Sequences, Series, Exponential and Logarithmic Functions

1

Unit Overview

In this unit you will study recursive and explicit representations of arithmetic and geometric sequences. You will also study exponential, logarithmic, and power functions and explore the key features of their graphs. In addition, you will look at transformations, compositions, and inverses of functions.

Key Terms

As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary

- converge
- diverge
- depreciation
- conjecture

Math Terms

- sigma notation
- sequence of partial sums
- mathematical induction
- polar grid
- common ratio
- series
- *n*th partial sum
- infinite sequence
- infinite series
- iteration
- recursive sequence
- explicit form
- exponential function
- interest rate
- exponential growth factor
- exponential decay factor
- half-life
- logarithm
- common logarithm
- strictly monotonic
- parent function
- even function
- odd function
- composition
- inverse function

ESSENTIAL QUESTIONS

? How are recursive relationships used to model and investigate long-term behavior involving sequential change?

? How are exponential, logarithmic, and power functions used to model real-world problems?

EMBEDDED ASSESSMENTS

These assessments, following Activities 3, 5, and 8, will provide you opportunities to demonstrate your understanding of sequences, exponential and logarithmic functions, and transformations and compositions of functions.

Getting Ready

Write your answers on notebook paper. Show your work.

1. Solve the system of equations:

$$\begin{cases} 3x + 7y = 6 \\ 2x + 9y = 4 \end{cases}$$

2. Given the equation $2x + 3y = 6$:
- **a.** Find the slope.
- **b.** Graph the equation.
- **c.** Find the slope of a line parallel to the line given by $2x + 3y = 6$.
- **d.** Graph the line that passes through (1, 3) and is parallel to the line given by $2x + 3y = 6$.

3. Simplify the following:
- **a.** $\sqrt{3}(2 + \sqrt{3})$
- **b.** $(3 + \sqrt{5})^2$
- **c.** $\left(\frac{2}{\sqrt{3}}\right)^3$

4. High temperatures for the first 7 days of February in Miami are displayed in the table below.

Day	1	2	3	4	5	6	7
Temp. (°F)	66	63	71	73	75	75	76

- **a.** Make a scatterplot of this data.
- **b.** Estimate a line of best fit for the data.

5. Tell the next term in each of the following, and explain the pattern that generates the sequence.
- **a.** 5, 7, 9, 11, . . .
- **b.** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
- **c.** 8, 7, 5, 2, −2, . . .

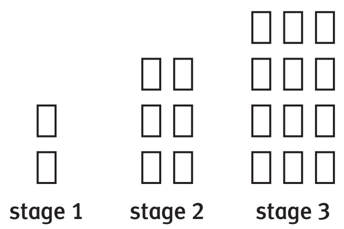
6. Simplify $\left(\frac{6x^2}{y^3}\right)^2$.

7. Factor $x^2 - 6x + 5$. Then solve $x^2 - 6x + 5 = 0$.

8. Find the x- and y-intercepts of the graph of $y = -2(x - 3) + 4$.

9. A simple interest loan with a principal of \$5000 is paid back after $2\frac{1}{2}$ years. The total payment is \$5875. What is the annual rate of interest on the loan?

10. Draw the fourth stage of the figure below. Explain how you would create any figure in the pattern and find the number of squares.



Arithmetic Sequences

ACTIVITY 1

DVD Promotions

Lesson 1-1 Sequences and Subscript Notation

My Notes

Learning Targets:

- Write an expression for a sequence.
- Use subscript notation.

SUGGESTED LEARNING STRATEGIES: Shared Reading, Summarizing, Paraphrasing, Close Reading, Interactive Word Wall, Quickwrite

Bopper's DVD Store has the latest in DVDs for rental or purchase. To attract more customers, Bopper's introduces the following promotion:

Bopper's DVD Store
Earn A Free DVD Rental!
Earn 3 DVD Points With Your 1st DVD Rental
Earn 2 DVD Points For Every Additional Rental
Redeem 24 DVD Points For 1 FREE DVD Rental!

1. Complete the following table to indicate the total number of DVD points after each indicated DVD rental.

Bopper's DVD Store						
DVD Rentals n	1	2	3	4	5	6
Total DVD Points B_n						

2. The table shows that the DVD points earned depend on the number of DVD rentals. Let B_n denote the total number of Bopper's DVD points earned after n rentals. What is the total number of DVD points for one, two, and three rentals?

$B_1 =$ $B_2 =$ $B_3 =$

The notation B_n is called ***subscript notation***. This notation can be used to describe a ***sequence***. In a sequence, B_n denotes the value of the n^{th} term in the sequence, as well as the number of DVD points earned after n rentals. The values $B_1, B_2, B_3, B_4, B_5, \ldots$ form a sequence of values. This sequence of values can be denoted as $\{B_n\}$.

READING MATH

B_1 and B_n are read as "*B* sub 1" and "*B* sub *n*," where "sub" represents *subscript*. In B_n, *n* is the term number, or *index*.

3. Use $\{B_n\}$ to answer the following.

a. List the first eight terms of the sequence $\{B_n\}$.

b. Make use of structure. Explain the meaning of B_7.

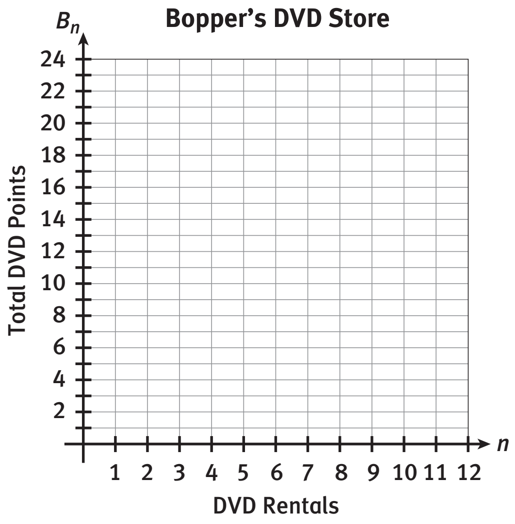
c. Write an algebraic expression for the n^{th} term, B_n, in terms of n, the number of DVD rentals at Bopper's DVD Store.

d. In the context of Bopper's DVD rentals, explain the meaning of n and the algebraic expression written in part c.

My Notes

4. On the coordinate grid below, the horizontal axis represents n, the number of DVD rentals, and the vertical axis represents B_n, the total number of DVD points earned from rentals at Bopper's.

a. Plot the sequence $\{B_n\}$ for $n = 1, 2, 3, \ldots 8$.



b. Using your graph, explain why $\{B_n\}$ is a function of n.

c. List as many properties of the graph of $\{B_n\}$ as possible.

5. How many rentals are needed to obtain a free DVD rental from Bopper's? Show the work that leads to your answer.

Check Your Understanding

6. Given the sequence 1, 8, 15, . . . :

a. Write an expression for a_n in terms of n, the term number.

b. Calculate a_{30}.

c. Given $a_n = 148$, solve for n.

7. Write an algebraic expression for the sequence {2, 5, 8, . . ., 20} in terms of n, the term number.

8. A sequence is defined by $a_1 = 5$, $a_{n+1} = 8 + a_n$. Write the first five terms in the sequence.

My Notes

A new video rental business, Fantastik Flicks, opens near Bopper's. The new store offers its own DVD point program to attract customers.

Fantastik Flicks DVD Store
Earn more DVD Points with each rental at Flicks!
Earn a free DVD rental faster at Flicks!
Earn 3 DVD Points with the first rental.
Earn 5 DVD Points with the second rental.
Earn 7 DVD Points with the third rental, and so on.
Free DVD rental for every 100 Points earned at Flicks!

9. Complete the following table to indicate the number of DVD points earned with the n^{th} rental and the total accumulated number of DVD points after each Fantastik Flicks DVD rental.

Fantastik Flicks DVD Store						
DVD Rentals, n	1	2	3	4	5	6
DVD Points Earned with n^{th} Rental, P_n	3	5	7	9	11	13
Total DVD Points After n Rentals, F_n						

10. Model with mathematics. The table in Item 9 indicates that there are two sequences associated with Fantastik Flicks DVD points, P_n and F_n.

a. At Fantastik Flicks, the number of points earned increases with each rental of a DVD. Write an algebraic expression for P_n, the number of points earned with the n^{th} rental, in terms of n.

b. Explain how the values in the third row of the table in Item 9 are obtained from the values in the second row.

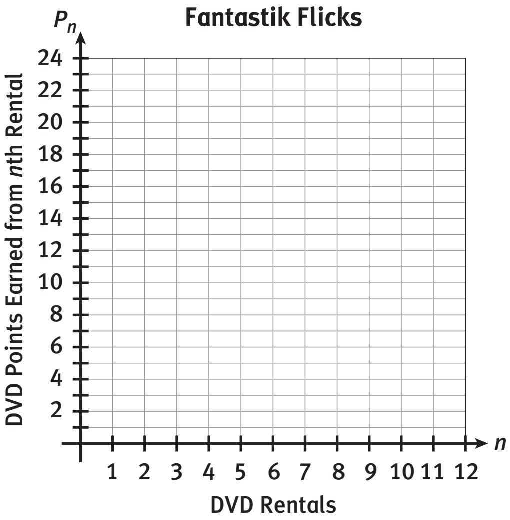
c. Find the value of the sum $P_1 + P_2 + P_3 + P_4 + P_5 + P_6 + P_7$.

d. What is the value of F_8? Show your work.

e. Write an equation that expresses F_{n+1} in terms of F_n and P_{n+1}.

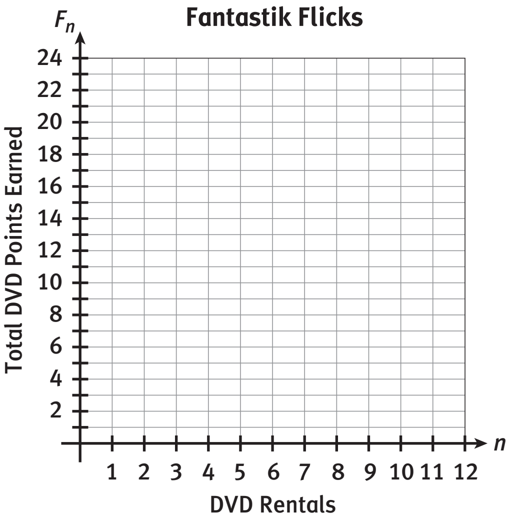
My Notes

11. On the coordinate grid below, the horizontal axis represents n, the number of DVD rentals at Fantastik Flicks, and the vertical axis represents P_n, the number of points earned per DVD rental. Use the grid to plot the sequence P_n for $n = 1, 2, \ldots, 8$.



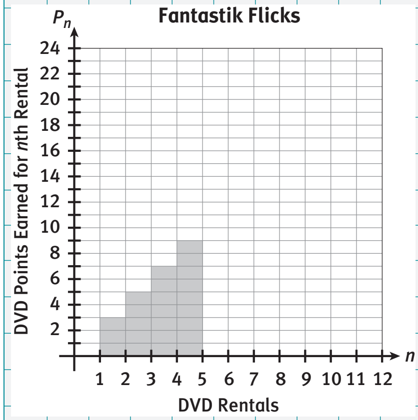
12. On the coordinate grid below, the vertical axis represents F_n, the total number of accumulated points earned after n rentals.

a. Plot the sequence F_n for $n = 1, 2, 3$, and 4.



b. Construct viable arguments. Does the graph of F_n appear to be linear? Explain your answer.

My Notes



13. The graph shows the first four terms of the sequence $\{P_n\}$ as a shaded area.

a. Find the area of this shaded region. Explain how this area relates to the Fantastik Flicks DVD rental point promotion.

b. Extend the plot of the sequence $\{P_n\}$ to $n = 5, 6, 7$, and 8. Find the area of the region under each of these plotted values and the new total area for each.

c. Investigate the relationship of the area under the plotted points of P_n to the accumulated number of Fantastik Flicks DVD points earned after n rentals for $n = 5, 6, 7$, and 8. Does the pattern discovered in part a also hold for $n = 5, 6, 7$, and 8? Explain.

Check Your Understanding

14. Suppose $B_n = 3n + 1$. Let F_n represent the sequence made from the sum of the first n terms of B_n.

a. What does the term F_5 represent?

b. What is the value of F_8? Show your work.

c. **Critique the reasoning of others.** Mark thinks that the graph of F_n is linear because $F_2 - F_1 = 7$. Is Mark correct? Explain your answer without graphing the sequence.

LESSON 1-1 PRACTICE

15. $\{A_n\}$ is an arithmetic sequence with $a_1 = 7$ and $d = -2.5$.

a. Write the first five terms of this sequence.

b. Write an expression for A_n in terms of n, the term number.

c. Write an expression for A_{n+1} using the expression for A_n.

16. **Make sense of problems.** Gary rents three DVDs each week at Fantastik Flicks. Write an algebraic expression for W_n, the total number of points earned in the nth week.

17. Distinguish between the notation $\{A_n\}$ and A_n. Are they equivalent?

18. For the sequence 3, −2, −7, −12, . . . , determine the value of A_{12}.

My Notes

Learning Targets:

- Use sigma notation to represent a series.
- Write the algebraic form of an arithmetic sequence.
- Calculate the *n*th term or *n*th partial sum of an arithmetic series.

SUGGESTED LEARNING STRATEGIES: Close Reading, Interactive Word Wall, Activating Prior Knowledge, Think-Pair-Share, Debriefing, Self Revision/Peer Revision, Group Presentation

READING MATH

Several math terms are denoted using letters of the Greek alphabet. The capital letter sigma, $\sum$, is used for summation. The lowercase sigma, σ, is used for standard deviation. $\sum_{j=1}^{n} P_j$ is read "The summation of P sub j for $j = 1$ to n."

Fantastik Flicks DVD points accumulate according to the sum $P_1 + P_2 + \ldots + P_{n-1} + P_n$ for the first n DVD rentals. ***Sigma notation*** can be used to streamline the writing of such sums. Using sigma notation, the sum

$P_1 + P_2 + \ldots + P_{n-1} + P_n$ is written as $\sum_{j=1}^{n} P_j = P_1 + P_2 + P_3 + \ldots + P_{n-1} + P_n$.

The notation $\sum_{j=1}^{n} P_j$ means the sum of the terms P_j, where j takes on the consecutive integer values starting with 1 and ending with n.

1. Let P_j denote the number of DVD points earned with the j^{th} rental at Fantastik Flicks. For each part, write the sum indicated by the sigma notation, and then determine the value of the sum.

a. $\sum_{j=1}^{5} P_j =$

b. $\sum_{j=3}^{5} P_j =$

Example A

Evaluate $\sum_{j=2}^{7} (2j + 3)$.

Step 1: The values of j are 2, 3, 4, 5, 6, and 7. Write a sum with six addends, and substitute each value of the variable.

$$\sum_{j=2}^{7} (2j + 3) = [2(2) + 3] + [2(3) + 3] + [2(4) + 3] + [2(5) + 3] + [2(6) + 3] + [2(7) + 3]$$

Step 2: Evaluate each expression. Then simplify.

$7 + 9 + 11 + 13 + 15 + 17 = 72$

Solution: $\sum_{j=2}^{7} (2j + 3) = 72$

My Notes

Try These A

a. Evaluate $\sum_{j=1}^{4}(5j-1)$.

b. Evaluate $\sum_{j=1}^{4}5j-1$.

c. Evaluate $\sum_{k=0}^{3}k^2$.

