Technology Exercises

98. The common cold is caused by a rhinovirus. The polynomial

$$-0.75x^4 + 3x^3 + 5$$

describes the billions of viral particles in our bodies after x days of invasion. Use a calculator to find the number of viral particles after 0 days (the time of the cold's onset), 1 day, 2 days, 3 days, and 4 days. After how many days is the number of viral particles at a maximum and consequently the day we feel the sickest? By when should we feel completely better?

99. Using data from the National Institute on Drug Abuse, the polynomial

$$0.0032x^3 + 0.0235x^2 - 2.2477x + 61.1998$$

approximately describes the percentage of U.S. high school seniors in the class of x who had ever used marijuana, where x is the number of years after 1980. Use a calculator to find the percentage of high school seniors from the class of 1980 through the class of 2000 who had used marijuana. Round to the nearest tenth of a percent. Describe the trend in the data.

Critical Thinking Exercises

In Exercises 100–103, perform the indicated operations.

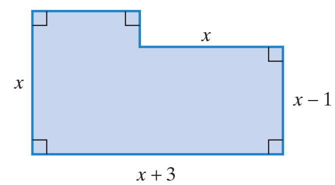
100. $(x - y)^2 - (x + y)^2$

101. $[(7x + 5) + 4y][(7x + 5) - 4y]$

102. $[(3x + y) + 1]^2$

103. $(x + y)(x - y)(x^2 + y^2)$

104. Express the area of the plane figure shown as a polynomial in standard form.



SECTION P.5 *Factoring Polynomials*

Objectives

1. Factor out the greatest common factor of a polynomial.
2. Factor by grouping.
3. Factor trinomials.
4. Factor the difference of squares.
5. Factor perfect square trinomials.
6. Factor the sum and difference of cubes.
7. Use a general strategy for factoring polynomials.
8. Factor algebraic expressions containing fractional and negative exponents.

A two-year-old boy is asked, "Do you have a brother?" He answers, "Yes." "What is your brother's name?" "Tom." Asked if Tom has a brother, the two-year-old replies, "No." The child can go in the direction from self to brother, but he cannot reverse this direction and move from brother back to self.

As our intellects develop, we learn to reverse the direction of our thinking. Reversibility of thought is found throughout algebra. For example, we can multiply polynomials and show that

$$(2x + 1)(3x - 2) = 6x^2 - x - 2.$$

We can also reverse this process and express the resulting polynomial as

$$6x^2 - x - 2 = (2x + 1)(3x - 2).$$

Factoring is the process of writing a polynomial as the product of two or more polynomials. The factors of $6x^2 - x - 2$ are $2x + 1$ and $3x - 2$.

In this section, we will be **factoring over the set of integers,** meaning that the coefficients in the factors are integers. Polynomials that cannot be factored using integer coefficients are called **irreducible over the integers,** or **prime.**

The goal in factoring a polynomial is to use one or more factoring techniques until each of the polynomial's factors is prime or irreducible. In this situation, the polynomial is said to be **factored completely.**

We will now discuss basic techniques for factoring polynomials.

1 Factor out the greatest common factor of a polynomial.

Common Factors

In any factoring problem, the first step is to look for the *greatest common factor*. The **greatest common factor,** abbreviated GCF, is an expression of the highest degree that divides each term of the polynomial. The distributive property in the reverse direction

$$ab + ac = a(b + c)$$

can be used to factor out the greatest common factor.

EXAMPLE 1 Factoring out the Greatest Common Factor

Factor: **a.** $18x^3 + 27x^2$ **b.** $x^2(x + 3) + 5(x + 3)$.

Solution

a. We begin by determining the greatest common factor. 9 is the greatest integer that divides 18 and 27. Furthermore, x^2 is the greatest expression that divides x^3 and x^2. Thus, the greatest common factor of the two terms in the polynomial is $9x^2$.

$18x^3 + 27x^2$

$= 9x^2(2x) + 9x^2(3)$ Express each term as the product of the greatest common factor and its other factor.

$= 9x^2(2x + 3)$ Factor out the greatest common factor.

b. In this situation, the greatest common factor is the common binomial factor $(x + 3)$. We factor out this common factor as follows:

$x^2(x + 3) + 5(x + 3) = (x + 3)(x^2 + 5)$. Factor out the common binomial factor.

Study Tip

The variable part of the greatest common factor always contains the *smallest* power of a variable or algebraic expression that appears in all terms of the polynomial.

Check Point 1 Factor:

a. $10x^3 - 4x^2$ **b.** $2x(x - 7) + 3(x - 7)$.

2 Factor by grouping.

Factoring by Grouping

Some polynomials have only a greatest common factor of 1. However, by a suitable rearrangement of the terms, it still may be possible to factor. This process, called **factoring by grouping,** is illustrated in Example 2.

EXAMPLE 2 Factoring by Grouping

Factor: $x^3 + 4x^2 + 3x + 12$.

Solution Group terms that have a common factor:

$$\boxed{x^3 + 4x^2} + \boxed{3x + 12}.$$

Common factor is x^2. Common factor is 3.

We now factor the given polynomial as follows.

$x^3 + 4x^2 + 3x + 12$

$= (x^3 + 4x^2) + (3x + 12)$ Group terms with common factors.

$= x^2(x + 4) + 3(x + 4)$ Factor out the greatest common factor from the grouped terms. The remaining two terms have $x + 4$ as a common binomial factor.

$= (x + 4)(x^2 + 3)$ Factor $(x + 4)$ out of both terms.

Thus, $x^3 + 4x^2 + 3x + 12 = (x + 4)(x^2 + 3)$. Check the factorization by multiplying the right side of the equation using the FOIL method. If the factorization is correct, you will obtain the original polynomial.

Discovery

In Example 2, group the terms as follows:

$(x^3 + 3x) + (4x^2 + 12)$.

Factor out the greatest common factor from each group and complete the factoring process. Describe what happens. What can you conclude?

Check Point 2 Factor: $x^3 + 5x^2 - 2x - 10$.

3 Factor trinomials.

Factoring Trinomials

To factor a trinomial of the form $ax^2 + bx + c$, a little trial and error may be necessary.

A Strategy for Factoring $ax^2 + bx + c$

(Assume, for the moment, that there is no greatest common factor.)

1. Find two **F**irst terms whose product is ax^2:

$$(\square x + \quad)(\square x + \quad) = ax^2 + bx + c.$$

2. Find two **L**ast terms whose product is c:

$$(x + \square)(x + \square) = ax^2 + bx + c.$$

3. By trial and error, perform steps 1 and 2 until the sum of the **O**utside product and **I**nside product is bx:

$$(\square x + \square)(\square x + \square) = ax^2 + bx + c.$$

I

O

(sum of O + I)

If no such combinations exist, the polynomial is prime.

EXAMPLE 3 Factoring Trinomials Whose Leading Coefficients Are 1

Factor: **a.** $x^2 + 6x + 8$ **b.** $x^2 + 3x - 18$.

Solution

a. The factors of the first term are x and x:

$$x^2 + 6x + 8 = (x \quad)(x \quad).$$

Factors of 8	8, 1	4, 2	−8, −1	−4, −2
Sum of Factors	9	6	−9	−6

This is the desired sum.

To find the second term of each factor, we must find two numbers whose product is 8 and whose sum is 6. From the table in the margin, we see that 4 and 2 are the required integers. Thus,

$$x^2 + 6x + 8 = (x + 4)(x + 2) \text{ or } (x + 2)(x + 4).$$

b. We begin with

$$x^2 + 3x - 18 = (x \quad)(x \quad).$$

Factors of −18	18, −1	−18, 1	9, −2	−9, 2	6, −3	−6, 3
Sum of factors	17	−17	7	−7	3	−3

This is the desired sum.

To find the second term of each factor, we must find two numbers whose product is −18 and whose sum is 3. From the table in the margin, we see that 6 and −3 are the required integers. Thus,

$$x^2 + 3x - 18 = (x + 6)(x - 3)$$
$$\text{or } (x - 3)(x + 6).$$

Check Point 3 Factor:

a. $x^2 + 13x + 40$ **b.** $x^2 - 5x - 14$.

EXAMPLE 4 Factoring a Trinomial Whose Leading Coefficient Is Not 1

Factor: $8x^2 - 10x - 3$.

Solution

Step 1 Find two *F*irst terms whose product is $8x^2$.

$$8x^2 - 10x - 3 \stackrel{?}{=} (8x \quad)(x \quad)$$
$$8x^2 - 10x - 3 \stackrel{?}{=} (4x \quad)(2x \quad)$$

Step 2 Find two *L*ast terms whose product is −3. The possible factorizations are $1(-3)$ and $-1(3)$.

