Section 9: Exponential and Logarithmic Functions

The following Mathematics Florida Standards will be covered in this section:	
MAFS.912.A-CED.1.1	Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions and simple rational, absolute, and exponential functions.
MAFS.912.A-CED.1.2	Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
MAFS.912.A-CED.1.3	Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or nonviable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.
MAFS.912.F-BF.2.3	Identify the effect on the graph of replacing $f(x)$ by $f(x)+k$, $kf(x)$, $f(kx)$, and $f(x+k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.
MAFS.912.F-BF.2.4.a	Find inverse functions. a. Solve an equation of the form $f(x) = c$ for a simple function, f, that has an inverse and write an expression for the inverse. For example, $f(x) = 2x3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.
MAFS.912.F-BF.2.a	Use the change of base formula.
MAFS.912.F-IF.2.4	For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.
MAFS.912.F-IF.3.7.d.e	Graph functions expressed symbolically and show key features of the graph by hand in simple cases and using technology for more complicated cases. d. Graph rational functions, identifying zeros and asymptotes when suitable factorizations are available and showing end behavior. e. Graph exponential and logarithmic functions, showing intercepts and end behavior, and trigonometric functions, showing period, midline, and amplitude and using phase shift.
MAFS.912.F-IF.3.8.b	Write a function defined by an expression in different but equivalent

	forms to reveal and explain different properties of the function. b. Use the properties of exponents to interpret expressions for exponential functions.
MAFS.912.A-REI.4.11	Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately (e.g., using technology to graph the functions, make tables of values, or find successive approximations).
MAFS.912.F-LE.1.4	For exponential models, express as a logarithm the solution to $ab^{ct} = d$, where a, c, and d are numbers and the base, b, is 2, 10, or e; evaluate the logarithm using technology.
MAFS.912.F-LE.2.5	Interpret the parameters in a linear or an exponential function in terms of a context.
MAFS.912.A-SSE.1.1.b	Interpret expressions that represent a quantity in terms of its context. b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1 + r)^n$ as the product of P and a factor not depending on P.
MAFS.912.A-SSE.2.3.c	Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression. c. Use the properties of exponents to transform expressions for exponential functions.

Topics in this Section

Section 9 – Topic 1
Real-World Exponential Growth and Decay – Part 1

Linear functions have a constant rate of change. We say that linear functions increase __________.

Exponential functions increase by a common ratio. We say that they increase ___________.

Exponential functions can model exponential __________ or exponential _________.

Consider the following table that models an exponential growth of the money in a bond fund.

Bond Fund	
Year	Amount
0	1500
1	1593
2	1692
3	1797
4	1908
5	2026

What is the starting amount of money in the fund?

What is the ratio that the amount in the fund is increasing by?

Consider the following table that models the exponential decay of the fish populations of Lake Placid.

Lake Placid	
Year	Number of Fish
0	14204
1	13494
2	12819
3	12178
4	11569

What is the beginning population of the fish in Lake Placid?

What is the ratio that the population of fish is decreasing by?

We can use the function $f(x) = a \cdot b^x$ to write the equations that model these examples of exponential growth or decay.

In the equation, a represents the ___________ amount.

In cases of exponential growth, the variable b is equal to $1 +$ (rate of ______________).

In cases of exponential decay, the variable b is equal to $1 -$ (rate of ______________).

Let's Practice!

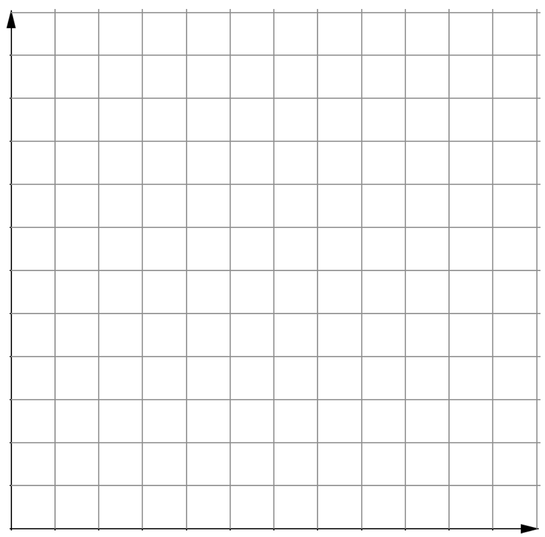
1. Recall our bond fund where the rate of increase was 6.2% and the initial amount was $\$1500$.

   a. Write the equation that represents the exponential growth of the bond function.

   b. How much money would be in the account at the end of 10 years?

   c. How much money would be in the account at the end of 20 years?

   d. Sketch the graph of the exponential growth of the money in the bond fund.