Check Your Understanding

2. Rewrite the following sum using sigma notation:

$1+5+9+13+17+21+25$

3. Trip is the head cheerleader. Each time his team scores, he does push-ups, one push-up for each point his team has on the scoreboard. At the Homecoming game, Trip's team scored a field goal, so he did three push-ups. Trip's team then scored seven points, so he did 10 push-ups. That day, Trip's team went on to score seven points five more times, and his team won, 45–17. How many push-ups did Trip do at the Homecoming game?

4. In Item 13 of the previous lesson, the number of Fantastik Flicks DVD points that accumulate with successive rentals can be modeled by the area that accumulates under the graph of the successive terms of P_n. Use the figures shown below to answer the following.

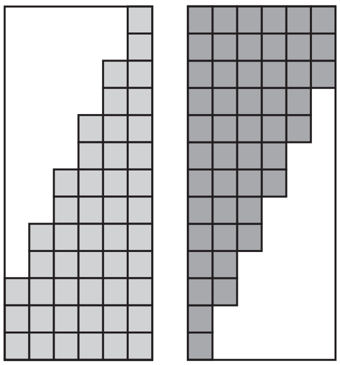


Figure 1 **Figure 2**

a. If Figure 1 represents a given number of DVDs rented from Fantastik Flicks and the total number of DVD points from these rentals, how many rentals are represented by this figure, and how many DVD points have been accumulated from these rentals?

Number of rentals: ________ Number of accumulated points: _______

My Notes

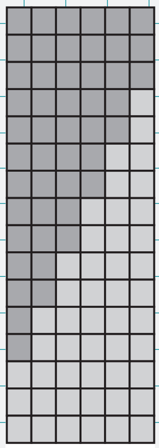


Figure 3

b. How does Figure 2 relate to Figure 1? Explain.

c. Figure 3 has a width that is equal to a given number of DVD rentals from Fantastik Flicks. How is the height of Figure 3 related to the first and last terms of the sum $\sum_{j=1}^{6} P_j$?

d. Find the area of the rectangle in terms of 6, P_1, and P_6.

e. Find the area of each of the two regions in Figure 3 in terms of 6, P_1, and P_6.

5. Use the results in Item 4 to answer the following.

a. Write an expression for the sum $\sum_{j=1}^{6} P_j$ in terms of 6, P_1, and P_6.

b. Reason abstractly. Express the sum $\sum_{j=1}^{n} P_j$ as a formula in terms of n, P_1, and P_n.

c. Complete the following table to verify that the formula in part b is true for other values of n besides $n = 6$.

DVD Rentals, n	Points Earned with n^{th} Rental, P_n	Total Points after n Rentals, F_n	Formula from Part b
1			
2			
3			
4			
5			
6			

My Notes

6. a. How many Fantastik Flicks DVD rentals are needed to obtain a free DVD rental? Show your work.

b. Fantastik Flicks has created a display for 260 DVDs. There are 8 DVDs in the top row of the display and the number of DVDs in each successive row will increase by a constant amount. If there are 10 rows in the display, how many DVDs will be in the 10^{th} row?

Check Your Understanding

7. Bopper's is designing a new DVD display. Fifteen DVDs will be placed on the top row of the display, and the number of DVDs in each successive row will increase by four from the top row to the bottom row of the display.

a. Let r_n denote the number of DVDs in the n^{th} row of the display. Write an expression for r_n in terms of n, the term number.

b. Let T_n denote the total number of DVDs in the display for n rows. Write a formula for T_n in terms of n.

c. Bopper's currently has 232 DVDs. How many rows will be in the display?

d. If Bopper's new DVD display will hold 135 DVDs in six rows, how many DVDs will be in the sixth row?

The Fantastik Flicks DVD point program provides an example of an arithmetic sequence $\{P_n\}$. An ***arithmetic sequence*** has the general algebraic form $a_n = a_1 + (n - 1)d$, where a_1 is the first term and d is the constant difference between consecutive terms.

READING MATH

When used as an adjective in "arithmetic sequence," the word *arithmetic* is pronounced with the accent on *-met-*, not on *-rith-*.

8. Use the table in Item 9 of Lesson 1-1 and examine the values for P_n and F_n.

a. Construct viable arguments. Explain why $\{P_n\}$ is an arithmetic sequence.

b. Explain why $\{F_n\}$ is a sequence but not an arithmetic sequence.

My Notes

The sum of all the terms of an arithmetic sequence forms an ***arithmetic series***. The sequence $\{F_n\}$ is called a ***sequence of partial sums*** of an arithmetic series, because each term of the sequence $\{F_n\}$ is a sum of the first n terms of an arithmetic sequence.

The following illustration shows the connection between an arithmetic sequence, an arithmetic series, and the partial sums.

Arithmetic Sequence	Arithmetic Series	Sequence of Partial Sums
1, 2, 3, 4, . . .	1 + 2 + 3 + 4 + . . .	$S_1 = 1$
		$S_2 = 1 + 2$
		$S_3 = 1 + 2 + 3$
		$S_4 = 1 + 2 + 3 + 4 \ldots$

Check Your Understanding

9. Write the sequence of the first five partial sums of the sequence $\{20 - 4n\}$, where $n = 1, 2, 3, \ldots$.

10. Write the sequence of the first five partial sums of the sequence $\{30 - 4n\}$, where $n = 1, 2, 3, \ldots$.

LESSON 1-2 PRACTICE

For Items 11–13, an arithmetic sequence has $a_1 = 4$ and $d = \frac{1}{2}$.

11. Write the general term for the sequence.

12. Write the associated arithmetic series using the first six terms, and express that sum using sigma notation.

13. Write the sequence of the first six partial sums of the sequence.

14. **Model with mathematics.** Consider a job offer with a starting salary of \$35,600 and a guaranteed raise of \$1200 per year for the next 5 years. What is the total amount of compensation at the end of the sixth year?

15. Is $\sum_{k=1}^{6}(3n+1) = \sum_{k=1}^{3}(3n+1) + \sum_{k=4}^{6}(3n+1)$? Verify your answer.

16. Evaluate $\sum_{k=1}^{6}(k^2 - 3)$.

17. Rewrite the following series using sigma notation:

$$\left(-\frac{5}{2}\right) + \left(-\frac{3}{2}\right) + \left(-\frac{1}{2}\right) + \frac{1}{2} + \frac{3}{2}$$

18. How many terms of the arithmetic sequence $-12, -3, 6, \ldots$ must be added to arrive at a sum of 363?

19. The sum of the first 24 terms of an arithmetic sequence is 300. If $P_1 = 47$, what is the value of P_{24}?

My Notes

Learning Targets:

- Understand the method of mathematical induction.
- Use mathematical induction to prove statements.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Think-Pair-Share, Debriefing, Self Revision/Peer Revision

Graphical representations of P_n and the area under the graph of the terms of P_n were used to develop the formula $F_n = \frac{1}{2}n(P_1 + P_n)$. The fact that this formula holds true for values of n greater than 6 could be verified by extending the table created in Item 18c for larger values of n. However, extending the table would not prove that the formula is true for all positive integers n unless the table went on forever.

1. Use the formula $p(n) = n^2 - n + 41$.

a. Generate p_1, p_2, p_3, p_4, and p_5. Are all of these prime numbers?

b. Attend to precision. Try other values of $n > 5$. Is p a prime number for your choices of n?

c. Construct viable arguments. Based upon your work in parts a and b, do you think that p is a prime number for all positive integers n? Explain why or why not.

d. There are positive integers n for which $p(n)$ is not a prime number. Try to find one of these integer values.

To prove that a property or algebraic rule is true for all positive integers, mathematicians have developed a method called ***mathematical induction***. The method is described below.

To prove that a property P_n is true for all positive integers n, two important steps are required.

Step 1: Verify that the property is true for the initial value P_1.
Step 2: Prove the inductive step: If P_n is true for $n = k$, then P_n must be true for the next value of n, $n = k + 1$.

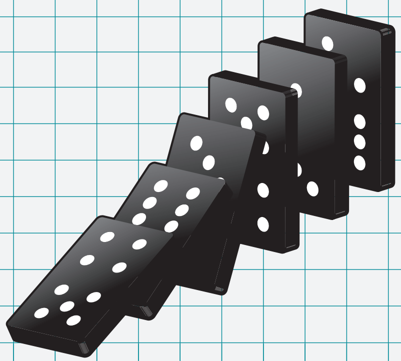
MATH TIP

Mathematical induction is sometimes incorrectly referred to as *inductive reasoning*, which is arguing logically from the specific to the general.

My Notes

Imagine dominoes lined up in a row so that when one domino is tipped, that domino will tip the next domino in the line. That domino will tip the next domino, which, in turn, will tip the next one. This process will continue as long as the dominoes are properly lined up.

The method of mathematical induction can be compared to dominoes properly lined up in a row.



2. **Make sense of problems.** Suppose dominoes are properly lined up in a row.
   a. Do the dominoes tip one another if the first domino is never tipped? How is this like Step 1 of the mathematical induction method described above?

   Dominoes properly lined up in a row are spaced close enough to each other so that when one domino in the row, for example, the domino in the k^{th} position, is tipped, it will tip the next domino in the $(k + 1)^{\text{st}}$ position.

   b. How is the condition of "dominoes properly lined up" like Step 2 of the mathematical induction method?

If a property expressed in terms of the positive integers n is true, mathematical induction proves this by showing that there is a starting point at which the property is true (like the first tipped domino).

If the property is true for $n = k$, this implies the truth of the property for $n = k + 1$ (like dominoes properly lined up).

By mathematical induction, the truth of the property progresses through the positive integers, starting with the first occurrence where the property is true and continuing on through all successive integers.

My Notes

3. **Make use of structure.** The formula for the sum of the first n positive integers is $1 + 2 + 3 + \ldots + (n - 1) + n = \sum_{j=1}^{n} j = \frac{n(n+1)}{2}$. The parts below illustrate the two steps of mathematical induction, a proof that this formula is correct for all positive integers n.

   a. Evaluate $\frac{n(n+1)}{2}$ for $n = 1$.

   Does the formula give the value of the sum $\sum_{j=1}^{n} j$ for $n = 1$?

   b. Suppose $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$. Will it be true for $k + 1$? Adding $k + 1$, the value of the $(k + 1)^{st}$ term, to both sides of $\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$ gives $k+1+\sum_{j=1}^{k} j = (k+1) + \frac{k(k+1)}{2}$. Verify algebraically that $(k+1) + \frac{k(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.

   c. Does $\frac{(k+1)(k+2)}{2}$ represent $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ when n is replaced by $k + 1$? Explain.

My Notes

d. Review your work in parts a–c and explain how this work proves that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ is true for all positive integers $n \geq 1$.

Check Your Understanding

The sum of the first n positive odd integers can be represented as

$1 + 3 + 5 + 7 + 9 + \ldots = \sum_{j=1}^{n}(2j - 1)$.

Complete Items 4 and 5 to prove $\sum_{j=1}^{n}(2j - 1) = n^2$ for all positive integers $n \geq 1$.

4. Carry out Step 1 of the mathematical induction method. Show your work.

5. Carry out Step 2 of the mathematical induction method. Show your induction assumption (let $n = k$) and the algebraic proof to establish that the formula is valid for $n = k + 1$.

6. Use mathematical induction to prove $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all positive integers $n \geq 1$.

LESSON 1-3 PRACTICE

Verify the following formulas using mathematical induction.

7. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

8. $\sum_{k=1}^{n} 2n = n(n+1)$

9. **Construct viable arguments.** Must the value of n in the initial step of a mathematical induction proof always be 1? Justify your answer.

ACTIVITY 1 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 1-1

1. Which of the following sequences are arithmetic? For those sequences that are arithmetic, identify d and write an expression for a_n in terms of n, the term number.
I. 100, 98, 96, 94, . . .
II. 64, 32, 16, 8, . . .
III. $2^1 + 1, 2^2 + 1, 2^3 + 1, 2^4 + 1, \ldots$
IV. $\frac{3}{4}, \frac{13}{12}, \frac{17}{12}, \frac{7}{4}, \ldots$

2. A marathon is 26.2 miles. A runner begins training by running 3 miles on the first day. He increases his distance by 0.8 mile each day thereafter. How many days does it take for him to run the distance of a marathon?

3. An essay-writing contest ranks the essays from 1 to 20. The prizes are \$100, \$97, \$94, and so on. How much does the person writing the 13th essay receive?

4. The arithmetic sequence with $a_1 = 3$ and $d = 2.4$ contains each term below except:
A. 12.6 **B.** 19.6
C. 27 **D.** 29.4
E. 34.2

5. Find x such that $x + 4$, $3x - 9$, and $2x + 8$ are consecutive terms of an arithmetic sequence.

6. Two sequences have the same common difference. How many terms could the sequences have in common? Justify your answer.

7. If a_1, a_2, a_3, a_4, a_5, and a_6 are the first six terms of an arithmetic sequence, is $3a_1, 3a_2, 3a_3, 3a_4, 3a_5, 3a_6, \ldots$ also an arithmetic sequence? Give an example or a counterexample to support your answer.

8. In an arithmetic sequence, a_3 is 27 and a_{12} is 90. Find a_{18}.

Lesson 1-2

9. Rewrite the following series using sigma notation: $27 + 22 + 17 + 12 + 7 + 2$.

10. Find the sum of the first 20 terms of the arithmetic sequence 27, 22, 17, ...

11. Evaluate $\sum_{j=1}^{5}(3j + 4)$.

12. Evaluate $\sum_{k=0}^{5}\left(\frac{1}{2}\right)^k$.

13. Find the sum of the first 20 terms of an arithmetic sequence with an 18th term of 8.1 and a common difference of 0.25.

14. Write the sequence of the first six partial sums of the sequence $\{3n - 4\}$, where $n = 1, 2, 3, \ldots$.

15. Will has \$210 in his bank account. Each Saturday, he deposits \$40. Describe what the partial sums of the sequence for this situation represent in terms of the context.

16. If $\sum_{i=1}^{K}(2 - i) = -2$, what is the value of K?

17. Claire and Jeremy are reading the same 240-page book. Claire has read 72 pages, and Jeremy has read 90 pages. On Monday, Claire begins to read 12 pages per day, and Jeremy continues to read 10 pages per day.

a. Write equations for the sequences C_n and J_n to represent how many total pages Claire and Jeremy have read n days after Sunday.

b. In this context, what should be the value of the last term of each sequence?

c. How many terms does each sequence have? How does this relate to who finished reading the book first?

18. The number of seats in the first three rows of the MIU theater is given in the table below.

Row Number	No. of Seats
1	18
2	20
3	22

The theater has 26 rows of seats, and the pattern in the table continues for all of the rows. Write an expression for the total number of seats in the first n rows and use this expression to calculate the seating capacity of the theater.

19. Write the sequence of the first five partial sums of the sequence $\{16 - 3n\}$, where $n = 1, 2, 3, \ldots$.

20. a. Find the value of n for which the following equation is true:

$$\sum_{k=1}^{n}(2k-1) = 100$$

b. Describe a characteristic of each term in the sequence.

21. Find the sum of $1 + 6 + 11 + \ldots + 96$.

A. 960
B. 970
C. 1,440
D. 1,940

Lesson 1-3

22. Verify the following formula using mathematical induction.

$$1 + 4 + 7 + \ldots + (3n-2) = \frac{n(3n-1)}{2}$$

23. Verify the following formula using mathematical induction.

$1 + 5 + 9 + \ldots + (4n-3) = n(2n-1)$

24. Consider the following equation:

$1 + 4 + 9 + \ldots + 3n - 2 = 4n - 3$

a. Is the equation true for $n = 1$?
b. Is the equation true for $n = 2$?
c. Is the equation true for all natural numbers?