Step 3 Try various combinations of these factors. The correct factorization of $8x^2 - 10x - 3$ is the one in which the sum of the *O*utside and *I*nside products is equal to $-10x$. Here is a list of the possible factorizations:

Possible Factorizations of $8x^2 - 10x - 3$	Sum of *O*utside and *I*nside Products (Should Equal $-10x$)
$(8x + 1)(x - 3)$	$-24x + x = -23x$
$(8x - 3)(x + 1)$	$8x - 3x = 5x$
$(8x - 1)(x + 3)$	$24x - x = 23x$
$(8x + 3)(x - 1)$	$-8x + 3x = -5x$
$(4x + 1)(2x - 3)$	$-12x + 2x = -10x$
$(4x - 3)(2x + 1)$	$4x - 6x = -2x$
$(4x - 1)(2x + 3)$	$12x - 2x = 10x$
$(4x + 3)(2x - 1)$	$-4x + 6x = 2x$

This is the required middle term.

Thus,

$$8x^2 - 10x - 3 = (4x + 1)(2x - 3) \quad \text{or} \quad (2x - 3)(4x + 1).$$

Show that this factorization is correct by multiplying the factors using the FOIL method. You should obtain the original trinomial.

Check Point 4 Factor: $6x^2 + 19x - 7$.

4 Factor the difference of squares.

Factoring the Difference of Two Squares

A method for factoring the difference of two squares is obtained by reversing the special product for the sum and difference of two terms.

The Difference of Two Squares

If A and B are real numbers, variables, or algebraic expressions, then

$$A^2 - B^2 = (A + B)(A - B).$$

In words: The difference of the squares of two terms factors as the product of a sum and a difference of those terms.

EXAMPLE 5 Factoring the Difference of Two Squares

Factor: **a.** $x^2 - 4$ **b.** $81x^2 - 49$.

Solution We must express each term as the square of some monomial. Then we use the formula for factoring $A^2 - B^2$.

a. $x^2 - 4 = x^2 - 2^2 = (x + 2)(x - 2)$

$$A^2 - B^2 = (A + B)(A - B)$$

b. $81x^2 - 49 = (9x)^2 - 7^2 = (9x + 7)(9x - 7)$

Check Point 5 Factor:

a. $x^2 - 81$ **b.** $36x^2 - 25$.

We have seen that a polynomial is factored completely when it is written as the product of prime polynomials. To be sure that you have factored completely, check to see whether the factors can be factored.

Study Tip

Factoring $x^4 - 81$ as

$$(x^2 + 9)(x^2 - 9)$$

is not a complete factorization. The second factor, $x^2 - 9$, is itself a difference of two squares and can be factored.

EXAMPLE 6 A Repeated Factorization

Factor completely: $x^4 - 81$.

Solution

$x^4 - 81 = (x^2)^2 - 9^2$ Express as the difference of two squares.

$= (x^2 + 9)(x^2 - 9)$ The factors are the sum and difference of the squared terms.

$= (x^2 + 9)(x^2 - 3^2)$ The factor $x^2 - 9$ is the difference of two squares and can be factored.

$= (x^2 + 9)(x + 3)(x - 3)$ The factors of $x^2 - 9$ are the sum and difference of the squared terms.

Check Point 6 Factor completely: $81x^4 - 16$.

5 Factor perfect square trinomials.

Factoring Perfect Square Trinomials

Our next factoring technique is obtained by reversing the special products for squaring binomials. The trinomials that are factored using this technique are called **perfect square trinomials.**

Factoring Perfect Square Trinomials

Let A and B be real numbers, variables, or algebraic expressions.

1. $A^2 + 2AB + B^2 = (A + B)^2$

   Same sign

2. $A^2 - 2AB + B^2 = (A - B)^2$

   Same sign

The two items in the box show that perfect square trinomials come in two forms: one in which the middle term is positive and one in which the middle term is negative. Here's how to recognize a perfect square trinomial:

1. The first and last terms are squares of monomials or integers.
2. The middle term is twice the product of the expressions being squared in the first and last terms.

EXAMPLE 7 Factoring Perfect Square Trinomials

Factor: **a.** $x^2 + 6x + 9$ **b.** $25x^2 - 60x + 36$.

Solution

a. $x^2 + 6x + 9 = x^2 + 2 \cdot x \cdot 3 + 3^2 = (x + 3)^2$ The middle term has a positive sign.

$A^2 + 2AB + B^2 = (A + B)^2$

b. We suspect that $25x^2 - 60x + 36$ is a perfect square trinomial because $25x^2 = (5x)^2$ and $36 = 6^2$. The middle term can be expressed as twice the product of $5x$ and 6.

$$25x^2 - 60x + 36 = (5x)^2 - 2 \cdot 5x \cdot 6 + 6^2 = (5x - 6)^2$$

$A^2 - 2AB + B^2 = (A - B)^2$

Check Point 7 Factor:

a. $x^2 + 14x + 49$ **b.** $16x^2 - 56x + 49$.

6 Factor the sum and difference of cubes.

Factoring the Sum and Difference of Two Cubes

We can use the following formulas to factor the sum or the difference of two cubes:

Factoring the Sum and Difference of Two Cubes

1. Factoring the Sum of Two Cubes

$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$

2. Factoring the Difference of Two Cubes

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

EXAMPLE 8 Factoring Sums and Differences of Two Cubes

Factor: **a.** $x^3 + 8$ **b.** $64x^3 - 125$.

Solution

a. $x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - x \cdot 2 + 2^2) = (x + 2)(x^2 - 2x + 4)$

$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

b. $64x^3 - 125 = (4x)^3 - 5^3 = (4x - 5)[(4x)^2 + (4x)(5) + 5^2]$

$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$= (4x - 5)(16x^2 + 20x + 25)$

Check Point 8 Factor:

a. $x^3 + 1$ **b.** $125x^3 - 8$.

7 Use a general strategy for factoring polynomials.

A Strategy for Factoring Polynomials

It is important to practice factoring a wide variety of polynomials so that you can quickly select the appropriate technique. The polynomial is factored completely when all its polynomial factors, except possibly for monomial factors, are prime. Because of the commutative property, the order of the factors does not matter.

A Strategy for Factoring a Polynomial

1. If there is a common factor, factor out the GCF.
2. Determine the number of terms in the polynomial and try factoring as follows:
   - a. If there are two terms, can the binomial be factored by one of the following special forms?

     Difference of two squares: $A^2 - B^2 = (A + B)(A - B)$

     Sum of two cubes: $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

     Difference of two cubes: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$
   - b. If there are three terms, is the trinomial a perfect square trinomial? If so, factor by one of the following special forms:

     $$A^2 + 2AB + B^2 = (A + B)^2$$
     $$A^2 - 2AB + B^2 = (A - B)^2.$$

     If the trinomial is not a perfect square trinomial, try factoring by trial and error.
   - c. If there are four or more terms, try factoring by grouping.
3. Check to see if any factors with more than one term in the factored polynomial can be factored further. If so, factor completely.

EXAMPLE 9 Factoring a Polynomial

Factor: $2x^3 + 8x^2 + 8x$.

Solution

Step 1 **If there is a common factor, factor out the GCF.** Because $2x$ is common to all terms, we factor it out.

$$2x^3 + 8x^2 + 8x = 2x(x^2 + 4x + 4) \quad \text{Factor out the GCF.}$$

Step 2 **Determine the number of terms and factor accordingly.** The factor $x^2 + 4x + 4$ has three terms and is a perfect square trinomial. We factor using $A^2 + 2AB + B^2 = (A + B)^2$.

$$2x^3 + 8x^2 + 8x = 2x(x^2 + 4x + 4)$$
$$= 2x(x^2 + 2 \cdot x \cdot 2 + 2^2)$$

$A^2 + 2AB + B^2$

$$= 2x(x + 2)^2 \quad A^2 + 2AB + B^2 = (A + B)^2$$

Step 3 **Check to see if factors can be factored further.** In this problem, they cannot. Thus,

$$2x^3 + 8x^2 + 8x = 2x(x + 2)^2.$$

Check Point 9 Factor: $3x^3 - 30x^2 + 75x$.

EXAMPLE 10 Factoring a Polynomial

Factor: $x^2 - 25a^2 + 8x + 16$.

Solution

Step 1 **If there is a common factor, factor out the GCF.** Other than 1 or –1, there is no common factor.

Step 2 **Determine the number of terms and factor accordingly.** There are four terms. We try factoring by grouping. Grouping into two groups of two terms does not result in a common binomial factor. Let's try grouping as a difference of squares.