Try It!

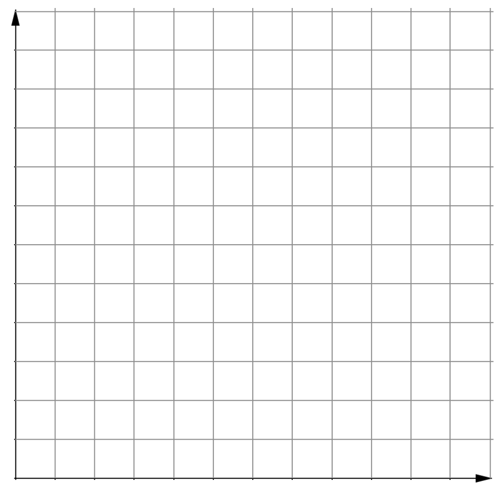
2. Recall our fish population with a rate of decrease of 5% and an initial population of 14204.

   a. Write the equation that represents the exponential decay of the fish population.

   b. What is the fish population at the end of 10 years?

   c. What is the fish population at the end of 20 years?

   d. Sketch the graph of the exponential decay of the fish population.



Section 9 – Topic 2
Real-World Exponential Growth and Decay – Part 2

Let's Practice!

1. The rabbit population in Central Park was 150 in the year 2000. The population is increasing by 11% each year.

   Consider the function that represents the exponential growth of the rabbit population.

   a. Define a variable for the function and state what the variable represents.

   b. What is a reasonable domain for the situation?

   c. Write the function that represents the exponential growth of the rabbit population.

   d. What will the rabbit population be in 2025?

Try It!

2. A new Honda Civic cost $\$24{,}500$ and loses 9% of its value the moment you drive it off the lot after a purchase. Over the next four years, the Civic will depreciate 10.5% each year. After four years, the car is valued at approximately 58% of its original cost.

   Consider the function that represents this situation.

   a. Define a variable for the function and state what the variable represents.

   b. What domain best fits this situation?

   c. Write a function to represent the situation.

   d. Use the function to prove that after four years the car is valued at approximately 58% of its original cost.

BEAT THE TEST!

1. Radium's most stable isotope, radium-226, has a half-life of approximately 1600 years. The half-life of a substance is the amount of years it takes for half of the substance to decay. In 2005, 40 grams of radium were stored. In six half-lives, there will be less than one gram of radium remaining.

   Part A: Select the appropriate definition for the variable x for the equation that models the amount of remaining radium, R.

   ☐ x is the number of half-lives that the radium has been stored.
   ☐ x is the number of years it takes for there to be less than one gram of radium remaining.

   Part B: Select the most appropriate domain for the equation that models the amount of remaining radium, R.

   ☐ $\{x|x \in \mathbb{R}\}$
   ☐ $\{x|x \geq 0\}$
   ☐ $\{x|0 \leq x \leq 6\}$

   Part C: Select the equation that can be used to model the amount of remaining radium, R.

   ☐ $R = 40(1-0.5)^x$
   ☐ $R = 40(1-0.5)^6$
   ☐ $R = 40(1+0.5)^x$

Section 9 – Topic 3
Interpreting Exponential Equations

A common occurrence of exponential functions in the real world is compound interest.

Compound interest is _______________ added to the principal of a deposit. This means that the interest also earns interest from that point forward. We call this __________________ interest.

Consider the following exponential equation that represents future value of an investment that is compounded yearly.

$$F = P(1+r)^t$$

This is an example of exponential ______________.

F represents the __________ value of the investment or loan.

P represents the ____________ amount invested or borrowed.

r represents ____________ of interest (in decimal form).

t represents time in ____________.

Let's Practice!