25. Suzanne's teacher writes the following statement on the board:

For all integers $n \geq 5$, $n^3 \geq n^2 + 100$.

a. Show that the statement is true for $n = 5$.
b. Show that the mathematical induction step can be written $n^3 + 3n^2 + n \geq n^2 + 100$.
c. Explain in your own words why the inequality in part b is a true statement.

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

26. For what types of problems is it useful to find an expression for a_n?

Geometric Sequences

She Sells Sea Shells

Lesson 2-1 Identifying Geometric Sequences

My Notes

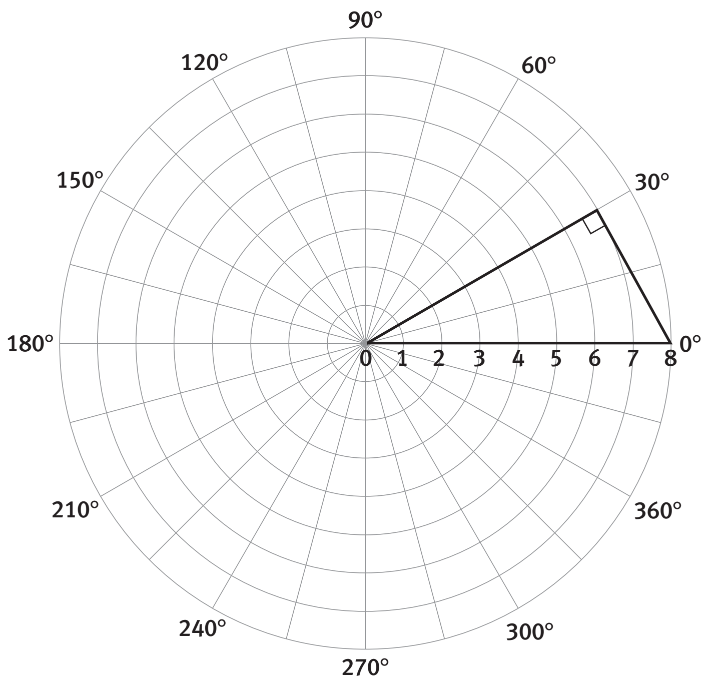
Learning Targets:

- Identify a geometric sequence.
- Determine the common ratio of a geometric sequence.

SUGGESTED LEARNING STRATEGIES: Shared Reading, Marking the Text, Activating Prior Knowledge, Think-Pair-Share

Shy Shelly Sellers sells seashells in her *Fourth North Shore Store*. She is planning a sign for the storefront. She wants a large neon spiral reminiscent of the cross section of a shell. Each triangle in the design is a 30°–60°–90° triangle, and the length of the longest segment in the figure is 8 feet.

1. Recreate Shelly's design on the ***polar grid*** below. The first triangle is already drawn with the longest hypotenuse from the origin along the positive x-axis. Each successive hypotenuse is angled 30° counterclockwise from the previous one. As you calculate each hypotenuse length, record it in the table below the graph.



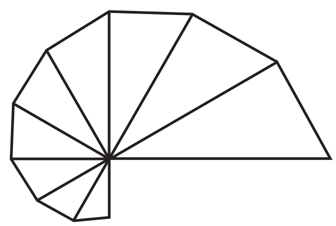
Triangle number	1	2	3	4	5	6	7	8	9
Exact hypotenuse length (ft)	8								
Decimal approximation to nearest hundredth									

MATH TIP

The length of the hypotenuse in a 30°–60°–90° triangle is twice the length of the shorter leg and the length of the longer leg is $\sqrt{3}$ times the length of the shorter leg.

MATH TERMS

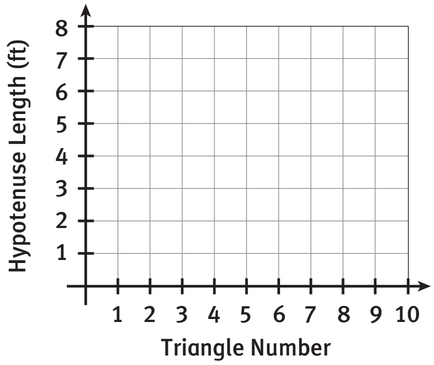
A **polar grid** is made up of concentric circles, the center of which is the **pole**. Coordinates for points on this grid are given in the form (r, θ), where r represents the distance from the pole and θ represents an angle measured from the positive x-axis, as shown.



My Notes

2. a. List the exact values of the hypotenuse lengths from the table in Item 1 as a sequence with $a_1 = 8$.

b. Plot the approximations on the grid.



3. Construct viable arguments. Do the lengths of the hypotenuses in Shelly's design form an arithmetic sequence? Explain your reasoning.

A ***geometric sequence*** is a sequence for which the ratio, r, of a term to its preceding term is a constant. The constant r is known as the ***common ratio***.

4. Complete each of the following ratios for a geometric sequence with common ratio r.

$$r = \frac{a_2}{\square} \qquad r = \frac{\square}{a_5} \qquad r = \frac{a_n}{\square}$$

My Notes

5. Explain why the sequence in Item 2 is a geometric sequence.

6. Each term in a geometric sequence can be written as a product of the first term and powers of the common ratio.

a. Complete the table for the terms in the sequence in Item 2.

n	a_n	a_n Written as a Product of a_1 and a Power of the Ratio
1	8	$8\left(\frac{\sqrt{3}}{2}\right)^0$
2		
3		
4		
5		
6		

b. Write an equation that gives a_n in terms of n.

My Notes

c. Use your equation from Item 6b to calculate a_{11} directly. Does the pattern in the table from Item 6a help to confirm your answer?

Check Your Understanding

7. How does a geometric sequence differ from an arithmetic sequence?

8. Identify the missing terms in the geometric sequence ______, ______, 22.5, 67.5, ______, 607.5.

9. Identify which of the sequences below are geometric. If the sequence is a geometric sequence, identify a_1 and r, write an expression for a_n, and calculate a_{15}.

I. $-5, -15, -45, -135, -405$

II. $-5, -1, 3, 7, 11$

III. $1, 4, 9, 16, 25$

IV. $6, -9, 13.5, -20.25, 30.375$

LESSON 2-1 PRACTICE

10. Find x such that $x - 4$, x, and $3x - 8$ are three consecutive terms in a geometric sequence.

11. Determine the first term of a geometric sequence with $r = 1.4$ and $a_5 = 76.832$.

12. Calculate n for a geometric sequence with $a_1 = \frac{1}{32}$, $r = 2$, and $a_n = 4$.

13. **Attend to precision.** A new scooter costs \$2,500. The depreciation rate is 30% per year. What is the value of this scooter after 5 years? Round to the nearest dollar.

14. Do the lengths of the short legs in the right triangles in Shelly's design form a geometric sequence? Make a prediction, calculate the actual lengths, and confirm or change your response.

My Notes

Learning Targets:

- Write the algebraic form of a geometric sequence.
- Calculate the sum of a finite geometric series.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Group Presentation, Debriefing, Self Revision/Peer Revision

1. A function of the form $f(x) = ab^x$ is an *exponential function*. Show why the expression that you wrote in Item 6 in Lesson 2-1 for a_n is an exponential function by identifying a and b.

2. Use the expression for a_n in Item 6 to verify the length of the ninth hypotenuse that you found in Item 1. (Do not use a_8.)

3. Suppose the spiral in Shelly's design is continued beyond the ninth triangle. Use the expression for a_n in Item 6 to calculate a_{115}.

My Notes

4. Given another geometric sequence with $a_1 = \frac{1}{2}$ and $a_2 = \frac{-3}{2}$, calculate r and a_{10}.

5. Write an expression for a_n in terms of a_1, r, and n that can be used for any geometric sequence.

MATH TERMS

The sum of the terms of a sequence is often used in applications. These sums are known as a **series**. The sum of the first n terms of a sequence is called the ***n***$^{\text{th}}$ **partial sum**.

6. a. **Attend to precision.** For each ***partial sum*** in the table, write an expression for $\frac{\sqrt{3}}{2} S_n$ in expanded form. Use exact values.

n	S_n	$\frac{\sqrt{3}}{2} S_n$
1	8	
2	$8 + 4\sqrt{3}$	
3	$8 + 4\sqrt{3} + 6$	
4	$8 + 4\sqrt{3} + 6 + 3\sqrt{3}$	
5	$8 + 4\sqrt{3} + 6 + 3\sqrt{3} + \frac{9}{2}$	

My Notes

b. Use your table from Part a to complete the table below.
- Complete the second column by writing an expression for $S_n - \frac{\sqrt{3}}{2}S_n$ as the difference of two terms.
- Complete the third column by factoring a_1 from each term in the second column.
- Complete the fourth column by expressing q in the third column as a power of r.

n	$S_n - \frac{\sqrt{3}}{2}S_n$	$a_1(1 - q)$	$a_1(1 - r^n)$
1	$8 - 4\sqrt{3}$	$8\left(1 - \frac{\sqrt{3}}{2}\right)$	$8\left(1 - \left(\frac{\sqrt{3}}{2}\right)^1\right)$
2			
3			
4			
5			

7. Use the results of the table in Item 6 to complete the following equation in terms of a_1, r, and n: $S_n - rS_n =$

My Notes

MATH TERMS

Recall that a *series* is the sum of the terms in a sequence. A **geometric series** is the sum of the terms in a geometric sequence.

8. Use appropriate tools strategically. Use factoring to solve the equation in Item 7 for S_n, and write the formula for the sum of a finite ***geometric series***.

9. Use the equation that you wrote in Item 8 to calculate the sum of the first 9 hypotenuses in Shelly's design.

Check Your Understanding

10. Find the sum of the first 5 terms of the geometric sequence if $a_2 = 8$ and $a_3 = 10$. Show your work.

11. Find the sum of the areas of the first 10 right triangles in Shelly's design. Show your work.

12. Shelly sends an email to three customers to invite them to a sale. The customers each forward the email to three of their friends. If this pattern continues, find the total number of emails sent after an email was forwarded six times.

LESSON 2-2 PRACTICE

13. Make use of structure. For the geometric sequence $a_n = \left(\frac{1}{2}\right)^n$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, and $S_4 = \frac{15}{16}$, predict S_5 and S_6.

14. Write a general term, S_n, for the geometric sequence above.

15. Find the sum of the first eight terms of the geometric sequence whose first term is -2.5 and ratio is 2.

16. Attend to precision. Evaluate $\sum_{k=0}^{9} 6(1.5)^k$. Round to the nearest hundredth.

17. Express the sum in Item 15 using sigma notation.

My Notes

Learning Targets:

- Determine if a sequence converges or diverges.
- Find the sum of an infinite geometric series.

SUGGESTED LEARNING STRATEGIES: Quickwrite, Think-Pair-Share, Summarizing, Paraphrasing, Interactive Word Wall, Debriefing, Self Revision/Peer Revision

1. As n increases, what is happening to the length of each successive hypotenuse in the triangles in Shelly's design?

If the terms of an infinite sequence approach some number L as n increases without bound, the sequence is said to ***converge***. If the sequence does not converge, it ***diverges***.

2. Does the sequence whose terms are the lengths of the hypotenuses in Shelly's design converge or diverge? If the sequence converges, what is the value that the terms appear to approach as n increases?

3. For each infinite geometric sequence below, answer the following questions.

a. Determine the common ratio for each sequence.

b. Which of the sequences converge and which diverge? For each sequence that converges, determine, if possible, the value to which the terms are approaching.

I. 0.025, 0.25, 2.5, 25, 250, . . .

II. 100, 50, 25, 12.5, 6.25, . . .

III. $-4.2, 4.2\ -4.2, 4.2, -4.2, \ldots$

IV. $\frac{1}{9}, -\frac{1}{3}, 1, -3, 9, \ldots$

V. $25, \frac{25}{\sqrt{5}}, \frac{25}{5}, \frac{25}{5\sqrt{5}}, \frac{25}{25}, \ldots$

VI. 32,000, 320, 3.2, 0.032, 0.00032, . . .

VII. $1, \sqrt{2}, 2, 2\sqrt{2}, 4, \ldots$

ACADEMIC VOCABULARY

Diverge can mean to move away from a location, and ***converge*** can mean to approach it. Think of a series that diverges as moving away from a specific value.

CONNECT TO AP

The value to which a sequence converges is called the limit of the sequence. Later in this course and in AP Calculus, you will learn about and use limits in a variety of ways.

My Notes

DISCUSSION GROUP TIPS

In your discussion groups, read the text carefully to clarify meaning. Reread definitions of terms as needed to help you comprehend the meanings of words, or ask your teacher to clarify vocabulary terms.

4. Create two infinite geometric sequences of your own, one that converges and one that diverges.

5. **Reason quantitatively.** How can the ratio of a geometric sequence be used to determine whether a sequence converges or diverges? If the sequence converges, what can be said about the value to which the terms are drawing near?

6. Calculate the first five partial sums for sequences I, II, and III in Item 3 and the sequences you created in Item 4. Which sequences of partial sums appear to converge and which appear to diverge?

7. If you calculated the 100^{th} partial sum for each of the sequences in Item 6, would any of your responses change? Explain your reasoning.

8. Suppose the spiral in Shelly's shell design continues and is allowed to overlap itself. As n increases, is there a length to which the hypotenuses are drawing near? Is there a value to which the sum of the lengths of the hypotenuses is drawing near?

9. **Express regularity in repeated reasoning.** What must be true for r, the common ratio of a geometric sequence, in order to have the sequence of the partial sums converge?

An ***infinite sequence*** is a sequence with an infinite number of terms.
An ***infinite series*** is the sum of the terms of an infinite sequence.

10. For some series, an infinite number of terms can be added to get a finite sum. Recall the formula for the sum of the first n terms in a geometric sequence: $S_n = \frac{a_1(1-r^n)}{(1-r)}$. Let $|r| < 1$. As n increases and gets very large, what happens to each of the following expressions?

a. r^n

b. $1 - r^n$

c. $\frac{a_1(1-r^n)}{(1-r)}$

11. Shelly wants to know the total length of material she will need for the display. Find the sum of the lengths of the hypotenuses in the triangles in her design if $a_1 = 8$ and the number of triangles in her design increases without bound.

My Notes

CONNECT TO AP

If a sequence diverges, then the corresponding series also diverges. However, if a sequence converges, the corresponding series may or may not converge. Determining whether or not an infinite series converges or diverges is a topic you will study in AP Calculus.

Example A

An ant located at the origin crawls 1 unit right, then $\frac{1}{2}$ unit up, then $\frac{1}{4}$ unit right, then $\frac{1}{8}$ unit up, and continues following this pattern indefinitely. What point is the ant approaching?

Step 1: Find the horizontal distance traveled.

The horizontal distances are $1, \frac{1}{4}, \frac{1}{16}, \ldots$. This is an infinite geometric sequence with $a_1 = 1$ and $r = \frac{1}{4}$. Therefore, the horizontal distance approaches $\frac{1}{1-\frac{1}{4}} = \frac{4}{3}$.

Step 2: Find the vertical distance traveled.

The vertical distances are $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$. This is an infinite geometric sequence with $a_1 = \frac{1}{2}$ and $r = \frac{1}{4}$. Therefore, the vertical distance approaches $\frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

Solution: The ant is approaching the point $\left(\frac{4}{3}, \frac{2}{3}\right)$.