$x^2 - 25a^2 + 8x + 16$

$= (x^2 + 8x + 16) - 25a^2$ — Rearrange terms and group as a perfect square trinomial minus $25a^2$ to obtain a difference of squares.

$= (x + 4)^2 - (5a)^2$ — Factor the perfect square trinomial.

$= (x + 4 + 5a)(x + 4 - 5a)$ — Factor the difference of squares. The factors are the sum and difference of the expressions being squared.

Step 3 **Check to see if factors can be factored further.** In this case, they cannot, so we have factored completely.

Check Point 10 Factor: $x^2 - 36a^2 + 20x + 100$.

8 Factor algebraic expressions containing fractional and negative exponents.

Factoring Algebraic Expressions Containing Fractional and Negative Exponents

Although expressions containing fractional and negative exponents are not polynomials, they can be simplified using factoring techniques.

EXAMPLE 11 Factoring Involving Fractional and Negative Exponents

Factor and simplify: $x(x + 1)^{-3/4} + (x + 1)^{1/4}$.

Solution The greatest common factor is $x + 1$ with the *smallest exponent* in the two terms. Thus, the greatest common factor is $(x + 1)^{-3/4}$.

$x(x + 1)^{-3/4} + (x + 1)^{1/4}$

$= (x + 1)^{-3/4}x + (x + 1)^{-3/4}(x + 1)$ — Express each term as the product of the greatest common factor and its other factor.

$= (x + 1)^{-3/4}[x + (x + 1)]$ — Factor out the greatest common factor.

$= \dfrac{2x + 1}{(x + 1)^{3/4}}$ — $b^{-n} = \dfrac{1}{b^n}$

Check Point 11 Factor and simplify: $x(x-1)^{-1/2}+(x-1)^{1/2}$.

EXERCISE SET P.5

Practice Exercises

In Exercises 1–10, factor out the greatest common factor.

1. $18x+27$
2. $16x-24$
3. $3x^2+6x$
4. $4x^2-8x$
5. $9x^4-18x^3+27x^2$
6. $6x^4-18x^3+12x^2$
7. $x(x+5)+3(x+5)$
8. $x(2x+1)+4(2x+1)$
9. $x^2(x-3)+12(x-3)$
10. $x^2(2x+5)+17(2x+5)$

In Exercises 11–16, factor by grouping.

11. $x^3-2x^2+5x-10$
12. $x^3-3x^2+4x-12$
13. x^3-x^2+2x-2
14. $x^3+6x^2-2x-12$
15. $3x^3-2x^2-6x+4$
16. x^3-x^2-5x+5

In Exercises 17–30, factor each trinomial, or state that the trinomial is prime.

17. x^2+5x+6
18. $x^2+8x+15$
19. $x^2-2x-15$
20. x^2-4x-5
21. $x^2-8x+15$
22. $x^2-14x+45$
23. $3x^2-x-2$
24. $2x^2+5x-3$
25. $3x^2-25x-28$
26. $3x^2-2x-5$
27. $6x^2-11x+4$
28. $6x^2-17x+12$
29. $4x^2+16x+15$
30. $8x^2+33x+4$

In Exercises 31–40, factor the difference of two squares.

31. x^2-100
32. x^2-144
33. $36x^2-49$
34. $64x^2-81$
35. $9x^2-25y^2$
36. $36x^2-49y^2$
37. x^4-16
38. x^4-1
39. $16x^4-81$
40. $81x^4-1$

In Exercises 41–48, factor any perfect square trinomials, or state that the polynomial is prime.

41. x^2+2x+1
42. x^2+4x+4
43. $x^2-14x+49$
44. $x^2-10x+25$
45. $4x^2+4x+1$
46. $25x^2+10x+1$
47. $9x^2-6x+1$
48. $64x^2-16x+1$

In Exercises 49–56, factor using the formula for the sum or difference of two cubes.

49. x^3+27
50. x^3+64
51. x^3-64
52. x^3-27
53. $8x^3-1$
54. $27x^3-1$
55. $64x^3+27$
56. $8x^3+125$

In Exercises 57–84, factor completely, or state that the polynomial is prime.

57. $3x^3-3x$
58. $5x^3-45x$
59. $4x^2-4x-24$
60. $6x^2-18x-60$
61. $2x^4-162$
62. $7x^4-7$
63. $x^3+2x^2-9x-18$
64. $x^3+3x^2-25x-75$
65. $2x^2-2x-112$
66. $6x^2-6x-12$
67. x^3-4x
68. $9x^3-9x$
69. x^2+64
70. x^2+36
71. x^3+2x^2-4x-8
72. x^3+2x^2-x-2
73. y^5-81y
74. y^5-16y
75. $20y^4-45y^2$
76. $48y^4-3y^2$
77. $x^2-12x+36-49y^2$
78. $x^2-10x+25-36y^2$
79. $9b^2x-16y-16x+9b^2y$
80. $16a^2x-25y-25x+16a^2y$
81. $x^2y-16y+32-2x^2$
82. $12x^2y-27y-4x^2+9$
83. $2x^3-8a^2x+24x^2+72x$
84. $2x^3-98a^2x+28x^2+98x$

In Exercises 85–94, factor and simplify each algebraic expression.

85. $x^{3/2}-x^{1/2}$
86. $x^{3/4}-x^{1/4}$
87. $4x^{-2/3}+8x^{1/3}$
88. $12x^{-3/4}+6x^{1/4}$
89. $(x+3)^{1/2}-(x+3)^{3/2}$
90. $(x^2+4)^{3/2}+(x^2+4)^{7/2}$
91. $(x+5)^{-1/2}-(x+5)^{-3/2}$
92. $(x^2+3)^{-2/3}+(x^2+3)^{-5/3}$
93. $(4x-1)^{1/2}-\frac{1}{3}(4x-1)^{3/2}$
94. $-8(4x+3)^{-2}+10(5x+1)(4x+3)^{-1}$

Application Exercises

95. Your computer store is having an incredible sale. The price on one model is reduced by 40%. Then the sale price is reduced by another 40%. If x is the computer's original price, the sale price can be represented by

$$(x-0.4x)-0.4(x-0.4x).$$

a. Factor out $(x-0.4x)$ from each term. Then simplify the resulting expression.

b. Use the simplified expression from part (a) to answer these questions: With a 40% reduction followed by a 40% reduction, is the computer selling at 20% of its original price? If not, at what percentage of the original price is it selling?

96. The polynomial $8x^2 + 20x + 2488$ describes the number, in thousands, of high school graduates in the United States x years after 1993.

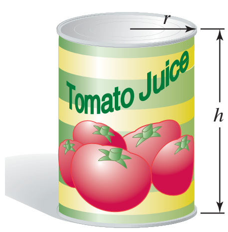
a. According to this polynomial, how many students will graduate from U.S. high schools in 2003?

b. Factor the polynomial.

c. Use the factored form of the polynomial in part (b) to find the number of high school graduates in 2003. Do you get the same answer as you did in part (a)? If so, does this prove that your factorization is correct? Explain.

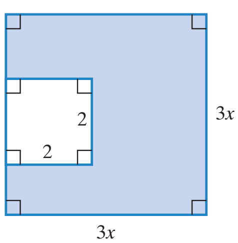
97. A rock is dropped from the top of a 256-foot cliff. The height, in feet, of the rock above the water after t seconds is described by the polynomial $256 - 16t^2$. Factor this expression completely.

98. The amount of sheet metal needed to manufacture a cylindrical tin can, that is, its surface area, S, is $S = 2\pi r^2 + 2\pi rh$. Express the surface area, S, in factored form.

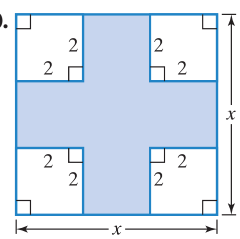


In Exercises 99–100, find the formula for the area of the shaded region and express it in factored form.

99.



100.



Writing in Mathematics

101. Using an example, explain how to factor out the greatest common factor of a polynomial.

102. Suppose that a polynomial contains four terms. Explain how to use factoring by grouping to factor the polynomial.

103. Explain how to factor $3x^2 + 10x + 8$.

104. Explain how to factor the difference of two squares. Provide an example with your explanation.

105. What is a perfect square trinomial and how is it factored?

106. Explain how to factor $x^3 + 1$.

107. What does it mean to factor completely?

Critical Thinking Exercises

108. Which one of the following is true?

a. Because $x^2 + 1$ is irreducible over the integers, it follows that $x^3 + 1$ is also irreducible.

b. One correct factored form for $x^2 - 4x + 3$ is $x(x - 4) + 3$.

c. $x^3 - 64 = (x - 4)^3$

d. None of the above is true.