1. Consider the expression $(1.039)^{2t}$ that represents the interest earned on a bond fund, where t is time in years.

   a. Estimate the yearly interest rate.

   b. Estimate the monthly interest rate.

   c. If a client invested $\$1500$ and the interest was calculated quarterly, write a function that could be used to determine the amount of money in the account over time.

Try It!

2. The expression $(1.042)^{5t}$ represents the interest charged by a credit card company, where t is time in years.

   a. Find the yearly rate of interest.

   b. Find the daily rate of interest.

   c. Assume that the interest is compounded daily. Ali charges $\$5000$ on the credit card. Write a function that can be used to determine the balance Ali owes until she begins paying down the charges.

   d. The company generously offers customers three months with no payment due. If Ali takes advantage of the offer, how much interest does she owe at the end of the three months? (Assume there are 30 days in a month.)

BEAT THE TEST!

1. The expression $(1.016)^{\frac{t}{4}}$ represents the interest that a bank pays on a savings account.

   Part A: What is the yearly interest rate?

   Part B: A customer invests $\$600$ that is compounded monthly. Write a function to show the future value of the account.

Section 9 – Topic 4
Euler's Number

Consider the following expression for compound interest, $P\left(1+\frac{r}{n}\right)^{nt}$, where P = the amount of money invested, r = percent rate as decimal, t = number of years, and n = the number of times compounded in a year.

Investing $\$1.00$ would change the formula to ______________.

A rate of 100% would change the formula to ____________.

Restricting to one year would change the formula to __________________.

Write the final exponential function for the compound interest.

$$e(n) = \_\_\_\_\_\_\_\_\_\_\_\_$$

What is the remaining variable and what does it represent?

Fill out the table for different values of n, round to the ten-thousandth for $e(n) = \left(1 + \frac{1}{n}\right)^n$.

	n	$e(n)$
Yearly	1	
Semiannually	2	
Quarterly	4	
Monthly	12	
Weekly	52	
Daily	365	
Hourly	8760	
Every Minute	525600	
Every Second	31536000	

- What number is the function approaching as n gets larger?

- This number is called **Euler's Number**.

- Is Euler's number rational or irrational?

- We can use a calculator to make estimations with Euler's Number.

Let's Practice!

1. Evaluate. Round your answer to the nearest tenth.

   a. e^7

   b. $e^{-5.6}$

Try It!

2. Evaluate. Round your answer to the nearest tenth.

   a. e^0

   b. $e^{-1.2}$

BEAT THE TEST!

For the questions below, choose the appropriate word or notation to make the statement true.

1. Euler's Number is a (real/imaginary) number that is (rational/irrational).

2. In the function $f(x) = e^x$, e represents Euler's number.

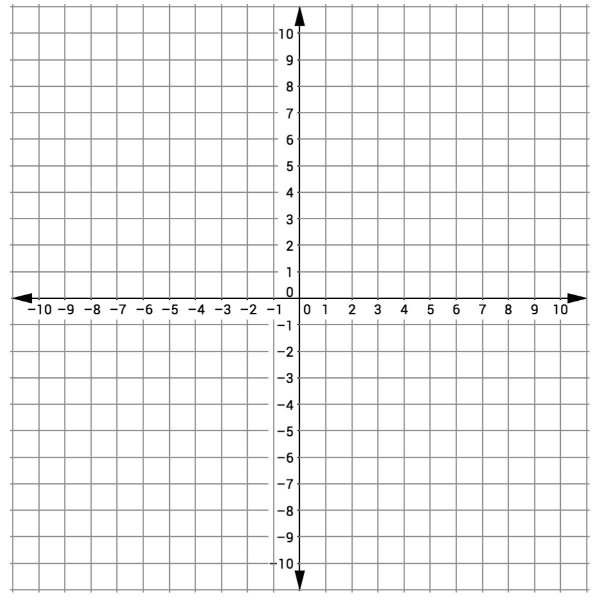
   Part A: The domain of f is $((-\infty, \infty), (-\infty, 0), (0, \infty))$.

   Part B: The range of f is $((-\infty, \infty), (-\infty, 0), (0, \infty))$.

Section 9 – Topic 5
Graphing Exponential Functions – Part 1

A function in the form of $f(x) = b^x$ is called an exponential function when $b > 0, b$ is not 1, and x is a real number.