My Notes

Try These A

a. For each infinite sequence in Item 34 that converges, find the sum of its corresponding series.

b. The repeating decimal 0.363636 . . . can be written as an infinite geometric series: $0.36(.01)^0 + 0.36(.01)^1 + 0.36(.01)^2 + \ldots$. Express the repeating decimal in sigma notation and as a fraction by finding the sum of the corresponding infinite series.

c. Evaluate $\sum_{k=1}^{\infty} 10(0.8)^k$.

Check Your Understanding

12. Find the value of $-3 + 2 - \frac{4}{3} + \frac{8}{9} - \ldots$.

13. **Critique the reasoning of others.** Amy says, "Each term of an infinite geometric series must be less than the previous term for the series to converge." Roger points out that if r is negative, this is not true. Rewrite Amy's statement so that it is true.

14. Give an example of an infinite geometric series that diverges.

LESSON 2-3 PRACTICE

15. Write the sum $8 + 2.4 + 0.72 + 0.216 + \ldots$ using summation notation.

16. **Make use of structure.** Express the repeating decimal 0.4444 . . . as a fraction.

17. Solve for r: $\sum_{n=1}^{\infty} 5r^n = 2.5$.

18. An infinite series converges to 10 with a common ratio of 0.6. What is a_1?

19. **Model with mathematics.** Suppose the ant in Example 1 crawled to (0, 1), and then turned to its right before crawling each additional distance. Describe how you would approach this problem.

ACTIVITY 2 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 2-1

1. Determine which of the sequences below are geometric. For each geometric sequence, calculate r, find an expression for a_n, and calculate a_{12}.

I. 100, 98, 96, 94, . . .

II. $2^0, 2^1, 2^2, 2^3, \ldots$

III. $3, 3\sqrt{6}, 18, 18\sqrt{6}, \ldots$

IV. $2^0 + 1, 2^1 + 2, 2^2 + 3, 2^3 + 4, \ldots$

2. Find a_1 for a geometric sequence if $r = 5$ and $a_6 = 1000$.

3. Each month, the balance in Freda's bank account is 1.003 times as large as the previous month's due to interest. If Freda does not deposit or withdraw any money from this account and she begins with \$2500, find the balance in this account at the beginning of the 36th month.

4. Complete the following analogy and explain your response. Arithmetic sequences are to linear functions as geometric sequences are to ______________.

5. Find the first term of the geometric sequence with a common ratio of 0.4 and $a_7 = 32$.

6. Find the common ratio of the geometric sequence with $a_1 = \frac{3}{5}$ and $a_6 = \frac{1875}{1024}$.

7. Write the first five terms of the geometric sequence with $a_3 = -135$ and $a_4 = 405$.

8. Find the eighth term of a geometric sequence with $a_1 = 6$ and $r = 1.2$.

9. Explain why $a_{n+1} = ra_n$ in any geometric sequence.

Lesson 2-2

10. Find the sum of the first 10 terms in each of the geometric sequences in Item 1.

11. Find a_n when $S_3 = \frac{122}{25}$ and $r = \frac{4}{5}$.

12. As a reward for inventing chess, Ja'qubi asked the Shah of Persia for 1 grain of wheat for the first of the 64 chessboard squares, 2 grains for the second, 4 grains for the third, 8 grains for the fourth, and so on, for all 64 squares. Calculate the number of wagons needed to transport the wheat if there are 20 million grains of wheat per ton and each wagon can carry 5 tons of grain.

13. A geometric sequence has $a_1 = 0.56$ and $r = 10$. Write the series representing the sum of the first six terms of the sequence, and express this sum using sigma notation.

14. Evaluate each sum.

a. $\sum_{k=0}^{4} 9\left(\frac{-1}{3}\right)^k$

b. $\sum_{k=1}^{6} 3(0.4)^k$

c. $\sum_{k=0}^{5} \left(\frac{1}{2}\right)^k$

15. Find the sum of $2 + 1 + \frac{1}{2} + \ldots + \frac{1}{256}$.

A. 3.976
B. 3.992
C. 4.500
D. 4.992

16. Which provides a greater annual amount: 8.5% interest compounded annually or 8% compounded quarterly?

17. Suppose you and your spouse have four children and suppose that each child has two children, and this pattern continues for 14 generations. How many people, starting with you and your spouse, are in your family tree?

18. Express this sum of the following series using sigma notation. Then, find the sum.

$4 + 4(2) + 4(2)^2 + 4(2)^3 + 4(2)^4 + 4(2)^5$

19. Find n such that $\sum_{k=1}^{n} (3)^k = 363$.

Lessons 2-3 and 2-4

20. Calculate the sum for each of the following infinite geometric series that converge.

I. $4 + 2 + 1 + \frac{1}{2} + \ldots$

II. $4^2 + 2^2 + 1^2 + \left(\frac{1}{2}\right)^2 + \ldots$

III. $\frac{-16}{81} + \frac{-8}{27} + \frac{-4}{9} + \frac{-2}{3} + \ldots$

IV. $\frac{\sqrt{10}}{4} + \frac{10}{16} + \frac{10\sqrt{10}}{64} + \frac{100}{256} + \ldots$

V. $1 + 1.1 + 1.21 + 1.331 + \ldots$

21. Express the repeating decimal 0.757575 . . . as an infinite series and write it as a fraction.

22. Not all infinite geometric series can be calculated. What must be true about an infinite geometric series if that series can be calculated?

23. Do the even-numbered terms of an infinite geometric sequence form another infinite geometric sequence? Justify your answer.

24. Find the common ratio, r, for an infinite series with an initial term of 4 that converges to a sum of $\frac{16}{3}$.

25. Consider the infinite geometric series

$\frac{8}{25} + \frac{4\sqrt{5}}{25} + \frac{2}{5} + \frac{\sqrt{5}}{5} + \ldots .$

a. What is the exact value of the common ratio of the series?

b. Does the series converge? Justify your answer without making calculations.

26. Find the sum of $1 + (0.2) + (0.2)^2 + (0.2)^3 + \ldots .$

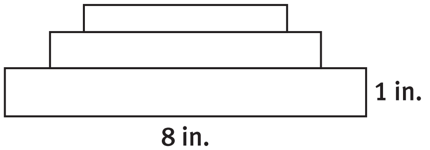
A. 1.2

B. 1.22

C. 1.25

D. 2

27. Helena makes a perspective drawing as shown below. Each rectangle is 75% as wide and as tall as the rectangle below it.



If the pattern continues indefinitely, what will be the total area of the figure?

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

28. Compare and contrast arithmetic and geometric sequences and series as to structure, notation, and patterns.

Modeling Recursive Relationships

ACTIVITY 3

Money Market Accounts

Lesson 3-1 Exploring a Recursive Relationship

My Notes

Learning Targets:

- Represent arithmetic and geometric sequences recursively.
- Determine the explicit form of a recursive sequence.

SUGGESTED LEARNING STRATEGIES: Chunking the Activity, Summarizing, Paraphrasing, Predict and Confirm, Think-Pair-Share, Activating Prior Knowledge

As you read, mark the text to identify key information and parts of sentences that help you make meaning from the text.

Maurice and Lester are twins who have just graduated from college. They have both been offered jobs where their take-home pay would be $2500 per month. Their parents have given Maurice and Lester two options for a graduation gift.

Option 1: If they choose to pursue a graduate degree, their parents will give each of them a gift of $35,000. However, they must pay for their tuition and living expenses out of the gift.

Option 2: If they choose to go directly into the workforce, their parents will give each of them a gift of $5000.

Maurice decides to go to graduate school for 2 years. He locks in a tuition rate by paying $11,500 for the 2 years in advance, and he figures that his monthly expenses will be $1000.

Lester decides to go straight into the workforce. Lester finds that after paying his rent, utilities, and other living expenses, he will be able to save $200 per month.

Their parents deposit the appropriate amount of money in a money market account for each twin. The money market accounts are currently paying a nominal interest rate of 3 percent, compounded monthly.

1. Before doing any calculations, predict which twin will have the greater balance in his money market account after 2 years. Will that twin always have more money in the account?

2. After Maurice withdraws $11,500 for tuition, how much money is left in his money market account?

DISCUSSION GROUP TIPS

If you do not understand something in group discussions, ask for help or raise your hand for help. Describe your questions as clearly as possible, using synonyms or other words when you do not know the precise words to use.

My Notes

MATH TIP

To compute the monthly interest rate when a yearly rate is given and the interest is compounded monthly, divide the interest rate by 12. For example, to calculate the balance in an account that pays 3 percent interest, compounded monthly, after one month, multiply the beginning balance by $\left(1+\frac{0.03}{12}\right)$, or 1.0025.

TECHNOLOGY TIP

Many calculators will perform recursive operations on the home screen by establishing the pattern and then pressing the ENTER key successively. A spreadsheet may also be used; a copied formula filled down a column can generate sequences quickly.

3. At the end of the first month, Maurice has earned a little interest from his money market account and pays his monthly bills out of this account. Find Maurice's current balance and show the work that supports your answer.

4. Complete the table below to record Maurice's monthly money market account balance after he collects interest and pays his bills.

Month	Computation	Account Balance
0		
1		
2		
3		
4		
5		

5. If Maurice's initial balance is $u_0 = 23{,}500$, u_n is the current month's balance, and u_{n-1} is last month's balance, write an expression for u_n in terms of u_{n-1}.

My Notes

MATH TERMS

The process of **iteration** is a repetitive application of the same rule.

6. Maurice decides to use a spreadsheet to determine the balance of his money market account when he graduates.

	A	B
1	**Month**	**Maurice's Account Balance**
2	0	$23,500
3	1	$22,558.75
4	2	$21,615.15
5	3	$20,669.19
6	4	$19,720.86
7	5	$18,770.16
8	6	
9	7	
10	8	
11	9	
12	10	
13	11	
14	12	
15	13	
16	14	
17	15	
18	16	
19	17	
20	18	
21	19	
22	20	
23	21	
24	22	
25	23	
26	Graduation	

a. If month 0 is identified as A2 and u_0 as B2 on the spreadsheet, how would Month 1 and $22,558.75 be identified?

b. Using the expression from Item 5, determine the relationship on the spreadsheet between B3 and B2.

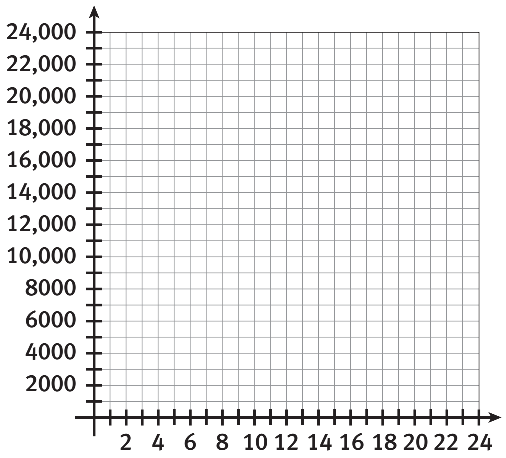
c. Determine the relationship on the spreadsheet between B4 and B3.

My Notes

d. Determine a relationship between B8 and B7, and find the account balance at 6 months.

e. Use appropriate tools strategically. Use a spreadsheet program or graphing calculator to complete the spreadsheet above. How much money is in Maurice's account upon graduation?

7. On the grid below, plot the balance for Maurice's money market account for each month of the 2 years he will attend graduate school. Label the axes.



8. As the number of months that Maurice attends graduate school increases, what is happening to his money market account balance?

9. Reason quantitatively. Consider the sequence of Maurice's monthly money market account balances. Is this an arithmetic sequence, a geometric sequence, or neither? Explain.

My Notes

Lester works during the time that Maurice attends graduate school. Each month, Lester saves $200 and deposits this amount into the $5000 money market account that his parents set up for him when he graduated.

10. Complete the table below to record Lester's money market account balance each month for the first 5 months that he works. Recall that Lester's money market account earns him 3 percent interest, compounded monthly.

Month	Account Balance
0	
1	
2	
3	
4	
5	

11. If Lester's initial balance is $u_0 = 5000$, u_n is the current month's balance, and u_{n-1} is last month's balance, write an expression for u_n in terms of u_{n-1}.

12. How much money does Lester have in his money market account after 1 year? How does his account balance compare to Maurice's account balance after 1 year?

13. Add a column C on the spreadsheet for Lester's money market account, for the 2 years he has been working. Enter 5000 for the value of C2. Plot the balance for Lester's money market account for each month of the first 2 years he will work on the grid of Item 7.

a. Write an equation that gives C3 in terms of C2.

b. **Model with mathematics.** When do the twins have approximately the same amount of money in their accounts? Explain your reasoning.

c. At the end of 2 years, which twin has the larger money market account balance, and how much more money does this twin have?

My Notes

Check Your Understanding

14. At the end of 2 years, Lester receives a raise and decides to save \$250 each month. Maurice receives a \$5000 graduation gift from his parents and deposits this amount into his money market account. Maurice goes to work and saves \$500 each month.

Complete the equations below for the money market account balance for each twin. Let the initial balance u_0 be the account balance at the end of 2 years. Write an expression for this month's account balance u_n in terms of u_{n-1}. Recall that the interest rate for the account is 3 percent, compounded monthly.

Maurice: $u_0 = \$5248.47$, $u_n =$ ______
Lester: $u_0 =$ ______, $u_n =$ ______

15. **Construct viable arguments.** Suppose Maurice has \$6953.11 in his account at the end of a certain month. How could you determine how much was in the account at the end of the previous month? Explain your method.

LESSON 3-1 PRACTICE

16. **Model with mathematics.** The number of bacteria in a Petri dish grows by 10 percent every hour. After the growth, about 100 bacteria die.

a. Suppose initially there are 500 bacteria. How many bacteria are alive after 1 hour?
b. How many bacteria are alive after 2 hours?
c. Write expressions for $u_{0.}$ and u_n for this situation.
d. What is happening to the bacteria population?

17. For the sequence 3, 4, 6, 9, 13, …, write expressions for u_0 and u_n.

18. An arithmetic sequence has a first term of 40 and a constant difference of -4. Write expressions for u_0 and u_n to represent this sequence.

19. **Use appropriate tools strategically.** Use a spreadsheet or calculator to determine how long it will take for the bacteria from Item 16 to die off completely.

My Notes

Learning Targets:

- Represent arithmetic and geometric sequences recursively.
- Determine the explicit form of a recursive sequence.

SUGGESTED LEARNING STRATEGIES: Chunking the Activity, Summarizing, Paraphrasing, Predict and Confirm, Think-Pair-Share, Activating Prior Knowledge

A ***recursive sequence*** of the form $\begin{cases} u_0 = \text{Initial amount} \\ u_n = r \cdot u_{n-1} + d \end{cases}$ has an ***explicit form*** $u_n = a + b \cdot r^n$, where a and b are constants, r is the same growth factor used in the recursive form, and n is the time in months.

1. Refer to the recursive sequence for Maurice's money market account that you wrote in Item 5 of Lesson 3-1. When Maurice's value for u_0 is substituted into the explicit form $u_n = a + b \cdot r^n$, what is the resulting equation?

2. When Maurice's value for u_1 is substituted into the explicit form $u_n = a + b \cdot r^n$, what is the resulting equation?

3. Find the solution to the system of equations you developed in Items 1 and 2. Show the work that supports your solution.

4. Using the values of a and b from Item 3, state the explicit form of the sequence for Maurice's money market account balance while he attended graduate school.

MATH TIP

You can solve a two-variable system of equations by multiplying one or both of the equations by a constant. The sum of the two equations will eliminate one of the variables, resulting in one equation with one variable.

My Notes

5. Use the explicit form of the sequence for Maurice's money market account balance while he attended graduate school to determine the balance at the end of 1 year and 2 years. Do these answers agree with the values found in the spreadsheet for Item 6 of Lesson 3-1?