In Exercises 109–112, factor completely.

109. $x^{2n} + 6x^n + 8$

110. $-x^2 - 4x + 5$

111. $x^4 - y^4 - 2x^3y + 2xy^3$

112. $(x - 5)^{-1/2}(x + 5)^{-1/2} - (x + 5)^{1/2}(x - 5)^{-3/2}$

In Exercises 113–114, find all integers b so that the trinomial can be factored.

113. $x^2 + bx + 15$

114. $x^2 + 4x + b$

Group Exercise

115. Without looking at any factoring problems in the book, create five factoring problems. Make sure that some of your problems require at least two factoring techniques. Next, exchange problems with another person in your group. Work to factor your partner's problems. Evaluate the problems as you work: Are they too easy? Too difficult? Can the polynomials really be factored? Share your response with the person who wrote the problems. Finally, grade each other's work in factoring the polynomials. Each factoring problem is worth 20 points. You may award partial credit. If you take off points, explain why points are deducted and how you decided to take off a particular number of points for the error(s) that you found.

SECTION P.6 Rational Expressions

Objectives

1. Specify numbers that must be excluded from the domain of rational expressions.
2. Simplify rational expressions.
3. Multiply rational expressions.
4. Divide rational expressions.
5. Add and subtract rational expressions.
6. Simplify complex rational expressions.

How do we describe the costs of reducing environmental pollution? We often use algebraic expressions involving quotients of polynomials. For example, the algebraic expression

$$\frac{250x}{100 - x}$$

describes the cost, in millions of dollars, to remove x percent of the pollutants that are discharged into a river. Removing a modest percentage of pollutants, say 40%, is far less costly than removing a substantially greater percentage, such as 95%. We see this by evaluating the algebraic expression for $x = 40$ and $x = 95$.

Evaluating $\frac{250x}{100 - x}$ for

$x = 40$:

Cost is $\frac{250(40)}{100 - 40} \approx 167.$

$x = 95$:

Cost is $\frac{250(95)}{100 - 95} = 4750.$

The cost increases from approximately \$167 million to a possibly prohibitive \$4750 million, or \$4.75 billion. Costs spiral upward as the percentage of removed pollutants increases.

Many algebraic expressions that describe costs of environmental projects are examples of rational expressions. First we will define rational expressions. Then we will review how to perform operations with such expressions.

Discovery

What happens if you try substituting 100 for x in

$$\frac{250x}{100 - x}?$$

What does this tell you about the cost of cleaning up all of the river's pollutants?

1 Specify numbers that must be excluded from the domain of rational expressions.

Rational Expressions

A **rational expression** is the quotient of two polynomials. Some examples are

$$\frac{x-2}{4}, \quad \frac{4}{x-2}, \quad \frac{x}{x^2-1}, \quad \text{and} \quad \frac{x^2+1}{x^2+2x-3}.$$

The set of real numbers for which an algebraic expression is defined is the **domain** of the expression. Because rational expressions indicate division and division by zero is undefined, we must exclude numbers from a rational expression's domain that make the denominator zero.

EXAMPLE 1 Excluding Numbers from the Domain

Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{4}{x-2}$ **b.** $\frac{x}{x^2-1}$.

Solution To determine the numbers that must be excluded from each domain, examine the denominators.

a. $\frac{4}{x-2}$ **b.** $\frac{x}{x^2-1} = \frac{x}{(x+1)(x-1)}$

This denominator would equal zero if $x = 2$.

This factor would equal zero if $x = -1$.

This factor would equal zero if $x = 1$.

For the rational expression in part (a), we must exclude 2 from the domain. For the rational expression in part (b), we must exclude both −1 and 1 from the domain. These excluded numbers are often written to the right of a rational expression.

$$\frac{4}{x-2},\ x \neq 2 \qquad \frac{x}{x^2-1},\ x \neq -1,\ x \neq 1$$

Check Point 1 Find all the numbers that must be excluded from the domain of each rational expression:

a. $\frac{7}{x+5}$ **b.** $\frac{x}{x^2-36}$.

2 Simplify rational expressions.

Simplifying Rational Expressions

A rational expression is **simplified** if its numerator and denominator have no common factors other than 1 or −1. The following procedure can be used to simplify rational expressions:

Simplifying Rational Expressions

1. Factor the numerator and denominator completely.
2. Divide both the numerator and denominator by the common factors.

EXAMPLE 2 Simplifying Rational Expressions

Simplify: **a.** $\frac{x^3+x^2}{x+1}$ **b.** $\frac{x^2+6x+5}{x^2-25}$.

Solution

a. $\frac{x^3 + x^2}{x + 1} = \frac{x^2(x + 1)}{x + 1}$ — Factor the numerator. Because the denominator is $x + 1$, $x \neq -1$.

$= \frac{x^2\cancel{(x + 1)}^{1}}{\cancel{x + 1}_{1}}$ — Divide out the common factor, $x + 1$.

$= x^2, x \neq -1$ — Denominators of 1 need not be written because $\frac{a}{1} = a$.

b. $\frac{x^2 + 6x + 5}{x^2 - 25} = \frac{(x + 5)(x + 1)}{(x + 5)(x - 5)}$ — Factor the numerator and denominator. Because the denominator is $(x + 5)(x - 5)$, $x \neq -5$ and $x \neq 5$.

$= \frac{\cancel{(x + 5)}^{1}(x + 1)}{\cancel{(x + 5)}_{1}(x - 5)}$ — Divide out the common factor, $x + 5$.

$= \frac{x + 1}{x - 5}, \quad x \neq -5, \quad x \neq 5$

Check Point 2 Simplify:

a. $\frac{x^3 + 3x^2}{x + 3}$ **b.** $\frac{x^2 - 1}{x^2 + 2x + 1}$.

3 Multiply rational expressions.

Multiplying Rational Expressions

The product of two rational expressions is the product of their numerators divided by the product of their denominators. Here is a step-by-step procedure for multiplying rational expressions:

Multiplying Rational Expressions

1. Factor all numerators and denominators completely.
2. Divide numerators and denominators by common factors.
3. Multiply the remaining factors in the numerator and multiply the remaining factors in the denominator.

EXAMPLE 3 Multiplying Rational Expressions

Multiply and simplify:

$$\frac{x - 7}{x - 1} \cdot \frac{x^2 - 1}{3x - 21}.$$

Solution

$$\frac{x-7}{x-1}\cdot\frac{x^2-1}{3x-21}$$ This is the given multiplication problem.

$$=\frac{x-7}{x-1}\cdot\frac{(x+1)(x-1)}{3(x-7)}$$ Factor all numerators and denominators. Because the denominator has factors of $x-1$ and $x-7$, $x \neq 1$ and $x \neq 7$.

$$=\frac{\overset{1}{\cancel{x-7}}}{\underset{1}{\cancel{x-1}}}\cdot\frac{(x+1)\overset{1}{\cancel{(x-1)}}}{3\underset{1}{\cancel{(x-7)}}}$$ Divide numerators and denominators by common factors.

$$=\frac{x+1}{3}, x \neq 1, x \neq 7$$ Multiply the remaining factors in the numerator and denominator.

These excluded numbers from the domain must also be excluded from the simplified expression's domain.

Check Point 3 Multiply and simplify:

$$\frac{x+3}{x^2-4}\cdot\frac{x^2-x-6}{x^2+6x+9}.$$

4 Divide rational expressions.

Dividing Rational Expressions

We find the quotient of two rational expressions by inverting the divisor and multiplying.

EXAMPLE 4 Dividing Rational Expressions

Divide and simplify:

$$\frac{x^2-2x-8}{x^2-9}\div\frac{x-4}{x+3}.$$

Solution

$$\frac{x^2-2x-8}{x^2-9}\div\frac{x-4}{x+3}$$ This is the given division problem.

$$=\frac{x^2-2x-8}{x^2-9}\cdot\frac{x+3}{x-4}$$ Invert the divisor and multiply.

$$=\frac{(x-4)(x+2)}{(x+3)(x-3)}\cdot\frac{x+3}{x-4}$$ Factor throughout. For nonzero denominators, $x \neq -3$, $x \neq 3$, and $x \neq 4$.

$$=\frac{\overset{1}{\cancel{(x-4)}}(x+2)}{\underset{1}{\cancel{(x+3)}}(x-3)}\cdot\frac{\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-4)}}}$$ Divide numerators and denominators by common factors.

$$=\frac{x+2}{x-3}, x \neq -3, x \neq 3, x \neq 4$$ Multiply the remaining factors in the numerator and the denominator.