Demonstrate on the graph below why $b \neq 1$ and b must be greater than 0.

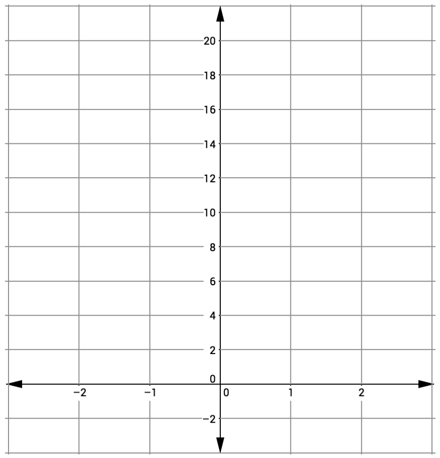


Let's Practice!

1. Graph the following exponential functions on the same coordinate plane.

$$f(x) = 2^x \text{ and } g(x) = 4^x$$

x	$f(x)$	$g(x)$
-2		
-1		
0		
1		
2		



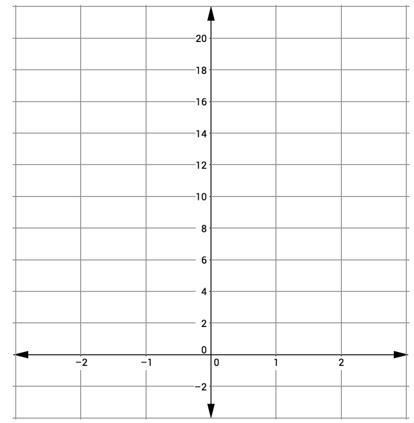
Where does $f(x) = g(x)$?

Try It!

2. Graph the following exponential functions on the same coordinate plane.

$$f(x) = \left(\frac{1}{2}\right)^x \text{ and } g(x) = \left(\frac{1}{4}\right)^x$$

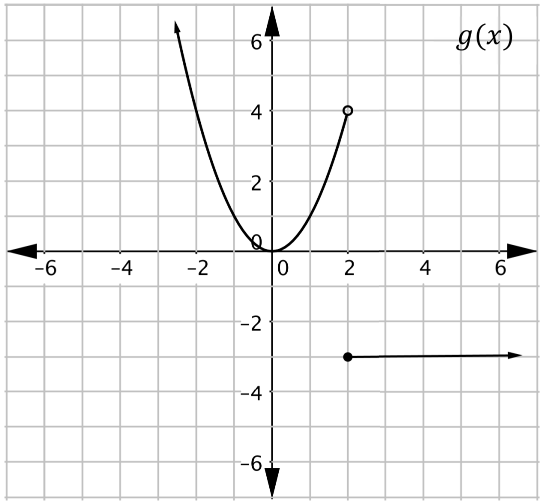
x	$f(x)$	$g(x)$
-2		
-1		
0		
1		
2		



Where does $f(x) = g(x)$?

BEAT THE TEST!

1. Solve $f(x) = g(x)$, to the nearest tenth, where $f(x) = 3^x - 2.2$ and $g(x)$ is shown.



Section 9 – Topic 6
Transformations of Exponential Functions

Let's apply our knowledge of transformations to exponential functions.

Let's Practice!

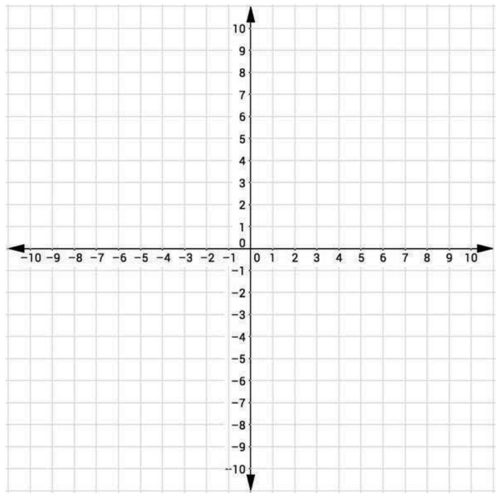
1. Compare and contrast $p(x) = \left(\frac{1}{4}\right)^x$ and $q(x) = p(x-2) + 3$.