6. How does the graph of the explicit form of the sequence for Maurice's money market account balance while he attended graduate school compare to the graph of the recursive form?

7. What is the sum of *a* and *b* in the explicit form of the sequence for Maurice's money market account balance? What does this sum represent?

My Notes

8. Recall that while Maurice was attending graduate school, Lester was working. The beginning balance for Lester's money market account was $5000, earning 3 percent interest, compounded monthly. Lester also deposited $200 into this account each month.

a. Find the explicit form of the sequence for Lester's money market account balance while Maurice was attending graduate school. Show the work that supports your answer.

b. Use the explicit form of the sequence for Lester's money market account balance to determine the balance at the end of 1 year and 2 years. Do these answers agree with the values found in Item 12 and the graph for Item 13 in Lesson 3-1?

c. What is the sum of *a* and *b* in the explicit form of the sequence for Lester's money market account balance, and what does this sum represent?

9. Use the explicit form of each twin's sequence to determine when Maurice and Lester have approximately the same amount of money in their accounts. Show the work that supports your answer.

My Notes

MATH TIP

To write a recursive expression, write a_1 and a rule for a_n based upon a_{n-1}. Sometimes a_0 is used instead of a_1 when the sequence represents a real situation, with a_0 as the initial value and a_1 as the first change.

10. Given the geometric sequence 3, 6, 12, 24, ...

a. Write an expression for a_n two ways: recursively and explicitly.

b. Explain why geometric sequences are a subset of the sequences in this activity where $u_n = ru_{n-1} + d$.

11. Given the arithmetic sequence 25, 22, 19, 16, ... Write an expression for a_n two ways: recursively and explicitly.

Check Your Understanding

12. Refer to Item 14 of Lesson 3-1. Find the explicit form of the sequence for Maurice and Lester's money market account balances after Maurice graduated from graduate school. Show your work.

13. Use the explicit form of the sequences found in Item 12, after Maurice finished graduate school, to confirm when the twins had approximately the same amount of money in their accounts. Show the work that supports your answer.

14. Write a recursive expression and an explicit expression for a_n for the sequence \$400, \$480, \$576, ...

15. Write a recursive expression and an explicit expression for a_n for the sequence \$400, \$400(1.003) + \$10, \$411.20(1.003) + \$10, ...

LESSON 3-2 PRACTICE

Catarina opens a savings account with \$50 and deposits \$20 each month. Her bank pays 3.6 percent interest, compounded monthly.

16. Write a recursive expression for a_n.

17. If $a_{15} = 358.68$, find a_{16}.

18. Write an explicit expression for a_n.

19. Use the explicit form for a_n to find a_{16}.

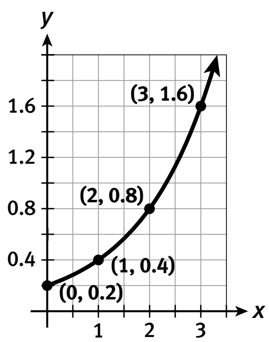
20. **Construct viable arguments.** Explain why it would be preferable to use an explicit formula rather than a recursive formula to find a_{20}.

ACTIVITY 3 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 3-1

1. For each of the sequences below, write a recursive expression for a_n.
   a. \$400, \$200, \$100, . . .
   b. \$400, \$400(1.003) + \$10, \$411.20(1.003) + \$10, . . .
   c. \$400, \$480, \$576, . . .
   d. \$400, \$418, \$436, . . .
2. For each of the following geometric sequences, write an expression for a_n recursively.
   a. 48, 24, 12, . . .
   b. 100, −150, 225, . . .
   c. \$600, \$602.40, \$604.81, . . .
3. For each of the following sequences, write an expression for a_n recursively.
   a. 2, 7, 12, . . .
   b. $\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}, \frac{5\sqrt{3}}{2}, \ldots$
   c. $(x+y), (x+y)^2, (x+y)^3, \ldots$
4. The first eight terms of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21. Write a recursive formula to find a_n for any term in the sequence.
5. If $a_n = 3a_{n-1}$ and $a_1 = \frac{4}{27}$, which statement is NOT correct?
   A. $a_2 = \frac{4}{9}$
   B. $a_4 = 4$
   C. $a_6 = 36$
   D. $a_8 = 108$
6. Write the first five terms of $a_n = -2a_{n-1}$ if $a_1 = 5$.
7. Write the recursive function that gives the values $f(1), f(2), f(3), \ldots$, where $f(x)$ is graphed below.



8. Write $n!$ as a recursive expression.
9. Supply the formula entries for the calculations of B4, B5, B6, and B7 in the spreadsheet.

	A	B
1	Open account	\$525
2	Monthly interest rate	0.5%
3	January	\$525
4	February	\$527.63
5	March	\$530.26
6	April	\$532.92
7	May	\$535. 58

10. Write a recursive expression for the set of ordered pairs (n, a_n).

x	y
0	3
1	9
2	15
3	21

11. A recursive expression is given as $a_n = a_{n-1} + 2n - 1$, where $a_1 = 1$.
    a. Find a_2, a_3, and a_4.
    b. What sequence of numbers is defined by this recursive expression?
    c. Does the structure of the recursive expression look familiar? Explain.

Lesson 3-2

12. Identify the explicit formula equivalent to the recursive formula $a_n = 5a_{n-1}$, with $a_4 = 12.5$.

A. $a_n = 10(5)^{n-1}$
B. $a_n = 10(0.5)^{n-1}$
C. $a_n = 0.1(5)^{n-1}$
D. $a_n = 0.1(0.5)^{n-1}$

13. For each of the following geometric sequences, write an expression for a_n explicitly.

I. 48, 24, 12, . . .
II. 100, −150, 225, . . .
III. \$600, \$602.40, \$604.81, . . .

14. For each of the following arithmetic sequences, write an expression for a_n explicitly.

I. −10, −4, 2, . . .
II. $\frac{\sqrt{3}}{2}, \frac{3\sqrt{3}}{2}, \frac{5\sqrt{3}}{2}, \ldots$
III. \$900, \$850, \$800, . . .

15. Pete starts with \$500 in a savings account that pays 3.6 percent interest, compounded monthly, and he deposits \$150 each month. Pete's sister, Rose, opens a savings account with \$800 that pays 3 percent interest, compounded monthly, and she deposits \$120 each month. How long will it take for Pete's account balance to catch up to Rose's account balance?

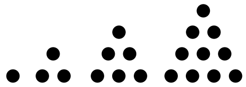
16. Eddie opens a money market account with \$4500. The account pays 3 percent interest, compounded monthly. Each month, Eddie takes out \$248 for a car payment.

a. Write a recursive expression for a_n.
b. Write an explicit expression for a_n.
c. Calculate a_{12}.
d. Explain the meaning of a_{12} in this problem.
e. Eddie does not want the balance in his account to go below \$500. How many months can Eddie go before he needs to add money to his account?

17. A recursive expression is defined as $a_{n+1} = 1.07a_n$.

a. Find r, the common ratio.
b. If $a_5 = 3.93$, find a_1.
c. Write the explicit expression for a_n.

18. The picture below represents the first five triangular numbers.



a. Use manipulatives to model and write the next three triangular numbers.
b. Write a recursive expression for a_n.
c. Is the sequence arithmetic, geometric, or neither?

19. The first four terms in a sequence are 200, 190, 180.5, and 171.475. Which of the following expressions defines this sequence?

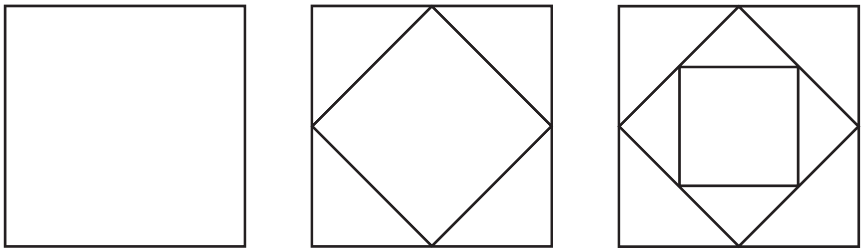
A. $a_n = 210 - 10n$
B. $a_n = 200(0.95)^{n-1}$
C. $a_n = a_{n-1} - 10$
D. $a_n = 200a_{n-1}$

20. **a.** Define the set of odd natural numbers by means of an explicit formula.
b. Define the set of odd natural numbers by means of a recursive formula.
c. Define the set of even natural numbers by means of an explicit formula.
d. Define the set of even natural numbers by means of a recursive formula.
e. What is the difference between the recursive formulas for the odd and the even integers?

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

21. A sequence a_n can often be written recursively or explicitly. Describe ways to identify, given the first several terms of a sequence, if a recursive or explicit formula can be determined.

A design was created for the Pacesetter Museum's advertising campaign. This design included a square divided into smaller regions. In each phase, the innermost quadrilateral is a square, and the triangles are isosceles right triangles. The design was created in phases, as shown.



1. In each phase, the number of nonoverlapping squares and triangles is determined. The list of these numbers forms a sequence.
   a. List the first five terms of this sequence.
   b. Is this sequence arithmetic, geometric, or neither? Explain how you know.
   c. Express the terms of this sequence two ways: recursively and explicitly.
   d. Suppose this pattern were continued. How many nonoverlapping regions would occur in Phase 10?
   e. Express the sum of the first 20 terms of this sequence using sigma notation, and calculate this sum.

2. In each phase, the length of the side of the smallest square drawn is determined. The list of these lengths forms a sequence. Let the length of the side of the square in Phase 1 be 20 cm. Use exact values in your responses. Add units to your answers.
   a. List the lengths represented by the first five terms of this sequence.
   b. Is this sequence arithmetic, geometric, or neither? Explain how you know.
   c. Express the terms of this sequence two ways: recursively and explicitly.

3. In each phase, the area of the smallest square is calculated. The list of these areas is a sequence.
   a. List the areas represented by the first five terms of this sequence, beginning with Phase 1.
   b. Suppose this pattern were continued. Determine the area of the innermost square in Phase 10.
   c. Express the sum of the first 10 terms of this sequence using sigma notation, and calculate this sum. What does this represent in terms of the situation?
   d. Suppose this pattern were continued without end. Find the sum of this infinite series, if possible, and its meaning in terms of the situation.
   e. Suppose the pattern developed in reverse, from a small square outward. Would the resulting infinite geometric series have a sum? Explain why or why not.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 2, 3)	• Clear and accurate understanding of arithmetic and geometric sequences and series, including writing series in sigma notation	• A functional understanding of arithmetic and geometric sequences and series	• Partial understanding of arithmetic and geometric sequences and series	• Little or no understanding of arithmetic and geometric sequences and series
Problem Solving (Items 1, 2, 3)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1c, 2c, 3d, 3e)	• Clear and accurate understanding of creating arithmetic and geometric sequences, including using sigma notation • Clear and accurate understanding of representations of geometric series, including infinite geometric series and when they converge	• Mostly accurate understanding of creating arithmetic and geometric sequences • A functional understanding of geometric series	• Partial understanding of sequences • Partial understanding of geometric series	• Inaccurate or incomplete understanding of sequences • Little or no understanding of geometric series
Reasoning and Communication (Items 1b, 1c, 2b, 2c, 3e)	• Precise use of appropriate math terms and language to evaluate sequences and series, including the representation of recursive sequences	• Correct characterization of sequences and series	• Misleading or confusing characterization of sequences and series	• Incomplete or inaccurate characterization of sequences and series

Exponential Functions

Pennsylvania Lottery

Lesson 4-1 Writing an Exponential Function

Learning Targets:

- Write, graph, analyze, and model with exponential functions.
- Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Shared Reading, Summarizing, Paraphrasing, Create Representations, Look for a Pattern, Quickwrite, Note Taking

Suppose your neighbor, Margaret Anderson, has just won the state lottery, and her first payment will be $50,000. Margaret is interested in options that involve spending part of her winnings and saving the balance so that she can accumulate a nest egg at the end of the 20-year period. The tasks that follow will help you analyze Margaret's situation.

1. **Model with mathematics.** If Margaret saves only her first lottery payment of $50,000 and deposits it in a savings account paying 5% interest, compounded annually, determine how much money she will have in her account at the end of the each year given in the table below.

Year	Years Since 2004	Account Balance
2004	0	$50,000
2005	1	$52,500
2006	2	$55,125
2007		
2008		
2009		
2014		

2. What patterns do you notice in the table?

Exponential functions are multiplicative. That is, when a change in the input is constant, there is a constant multiplicative change in the output. The general form of an ***exponential function*** can be expressed as $f(x) = a(b)^x$, where a is a nonzero constant and b is a positive constant, $b \neq 1$.

3. Why is an exponential function appropriate for representing the data in the table?

4. What is the constant multiplier for the exponential function representing the data in the table? Explain how to find the constant multiplier for a set of data.

My Notes

CONNECT TO HISTORY

Harry Casey was the first winner of the Pennsylvania lottery in 1972. He won $1 million, which was paid in 20 annual installments of $50,000. Harry retired immediately, spent $50,000 each year, received his last check in 1991, and was broke by the spring of 1992.

My Notes

5. Complete the table below.

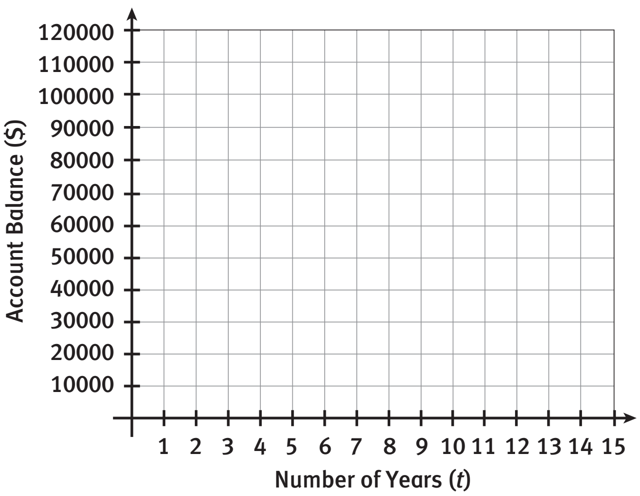
Years Since 2004	Change in Account Value from Previous Year	Annual Growth Factor
1	$2500.00	1.05
2		
3		
4		
5		

6. As described in Item 1, the amount of money in Margaret's account, her account balance $A(t)$, is a function of the number of years t that have elapsed since 2004. Write an equation that defines $A(t)$.

MATH TIP

The end behavior of the graph of $A(t)$ continues to increase without bound. As $t \to \infty$, $A(t) \to \infty$.

7. Use appropriate tools strategically. Using a graphing calculator, graph $A(t)$ in an appropriate viewing window. Sketch the function on the grid below and estimate the following values.



a. the money in Margaret's account after 10 years

b. the years needed to double her initial investment

c. Attend to precision. Use your equation from Item 6 to find precise answers for parts a and b.

My Notes

MATH TERMS

In an exponential function, the constant multiplier, or scale factor, is known as an **exponential growth factor** when the constant is greater than 1. The constant multiplier is known as an **exponential decay factor** when the constant is between 0 and 1.

The parameters of the function A represent particular features of the situation. The $50,000 value represents the amount of money that was deposited to open Margaret's savings account. This value is known as the initial amount, or the ***principal***, P. For a 5% interest rate, the value $1 + 0.05$, or 1.05, represents the amount by which the current balance is multiplied to get the following year's balance. For any ***interest rate*** r, $1 + r$ is the annual ***growth factor***.