Check Point 4 Divide and simplify:

$$\frac{x^2-2x+1}{x^3+x}\div\frac{x^2+x-2}{3x^2+3}.$$

5 Add and subtract rational expressions.

Adding and Subtracting Rational Expressions with the Same Denominator

We add or subtract rational expressions with the same denominator by (1) adding or subtracting the numerators, (2) placing this result over the common denominator, and (3) simplifying, if possible.

EXAMPLE 5 Subtracting Rational Expressions with the Same Denominator

Subtract: $\frac{5x+1}{x^2-9} - \frac{4x-2}{x^2-9}$.

Study Tip

Example 5 shows that when a numerator is being subtracted, we must subtract every term in that expression.

Solution

$$\frac{5x+1}{x^2-9} - \frac{4x-2}{x^2-9} = \frac{5x+1-(4x-2)}{x^2-9}$$

Subtract numerators and include parentheses to indicate that both terms are subtracted. Place this difference over the common denominator.

$$= \frac{5x+1-4x+2}{x^2-9}$$

Remove parentheses and then change the sign of each term.

$$= \frac{x+3}{x^2-9}$$

Combine like terms.

$$= \frac{\overset{1}{\cancel{x+3}}}{\underset{1}{\cancel{(x+3)}}(x-3)}$$

Factor and simplify ($x \neq -3$ and $x \neq 3$).

$$= \frac{1}{x-3}, \; x \neq -3, x \neq 3$$

Check Point 5 Subtract: $\frac{x}{x+1} - \frac{3x+2}{x+1}$.

Adding and Subtracting Rational Expressions with Different Denominators

Rational expressions that have no common factors in their denominators can be added or subtracted using one of the following properties:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}, \; b \neq 0, d \neq 0.$$

The denominator, bd, is the product of the factors in the two denominators. Because we are looking at rational expressions that have no common factors in their denominators, the product bd gives the least common denominator.

EXAMPLE 6 Subtracting Rational Expressions Having No Common Factors in Their Denominators

Subtract: $\frac{x+2}{2x-3} - \frac{4}{x+3}$.

Solution We need to find the least common denominator. This is the product of the distinct factors in each denominator, namely $(2x-3)(x+3)$. We can therefore use the subtraction property given previously as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{x+2}{2x-3} - \frac{4}{x+3} = \frac{(x+2)(x+3) - (2x-3)4}{(2x-3)(x+3)}$$

Observe that $a = x + 2$, $b = 2x - 3$, $c = 4$, and $d = x + 3$.

$$= \frac{x^2 + 5x + 6 - (8x - 12)}{(2x-3)(x+3)}$$

Multiply.

$$= \frac{x^2 + 5x + 6 - 8x + 12}{(2x-3)(x+3)}$$

Remove parentheses and then change the sign of each term.

$$= \frac{x^2 - 3x + 18}{(2x-3)(x+3)}, x \neq \frac{3}{2}, x \neq -3$$

Combine like terms in the numerator.

Check Point 6 Add: $\frac{3}{x+1} + \frac{5}{x-1}$.

The **least common denominator,** or LCD, of several rational expressions is a polynomial consisting of the product of all prime factors in the denominators, with each factor raised to the greatest power of its occurrence in any denominator. When adding and subtracting rational expressions that have different denominators with one or more common factors in the denominators, it is efficient to find the least common denominator first.

Finding the Least Common Denominator

1. Factor each denominator completely.
2. List the factors of the first denominator.
3. Add to the list in step 2 any factors of the second denominator that do not appear in the list.
4. Form the product of each different factor from the list in step 3. This product is the least common denominator.

EXAMPLE 7 Finding the Least Common Denominator

Find the least common denominator of

$$\frac{7}{5x^2 + 15x} \quad \text{and} \quad \frac{9}{x^2 + 6x + 9}.$$

Solution

Step 1 Factor each denominator completely.

$$5x^2 + 15x = 5x(x + 3)$$
$$x^2 + 6x + 9 = (x + 3)^2$$

Step 2 List the factors of the first denominator.

$$5, x, (x + 3)$$

Step 3 **Add any unlisted factors from the second denominator.** The second denominator is $(x+3)^2$ or $(x+3)(x+3)$. One factor of $x+3$ is already in our list, but the other factor is not. We add $x+3$ to the list. We have

$$5, x, (x+3), (x+3).$$

Step 4 **The least common denominator is the product of all factors in the final list.** Thus,

$$5x(x+3)(x+3), \quad \text{or} \quad 5x(x+3)^2$$

is the least common denominator.

Check Point 7 Find the least common denominator of

$$\frac{3}{x^2-6x+9} \quad \text{and} \quad \frac{7}{x^2-9}.$$

Finding the least common denominator for two (or more) rational expressions is the first step needed to add or subtract the expressions.

Adding and Subtracting Rational Expressions That Have Different Denominators with Shared Factors

1. Find the least common denominator.
2. Write all rational expressions in terms of the least common denominator. To do so, multiply both the numerator and the denominator of each rational expression by any factor(s) needed to convert the denominator into the least common denominator.
3. Add or subtract the numerators, placing the resulting expression over the least common denominator.
4. If necessary, simplify the resulting rational expression.

EXAMPLE 8 Adding Rational Expressions with Different Denominators

Add: $\dfrac{x+3}{x^2+x-2} + \dfrac{2}{x^2-1}$.

Solution

Step 1 **Find the least common denominator.** Start by factoring the denominators.

$$x^2+x-2 = (x+2)(x-1)$$
$$x^2-1 = (x+1)(x-1)$$

The factors of the first denominator are $x+2$ and $x-1$. The only factor from the second denominator that is not listed is $x+1$. Thus, the least common denominator is

$$(x+2)(x-1)(x+1).$$

Step 2 **Write all rational expressions in terms of the least common denominator.** We do so by multiplying both the numerator and the denominator by any factor(s) needed to convert the denominator into the least common denominator.

$$\frac{x+3}{x^2+x-2}+\frac{2}{x^2-1}$$

$$=\frac{x+3}{(x+2)(x-1)}+\frac{2}{(x+1)(x-1)}$$

The least common denominator is $(x+2)(x-1)(x+1)$.

$$=\frac{(x+3)(x+1)}{(x+2)(x-1)(x+1)}+\frac{2(x+2)}{(x+2)(x-1)(x+1)}$$

Multiply each numerator and denominator by the extra factor required to form $(x+2)(x-1)(x+1)$, the least common denominator.

Step 3 **Add numerators, putting this sum over the least common denominator.**

$$=\frac{(x+3)(x+1)+2(x+2)}{(x+2)(x-1)(x+1)}$$

$$=\frac{x^2+4x+3+2x+4}{(x+2)(x-1)(x+1)}$$

Perform the multiplications in the numerator.

$$=\frac{x^2+6x+7}{(x+2)(x-1)(x+1)},\ x\neq-2,\ x\neq1,\ x\neq-1$$

Combine like terms in the numerator.

Step 4 **If necessary, simplify.** Because the numerator is prime, no further simplification is possible.

Check Point 8 Subtract: $\dfrac{x}{x^2-10x+25}-\dfrac{x-4}{2x-10}$.

6 Simplify complex rational expressions.

Complex Rational Expressions

Complex rational expressions, also called **complex fractions,** have numerators or denominators containing one or more rational expressions. Here are two examples of such expressions:

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

Separate rational expressions occur in the numerator and denominator.

$$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}.$$

Separate rational expressions occur in the numerator.

One method for simplifying a complex rational expression is to combine its numerator into a single expression and combine its denominator into a single expression. Then perform the division by inverting the denominator and multiplying.

EXAMPLE 9 Simplifying a Complex Rational Expression

Simplify: $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$.

Solution

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x}{x}+\frac{1}{x}}{\frac{x}{x}-\frac{1}{x}},\ x \neq 0$$

The terms in the numerator and in the denominator are each combined by performing the addition and subtraction. The least common denominator is x.

$$= \frac{\frac{x+1}{x}}{\frac{x-1}{x}}$$

Perform the addition in the numerator and the subtraction in the denominator.

$$= \frac{x+1}{x} \div \frac{x-1}{x}$$

Rewrite the main fraction bar as ÷.

$$= \frac{x+1}{x} \cdot \frac{x}{x-1}$$

Invert the divisor and multiply ($x \neq 0$ and $x \neq 1$).

$$= \frac{x+1}{\not{x}_1} \cdot \frac{\overset{1}{\not{x}}}{x-1}$$

Divide a numerator and denominator by the common factor, x.

$$= \frac{x+1}{x-1},\ x \neq 0,\ x \neq 1$$

Multiply the remaining factors in the numerator and in the denominator.