2. Complete the table below for $g(x) = 3 \cdot 2^x + 1$ and $h(x) = 2g(x+1) - 2$.

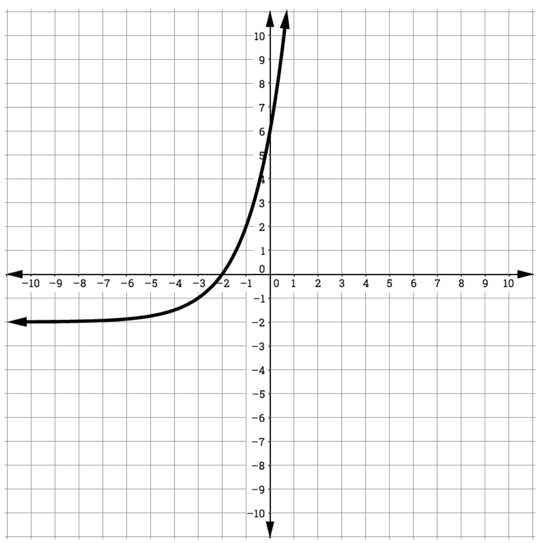
x	$g(x)$	$h(x)$
-2		
0		
2		

Try It!

3. Graph $m(x) = 2^x$ and $n(x) = -m(x-3) + 2$.



4. The graph below models $b(x) = c(x-2) + 3$. Model the graph for $c(x)$ on the same coordinate plane.



5. The table below models a transformation on $f(x)$. Complete the missing values of each ordered pair.

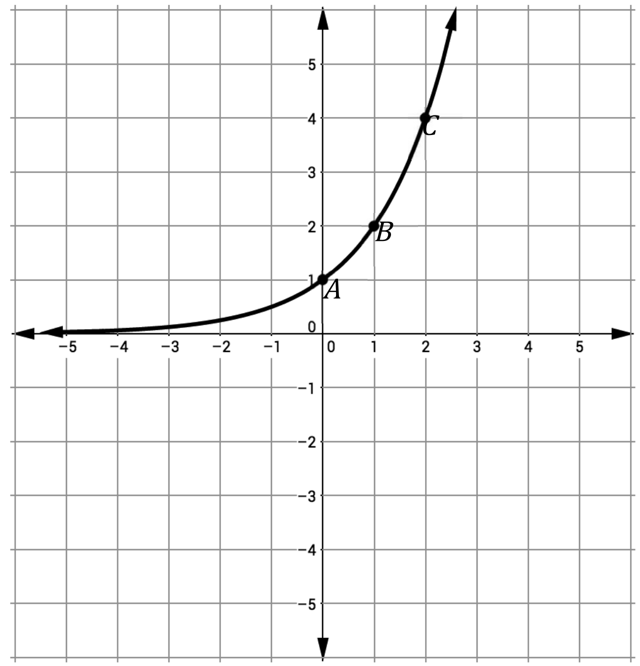
$f(x)$		$f(x+3)-1$	
$x = -6$	$y = 37$	$x' =$	$y' =$
$x =$	$y =$	$x' = -2$	$y' = \frac{1}{2}$
$x = -4$	$y =$	$x' =$	$y' = 16$

BEAT THE TEST!

1. For functions $f(x) = -2^x$ and $g(x) = f(x+3) + 5$, select all the true statements.

   - ☐ The graphs of $f(x)$ and $g(x)$ are both decreasing.
   - ☐ The graphs both have a y –intercept at $(0, -1)$.
   - ☐ The graph of $g(x)$ has an asymptote at $y = 5$.
   - ☐ The graph of $g(x)$ is shifted 3 units to the left.
   - ☐ Both graphs have x –intercepts.

2. A function $f(x)$ is shown below. The function is transformed to create the function $h(x)$ such that $h(x) = f(x+2) - 3$.



Complete the table to show the points A', B', C'.

