.

We can always verify our answers with a graph.

Consider the graph of the quadratic equation $x^2 - 4x + 3 = 0$.



Does the graph verify the solutions we found using the quadratic formula?

Use the quadratic formula to solve $2w^2 + w = 5$.

Try It!

Use the quadratic formula to solve the following quadratic equations.

$x^2 - x = 20$

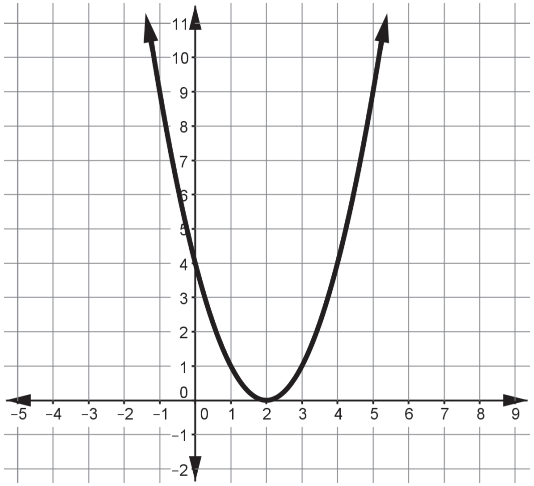
$3q^2 - 11 = 20q$

STUDY EDGE TIP

Quadratic equations can always be solved with the quadratic formula. There are other methods that work, but when in doubt, use the quadratic formula.

Let's use the quadratic formula to discuss the nature of the solutions.

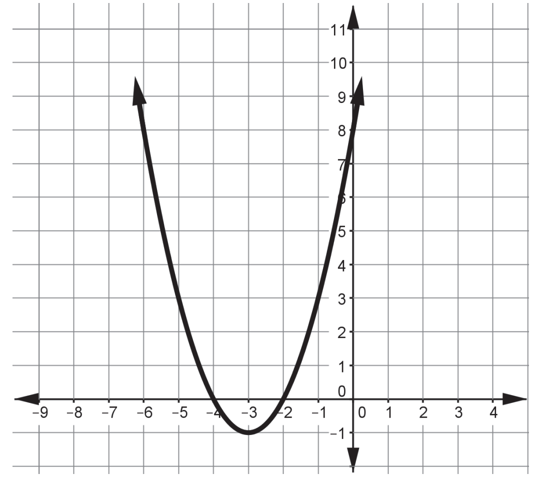
Consider the graph of the function $f(x) = x^2 - 4x + 4$.



Where does this parabola intersect the x-axis?

Use the quadratic formula to find the zero(s) of the function.

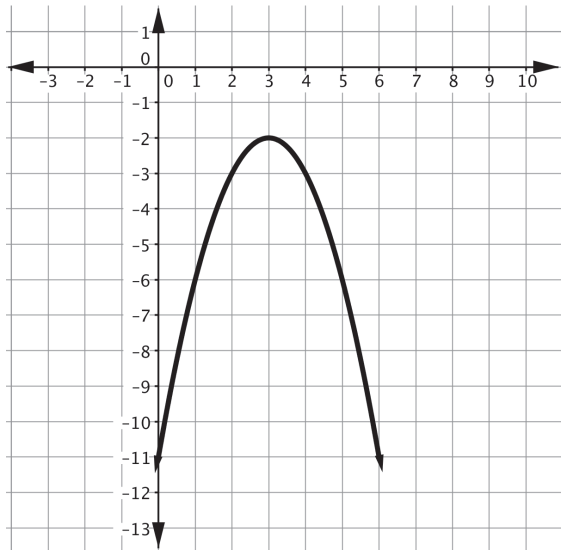
Consider the graph of the function $f(x) = x^2 + 6x + 8$.



Where does this parabola intersect the x-axis?

Use the quadratic formula to find the zero(s) of the function.

Consider the graph of the function $f(x) = -x^2 + 6x - 11$.



Where does this parabola intersect the x-axis?

Use the quadratic formula to find the zero(s) of the function.

STUDY EDGE TIP

When using the quadratic formula, if the discriminant of the quadratic (part under the radical) results in a negative number then solutions are non-real, complex solutions.

Try It!

Determine if the following quadratic equations have complex or real solution(s).

$2x^2 - 3x - 10 = 0$

$x^2 - 6x + 9 = 0$

$x^2 - 8x + 20 = 0$

BEAT THE TEST!