Check Point 9 Simplify: $\dfrac{\dfrac{1}{x}-\dfrac{3}{2}}{\dfrac{1}{x}+\dfrac{3}{4}}$.

A second method for simplifying a complex rational expression is to find the least common denominator of all the rational expressions in its numerator and denominator. Then multiply each term in its numerator and denominator by this least common denominator. Here we use this method to simplify the complex rational expression in Example 9.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\left(1+\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)} \cdot \frac{x}{x}$$

The least common denominator of all the rational expressions is x. Multiply the numerator and denominator by x. Because $\frac{x}{x} = 1$, we are not changing the complex fraction ($x \neq 0$).

$$= \frac{1 \cdot x + \frac{1}{x} \cdot x}{1 \cdot x - \frac{1}{x} \cdot x}$$

Use the distributive property. Be sure to distribute x to every term.

$$= \frac{x+1}{x-1},\ x \neq 0,\ x \neq 1$$

Multiply. The complex rational expression is now simplified.

EXERCISE SET P.6

Practice Exercises

In Exercises 1–6, find all numbers that must be excluded from the domain of each rational expression.

1. $\frac{7}{x-3}$

2. $\frac{13}{x+9}$

3. $\frac{x+5}{x^2-25}$

4. $\frac{x+7}{x^2-49}$

5. $\frac{x-1}{x^2+11x+10}$

6. $\frac{x-3}{x^2+4x-45}$

In Exercises 7–14, simplify each rational expression. Find all numbers that must be excluded from the domain of the simplified rational expression.

7. $\frac{3x-9}{x^2-6x+9}$

8. $\frac{4x-8}{x^2-4x+4}$

9. $\frac{x^2-12x+36}{4x-24}$

10. $\frac{x^2-8x+16}{3x-12}$

11. $\frac{y^2+7y-18}{y^2-3y+2}$

12. $\frac{y^2-4y-5}{y^2+5y+4}$

13. $\frac{x^2+12x+36}{x^2-36}$

14. $\frac{x^2-14x+49}{x^2-49}$

In Exercises 15–32, multiply or divide as indicated.

15. $\frac{x-2}{3x+9} \cdot \frac{2x+6}{2x-4}$

16. $\frac{6x+9}{3x-15} \cdot \frac{x-5}{4x+6}$

17. $\frac{x^2-9}{x^2} \cdot \frac{x^2-3x}{x^2+x-12}$

18. $\frac{x^2-4}{x^2-4x+4} \cdot \frac{2x-4}{x+2}$

19. $\frac{x^2-5x+6}{x^2-2x-3} \cdot \frac{x^2-1}{x^2-4}$

20. $\frac{x^2+5x+6}{x^2+x-6} \cdot \frac{x^2-9}{x^2-x-6}$

21. $\frac{x^3-8}{x^2-4} \cdot \frac{x+2}{3x}$

22. $\frac{x^2+6x+9}{x^3+27} \cdot \frac{1}{x+3}$

23. $\frac{x+1}{3} \div \frac{3x+3}{7}$

24. $\frac{x+5}{7} \div \frac{4x+20}{9}$

25. $\frac{x^2-4}{x} \div \frac{x+2}{x-2}$

26. $\frac{x^2-4}{x-2} \div \frac{x+2}{4x-8}$

27. $\frac{4x^2+10}{x-3} \div \frac{6x^2+15}{x^2-9}$

28. $\frac{x^2+x}{x^2-4} \div \frac{x^2-1}{x^2+5x+6}$

29. $\frac{x^2-25}{2x-2} \div \frac{x^2+10x+25}{x^2+4x-5}$

30. $\frac{x^2-4}{x^2+3x-10} \div \frac{x^2+5x+6}{x^2+8x+15}$

31. $\frac{x^2+x-12}{x^2+x-30} \cdot \frac{x^2+5x+6}{x^2-2x-3} \div \frac{x+3}{x^2+7x+6}$

32. $\frac{x^3-25x}{4x^2} \cdot \frac{2x^2-2}{x^2-6x+5} \div \frac{x^2+5x}{7x+7}$

In Exercises 33–54, add or subtract as indicated.

33. $\frac{4x+1}{6x+5} + \frac{8x+9}{6x+5}$

34. $\frac{3x+2}{3x+4} + \frac{3x+6}{3x+4}$

35. $\frac{x^2-2x}{x^2+3x} + \frac{x^2+x}{x^2+3x}$

36. $\frac{x^2-4x}{x^2-x-6} + \frac{4x-4}{x^2-x-6}$

37. $\frac{4x-10}{x-2} - \frac{x-4}{x-2}$

38. $\frac{2x+3}{3x-6} - \frac{3-x}{3x-6}$

39. $\frac{x^2+3x}{x^2+x-12} - \frac{x^2-12}{x^2+x-12}$

40. $\frac{x^2-4x}{x^2-x-6} - \frac{x-6}{x^2-x-6}$

41. $\frac{3}{x+4} + \frac{6}{x+5}$

42. $\frac{8}{x-2} + \frac{2}{x-3}$

43. $\frac{3}{x+1} - \frac{3}{x}$

44. $\frac{4}{x} - \frac{3}{x+3}$

45. $\frac{2x}{x+2} + \frac{x+2}{x-2}$

46. $\frac{3x}{x-3} - \frac{x+4}{x+2}$

47. $\frac{x+5}{x-5} + \frac{x-5}{x+5}$

48. $\frac{x+3}{x-3} + \frac{x-3}{x+3}$

49. $\frac{4}{x^2+6x+9} + \frac{4}{x+3}$

50. $\frac{3}{5x+2} + \frac{5x}{25x^2-4}$

51. $\frac{3x}{x^2+3x-10} - \frac{2x}{x^2+x-6}$

52. $\frac{x}{x^2-2x-24} - \frac{x}{x^2-7x+6}$

53. $\frac{4x^2+x-6}{x^2+3x+2} - \frac{3x}{x+1} + \frac{5}{x+2}$

54. $\frac{6x^2+17x-40}{x^2+x-20} + \frac{3}{x-4} - \frac{5x}{x+5}$

In Exercise 55–64, simplify each complex rational expression.

55. $\dfrac{\frac{x}{3} - 1}{x - 3}$

56. $\dfrac{\frac{x}{4} - 1}{x - 4}$

57. $\dfrac{1 + \frac{1}{x}}{3 - \frac{1}{x}}$

58. $\dfrac{8 + \frac{1}{x}}{4 - \frac{1}{x}}$

59. $\dfrac{\frac{1}{x} + \frac{1}{y}}{x + y}$

60. $\dfrac{1 - \frac{1}{x}}{xy}$

61. $\dfrac{x - \frac{x}{x + 3}}{x + 2}$

62. $\dfrac{x - 3}{x - \frac{3}{x - 2}}$

63. $\dfrac{\frac{3}{x - 2} - \frac{4}{x + 2}}{\frac{7}{x^2 - 4}}$

64. $\dfrac{\frac{x}{x - 2} + 1}{\frac{3}{x^2 - 4} + 1}$

Application Exercises

65. The rational expression

$$\frac{130x}{100 - x}$$

describes the cost, in millions of dollars, to inoculate x percent of the population against a particular strain of flu.

a. Evaluate the expression for $x = 40$, $x = 80$, and $x = 90$. Describe the meaning of each evaluation in terms of percentage inoculated and cost.

b. For what value of x is the expression undefined?

c. What happens to the cost as x approaches 100%? How can you interpret this observation?

66. Doctors use the rational expression

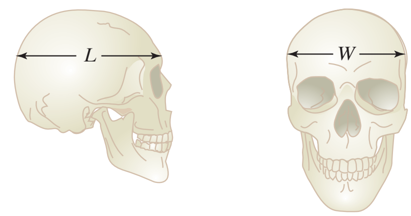
$$\frac{DA}{A + 12}$$

to determine the dosage of a drug prescribed for children. In this expression, A = child's age, and D = adult dosage. What is the difference in the child's dosage for a 7-year-old child and a 3-year-old child? Express the answer as a single rational expression in terms of D. Then describe what your answer means in terms of the variables in the rational expression.

67. Anthropologists and forensic scientists classify skulls using

$$\frac{L + 60W}{L} - \frac{L - 40W}{L}$$

where L is the skull's length and W is its width.



a. Express the classification as a single rational expression.

b. If the value of the rational expression in part (a) is less than 75, a skull is classified as long. A medium skull has a value between 75 and 80, and a round skull has a value over 80. Use your rational expression from part (a) to classify a skull that is 5 inches wide and 6 inches long.