Point	x –coordinate	y –coordinate
A'		
B'		
C'		

Section 9 – Topic 7
Key Features of Exponential Functions

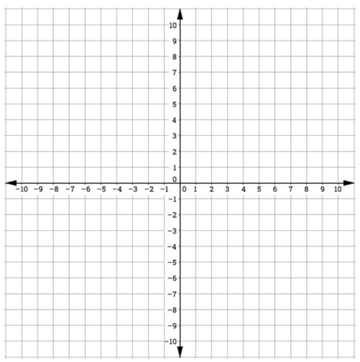
The key features of exponential functions are:

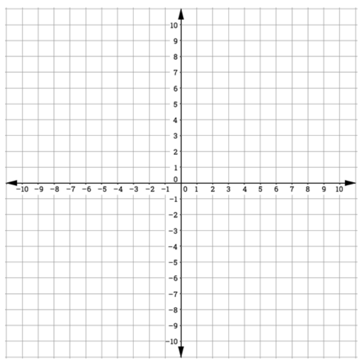
- Intercepts
- Intervals where the function is increasing or decreasing
- Intervals where the function is positive or negative
- End Behavior

How many x –intercept(s) does an exponential function have?

How many y –intercept(s) does an exponential function have?

Exponential functions are either increasing or decreasing. Sketch graphs of each type.





Let's Practice!

1. Determine the following features for $f(x) = \left(\frac{1}{2}\right)^{x}$.

   a. x –intercept:

   b. y –intercept:

   c. Increasing interval(s):

   d. Decreasing interval(s):

   e. Positive interval(s):

   f. Negative interval(s):

   g. End Behavior:

Try It!

2. Give an algebraic representation of an exponential function for each of the following features.

   a. No x –intercept:

   b. y –intercept at $(0, -3)$:

   c. Decreasing interval over $(-\infty, \infty)$:

   d. Positive interval over $(2, \infty)$:

   e. Negative interval over $(-\infty, 3)$:

   f. End behavior: As $x \to -\infty$, $y \to \infty$:

BEAT THE TEST!

1. Complete the following table by describing key features of exponential functions.

Exponential functions have one x –intercept.	○ Always ○ Sometimes ○ Never
Exponential functions have one y –intercept.	○ Always ○ Sometimes ○ Never
Exponential functions are increasing.	○ Always ○ Sometimes ○ Never
Exponential functions have positive intervals over $(-\infty, \infty)$.	○ Always ○ Sometimes ○ Never
Exponential functions have symmetry.	○ Always ○ Sometimes ○ Never
In exponential functions, as $x \to \infty$, $y \to -\infty$.	○ Always ○ Sometimes ○ Never

Section 9 – Topic 8
Logarithmic Functions – Part 1

Let's Practice!

Solve the following exponential equations.

1. $6^x = 36$

2. $5^{5y-3} = 25^{11+3y}$

Try It!

Solve the following exponential equations.

3. $4^x = 64$

4. $9^{x+13} = 3^{5x-4}$

Now, let's explore how to solve exponential equations that do not have the same base.

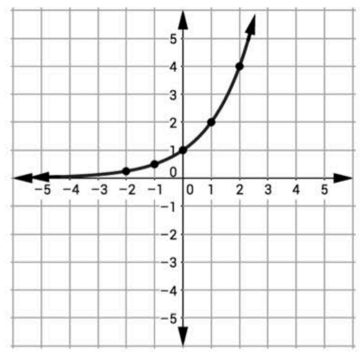
The exponential function $f(x) = b^x$ is a one-to-one function. Therefore, it has an inverse.

Consider $f(x) = 2^x$. Create a table of values below for $f^{-1}(x)$.

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f^{-1}(x)$

The graph of $f(x)$ is shown below. Graph $f^{-1}(x)$ on the same coordinate plane.



Consider the following steps for writing the inverse of $f(x) = 2^x$.

- Replace $f(x)$ with y: $y = 2^x$
- Interchange x and y: $x = 2^y$
- Solve for y:

We are not able to do this using our current knowledge.

A new notation, called _______________ notation, helps us write the inverse function of an exponential function.

Logarithmic Definition: If $b > 0$ and $b \neq 1$, then

$$x = b^y \text{ means } y = log_b x$$

for every $x > 0$ and every real number y.

Let's Practice!