8. Using parameters P and r, define a general function $A(t)$, where t is the number of years since the principal was deposited in Margaret's savings account.

9. Write a function for Margaret's account balance at the same annual interest rate of 5%, but with a principal of $30,000.

10. Margaret wants to compare how her investment grows over time when the principal changes.

a. Write the equation to find the time it will take to double the $30,000 initial investment.

b. How long would it take for Margaret to double her investment if she deposited $30,000 instead of $50,000? Explain how you arrived at this conclusion.

c. From the results of Items 7c and 10b, and any other principal amounts you choose to investigate, what conclusion can you make regarding the doubling time for any principal amount P at an annual interest rate of 5%?

Check Your Understanding

11. Write a function for Margaret's account balance at the annual interest rate of 4% with a principal of $50,000.

12. How long would it take to double Margaret's initial investment of $50,000 if the annual interest rate were 4%?

13. Explain why $A(t) = P(1.05)^t$ forms a geometric sequence for $t = 1, 2, 3, \ldots$.

My Notes

Margaret will invest in an account that offers a 5% annual interest rate, compounded annually. However, she may not invest all of the $50,000.

14. Write functions $A(t)$, $B(t)$, and $C(t)$ for the amount of money Margaret would have in her account if she makes initial investments of $10,000, $25,000, and $50,000.

15. For each function, find the amount of money Margaret would have in her account after 10 years and after 20 years.

16. **Use appropriate tools strategically.** Use a graphing calculator. Graph each function for the first 20 years of the investment on one graph.

a. What is the relationship between the y-intercepts of the graphs and the investments?

b. Describe the end behavior of each graph as t increases.

MATH TIP

Determine an appropriate viewing window on which to view the graphs. Use your input and output values from Item 16.

Margaret makes another small investment with some of her money. The investment has an annual interest rate that is compounded annually. A graph of her account balance over time in years passes through the points (1, 3240) and (2, 3499.20).

17. **Reason quantitatively.** What interest rate does the account earn? How did you determine this?

18. What was Margaret's principal for the account? Explain your reasoning.

My Notes

19. Graph the functions $f(x) = a(b)^x$ and $g(x) = a(b)^{-x}$ on a graphing calculator. Choose various positive values for a and b, where $b \neq 1$. Then determine if the descriptions below are true for the functions by writing $f(x)$, $g(x)$, *both*, or *neither* next to each.

a. increasing on the interval $(-\infty, \infty)$
b. decreasing on the interval $(-\infty, \infty)$
c. x-intercept: $(a, 0)$
d. y-intercept: $(0, a)$
e. horizontal asymptote: $y = 0$
f. domain: all real numbers
g. range: $y > 0$
h. range: $y < 0$

20. Make use of structure. Consider the functions $f(x)$ and $g(x)$ from Item 19. How is the graph of $f(x)$ related to the graph of $g(x)$? Explain by writing an equation that defines $f(x)$ in terms of $g(x)$.

Check Your Understanding

21. Chad invests $12,000 at a 5% annual interest rate, compounded annually. Write a function $A(t)$ that finds the amount Chad has in his account after t years. Explain what the y-intercept represents. Describe the end behavior of the graph of $A(t)$ as t increases.

22. Reason quantitatively. After t years, Chad has $16,081.15. How many years did it take for the account to reach this balance? Explain how to find the number of years it took to reach the balance.

LESSON 4-1 PRACTICE

23. Michael opened a savings account at an annual interest rate of 5%. At the end of 3 years, the account balance is $4630.50. If Michael did not add any other amounts to this account, how much was his initial deposit?

24. Find the annual interest rate when the amount of $25,000 grows to $26,625 in the first year and $28,355.63 in the second year.

25. Write and solve an equation to determine the balance after 25 years in an account that had an initial investment of $18,000 at 5% interest, compounded annually.

26. Any principal amount invested at 5% annual interest takes 15 years to double. How many years does it take for the principal amount to triple?

27. Express regularity in repeated reasoning. Repeat Item 19 assuming a negative value of a.

My Notes

Learning Targets:

- Write, graph, analyze, and model with exponential functions.
- Calculate compound interest.
- Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations, Quickwrite, Group Presentation, Debrief

1. **Model with mathematics.** Consider two investments made at the same time. In the first investment, $50,000 is deposited in an account that offers an annual interest rate of 5% compounded annually. In the second investment, $30,000 is deposited in an account that offers an annual interest rate of 8.5% compounded annually.
   a. Use a graphing calculator. Graph the balance in each account for the first 20 years of the investments. Write a function for each investment.

   b. Over the first 20 years, for which years is the amount of money greater in the account that began with an investment of $50,000? For which years is the amount of money greater in the account that began with an investment of $30,000?

2. Over a long period of time, which parameter, principal or interest, has a greater effect on the amount of money in an account that has interest compounded annually? Explain your reasoning.

Most savings institutions offer compounding intervals other than annual compounding. For example, a bank that offers *quarterly compounding* computes interest on an account every quarter, that is, every 3 months. Instead of computing the interest once each year, interest is computed four times each year. If a bank advertises that it is offering 8% annual interest, compounded quarterly, 8% is not the growth factor. Instead, the bank will use $\frac{8\%}{4} = 2\%$ to determine the quarterly growth factor. In this example, 8% is the *nominal interest rate*, and 2% is the *quarterly interest rate*.

3. What is the quarterly interest rate for an account with a nominal rate of 5%, compounded quarterly?

My Notes

4. Suppose that Margaret invested $50,000 at 5% interest, compounded quarterly.

a. In the table below, determine Margaret's account balance after the specified periods of time since her initial investment.

Time Since Initial Investment	Account Balance
3 months	
6 months	
9 months	
1 year	
4 years	
t years	

b. Write the amount A in the account as a function of t, the number of years since the investment was made.

5. How long will it take for Margaret's investment, as described in Item 4, to double?

6. **Make sense of problems.** How much less time is required for Margaret's initial amount of money to double at the rate of 5% compounded quarterly, than at the rate of 5% compounded annually?

7. For the compounding periods given below, determine the amount of money that Margaret would have in an account after 20 years if the principal is $50,000 and the nominal rate is 5%.

a. yearly

b. quarterly

c. monthly

d. daily (assume that there are 365 days in a year)

8. What is the effect of the compounding period on the amount of money after 20 years as the number of times the interest is compounded each year increases?

My Notes

9. **Attend to precision.** Consider an initial investment of $1 and an interest rate of 100%. Find the amount of money in this account after one year with the following number of compounding periods per year. Record your answers to four decimal places in the table.

Compounding Periods per Year	Account Balance
1	
10	
100	
1,000	
10,000	
100,000	
1,000,000	

10. As the number of times the account in Item 9 is compounded per year increases, what appears to be happening to the amount of money in the account after 1 year?

Check Your Understanding

11. A bank offers a certificate of deposit, or CD, with a monthly interest rate of 2.5%. What is the nominal interest rate for the CD?
12. Which yields more interest after 5 years: $4000 invested at an annual interest rate of 5% compounded monthly, or $4000 invested at an annual interest rate of 4% compounded daily?

LESSON 4-2 PRACTICE

13. Write and solve an equation to determine the balance after 10 years in an account that had an initial investment of $25,000 at 3.5% interest, compounded quarterly.
14. **Make sense of problems.** How much additional interest could $2500 earn in 10 years, compounded quarterly, if the annual interest rate were $3\frac{1}{4}\%$ as opposed to 3%?
15. How much money needs to be deposited into an account that earns 4% annual interest, compounded monthly, to have a balance of $5000 after 5 years?

My Notes

Learning Targets:

- Write, graph, analyze, and model with exponential functions.
- Calculate compound interest.
- Solve exponential equations.

SUGGESTED LEARNING STRATEGIES: Note Taking, Interactive Word Wall, Create Representations, RAFT

The exponential function $A(t) = Pe^{rt}$, where P is the principal, r is the interest rate, t is time, and e is a constant with a value of 2.718281828459…, is used to calculate a quantity (most frequently money) that is *compounded continuously* (that is, the number of compounding periods approaches infinity).

1. Find the amount of money in an account after 20 years if the principal is \$50,000 and the nominal rate is 5% compounded continuously. Compare this answer to your answers in Item 7 of Lesson 4-2.

2. Margaret would like information on a few different investment options. She wants to invest either all or half the amount of her first \$50,000 lottery check. Write a proposal to Margaret giving her advice on where to invest her money. Include an explanation of why you are making these recommendations. Include options for both a \$50,000 and a \$25,000 initial investment. Use the following account information to help make your recommendations.

Big Bucks Bank:	Annual rate of 4% on amounts greater than or equal to \$30,000 Annual rate of 3.7% on amounts less than \$30,000
Serious Savings:	Nominal rate of 3.67% compounded weekly
Infinite Investments:	Nominal rate of 3.5% compounded continuously

CONNECT TO STATISTICS

In 1683, Jacob Bernoulli looked at the problem of continuously compounded interest and tried to find the limit of $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$. Bernoulli used the *Binomial Theorem* to show that this limit had to lie between 2 and 3. In 1731, Leonhard Euler first used the notation e to represent this limit; he gave an approximation of the irrational number e to 18 decimal places.

The number e is believed to be the first number to be defined using a limit and has since been calculated to thousands of decimal places. This number is very important in advanced mathematics and frequently appears in statistics, science, and business formulas.

My Notes

CONNECT TO FINANCE

Depreciation is the reduction in the value of an asset due to usage, passage of time, wear and tear, technological outdating or obsolescence, depletion, or other such factors.

3. Margaret plans to purchase a boat that will cost her $10,000. The boat continuously ***depreciates*** at an annual rate of 17%.

a. Write an exponential function for the value. How much will the boat be worth in 15 years?

b. Model with mathematics. How long will it take for the boat to be worth half of its original value?

4. Use a graphing calculator to graph a function with 17% growth and a function with 17% depreciation. Compare and contrast the graph of the exponential growth function with that of the exponential decay function.

Check Your Understanding

5. Explain why investing $1000 at 4% interest, compounded continuously, for 2 years is equivalent to investing $1000 at 8% interest for 1 year.

6. Make use of structure. Recall that the constant multiplier of an exponential function is known as an *exponential decay factor* when the constant is between 0 and 1. Rewrite the function $f(x) = 100e^{-x}$ with a constant multiplier between 0 and 1.

7. Critique the reasoning of others. Edgar says that the function $f(x) = -50e^{-x}$ is a decreasing function because the negative sign on the exponent always represents exponential decay. Explain Edgar's mistake.

LESSON 4-3 PRACTICE

8. An account that was invested at 5% with continuous compounding for 10 years now contains $5900. What was the initial investment?

9. A new car is purchased for $25,000. It depreciates continuously at a rate of 12%. Write an exponential function that represents the value of the car after *t* years of ownership. When will the car have a value of $0. Explain.

10. The ***half-life*** of the radioactive substance C-14 is about 5730 years. This means after every 5730 years, the amount present is half as much as before. Solve the equation $0.5 = e^{-5730r}$ to find the decay rate *r*. Round your answer to five decimal places.

MATH TERMS

The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one-half.

ACTIVITY 4 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 4-1

1. Write and solve an equation to determine the balance after 25 years in an account that had an initial investment of \$18,000 at 3% interest, compounded annually.
2. Determine the balance after 10 years in an account that had an initial investment of \$25,000 at 5% interest, compounded annually.
3. On a graphic calculator, enter $y_1 = 2000(1.04)^t$ and $y_2 = 1000(1.08)^t$. Open the table function. Record the value of each at $t = 1$. Which function has the greater value?
4. In Item 3, between which two values of t are the two functions approximately equal?
5. What will a \$150,000 house be worth in 10 years if the inflation rate remains constant at 3%?
6. Katie has \$15,000 to invest in an account earning 4.75% interest, compounded monthly. How much interest has Katie earned after 5 years?
7. If Katie withdraws \$10,000 at the end of the second year, how much interest will she lose?
8. How much interest is earned on an investment of \$5200 earning 7.5% interest, compounded annually, over a period of 3 years?
   A. \$22.50
   B. \$1259.94
   C. \$2250
   D. \$459.95
9. Find the annual interest rate when the amount of \$15,000 grows to \$15,525 in the first year and \$16,068.38 in the second year.
10. Which piece of information is missing to determine the balance with \$5000 at 5%?
    A. principal
    B. amount
    C. rate
    D. time

11. A company that makes mobile apps has profits of \$7000 in its first year. The CEO says that this will triple each year over the next 5 years.
    a. Write an exponential function that represents this situation.
    b. In the fifth year after the CEO's statement, the company had a profit of 1.5 million dollars. Was the CEO's goal realized that year?

Lesson 4-2

12. Determine the interest earned after 5 years in an account that had an initial investment of \$25,000 at 3.5% interest, compounded daily.
13. Determine the balance after 10 years in an account that had an initial investment of \$25,000 at 3.5% interest, compounded quarterly.
14. Determine the amount of money that needs to be invested at 6% compounded monthly, to have \$20,000 in 15 years.
15. Complete the table by finding each balance.

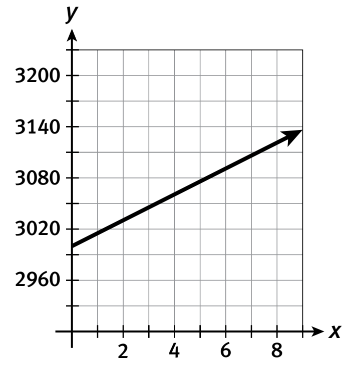
\$10,000	5% annually	5% quarterly	5% monthly	5% weekly
in 5 years				
in 10 years				

16. Harry has \$3100 to invest. In 2 years, he needs \$3500. Which investment will allow Harry to meet his goal?
    A. 4% compounded weekly
    B. 5% compounded monthly
    C. 6% compounded quarterly
    D. 8% compounded annually

17. How much additional interest could \$15,000 earn in 20 years, compounded monthly, if the annual interest rate were $5\frac{1}{4}\%$ as opposed to 5%?

18. If a balance of \$20,000 grew to \$29,816.66 in 10 years, determine the rate of interest.
A. 4% compounded weekly
B. 5% compounded weekly
C. 4% compounded monthly
D. 5% compounded monthly

19. The graph shows the value of an account after *t* months.



a. What is the initial deposit of the account?
b. What is the balance at 7 months?
c. Estimate the rate of interest.

Lesson 4-3

20. Determine the balance after 20 years in an account that had an initial investment of \$25,000 at 5% interest, compounded continuously.

21. Graph the ordered pairs (1, 2400), (2, 2308), and (3, 2219). Find the rate of change and the initial amount from the graph.

22. Find the interest rate when an investment of \$200, compounded continuously for 5 years, 3 months, is valued at \$253.30.

23. The population of deer on an island is growing exponentially. The first year the population was measured, there were 500 deer. Five years later, there were 552.
a. Write an exponential function that represents the number of deer on the island given the years since the initial population count.
b. How long will it take for the number of deer in Item 23a to double?

24. Write and solve an equation to determine the balance after 15 years in an account that had an initial investment of \$25,000 at 3.5% interest, compounded continuously.

25. The Tamerix tree was introduced to a region in 2006 and has been spreading exponentially. The initial population was 20 trees, and the population is increasing at an annual rate of 15%.
a. Create a continuous exponential function that represents the number of Tamerix trees in the region in a given year since the first population was measured.
b. In what year will the population reach 300?

26. A new car was purchased in 2005 for \$20,000. It depreciates at a rate of 9%.
a. Write a continuous exponential function that represents the value of the car after *t* years of ownership.
b. When will the car have a value of \$10,000?