68. The polynomial

$$6t^4 - 207t^3 + 2128t^2 - 6622t + 15{,}220$$

describes the annual number of drug convictions in the United States t years after 1984. The polynomial

$$28t^4 - 711t^3 + 5963t^2 - 1695t + 27{,}424$$

describes the annual number of drug arrests in the United States t years after 1984. Write a rational expression that describes the conviction rate for drug arrests in the United States t years after 1984.

69. The average speed on a round-trip commute having a one-way distance d is given by the complex rational expression

$$\frac{2d}{\frac{d}{r_1} + \frac{d}{r_2}}$$

in which r_1 and r_2 are the speeds on the outgoing and return trips, respectively. Simplify the expression. Then find the average speed for a person who drives from home to work at 30 miles per hour and returns on the same route averaging 20 miles per hour. Explain why the answer is not 25 miles per hour.

Writing in Mathematics

70. What is a rational expression?

71. Explain how to determine which numbers must be excluded from the domain of a rational expression.

72. Explain how to simplify a rational expression.

73. Explain how to multiply rational expressions.

74. Explain how to divide rational expressions.

75. Explain how to add or subtract rational expressions with the same denominators.

76. Explain how to add rational expressions having no common factors in their denominators. Use $\frac{3}{x+5} + \frac{7}{x+2}$ in your explanation.

77. Explain how to find the least common denominator for denominators of $x^2 - 100$ and $x^2 - 20x + 100$.

78. Describe two ways to simplify $\dfrac{\frac{3}{x} + \frac{2}{x^2}}{\frac{1}{x^2} + \frac{2}{x}}$.

Explain the error in Exercises 79–81. Then rewrite the right side of the equation to correct the error that now exists.

79. $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$

80. $\frac{1}{x} + 7 = \frac{1}{x+7}$

81. $\frac{a}{x} + \frac{a}{b} = \frac{a}{x+b}$

82. A politician claims that each year the conviction rate for drug arrests in the United States is increasing. Explain how to use the polynomials in Exercise 68 to verify this claim.

Technology Exercises

83. How much are your monthly payments on a loan? If P is the principal, or amount borrowed, i is the monthly interest rate, and n is the number of monthly payments, then the amount, A, of each monthly payment is

$$A = \frac{Pi}{1 - \frac{1}{(1+i)^n}}.$$

a. Simplify the complex rational expression for the amount of each payment.

b. You purchase a \$20,000 automobile at 1% monthly interest to be paid over 48 months. How much do you pay each month? Use the simplified rational expression from part (a) and a calculator. Round to the nearest dollar.

Critical Thinking Exercises

84. Which one of the following is true?

a. $\frac{x^2 - 25}{x - 5} = x - 5$

b. $\frac{x}{y} \div \frac{y}{x} = 1$, if $x \neq 0$ and $y \neq 0$.

c. The least common denominator needed to find $\frac{1}{x} + \frac{1}{x+3}$ is $x + 3$.

d. The rational expression

$$\frac{x^2 - 16}{x - 4}$$

is not defined for $x = 4$. However, as x gets closer and closer to 4, the value of the expression approaches 8.

In Exercises 85–86, find the missing expression.

85. $\frac{3x}{x-5} + \frac{\square}{5-x} = \frac{7x+1}{x-5}$

86. $\frac{4}{x-2} - \boxed{\frac{}{}} = \frac{2x+8}{(x-2)(x+1)}$

87. In one short sentence, five words or less, explain what

$$\frac{\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}}{\frac{1}{x^4} + \frac{1}{x^5} + \frac{1}{x^6}}$$

does to each number x.

CHAPTER SUMMARY, REVIEW, AND TEST

Summary: Basic Formulas

Definition of Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Distance between Points *a* and *b* on a Number Line

$$|a - b| \quad \text{or} \quad |b - a|$$

Properties of Algebra

Commutative	$a + b = b + a, \quad ab = ba$
Associative	$(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$
Distributive	$a(b + c) = ab + ac$
Identity	$a + 0 = a \quad a \cdot 1 = a$
Inverse	$a + (-a) = 0 \quad a \cdot \frac{1}{a} = 1, a \neq 0$

Properties of Exponents

$$b^{-n} = \frac{1}{b^n}, \quad b^0 = 1, \quad b^m \cdot b^n = b^{m+n},$$

$$(b^m)^n = b^{mn}, \quad \frac{b^m}{b^n} = b^{m-n}, \quad (ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Product and Quotient Rules for *n*th Roots

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab} \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

Rational Exponents

$$a^{1/n} = \sqrt[n]{a}, \quad a^{-1/n} = \frac{1}{a^{1/n}} = \frac{1}{\sqrt[n]{a}},$$

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}, \quad a^{-m/n} = \frac{1}{a^{m/n}}$$

Special Products

$$(A + B)(A - B) = A^2 - B^2$$
$$(A + B)^2 = A^2 + 2AB + B^2$$
$$(A - B)^2 = A^2 - 2AB + B^2$$
$$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$$
$$(A - B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$$

Factoring Formulas

$$A^2 - B^2 = (A + B)(A - B)$$
$$A^2 + 2AB + B^2 = (A + B)^2$$
$$A^2 - 2AB + B^2 = (A - B)^2$$
$$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$$
$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Review Exercises

You can use these review exercises, like the review exercises at the end of each chapter, to test your understanding of the chapter's topics. However, you can also use these exercises as a prerequisite test to check your mastery of the fundamental algebra skills needed in this book.

P.1

1. Consider the set:

$$\left\{-17, -\tfrac{9}{13}, 0, 0.75, \sqrt{2}, \pi, \sqrt{81}\right\}.$$

List all numbers from the set that are **a.** natural numbers, **b.** whole numbers, **c.** integers, **d.** rational numbers, **e.** irrational numbers.

In Exercises 2–4, rewrite each expression without absolute value bars.

2. $|-103|$

3. $|\sqrt{2} - 1|$

4. $|3 - \sqrt{17}|$

5. Express the distance between the numbers -17 and 4 using absolute value. Then evaluate the absolute value.

In Exercises 6–7, evaluate each algebraic expression for the given value of the variable.

6. $\frac{5}{9}(F - 32); F = 68$ **7.** $\frac{8(x + 5)}{3x + 8}, x = 2$

In Exercises 8–13, state the name of the property illustrated.

8. $3 + 17 = 17 + 3$ **9.** $(6 \cdot 3) \cdot 9 = 6 \cdot (3 \cdot 9)$

10. $\sqrt{3}(\sqrt{5} + \sqrt{3}) = \sqrt{15} + 3$

11. $(6 \cdot 9) \cdot 2 = 2 \cdot (6 \cdot 9)$

12. $\sqrt{3}(\sqrt{5} + \sqrt{3}) = (\sqrt{5} + \sqrt{3})\sqrt{3}$

13. $(3 \cdot 7) + (4 \cdot 7) = (4 \cdot 7) + (3 \cdot 7)$

In Exercises 14–15, simplify each algebraic expression.

14. $3(7x - 5y) - 2(4y - x + 1)$

15. $\frac{1}{5}(5x) + [(3y) + (-3y)] - (-x)$

P.2

Evaluate each exponential expression in Exercises 16–19.

16. $(-3)^3(-2)^2$ **17.** $2^{-4} + 4^{-1}$

18. $5^{-3} \cdot 5$ **19.** $\frac{3^3}{3^6}$

Simplify each exponential expression in Exercises 20–23.

20. $(-2x^4y^3)^3$ **21.** $(-5x^3y^2)(-2x^{-11}y^{-2})$

22. $(2x^3)^{-4}$ **23.** $\frac{7x^5y^6}{28x^{15}y^{-2}}$

In Exercises 24–25, write each number in decimal notation.

24. 3.74×10^4 **25.** 7.45×10^{-5}

In Exercises 26–27, write each number in scientific notation.

26. 3,590,000 **27.** 0.00725

In Exercises 28–29, perform the indicated operation and write the answer in decimal notation.

28. $(3 \times 10^3)(1.3 \times 10^2)$ **29.** $\frac{6.9 \times 10^3}{3 \times 10^5}$

30. If you earned \$1 million per year ($\$10^6$), how long would it take to accumulate \$1 billion ($\$10^9$)?

31. If the population of the United States is 2.8×10^8 and each person spends about \$150 per year going to the movies (or renting movies), express the total annual spending on movies in scientific notation.

P.3

Use the product rule to simplify the expressions in Exercises 32–35. In Exercises 34–35, assume that variables represent nonnegative real numbers.