5. Write each of the following as an exponential equation.

 a. $log_2 16 = 4$

 b. $log_{13} y = 2$

6. Write each of the following as a logarithmic equation.

 a. $5^3 = 125$

 b. $4^{-2} = \frac{1}{16}$

7. Solve the following logarithmic equations.

 a. $log_2 x = 3$

 b. $log_x 49 = 2$

Try It!

8. Write each of the following as an exponential equation.

   a. $log_5 25 = 2$

   b. $log_2 8 = x$

9. Write each of the following as a logarithmic equation.

   a. $4^3 = 64$

   b. $6^{-2} = \frac{1}{36}$

10. Solve the following logarithmic equations.

   a. $log_2 x = -1$

   b. $log_x 49 = -2$

Section 9 – Topic 9
Logarithmic Functions – Part 2

What do the following logarithmic expressions equal?

$log_b 1 =$

$log_b b^x =$

In the logarithm equation, $y = log_b x$, can x be a negative number? Justify your answer.

Show that the following property is true.

$b^{log_b x} = x$

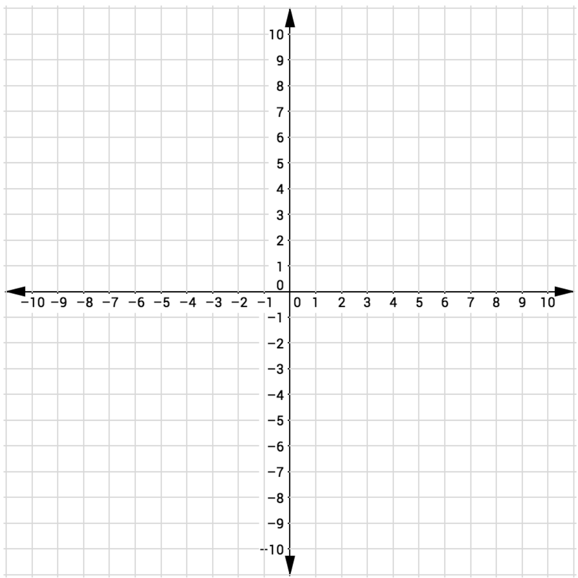
Remember that a logarithm is a way to write an inverse function for an exponential function.

Let's Practice!

1. Given the function $f(x) = 3^x$, find $f^{-1}(x)$.

   a. Use compositions to prove that your answer is correct.

   b. Graph $f(x)$ and $f^{-1}(x)$.

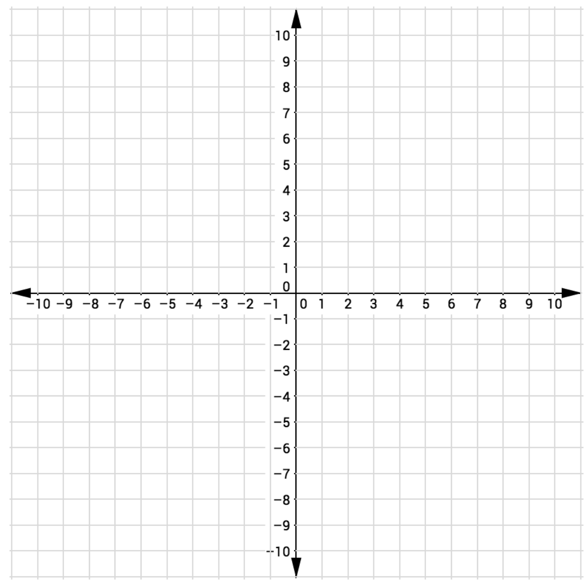


Try It!

2. Given the function $f(x) = \left(\frac{1}{2}\right)^x$, find $f^{-1}(x)$.

   a. Use compositions to prove that your answer is correct.

   b. Graph $f(x)$ and $f^{-1}(x)$.



When we were graphing the inverse of an exponential function, we were graphing a logarithmic function. Let's make observations from those graphs.

Let's Practice!

3. Consider $f(x) = log_b x, b > 0, b \neq 1$.

    a. Is $f(x)$ a one-to-one function.

    b. What is the x –intercept?