1. Which of the following quadratic equations have real solutions? Select all that apply.

   - ☐ $-3x^2 + 5x = 11$
   - ☐ $-x^2 - 12x + 6 = 0$
   - ☐ $2x^2 + x + 6 = 0$
   - ☐ $5x^2 - 10x = 3$
   - ☐ $x^2 - 2x = -8$

2. Your neighbor's garden measures 12 meters by 16 meters. He plans to install a pedestrian pathway all around it, increasing the total area to 285 square meters. The new area can be represented by $4w^2 + 56w + 192$. Use the quadratic formula to find the width, w, of the pathway.

   Part A: Write an equation that can be used to solve for the width of the pathway.

   Part B: Use the quadratic formula to solve for the width of the pathway.

Section 6 – Video 3
Factoring Quadratic Expressions

We can factor quadratic expressions by using the same distributive property that we practiced in Section 1.

Think back to the area model we used for the distributive property:

$$3(x + 2y - 7z)$$

3			
	x	$2y$	$-7z$

We can also use the area model with the distributive property to factor out the greatest common factor (GCF) of an expression.

$$10x^3 - 14x^2 + 12x$$

$10x^3$	$-14x^2$	$+12x$

Here is another multiplication problem we can express using the area model.

$$(2x + 5)(x + 3)$$

$2x$		
$+5$		
	x	$+3$

We can use this same model with the distributive property to factor a quadratic expression. Notice the following four patterns:

- The first term of the trinomial can always be found in the ____________ ____________ rectangle.
- The last term of the trinomial can always be found in the ____________ ____________ rectangle.
- The second term of the trinomial is the ____________ of the ____________ ____________ and ____________ ____________ rectangles.
- The ______________ of the ____________ is always equal.

  For example: $2x^2 \cdot 15 = 30x^2$
  $5x \cdot 6x = 30x^2$

We can use these four patterns to factor any quadratic expression.

Let's Practice!

Factor each quadratic expression.

$2x^2 + 3x - 5$

$3x^2 - 8x + 4$

STUDY EDGE TIP

You can check your answer to every factor by using the distributive property. The product of the factors should always result in the trinomial that you started with (your original problem).

Try It!

Factor each quadratic expression.

$a^2 + 11a + 24$

$4w^2 - 21w + 20$

BEAT THE TEST!

1. Identify all factors of the expression $18x^2 - 9x - 5$.

- ☐ $2x + 5$
- ☐ $6x - 5$
- ☐ $18x - 5$
- ☐ $3x + 5$
- ☐ $3x + 1$

Section 6 – Video 4
Solving Quadratics by Factoring – Part 1

Solve a quadratic equation by factoring:

- Once a quadratic equation is factored, we can use the ***zero product property*** to solve the equation.
- The zero product property states that if the product of two factors is zero, then one (or both) of the factors must be _________________
  - If $ab = 0$, then either $a = 0$, $b = 0$, or $a = b = 0$.

To solve a quadratic equation by factoring:

1. Set the equation equal to zero.
2. Factor the quadratic.
3. Set each factor equal to zero and solve.
4. Write the solution set.

Let's Practice!

Solve for b by factoring $b^2 + 8b + 15 = 0$.

Solve for f by factoring $10f^2 + 17f + 3 = 0$.

Try it!

Solve each quadratic equation by factoring.

$x^2 - 11x + 18 = 0$

$6j^2 - 19j + 14 = 0$

BEAT THE TEST!

1. Which of the following quadratic equations has the solution set $\left\{-\frac{5}{3}, 6\right\}$?

- ☐ $(x-6)(3x+5)=0$
- ☐ $(3x-5)(x-6)=0$
- ☐ $(5x+3)(x-6)=0$
- ☐ $(5x+3)(x+6)=0$
- ☐ $(3x+5)(2x-12)=0$
- ☐ $(-3x+5)(x-6)=0$

2. Tyra solved the quadratic equation $x^2-10x-24=0$ by factoring. Her work is shown below:

Step 1: $x^2-10x-24=0$
Step 2: $x^2-4x-6x-24=0$
Step 3: $(x^2-4x)+(-6x-24)=0$
Step 4: $x(x-4)-6(x-4)=0$
Step 5: $(x-4)(x-6)=0$
Step 6: $x-4=0, x-6=0$
Step 7: $x=4$ or $x=6$
Step 8: $\{4, 6\}$

Tyra did not find the correct solutions. Identify the step(s) where she made mistakes and explain how to correct Tyra's work.

Section 6 – Video 5
Solving Quadratics by Factoring – Part 2

Many quadratic equations will not be in standard form:

- The equation won't always equal zero.
- There may be a greatest common factor (GCF) within all of the terms.

Solve for p: $p^2 + 36 = 13p$

Solve for m: $3m^2 + 30m - 168 = 0$

Solve for x: $(x + 4)(x - 5) = -8$

STUDY EDGE TIP

When solving a quadratic, if the quadratic is factored but not equal to zero, then you've got some work to do!