32. $\sqrt{300}$ **33.** $\sqrt{12x^2}$

34. $\sqrt{10x} \cdot \sqrt{2x}$ **35.** $\sqrt{r^3}$

Use the quotient rule to simplify the expressions in Exercises 36–37.

36. $\sqrt{\frac{121}{4}}$

37. $\frac{\sqrt{96x^3}}{\sqrt{2x}}$ (Assume that $x > 0$.)

In Exercises 38–40, add or subtract terms whenever possible.

38. $7\sqrt{5} + 13\sqrt{5}$ **39.** $2\sqrt{50} + 3\sqrt{8}$

40. $4\sqrt{72} - 2\sqrt{48}$

In Exercises 41–44, rationalize the denominator.

41. $\frac{30}{\sqrt{5}}$ **42.** $\frac{\sqrt{2}}{\sqrt{3}}$

43. $\frac{5}{6 + \sqrt{3}}$ **44.** $\frac{14}{\sqrt{7} - \sqrt{5}}$

Evaluate each expression in Exercises 45–48 or indicate that the root is not a real number.

45. $\sqrt[3]{125}$ **46.** $\sqrt[5]{-32}$

47. $\sqrt[4]{-125}$ **48.** $\sqrt[4]{(-5)^4}$

Simplify the radical expressions in Exercises 49–53.

49. $\sqrt[3]{81}$ **50.** $\sqrt[3]{y^5}$

51. $\sqrt[4]{8} \cdot \sqrt[4]{10}$ **52.** $4\sqrt[3]{16} + 5\sqrt[3]{2}$

53. $\frac{\sqrt[4]{32x^5}}{\sqrt[4]{16x}}$ (Assume that $x > 0$.)

In Exercises 54–59, evaluate each expression.

54. $16^{1/2}$ **55.** $25^{-1/2}$

56. $125^{1/3}$ **57.** $27^{-1/3}$

58. $64^{2/3}$ **59.** $27^{-4/3}$

In Exercises 60–62, simplify using properties of exponents.

60. $(5x^{2/3})(4x^{1/4})$ **61.** $\frac{15x^{3/4}}{5x^{1/2}}$

62. $(125x^6)^{2/3}$

63. Simplify by reducing the index of the radical: $\sqrt[6]{y^3}$.

P.4

In Exercises 64–65, perform the indicated operations. Write the resulting polynomial in standard form and indicate its degree.

64. $(-6x^3 + 7x^2 - 9x + 3) + (14x^3 + 3x^2 - 11x - 7)$

65. $(13x^4 - 8x^3 + 2x^2) - (5x^4 - 3x^3 + 2x^2 - 6)$

In Exercises 66–72, find each product.

66. $(3x - 2)(4x^2 + 3x - 5)$ **67.** $(3x - 5)(2x + 1)$

68. $(4x + 5)(4x - 5)$ **69.** $(2x + 5)^2$

70. $(3x - 4)^2$ **71.** $(2x + 1)^3$

72. $(5x - 2)^3$

In Exercises 73–74, perform the indicated operations. Indicate the degree of the resulting polynomial.

73. $(7x^2 - 8xy + y^2) + (-8x^2 - 9xy - 4y^2)$

74. $(13x^3y^2 - 5x^2y - 9x^2) - (-11x^3y^2 - 6x^2y + 3x^2 - 4)$

In Exercises 75–79, find each product.

75. $(x + 7y)(3x - 5y)$ **76.** $(3x - 5y)^2$

77. $(3x^2 + 2y)^2$ **78.** $(7x + 4y)(7x - 4y)$

79. $(a - b)(a^2 + ab + b^2)$

P.5

In Exercises 80–96, factor completely, or state that the polynomial is prime.

80. $15x^3 + 3x^2$ **81.** $x^2 - 11x + 28$

82. $15x^2 - x - 2$ **83.** $64 - x^2$

84. $x^2 + 16$ **85.** $3x^4 - 9x^3 - 30x^2$

86. $20x^7 - 36x^3$ **87.** $x^3 - 3x^2 - 9x + 27$

88. $16x^2 - 40x + 25$ **89.** $x^4 - 16$

90. $y^3 - 8$ **91.** $x^3 + 64$

92. $3x^4 - 12x^2$ **93.** $27x^3 - 125$

94. $x^5 - x$ **95.** $x^3 + 5x^2 - 2x - 10$

96. $x^2 + 18x + 81 - y^2$

In Exercises 97–99, factor and simplify each algebraic expression.

97. $16x^{-3/4} + 32x^{1/4}$

98. $(x^2 - 4)(x^2 + 3)^{1/2} - (x^2 - 4)^2(x^2 + 3)^{3/2}$

99. $12x^{-1/2} + 6x^{-3/2}$

P.6

In Exercises 100–102, simplify each rational expression. Also, list all numbers that must be excluded from the domain.

100. $\dfrac{x^3 + 2x^2}{x + 2}$ **101.** $\dfrac{x^2 + 3x - 18}{x^2 - 36}$

102. $\dfrac{x^2 + 2x}{x^2 + 4x + 4}$

In Exercises 103–105, multiply or divide as indicated.

103. $\dfrac{x^2 + 6x + 9}{x^2 - 4} \cdot \dfrac{x + 3}{x - 2}$ **104.** $\dfrac{6x + 2}{x^2 - 1} \div \dfrac{3x^2 + x}{x - 1}$

105. $\dfrac{x^2 - 5x - 24}{x^2 - x - 12} \div \dfrac{x^2 - 10x + 16}{x^2 + x - 6}$

In Exercises 106–109, add or subtract as indicated.

106. $\dfrac{2x - 7}{x^2 - 9} - \dfrac{x - 10}{x^2 - 9}$ **107.** $\dfrac{3x}{x + 2} + \dfrac{x}{x - 2}$

108. $\dfrac{x}{x^2 - 9} + \dfrac{x - 1}{x^2 - 5x + 6}$

109. $\dfrac{4x - 1}{2x^2 + 5x - 3} - \dfrac{x + 3}{6x^2 + x - 2}$

In Exercises 110–112, simplify each complex rational expression.

110. $\dfrac{\frac{1}{x} - \frac{1}{2}}{\frac{1}{3} - \frac{x}{6}}$ **111.** $\dfrac{3 + \frac{12}{x}}{1 - \frac{16}{x^2}}$ **112.** $\dfrac{3 - \frac{1}{x + 3}}{3 + \frac{1}{x + 3}}$

Chapter P Test

1. List all the rational numbers in this set:

$$\{-7, -\tfrac{4}{5}, 0, 0.25, \sqrt{3}, \sqrt{4}, \tfrac{22}{7}, \pi\}.$$

In Exercises 2–3, state the name of the property illustrated.

2. $3(2 + 5) = 3(5 + 2)$ **3.** $6(7 + 4) = 6 \cdot 7 + 6 \cdot 4$

4. Express in scientific notation: 0.00076.

Simplify each expression in Exercises 5–11.

5. $9(10x - 2y) - 5(x - 4y + 3)$

6. $\dfrac{30x^3y^4}{6x^9y^{-4}}$

7. $\sqrt{6r}\sqrt{3r}$ (Assume that $r \geq 0$.)

8. $4\sqrt{50} - 3\sqrt{18}$ **9.** $\dfrac{3}{5 + \sqrt{2}}$

10. $\sqrt[3]{16x^4}$ **11.** $\dfrac{x^2 + 2x - 3}{x^2 - 3x + 2}$

12. Evaluate: $27^{-5/3}$.

In Exercises 13–14, find each product.

13. $(2x - 5)(x^2 - 4x + 3)$ **14.** $(5x + 3y)^2$

In Exercises 15–20, factor completely, or state that the polynomial is prime.

15. $x^2 - 9x + 18$

16. $x^3 + 2x^2 + 3x + 6$

17. $25x^2 - 9$

18. $36x^2 - 84x + 49$

19. $y^3 - 125$

20. $x^2 + 10x + 25 - 9y^2$

21. Factor and simplify:

$$x(x + 3)^{-3/5} + (x + 3)^{2/5}.$$

In Exercises 22–25, perform the operations and simplify, if possible.

22. $\dfrac{2x + 8}{x - 3} \div \dfrac{x^2 + 5x + 4}{x^2 - 9}$

23. $\dfrac{x}{x + 3} + \dfrac{5}{x - 3}$

24. $\dfrac{2x + 3}{x^2 - 7x + 12} - \dfrac{2}{x - 3}$

25. $\dfrac{\frac{1}{x} - \frac{1}{3}}{\frac{1}{x}}$