SECTION 9.2: ARITHMETIC SEQUENCES and PARTIAL SUMS

PART A: WHAT IS AN ARITHMETIC SEQUENCE?

The following appears to be an example of an arithmetic (stress on the "me") sequence:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 5 \\ a_3 &= 8 \\ a_4 &= 11 \\ &\vdots \end{aligned}$$

We begin with 2. After that, we successively **add** 3 to obtain the other terms of the sequence.

An arithmetic sequence is determined by:

• Its initial term

Here, it is a_1, although, in other examples, it could be a_0 or something else. Here, $a_1 = 2$.

• Its common difference

This is denoted by d. It is the number that is always added to a previous term to obtain the following term. Here, $d = 3$.

Observe that: $d = a_2 - a_1 = a_3 - a_2 = \ldots = a_{k+1} - a_k \left(k \in \mathbf{Z}^+ \right) = \ldots$

The following information completely determines our sequence:

The sequence is arithmetic.
(Initial term) $a_1 = 2$
(Common difference) $d = 3$

In general, a recursive definition for an arithmetic sequence that begins with a_1 may be given by:

$$\begin{cases} a_1 \text{ given} \\ a_{k+1} = a_k + d \quad \left(k \geq 1; \text{"}k \text{ is an integer" is implied}\right) \end{cases}$$

<u>Example</u>

The arithmetic sequence 25, 20, 15, 10, …
can be described by:

$$\begin{cases} a_1 = 25 \\ d = -5 \end{cases}$$

PART B : FORMULA FOR THE GENERAL n^{th} TERM OF AN ARITHMETIC SEQUENCE

Let's begin with a_1 and keep adding d until we obtain an expression for a_n, where $n \in \mathbf{Z}^+$.

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 + d \\ a_3 &= a_1 + 2d \\ a_4 &= a_1 + 3d \\ &\vdots \\ a_n &= a_1 + (n-1)d \end{aligned}$$

The general n^{th} term of an arithmetic sequence with initial term a_1 and common difference d is given by:

$$a_n = a_1 + (n-1)d$$

Think: We take $n-1$ steps of size d to get from a_1 to a_n.

Note: Observe that the expression for a_n is linear in *n*. This reflects the fact that arithmetic sequences often arise from linear models.

Example

Find the 100th term of the arithmetic sequence: 2, 5, 8, 11, …
(Assume that 2 is the "first term.")

Solution

$$\begin{aligned} a_n &= a_1 + (n-1)d \\ a_{100} &= 2 + (100-1)(3) \\ &= 2 + (99)(3) \\ &= 299 \end{aligned}$$

PART C : FORMULA FOR THE n^{th} PARTIAL SUM OF AN ARITHMETIC SEQUENCE

The n^{th} partial sum of an arithmetic sequence with initial term a_1 and common difference d is given by:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Think: The (cumulative) sum of the first n terms of an arithmetic sequence is given by the number of terms involved times the average of the first and last terms.

Example

Find the 100^{th} partial sum of the arithmetic sequence: 2, 5, 8, 11, …

Solution

We found in the previous Example that: $a_{100} = 299$

$$\begin{aligned} S_n &= n\left(\frac{a_1 + a_n}{2}\right) \\ S_{100} &= (100)\left(\frac{2 + 299}{2}\right) \\ &= (100)\left(\frac{301}{2}\right) \\ &= 15{,}050 \end{aligned}$$

i.e., $2 + 5 + 8 + \ldots + 299 = 15{,}050$

This is much easier than doing things brute force on your calculator!

Read the Historical Note on p.628 in Larson for the story of how Gauss quickly computed the sum of the first 100 positive integers, $\sum_{k=1}^{100} k = 1 + 2 + 3 + \ldots + 100$. Use our formula to confirm his result. Gauss's trick is actually used in the proof of our formula; see p.694 in Larson. We will touch on a related question in Section 9.4.

SECTION 9.3: GEOMETRIC SEQUENCES, PARTIAL SUMS, and SERIES

PART A: WHAT IS A GEOMETRIC SEQUENCE?