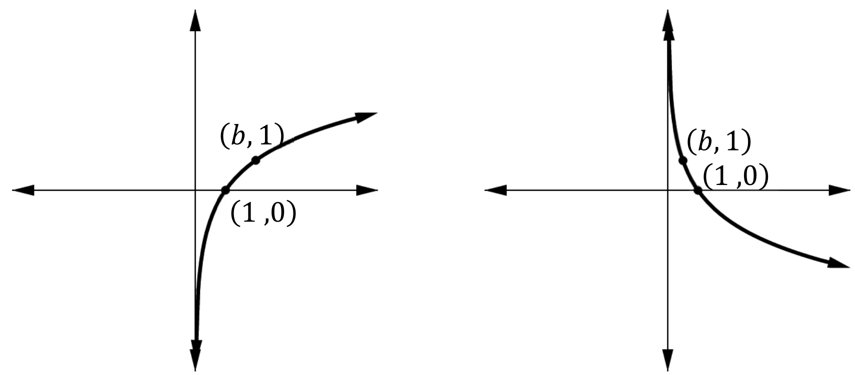
    c. What is the y –intercept?

    d. What is the domain of the function?

    e. What is the range of the function?

Try It!

4. The following are graphs of $f(x) = log_b x$. Label the graph where $b > 1$ and the graph where $0 < b < 1$.



BEAT THE TEST!

1. What is the inverse of the function $f(x) = log_7 x$?

Ⓐ $f^{-1}(x) = x^7$
Ⓑ $f^{-1}(x) = 7^x$
Ⓒ $f^{-1}(x) = log_x 7$
Ⓓ $f^{-1}(x) = -log_7 x$

Section 9 – Topic 10
Common and Natural Logarithms

Logarithms to __________ ______ are called _________________ logarithms. Because we use a base 10 number system, these are frequently used.

$log_{10}x$ means $log\, x$

Let's Practice!

1. Find the exact value of each logarithm.

   a. $log\, 1000$

   b. $log\, \sqrt{100}$

2. Find the approximate value of each logarithm. Round your answer to the nearest thousandth.

   a. $log\, 5$

   b. $log\, 101$

Natural events that occur in the real world are often described with ____________ logarithms. Natural logarithms have a base ____ (Euler's Number), which is approximately equal to the constant 2.7183. We use the abbreviation $ln\ x$.

$$log_e x \text{ means } ln\ x$$

Let's Practice!

3. Find the approximate value of each logarithm. Round your answer to the nearest thousandth.

   a. $ln\ 5$

   b. $ln\ 101$

In some situations, we need to approximate a value for a logarithm that is not base 10 or base e. There is a formula that can help us with such situations.

Change of Base Formula:

$$log_b a = \frac{log\ a}{log\ b} = \frac{ln\ a}{ln\ b}$$

Let's Practice!

4. Use logarithms to solve the following exponential equations.

   a. $4^x = 5$

   b. $3^y = 2$

Logarithms are also very helpful in solving real world problems.

Recall the formula for compound interest, $A = P\left(1 + \frac{r}{n}\right)^{nt}$, where A represents the future value of the investment or loan, P represents the initial amount invested or borrowed, r represents the annual rate of interest, n represents the number of times the interest is compounded per year, and t represents time in years.

Approximately how long will it take $\$2500$ to double if it is invested at 5% interest compounded monthly?

5. DDT is a pesticide that was banned in the 1970s. One of the reasons is due to its toxicity to aquatic life. The following formula $y = Ie^{-0.0046t}$ represents the exponential decay of DDT in an aquatic environment, where y is the remaining amount of DDT, I is the initial amount of DDT and t is the time in years.

   Approximately how many years will it take 50 grams of DDT to decay to 1 gram of DDT?

Try It!

6. The formula to determine the concentration of hydrogen ions in a substance is given by $H^+ = 10^{(-pH)}$, where H^+ is the amount of hydrogen ions in moles per liter and pH is the acidity level of the substance. Human blood has 3.6×10^{-8} hydrogen ions moles per liter.

   What is the approximate acidity level of human blood?

BEAT THE TEST!

1. Newton's Law of Cooling states that the rate of change of the temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surroundings. The function, $T(t) = 70 + 130e^{-0.0486t}$, describes the temperature of coffee that has been heated to $200°F$ and is now cooling in a room at $70°F$, where t is time in minutes.

   How many minutes, rounded to the nearest tenth, will the coffee take to cool to $105°F$?