Try It!

Solve for d: $6d^2 + 5d = 1$

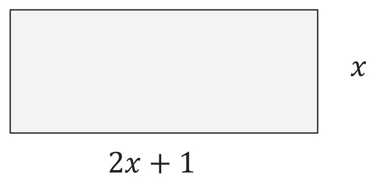
Solve for y: $200y^2 = 900y - 1000$

BEAT THE TEST!

1. What are the solutions to $40x^2 - 30x = 135$? Select all that apply.

- ☐ $-\frac{9}{2}$
- ☐ $-\frac{9}{4}$
- ☐ $-\frac{3}{2}$
- ☐ $-\frac{3}{4}$
- ☐ $\frac{3}{4}$
- ☐ $\frac{3}{2}$
- ☐ $\frac{9}{4}$

2. The area of the rectangle is 105 square units.



What is the value of x?

Section 6 – Video 6
Solving Quadratics by Factoring – Special Cases

There are a few special cases when solving quadratics by factoring.

Perfect Square Trinomials

- $x^2 + 6x + 9$ is an example of ***perfect square trinomial***. We see this when we factor.

- A perfect square trinomial is created when you square a ________________________.

Recognizing a Perfect Square Trinomial

A quadratic expression can be factored as a perfect square trinomial if it can be re-written in the form $a^2 + 2ab + b^2$.

Factoring a Perfect Square Trinomial

- If $a^2 + 2ab + b^2$ is a perfect square trinomial, then $a^2 + 2ab + b^2 = (a + b)^2$.
- If $a^2 - 2ab + b^2$ is a perfect square trinomial, then $a^2 - 2ab + b^2 = (a - b)^2$.

Let's Practice!

Determine whether the following expressions are perfect square trinomials.

$16x^2 + 44x + 121$

$x^2 - 8x + 64$

Solve for q: $q^2 - 10q + 25 = 0$

Solve for w: $4w^2 + 49 = -28w$

What do you notice about the number of solutions to perfect square quadratic equations?

Sketch the graph of a quadratic equation that is a perfect square trinomial.

Try It!

Solve for x: $x^2 - 16 = 0$

Difference of Squares

Use the distributive property to multiply the following binomials.

$(x+5)(x-5)$

$(2x+7)(2x-7)$

$(5x+1)(5x-1)$

Describe any patterns you notice.

- When we have a binomial in the form $a^2 - b^2$, it is called the ***difference of two squares***. We can factor this as $(a+b)(a-b)$.

Let's Practice!

Solve the equation $49k^2 = 64$.

Try It!

Solve each quadratic equation.

$0 = 121p^2 - 100$

$p^2 - 144 = 0$

BEAT THE TEST!

1. Which of the following expressions are equivalent to $8a^3 - 98a$? Select all that apply.

   - ☐ $2(4a^3 - 49a)$
   - ☐ $2a(4a^2 - 49)$
   - ☐ $2a(4a^3 - 49a)$
   - ☐ $(2a - 7)(2a + 7)$
   - ☐ $2(2a - 7)(2a + 7)$
   - ☐ $2a(2a - 7)(2a + 7)$

2. A bird flies to the ground and lands, only to be scared away by a cat. The bird's motion can be described by the equation $h(t) = 4t^2 - 20t + 25$, where t represents time in seconds and $h(t)$ represents the bird's height.

   Part A: At what time will the bird land on the ground?

   Part B: What ordered pair represents your solution?

Section 6 – Video 7
Solving Quadratics by Taking Square Roots

How would you solve a quadratic equation like the one below?

$$2x^2 - 36 = 0$$

When quadratic equations are in the form $ax^2 + c = 0$, solve by taking the square root.

1. Get the variable on the left and the constant on the right.

2. Then take the square root of both sides of the equation.

   ➢ Don't forget the negative root!

Solve for x by taking the square root.

$2x^2 - 36 = 0$

Try It!

Solve $x^2 - 121 = 0$.

Solve $-5x^2 + 80 = 0$.

BEAT THE TEST!

1. What is the smallest solution to the equation $2x^2 + 17 = 179$?

   Ⓐ -9
   Ⓑ -3
   Ⓒ 3
   Ⓓ 9

2. A rescuer on a helicopter that is 50 feet above the sea drops a lifebelt. The distance from the lifebelt to the sea can be modeled by the equation $h(t) = -16t^2 + s$, where $h(t)$ represents the lifebelt's height from the sea at any given time, t is the time in seconds, and s is the initial height from the sea, in feet.