The following appears to be an example of a geometric sequence:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 6 \\ a_3 &= 18 \\ a_4 &= 54 \\ &\vdots \end{aligned}$$

We begin with 2. After that, we successively **multiply** by 3 to obtain the other terms of the sequence. Recall that, for an arithmetic sequence, we successively **add**.

A geometric sequence is determined by:

• Its initial term

Here, it is a_1, although, in other examples, it could be a_0 or something else. Here, $a_1 = 2$.

• Its common ratio

This is denoted by r. It is the number that we always multiply the previous term by to obtain the following term. Here, $r = 3$.

Observe that: $r = \dfrac{a_2}{a_1} = \dfrac{a_3}{a_2} = \ldots = \dfrac{a_{k+1}}{a_k} \left(k \in \mathbf{Z}^+\right) = \ldots$

The following information completely determines our sequence:

The sequence is geometric.
(Initial term) $a_1 = 2$
(Common ratio) $r = 3$

In general, a recursive definition for a geometric sequence that begins with a_1 may be given by:

$$\begin{cases} a_1 \text{ given} \\ a_{k+1} = a_k \cdot r \quad \left(k \geq 1; \text{"}k \text{ is an integer" is implied}\right) \end{cases}$$

We assume $a_1 \neq 0$ and $r \neq 0$.

<u>Example</u>

The geometric sequence 2, 6, 18, 54, …
can be described by:

$$\begin{cases} a_1 = 2 \\ r = 3 \end{cases}$$

PART B : FORMULA FOR THE GENERAL n^{th} TERM OF A GEOMETRIC SEQUENCE

Let's begin with a_1 and keep multiplying by r until we obtain an expression for a_n, where $n \in \mathbf{Z}^+$.

$$\begin{aligned} a_1 &= a_1 \\ a_2 &= a_1 \cdot r \\ a_3 &= a_1 \cdot r^2 \\ a_4 &= a_1 \cdot r^3 \\ &\vdots \\ a_n &= a_1 \cdot r^{n-1} \end{aligned}$$

The general n^{th} term of a geometric sequence with initial term a_1 and common ratio r is given by:

$$a_n = a_1 \cdot r^{n-1}$$

Think: As with arithmetic sequences, we take $n-1$ steps to get from a_1 to a_n.

Note: Observe that the expression for a_n is exponential in *n*. This reflects the fact that geometric sequences often arise from exponential models, for example those involving compound interest or population growth.

Example

Find the 6^{th} term of the geometric sequence: $2,\ -1,\ \frac{1}{2},\ \ldots$
(Assume that 2 is the "first term.")

Solution

Here, $a_1 = 2$ and $r = -\frac{1}{2}$.

$$\begin{aligned} a_n &= a_1 \cdot r^{n-1} \\ a_6 &= (2)\left(-\frac{1}{2}\right)^{6-1} \\ &= (2)\left(-\frac{1}{2}\right)^{5} \\ &= (2)\left(-\frac{1}{32}\right) \\ &= -\frac{1}{16} \end{aligned}$$

Observe that, as $n \to \infty$, the terms of this sequence approach 0.

$$\boxed{\text{Assume } a_1 \neq 0\text{. Then, } \left(a_1 \cdot r^{n-1} \to 0 \text{ as } n \to \infty\right) \Leftrightarrow \underbrace{\left(-1 < r < 1\right)}_{\text{i.e., } |r|<1}}$$

PART C : FORMULA FOR THE n^{th} PARTIAL SUM OF A GEOMETRIC SEQUENCE

The n^{th} partial sum of a geometric sequence with initial term a_1 and common ratio r (where $r \neq 1$) is given by:

$$S_n = \frac{a_1 - a_1 r^n}{1-r} \quad \text{or} \quad a_1\left(\frac{1-r^n}{1-r}\right)$$

You should get used to summation notation:

Remember that S_n for a sequence starting with a_1 is given by:

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

Because $a_k = a_1 \cdot r^{k-1}$ for our geometric series:

$$S_n = \sum_{k=1}^{n} a_1 r^{k-1} = a_1 + \underbrace{a_1 r}_{a_2} + \underbrace{a_1 r^2}_{a_3} + \ldots + \underbrace{a_1 r^{n-1}}_{a_n}$$

$$= \frac{a_1 - a_1 r^n}{1-r} \quad \text{(according to our theorem in the box above)}$$

Note: The book Concrete Mathematics by Graham, Knuth, and Patashnik suggests a way to remember the numerator: "first in – first out." This is because a_1 is the "first" term **included** in the sum, while $a_1 r^n$ is the first term in the corresponding infinite geometric series that is **excluded** from the sum.

Technical Note: The key is that $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$. You can see that this is true by multiplying both sides by $(1-r)$. Also see the proof on p.694 of Larson.

Technical Note: If $r = 1$, then we are dealing with a constant sequence and essentially a multiplication problem. For example, the 4^{th} partial sum of the series $7+7+7+7+\ldots$ is $7+7+7+7 = (4)(7) = 28$. In general, the n^{th} partial sum of the series $a_1 + a_1 + a_1 + \ldots$ is given by na_1.

Example

Find the 6th partial sum of the geometric sequence $2,\ -1,\ \frac{1}{2},\ \ldots$

Solution

Recall that $a_1 = 2$ and $r = -\frac{1}{2}$ for this sequence.

We found in the previous Example that: $a_6 = -\frac{1}{16}$

We will use our formula to evaluate:

$$S_6 = 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$$

Using our formula directly:

$$S_n = \frac{a_1 - a_1 r^n}{1-r} \quad \text{or} \quad a_1\left(\frac{1-r^n}{1-r}\right)$$

If we use the second version on the right …

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

$$S_6 = 2\left(\frac{1-\left(-\frac{1}{2}\right)^6}{1-\left(-\frac{1}{2}\right)}\right)$$

$$= 2\left(\frac{1-\frac{1}{64}}{1+\frac{1}{2}}\right)$$

$$= 2\left(\frac{\frac{63}{64}}{\frac{3}{2}}\right)$$

$$= 2\left(\frac{{}^{21}\cancel{63}}{{}_{32}\cancel{64}} \cdot \frac{\cancel{2}^{1}}{\cancel{3}_{1}}\right)$$

$$= \cancel{2}\left(\frac{21}{\cancel{32}_{16}}\right)$$

$$= \frac{21}{16}$$

We can also use the first version and the "first in – first out" idea:

$$S_6 = 2 - 1 + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}$$

"First out" is: $a_7 = \dfrac{1}{32}$

$$S_n = \frac{a_1 - a_1 r^n}{1 - r}$$

$$S_6 = \frac{2 - \frac{1}{32}}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{\frac{63}{32}}{\frac{3}{2}} \qquad \left(\leftarrow \frac{64}{32} - \frac{1}{32}\right)$$

$$= \frac{{}^{21}\cancel{63}}{{}_{16}\cancel{32}} \cdot \frac{\cancel{2}^{1}}{\cancel{3}_{1}}$$

$$= \frac{21}{16}$$

PART D: INFINITE GEOMETRIC SERIES

An infinite series <u>converges</u> (i.e., has a <u>sum</u>) $\Leftrightarrow$ The S_n partial sums approach a real number $\left(\text{as } n \to \infty\right)$, which is then called the sum of the series.

In other words, if $\lim_{n\to\infty} S_n = S$, where S is a real number, then S is the sum of the series.