27. Complete the table below for each interest rate for \$1000, compounded continuously.

	4%	5%	6%	8%
\$1000	1040.81	1051.27	1061.83	1083.29
2 yr	1083.29			
5 yr	1221.40			

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

28. How do exponential functions relate to geometric sequences?

Logarithms

Power Trip

Lesson 5-1 Common and Natural Logarithms

My Notes

Learning Targets:

- Explore the inverse relationship between exponents and logarithms.
- Graph logarithmic functions and analyze key features of the graphs.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Discussion Groups, Interactive Word Wall, Note Taking, Create Representations

As you work in groups, read the problem scenario carefully and explore together the information provided. Discuss your understanding of the problem and ask peers or your teacher to clarify any areas that are not clear.

The concentration of hydrogen ions (H^+) in a solution determines how acidic or basic the solution is. Pure water is neither acidic nor basic. A solution with a higher concentration of H^+ than pure water is acidic. A solution with a lower concentration of H^+ than pure water is basic.

For a strong acid, the concentration of H^+ may be 1 mole/liter. For a strong base, the concentration of H^+ may be 0.00000000000001 mole/liter. So the H^+ concentration of a strong acid may be 100,000,000,000,000 times greater than the H^+ concentration of a strong base.

To deal with such wide variations in concentration, scientists have developed a shorthand way of representing how acidic or basic a solution is. This method is based on *logarithms*. Logarithms can be used to simplify computations involving very large or very small numbers. You will begin to explore logarithms in this activity.

CONNECT TO CHEMISTRY

An *ion* is an atom or group of atoms with a net electric charge. A hydrogen ion is a hydrogen atom with a net charge of $+1$.

A *mole* is a quantity that is equal to about 6.022×10^{23} units of a substance. A mole of hydrogen ions is equal to about 6.022×10^{23} hydrogen ions.

1. What type of sequence is represented by each pattern shown below? Explain how you know.

A	0	1	2	3	4	5	6	7	8	9	. . .
B	1	10	100	1000	10,000	100,000	1,000,000	10,000,000	100,000,000	1,000,000,000	. . .
C	1	2	4	8	16	32	64	128	256	512	. . .

My Notes

2. Use patterns A and B to answer the following.
   a. Circle two columns in the table so that the sum in row A is 9 or less. Find the product of the two numbers you circled in pattern B and circle the column that contains that product.

   b. Explain how the circled numbers in pattern A are related.

   c. Explain how the answer to Part b is related to the product from pattern B.

   d. If the numbers 2 and 9 are selected from pattern A, how can they be used to find the product of 100 and 1,000,000,000?

3. Use patterns A and C to answer the following.
   a. Find the product of the two numbers you circled in pattern C and circle the column that contains that product.

   b. Look at the circled numbers from pattern A. How are these numbers related to the circled numbers from pattern C and their product?

   c. If the numbers 2 and 9 are selected from pattern A, how can they be used to find the product of 4 and 512?

The process you went through in Items 1–3 is the basis for the concept of logarithms. If b and x are positive real numbers with $b \neq 1$, then the ***logarithm*** of x ***base*** b is written as $\log_b x$ and is defined as:

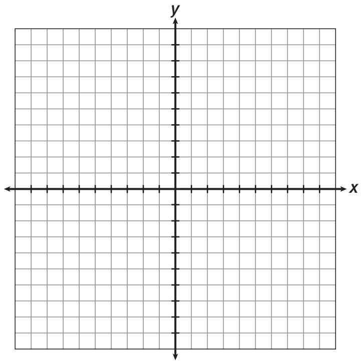
$$\log_b x = y \text{ if and only if } b^y = x.$$

In Item 1, the numbers in pattern A are the logarithms to base 10 of the corresponding numbers in pattern B. The numbers in pattern A are also the logarithms to base 2 of the corresponding numbers in pattern C.

Although any number may be used as the base of a logarithm, the two most frequently used bases are the numbers 10 and e.

4. Use your graphing calculator to graph the functions $y = 10^x$ and $y = \log x$.

a. Sketch a graph of each function.



b. The functions $y = \log x$ and $y = 10^x$ are *inverses*. How does the graph support this statement?

c. The point (2, 100) lies on the graph of $y = 10^x$ because $10^2 = 100$. Explain how you can use this point and the properties of inverse functions to determine the value of log 100.

5. a. What are the domain and range of $y = \log x$?

b. What are the domain and range of $y = 10^x$?

c. What is the relationship between the domains and the ranges of the two functions?

My Notes

MATH TERMS

A **logarithm** is the power to which a **base** is raised: $\log_b x = y$ if and only if $b^y = x$, where $b > 0$, $b \neq 1$, and $x > 0$.

The logarithm to base 10 is called the **common logarithm**. $\log_{10} x = y$ means $10^y = x$. $\log_{10} x$ is usually written as just $\log x$.

CONNECT TO HISTORY

John Napier of Scotland (1550–1617) is credited with the invention of logarithms.

My Notes

6. Use the concept of exponents to explain why the function $y = \log_b x$ has a vertical asymptote at $x = 0$.

7. Is $y = \log x$ increasing or decreasing over its entire domain or is there a turning point?

MATH TERMS

Functions that either increase or decrease over their entire domain are called **strictly monotonic**.

8. a. What happens to the value of $y = \log x$ as x approaches infinity?

b. What happens to the value of $y = \log x$ as x approaches 0 for $x > 0$?

MATH TERMS

The **natural logarithm** is the logarithm to base *e*.

The function $y = \log_e x$, usually written as $y = \ln x$, is called the ***natural logarithm***. In exponential form, $\ln x = y$ means $e^y = x$.

9. Use the definition of logarithms to complete the following tables.

$y = \log x$

x	y
1	
10	
10^2	
10^5	

$y = \ln x$

x	y
1	
e	
e^2	
e^5	

10. Express regularity in repeated reasoning. Use the information you found in the table above to complete the properties of logarithms and rewrite each property in exponential form.

Properties of Logarithms	Exponential Form
$\log_b 1 =$	$b^0 =$
$\log_b b =$	$b^1 =$
$\log_b b^x =$	$b^x =$
$\ln 1 =$	$e^0 =$
$\ln e =$	$e^1 =$
$\ln e^x =$	$e^x =$

11. Use properties of logarithms to evaluate.

a. $\log_3 27$ **b.** $\log_4 2$ **c.** $\log_{10}\left(\frac{1}{100}\right)$

12. The pH scale is a logarithmic scale from 0 to 14 used to determine how acidic a solution is. The pH of a solution is determined by using the formula pH $= -\log[H^+]$, where $[H^+]$ is the hydrogen ion concentration of the solution in moles/liter.

   a. What is the pH of a solution with a hydrogen ion concentration of 0.001 mole/liter? Explain how you know. (Try graphing $y = -\log x$, and look for the value of y when $x = 0.001$.)

   b. What is the hydrogen ion concentration of a solution with a pH of 7? Explain how you know.

   c. The hydrogen ion concentration of a solution with a pH of 4 is how many times the hydrogen ion concentration of a solution with a pH of 5? Explain.

13. Let $f(x) = \ln x$ and $g(x) = e^x$.

   a. Find $f(g(x))$ and $g(f(x))$.

   b. What do your results from Part a indicate about the relationship between $f(x) = \ln x$ and $g(x) = e^x$?

14. Verify that $f(x) = \frac{1}{2} + \ln x$ and $g(x) = e^{x-\frac{1}{2}}$ are inverses.

My Notes

MATH TIP

$[H^+]$ can be written as bx where b and x are any real numbers. Graph the pH formula $y = -\log bx$ for different values of b. Notice the end behavior of the graph: as $x \to \infty, y \to \infty$ and as $x \to 0$, $y \to -\infty$.

MATH TIP

You can also graph $Y1 = -\log x$ and $Y2 = 7$, and determine where they intersect.

MATH TIP

Recall that two functions are inverses when

$f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

My Notes

15. Given the function $f(x) = \log x$, find the following.
a. Find x when $f(x) = 7$. **b.** Find $f(x)$ for $x = 158$.

Check Your Understanding

Write each logarithmic equation as an exponential equation.

16. $\log 1000 = 3$

17. $\log_2 32 = 5$

18. Critique the reasoning of others. A student claims that the expression $\log_3 9$ represents the exponent to which the base 9 must be raised to equal 3. Is the student's claim correct? Explain.

19. An exponential function of the form $y = b^x$, where $b > 0$ and $b \neq 1$, has a y-intercept of 1. Based on this information, what can you conclude about the x-intercept of a logarithmic function of the form $y = \log_b x$? Justify your conclusion.

20. Explain how you could use the graph of $y = 4^x$ to graph the function $y = \log_4 x$.

21. Explain how scientists use logarithms to make it easier to describe and compare the acidity of solutions.

LESSON 5-1 PRACTICE

Evaluate each logarithm.

22. $\log_3\left(\frac{1}{9}\right)$

23. $\log_4 64$

24. a. Complete the table for the function $y = \log_2 x$, a real-world model.

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
y							

b. Graph the function.
c. Describe key features of the function, including the domain, range, x-intercept, asymptote, and end behavior.

25. If a solution has a pH less than 7, then the solution is acidic. If the pH is greater than 7, then the solution is basic. Honey has a hydrogen ion concentration of 0.0001 mole/liter. Is honey acidic or basic? Use a logarithmic function and its graph to support your answer.

26. Construct viable arguments. Explain how you could use multiple representations to convince someone that $y = e^x$ and $y = \ln x$ are inverse functions.

My Notes

Learning Targets:

- Apply the Change of Base Formula.
- Use properties of logarithms to evaluate and transform expressions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Look for a Pattern, Think-Pair-Share, Think Aloud, Construct an Argument

Common logarithm and natural logarithm functions are typically built into calculator systems. However, it is possible to use a calculator to evaluate logarithms in other bases by using the Change of Base Formula.

Change of Base Formula

Let a, b, and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_b x = \dfrac{\log_a x}{\log_a b}$.

Example A

Use a calculator and the Change of Base Formula to find an approximation of $\log_5 28$.

$\log_5 28 = \dfrac{\log 28}{\log 5} \approx 2.070$

Try These A

Use the Change of Base Formula to rewrite. For Items a and b, use a calculator to find an approximation.

a. $\log_5 8$ with a base of 10

b. $\log_2 12$ with a base of e

c. $\log_b x$ with a base of 10

d. $\log_b x$ with a base of e

CONNECT TO BIOLOGY

The doubling time of a population is the length of time it takes a population to double. The smaller the doubling time, the faster the population is growing.

1. **Reason quantitatively.** A biologist is studying a colony of bacteria with a doubling time of 1 hour. The colony initially contains 20 bacteria.
   a. Write the equation of an exponential function $n(t)$ that can be used to determine the number of bacteria n in the colony after t hours.

   b. Use your function to write an equation that can be used to determine how many hours must pass before the population of the colony is 10,000 bacteria.

My Notes

c. Rewrite your equation from Part b so that 2^t is isolated on one side.

d. Write your equation from Part c as an equivalent logarithmic equation.

e. Use the Change of Base Formula to solve the equation from Part d for t. Interpret the solution in the context of the problem.

CONNECT TO AP

You can prove the Change of Base Formula $b^{\log_b x} = x$ because exponents and logarithms are inverses. Take the log base a of both sides: $\log_a b^{\log_b x} = \log_a x$. By the Power Property of Logarithms, $(\log_b x)\log_a b = \log_a x$. Dividing both sides by $\log_a b$ gives $\log_b x = \dfrac{\log_a x}{\log_a b}$.

Each property of exponents has a corresponding property for logarithms.

Properties	Properties of Exponents	Properties of Logarithms
Product of a Power	$b^m b^n = b^{m+n}$	$\log_b (mn) = \log_b m + \log_b n$
Quotient of Power	$\dfrac{b^m}{b^n} = b^{m-n}$	$\log_b \left(\dfrac{m}{n}\right) = \log_b m - \log_b n$
Power of a Power	$(b^m)^n = b^{mn}$	$\log_b m^n = n \log_b m$
One to One	if $b^m = b^n$, then $m = n$	if $\log_b m = \log_b n$, then $m = n$

2. Use your graphing calculator to graph $y = \log 3x$ and $y = \log 3 + \log x$. How do the graphs compare?

3. Write each property for the natural logarithm that corresponds to the property for common logarithms in the table below.

Properties of Logarithms	Properties of Natural Logarithms
$\log_b (mn) = \log_b m + \log_b n$	
$\log_b \left(\dfrac{m}{n}\right) = \log_b m - \log_b n$	
$\log_b m^n = n \log_b m$	
if $\log_b m = \log_b n$, then $m = n$	

My Notes

Example B

Use the properties of logarithms to expand $\ln\left(\frac{3xy}{z}\right)$.

$$\ln\left(\frac{3xy}{z}\right) = \ln 3xy - \ln z$$
$$= \ln 3 + \ln x + \ln y - \ln z$$

Try These B

Use the properties of logarithms to expand each expression.

a. $\log(5xy^3)$

b. $\ln\left(\frac{xy}{z^4}\right)$

c. $\ln\left(\frac{x}{\sqrt{x^2+1}}\right)$

d. $\log_2 \sqrt{2x(x^2+2)}$

MATH TIP

Logarithms of sums and differences, $\log_a(m \pm n)$, *cannot* be rewritten using general properties of logarithms.

Example C

Use the properties of logarithms to write $\frac{1}{2}(3\log x + \log(x+1) - \log x)$ as a single logarithm.

$\frac{1}{2}(3\log x + \log(x+1) - \log x)$	
$\frac{1}{2}(\log x^3 + \log(x+1) - \log x)$	Power Property of Logarithms
$\frac{1}{2}\left(\log\left(\frac{x^3(x+1)}{x}\right)\right)$	Product and Quotient Properties
$\frac{1}{2}\log(x^3+x^2)$	Divide.
$\log(x^3+x^2)^{\frac{1}{2}}$	Power Property of Logarithms
$\log\sqrt{x^3+x^2}$	Write as a radical expression.

Try These C

Write each expression as a single logarithm.

a. $3\ln(x-1) - \ln x$

b. $\log x - 4[\log(x-2) + \log(x+2)]$

c. $\frac{1}{2}[\ln(x+2) - \ln(x^2-4)]$

d. **Make use of structure.** Explain why you cannot write the expression $\log_2(x+5) + \log_5(x+2)$ as a single logarithm.

My Notes

Check Your Understanding

Use properties of logarithms to find the value of each expression without using a calculator. List a property or give an explanation for each step in your work.

4. $\log_3 \left(\frac{1}{9}\right)^4$

5. $\log 25 + 2 \log 2$

6. Explain how you could use the Change of Base Formula to help you graph the function $y = \log_4 x + 3$ on a graphing calculator.

7. Use properties of logarithms to explain why the functions $f(x) = \log(2x^2)$ and $g(x) = 2 \log x + \log 2$ have the same graph.

8. A student expanded the expression $\log (3x^2)$ as follows. Identify the error that the student made. Fix the error and expand the expression correctly.

$\log (3x^2) = 2 \log (3x)$	Power Property of Logarithms
$= 2(\log 3 + \log x)$	Product Property of Logarithms
$= 2 \log 3 + 2 \log x$	Distributive Property

LESSON 5-2 PRACTICE

Use a calculator and the Change of Base Formula to find an approximation.

9. $\log_7 4$

10. $\log_3 30$

Use the properties of logarithms to expand each expression.

11. $\ln xyz$

12. $\ln \sqrt{x - 1}$

Write each expression as a single logarithm.