   How long will it take for the lifebelt to reach the sea? Round your answer to the nearest tenth of a second.

   ☐ seconds

Section 6 – Video 8
Solving Quadratics by Completing the Square

Sometimes, you won't be able to solve a quadratic equation by factoring. However, you can rewrite the quadratic equation so that you can ***complete the square*** to factor and solve.

What value could be added to the quadratic to make it a perfect square trinomial?

$x^2 + 6x +$ ______

$x^2 - 30x +$ ______

$x^2 + 8x + 3 +$ ______

$x^2 - 4x + 57 +$ ______

Try It!

What value could be added to the quadratic to make it a perfect square trinomial?

$x^2 - 14x +$ ______

$x^2 + 18x +$ ______

$x^2 - 22x + 71 +$ ______

$x^2 - 10x + 32 +$ ______

Let's see how this can be used to solve quadratic equations.

Recall from a previous video how we factored perfect square trinomials. If $ax^2 + bx + c$ is a perfect square trinomial, then $ax^2 + bx + c = \left(\sqrt{a}x + \sqrt{c}\right)^2$ and $ax^2 - bx + c = \left(\sqrt{a}x - \sqrt{c}\right)^2$.

$x^2 + 6x = 0$

$x^2 + 6x + \_\_\_\_\_\_ = 0 + \_\_\_\_\_\_$

$3x^2 + 6x = 0$

$2x^2 - 6x = -6$

Try It!

Complete the square to solve the following equations.

$x^2 - 16x = 0$

$-5x^2 + 30x = 0$

$2x^2 + 4x = -3$

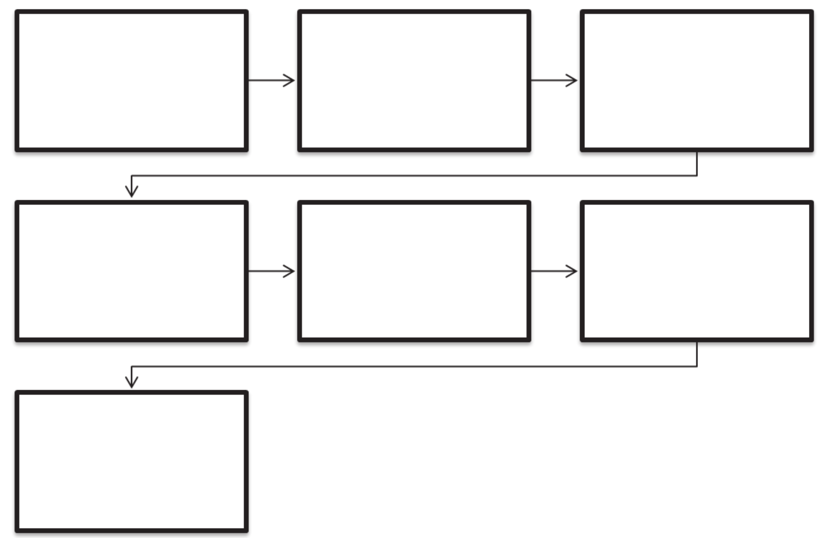
To summarize, here are the steps for solving a quadratic by completing the square:

1. Write the equation in standard form.
2. If a does not equal one, divide every term in the equation by a.
3. Subtract c from both sides.
4. Divide b by two and square the result. Add this value to both sides of the equation to create a perfect square trinomial.
5. Rewrite the equation as a perfect square trinomial.
6. Factor the trinomial.
7. Take the square root of both sides.
8. Solve for x.

BEAT THE TEST!

1. Demonstrate how to solve $2x^2 + 24x - 29 = 0$ by completing the square. Place the equations in the correct order.

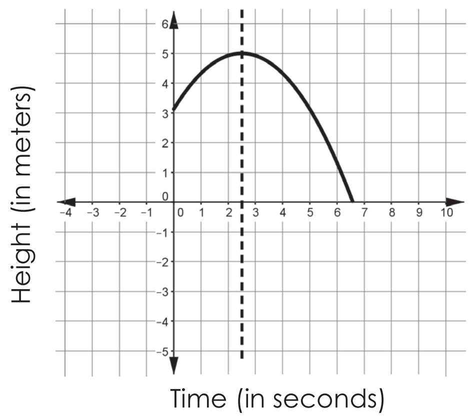
A) $x^2 + 12x + 36 = 14.5 + 36$	E) $(x + 6)^2 = 50.5$
B) $x + 6 = \pm\sqrt{50.5}$	F) $x^2 + 12x - 14.5 = 0$
C) $x^2 + 12x = 14.5$	G) $\sqrt{(x + 6)^2} = \pm\sqrt{50.5}$
D) $x = -6 \pm \sqrt{50.5}$	



Section 6 – Video 9
Quadratics in Action

Let's consider solving some real-world situations that involve quadratic functions.