<u>Example</u>

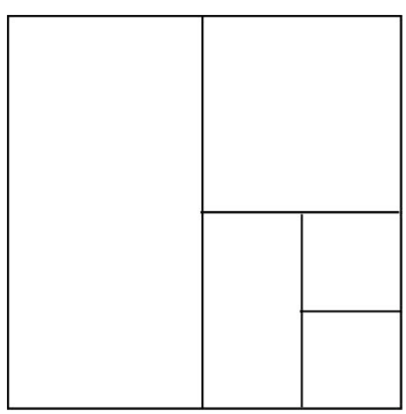
Consider the geometric series: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$

Let's take a look at the partial (cumulative) sums:

$$\underbrace{\underbrace{\underbrace{\underbrace{\frac{1}{2}}_{S_1=\frac{1}{2}} + \frac{1}{4}}_{S_2=\frac{3}{4}} + \frac{1}{8}}_{S_3=\frac{7}{8}} + \frac{1}{16}}_{S_4=\frac{15}{16}} + ...$$

It appears that the partial sums are approaching 1. In fact, they are; we will have a formula for this. This series has a sum, and it is 1.

The figure below may make you a believer:



<u>Example</u>

The geometric series $2+6+18+54+\ldots$ has no sum, because: $\lim_{n\to\infty} S_n = \infty$

<u>Example</u>

The geometric series $1-1+1-1+\ldots$ has no sum, because the partial sums do not approach a single real number. Observe:

$$\underbrace{\underbrace{\underbrace{\underbrace{1}_{S_1=1} - 1}_{S_2=0} + 1}_{S_3=1} - 1}_{S_4=0} + \ldots$$

An infinite geometric series converges $\Leftrightarrow \underbrace{(-1<r<1)}_{\text{i.e., } \lvert r \rvert < 1}$

Take another look at the Examples of this Part.

It is true that an infinite **geometric** series converges $\Leftrightarrow$ Its terms approach 0.

<u>Warning</u>: However, this cannot be said about series in general. For example, the famous <u>harmonic series</u> $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$ does **not** converge, even though the terms of the series approach 0. In order for a series to converge, it is necessary **but not sufficient** for the terms to approach 0.

No infinite arithmetic sequence (such as $2+5+8+11+\ldots$) can have a sum, unless you include $0+0+0+\ldots$ as an arithmetic sequence.

The sum of a convergent infinite geometric series with initial term a_1 and common ratio r, where $\underbrace{-1 < r < 1}_{\text{i.e., } |r|<1}$, is given by:

$$S = \frac{a_1}{1-r}$$

Technical Note: This comes from our partial sum formula $S_n = \dfrac{a_1 - a_1 r^n}{1-r}$ and the fact that $\left(a_1 r^n \to 0 \text{ as } n \to 0\right)$ if $\underbrace{-1 < r < 1}_{\text{i.e., } |r|<1}$.

Example

Write $0.\overline{81}$ as a nice (simplified) fraction of the form $\dfrac{\text{integer}}{\text{integer}}$.

Recall how the repeating bar works: $0.\overline{81} = 0.81818181...$

Note: In Arithmetic, you learned how to use long division to express a "nice" fraction as a repeated decimal; remember that rational numbers can always be expressed as either a terminating or a "nicely" repeating decimal. Now, after all this time, you will learn how to do the reverse!

Solution

$0.\overline{81}$ can be written as: $0.81 + 0.0081 + 0.000081 + ...$

Observe that this is a geometric series with initial term $a_1 = 0.81$ and common ratio $r = \dfrac{0.0081}{0.81} = \dfrac{1}{100} = 0.01$; because $|r| < 1$, the series converges.

The sum of the series is given by:

$$S = \frac{a_1}{1-r} = \frac{0.81}{1-0.01} = \frac{0.81}{0.99} = \frac{81}{99} = \mathbf{\frac{9}{11}}$$

Again, you should get used to summation notation:

$$\begin{aligned} S &= \sum_{k=1}^{\infty} a_1 r^{k-1} \\ &= \sum_{k=1}^{\infty} (0.81)(0.01)^{k-1} \\ &= \frac{9}{11} \end{aligned}$$

If you make the substitution $i = k - 1$, the summation form can be rewritten as:

$$\begin{aligned} S &= \sum_{k=1}^{\infty} (0.81)(0.01)^{k-1} \\ &= \sum_{i=0}^{\infty} (0.81)(0.01)^{i} \end{aligned}$$

In Calculus, 0 is more common than 1 as a lower limit of summation.