Consider an object being launched into the air. We compare the height versus time elapsed. Consider these questions:



- *From what height was the object launched?*
- *How long does it take the object to reach its maximum height?*
- *What is the maximum height?*
- *How long does it take until the object hits the ground?*
- *At what time will the object reach a certain height or how high will the object be after a certain time?*

	Question	How to Answer it
1.	From what height was the object launched?	This is typically the y-intercept. In the standard form, $ax^2 + bx + c$, c is the y-intercept.
2.	How long did it take the object to reach its maximum height?	This is the x-coordinate of the vertex, $x = \frac{-b}{2a}$, where values of a and b come from the standard form of a quadratic equation. $x = \frac{-b}{2a}$ is also the equation that represents the axis of symmetry.
3.	What was the maximum height?	This is the y-coordinate of the vertex. Plug in the x-coordinate from the step above and evaluate to find y. In vertex form, the height is k and the vertex is (h, k).
4.	At what time(s) was the object on the ground?	The x-intercept(s) are the solution(s), or zero(s), of the quadratic function. Solve by factoring, using the quadratic formula, or by completing the square. In a graph, look at the x-intercept(s).
5.	At what time did the object reach a certain height or how high was the object after a certain time?	In function $H(t) = at^2 + bt + c$, if height is given, then substitute the value for $H(t)$. If time is given, then substitute for t.

Let's Practice!

An athlete throws a javelin. The javelin's height above the ground, in feet, after it has traveled a horizontal distance for t seconds is given by the equation:

$$h(t) = -0.08t^2 + 0.64t + 5.15$$

What would the graph of $h(t)$ versus t look like?

From what height was the javelin thrown?

When did the javelin reach its maximum height?

What is the maximum height reached by the javelin?

How high is the javelin three seconds after it was launched?

When is the javelin three feet above the ground?

How long does it take until the javelin hits the ground?

Try It!

Ferdinand is playing golf. He hits a shot off the tee box that has a height modeled by the function $h(t) = -16t^2 + 80t$, where $h(t)$ is the height of the ball, in feet, and t is the time in seconds it has been in the air.

What would the graph of $h(t)$ versus t look like?

Why is the y-intercept at the origin?

When does the ball reach its maximum height? What is the maximum height of the ball?

What is the height of the ball at 3.5 seconds? When is the ball at the same height?

When is the ball 65 feet in the air? Explain.

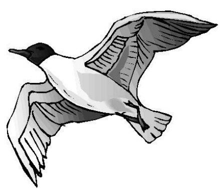
How long does it take until the golf ball hits the ground?

Suppose Ferdinand hits a second ball from a tee box that was elevated eight feet above the fairway. What effect does this have on the function? Write a function that describes the new path of the ball. Compare and contrast both functions.

These are called "U-shaped parabolas." In these parabolas, look for minimums rather than maximums.

Let's Practice!

The height of a seagull over time as it bobs up and down over the ocean has a shape, or trajectory, of a parabola or multiple parabolas.



The Pak family was enjoying a great day at the beach. At lunchtime, they took out food. A seagull swooped down, grabbed some of the food, and flew back up again. Its height above the ground, in meters, after it has traveled a horizontal distance for *t* seconds is given by the function:

$$h(t) = (t-7)^2$$

Describe the graph of height versus time.

How high was the seagull flying before he dove down to take the Pak family's food?

What is the minimum height of the seagull? How much time did it take the seagull to dive down for the food?

What is the height of the seagull after nine seconds? Describe the scenario.

BEAT THE TEST!

1. Baymeadows Pointe is throwing a huge fireworks celebration for the 4th of July. The president of the neighborhood association launched a bottle rocket upward from the ground with an initial velocity of 160 feet per second. Consider the formula for vertical motion of an object: $h(t) = 0.5at^2 + vt + s$, where the gravitational constant, a, is -32 feet per square second, v is the initial velocity, s is the initial height, and $h(t)$ is the height in feet modeled as a function of time, t.

   Part A: What function describes the height, h, of the bottle rocket after t seconds have elapsed?

   Part B: Sketch a graph of the height of the bottle rocket as a function of time, and give a written description of the graph.

   Part C: What is the bottle rocket's maximum height?

   Part D: What is the height of the bottle rocket after three seconds? When is it at this height again?

   Part E: Suppose the bottle rocket is launched from the top of a 200-foot-tall building. How does this change the height versus time function for the bottle rocket? What does the new graph tell you about the situation?