13. $\ln x - 3 \ln (x + 1)$

14. $2 \log 6 + \log x$

15. Present a numerical example that illustrates the Quotient Property of Logarithms.

16. **Make sense of problems.** The number of visitors to a website is tripling each month. This month, the website had 1500 visitors. Explain how you can use a logarithm and the Change of Base Formula to predict how many months it will take for the number of visitors to the website to reach 300,000.

My Notes

Learning Targets:

- Solve exponential equations by taking the logarithm of both sides.
- Use properties of exponents and logarithms to solve logarithmic equations.

SUGGESTED LEARNING STRATEGIES: Close Reading, Levels of Questions, Note Taking, Think-Pair-Share, Identify a Subtask

In a previous activity, you solved exponential equations with the aid of a graphing calculator. You can use the properties of logarithms and exponents to solve these same equations algebraically.

Example A

Jose invested \$100,000 in a retirement account that averages 6.5% interest compounded continuously. How long will it take for him to double his money?

$200{,}000 = 100{,}000e^{0.065t}$

$2 = e^{0.065t}$	Divide both sides by 100,000.
$\ln 2 = \ln e^{0.065t}$	Take the ln of both sides.
$\ln 2 = 0.065t \ln e$	Power Property of Logarithms
$\ln 2 = 0.065t$	Inverse Property of Exponents and Logarithms: $\ln e = 1$
$\frac{\ln 2}{0.065} = t$	Solve for t.
$t \approx 10.664$	

Jose will double his money in about 10.664 years.

Try These A

Use the information from the example to determine how long it will it take for Jose to have the following amounts of money in his account.

a. \$120,000 **b.** \$250,000 **c.** \$1,000,000

My Notes

Example B

Solve $4(2^{2x-1}) + 5 = 29$.

$4(2^{2x-1}) + 5 = 29$	
$4(2^{2x-1}) = 24$	Isolate the exponential term.
$2^{2x-1} = 6$	
$\log_2 2^{2x-1} = \log_2 6$	Take the logarithm of both sides.
$(2x - 1)\log_2 2 = \log_2 6$	Power Property of Logarithms
$2x - 1 = \log_2 6$	Inverse Property of Exponents and Logarithms: $\log_2 2 = \log_2 2^1 = 1$
$x = \dfrac{\log_2 6 + 1}{2}$	Solve for x.
$x = \dfrac{\frac{\ln 6}{\ln 2} + 1}{2}$	Change the base to 10 or e.
$x \approx 1.792$	

TECHNOLOGY TIP

Logarithms on the graphing calculator may appear with an opening parenthesis.

log(

ln(

To assure correct order of operations, close the parentheses when performing additional operations, for example: ln(6)/ln(2).

Try These B

Solve.

a. $10^{x-1} = 270$ **b.** $5e^{x+1} - 3 = 12$ **c.** $3(4^{2x}) = 7$

d. Make use of structure. $e^{2x} - 4e^x - 5 = 0$ (*Hint:* Factor.)

Example C

Solve $3 \log_2 3x = 9$.

$3 \log_2 3x = 9$	
$\log_2 3x = 3$	Isolate the logarithm.
$2^{\log_2 3x} = 2^3$	Take the base 2 to both sides.
$3x = 8$	Inverse Property of Exponents and Logarithms: $2^{\log_2 3x} = 3x$
$x = \dfrac{8}{3}$	

Try These C

Simplify and solve.

a. $2 \ln 3x = 12$ **b.** $\log x^2 = 10$

c. $3 \log (x - 4) = 13$ **d.** $\ln \sqrt{x + 2} = 7$

My Notes

1. Sound pressure level L in decibels is given by $L = 20 \log \frac{p}{p_0}$, where p is the sound pressure and p_0 is a reference pressure. The sound pressure level of a jet engine is 145 decibels, when p_0 is the sound pressure at the threshold of hearing, 2×10^{-5} pascal.
   a. Write and solve an equation to find the sound pressure p of the jet engine to the nearest pascal.

   b. How many times the sound pressure at the threshold of hearing is the sound pressure of the jet engine?

CONNECT TO PHYSICS

A pascal is a unit of pressure equal to a force of 1 newton per square meter.

The properties of logarithms can also be used to solve equations. You must check to see that solutions are not *extraneous*, however.

MATH TIP

An extraneous solution is a root of a transformed equation that is *not* a root of the original equation because it was not in the domain of the original equation.

Example D

Solve $\ln(2x - 3) + \ln(x + 2) = 2 \ln x$ for x.

$\ln(2x - 3) + \ln(x + 2) = 2 \ln x$	
$\ln((2x - 3)(x + 2)) = \ln x^2$	Use properties of logarithms to simplify.
$\ln(2x^2 + x - 6) = \ln x^2$	Multiply.
$2x^2 + x - 6 = x^2$	One to One Property
$x^2 + x - 6 = 0$	
$(x + 3)(x - 2) = 0$	$x = -3$ and $x = 2$

Substitute the values in the original equation to see if they are solutions.

For $x = -3$, $\ln(2(-3) - 3)) + \ln((-3) + 2)) = 2 \ln(-3)$.

Note that $2 \ln(-3)$ is undefined. Because negative numbers are not in the domain of the natural log function, -3 is *not* a solution.

Check $x = 2$ in the same way to verify that is the only solution.

Try These D

Solve each equation for x. Check your solutions in the original equation.

a. $\log_2 x - \log_2(x - 1) = 1$

b. $\ln x + \ln(x^2 - 8) = \ln 8x$

c. $\log(x + 5) = \log(x - 1) - \log(x + 1)$

My Notes

Check Your Understanding

2. You can solve the equation $3^{x+2} = 50$ by taking the logarithm of both sides. Does it matter which base you use for the logarithms?

3. Explain how the Inverse Property of Exponents and Logarithms can help you solve the equation $\log_4 (x - 3) = 2$.

4. Explain why the equation $\log (x + 3) + \log (x - 3) = \log (9 - 3x)$ has no solution.

5. **Attend to precision.** Write a set of step-by-step instructions for a student who is absent from class explaining how to solve the equation $\ln x - \ln 2 = \ln (x - 6)$.

LESSON 5-3 PRACTICE

Solve.

6. $8^{-x-2} = 237$

7. $e^{2x} - 5e^x + 6 = 0$

8. $4 \ln 3x = 16$

9. $\log (x - 2) = 4$

10. $\ln (x + 2) - \ln x = \ln (x + 5)$

11. **Reason quantitatively.** A biologist is studying a colony of bacteria with a doubling time of 20 minutes. The colony initially contains 16 bacteria.

a. Write an equation that can be used to determine t, the number of hours it will take for the population of the bacteria colony to reach 10,000.

b. Solve the equation and interpret the solution.

c. Explain how you know that your answer to part b is reasonable.

12. Laura invested \$3000 in an account that earns continuously compounded interest. After three years, she had \$3332.13 in the account. Write and solve an equation to find the annual interest rate to the nearest tenth of a percent.

ACTIVITY 5 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 5-1

Write each logarithmic equation as an exponential equation.

1. $\log 0.01 = -2$
2. $\log_2 8 = 3$
3. $\log_4 x = 0$

Write each exponential equation as a logarithmic equation.

4. $10^5 = 100{,}000$
5. $2^{-1} = \frac{1}{2}$
6. $4^x = 64$

Use the logarithmic function $f(x) = -\log_3 x$ for Items 7–13.

7. Make a table of values for the function, using the x-values $\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9, and 27.
8. Graph the function on a coordinate plane.
9. What are the domain and range of the function?
10. What are the intercepts of the function, if any?
11. Identify the asymptote of the function.
12. a. What happens to the value of $f(x)$ as x approaches infinity?
    b. What happens to the value of $f(x)$ as x approaches 0 for $x > 0$?
13. a. Let g be the inverse function of f. What is the equation of g?
    b. Show both graphically and algebraically that f and g are inverse functions.
14. Explain why $\ln 1 = 0$.

Evaluate each expression without using a calculator.

15. $\log_3 81$
16. $\log 0.001$
17. $\log_{16}\left(\frac{1}{4}\right)$
18. $\log_5 125$
19. Sound pressure level L in decibels is given by $L = 20 \log \frac{p}{p_0}$, where p is the sound pressure and p_0 is a reference pressure. The sound pressure of a vacuum cleaner is 10,000 times the reference pressure. Write and solve an equation to find the sound pressure level of the vacuum cleaner in decibels. Alternatively, graph the function L.

Lesson 5-2

Use a calculator and the Change of Base Formula to find an approximation to the nearest thousandth.

20. $\log_2 24$
21. $\log_5 36$
22. $\log_3 400$
23. $\log_8 0.03$
24. Which of the following can be used to find $\log_3 8$?
    A. $\frac{\ln 8}{\ln 3}$
    B. $\frac{\log_8 3}{\log 8}$
    C. $\frac{\ln 3}{\ln 8}$
    D. $\frac{\ln 3}{8}$

Use the properties of logarithms to expand each expression.

25. $\log (3x^2)$
26. $\log \frac{4x}{y^2}$
27. $\ln \frac{x}{yz}$
28. $\ln \sqrt{\frac{x^2}{y^2}}$
29. Explain how to graph the function $y = 3 \log_3 (x + 2)$ on a graphing calculator.

Write each expression as a single logarithm.

30. $2 \log x + \log (x + 3)$

31. $2[\log (x - 8) - \log 8]$

32. $\ln (x - 2) - \ln (x + 2) + \ln x$

33. Which expression is equivalent to $\ln (3x^2)$?
A. $(\ln 3)(\ln x^2)$
B. $2 \ln x + \ln 3$
C. $3(\ln x + \ln x)$
D. $2(\ln 3 + \ln x)$

34. The population of a city is growing at a rate of 3 percent per year. The city's current population is 520,000.
a. Write an exponential equation that can be used to determine how many years it will take before the city's population reaches 1,000,000.
b. Write the equation as an equivalent logarithmic equation.
c. Use the Change of Base Formula to solve the equation. Interpret the solution in the context of the problem.

Lesson 5-3

Solve.

35. $200 + 10^{x+3} = 1600$

36. $30(3^{-2x}) = 20$

37. $100e^{-2x} = 25$

38. $\dfrac{500}{1 + e^{-x}} = 275$

39. $\log x^2 = 6$

40. $5 \ln 2x = 25$

41. $18 - \log_2 (x - 4) = 7$

42. $6 \ln (x + 8) = 24$

Solve. Check your solutions.

43. $\log x + \log (x + 3) = 1$

44. $\ln (x + 2) + \ln (x - 8) = \ln (-5x - 4)$

45. $\log (x^2 - 2x) - \log (-x) = 2 \log x$

46. The equation $T = T_a + (T_0 - T_a)e^{-kt}$ can be used to determine the temperature T of an object after t minutes, where T_0 is the initial temperature of the object, T_a is the ambient or room temperature, and k is a constant. How many minutes will it take a cup of hot tea with an initial temperature of 212°F to cool to 150°F? The room temperature is 72°F, and the value of the constant k is 0.0576 per minute.

47. The pH of a solution is given by $\text{pH} = -\log[H^+]$, where $[H^+]$ is the hydrogen ion concentration of the solution in moles/liter. A chemist mixes a solution that has a pH of 5.5. Next, the chemist mixes a solution with a hydrogen ion concentration that is twice that of the first solution. What is the pH of the second solution?

48. The half-life of a medication is the amount of time it takes for half of the original amount to be eliminated from the patient's bloodstream. A patient is given a dose of 400 mg of a medication. After four hours, the patient's blood is tested, showing that about 85 mg of the medication remains in the patient's bloodstream. Write and solve an equation to find the half-life of the medication to the nearest minute.

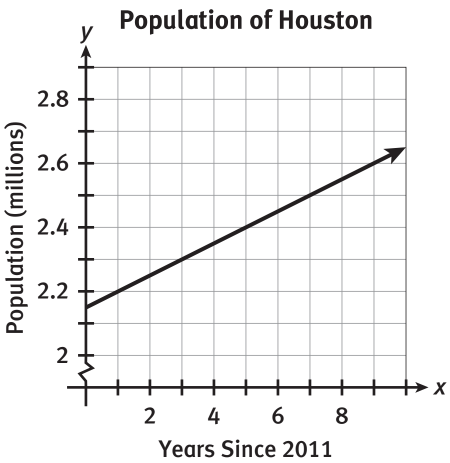
MATHEMATICAL PRACTICES
Look For and Make Use of Structure

49. Find the value of each expression without using a calculator. List a property or give an explanation for each step in your work.
a. $2 \log 2 - \log 40$
b. $2(\log_6 2 + \log_6 3)$
c. $3 \log_3 2 - \log_3 8$
d. $\log_2 36 - \log_2 9$

According to 2010 census data, Houston, Texas, is the fourth-largest city in the United States. In 2011, its population was 2,145,000, an increase of 2.2% compared to the previous year.

1. Assuming that Houston's population continues to grow at a rate of 2.2% per year, write the equation of a function $f(x)$ that can be used to model Houston's population in millions, where x is the number of years since 2011.

2. Graph the function on the coordinate grid.



3. What is the y-intercept of the function? What does the y-intercept represent in this situation?

4. Based on the model, what will the population of Houston be in 2023? Explain how you determined your answer.

5. **a.** Write an equation that can be used to predict when the population of Houston will reach 3 million.

b. Solve your equation. For each step, list a property or give an explanation. Then interpret the solution.

c. Describe how the inverse relationship between exponents and logarithms helped you solve the equation.

6. According to the 2010 census, Chicago is the third-largest city in the United States. In 2011, its population was 2,707,000, an increase of 0.4% compared to the previous year.

a. Assuming that the populations of Chicago and Houston are growing exponentially, write an equation that can be used to predict when the population of Houston will equal that of Chicago.

b. Solve your equation. For each step, list a property or give an explanation. Then interpret the solution.

7. The function $g(x) = 112 \ln(0.121x) + 2011$ models the year in which the population of New York City will equal x million people. Write and solve an equation to estimate the population of New York City in 2020. For each step, list a property or give an explanation.

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Items 1, 6)	• Clear and accurate understanding of exponential growth (annual rate model)	• A functional understanding of exponential growth (possible use of alternate exponential model)	• Partial understanding of exponential growth model	• Little or no understanding of exponential growth
Problem Solving (Items 1, 2, 4, 5, 6)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2, 3, 6)	• Clear and accurate representation of the exponential model graphically (nearly linear looking, yet exponential, with an initial slope of $0.0467 \approx 0.05$) • Clear and accurate understanding of exponential growth	• A functional representation of the growth model (maybe with just a single other point used.) • Mostly accurate understanding of exponential growth	• Partial representation of the growth model • Partial understanding of exponential growth	• Little or no understanding of the growth model • Inaccurate or incomplete understanding of exponential growth
Reasoning and Communication (Items 4, 5, 6, 7)	• Precise use of appropriate math terms and language to solve exponential equations logarithmically	• Correct characterization of solving exponential equations	• Misleading or confusing characterization of solving exponential equations	• Incomplete or inaccurate characterization of solving exponential equations

Transformations of Functions

I Doubt It

Lesson 6-1 Transforming Functions

My Notes

Learning Targets:

- Graph transformations of functions and write the equations of the transformed functions.
- Describe the symmetry of the graphs of even and odd functions.

SUGGESTED LEARNING STRATEGIES: Close Reading, Create Representations, Look for a Pattern, Group Presentation, Self Revision/ Peer Revision

Functions can be organized into families, with the most basic function of each family known as the ***parent function***.

MATH TERMS

A **parent function** is the most basic function of a particular type.

1. Sketch a graph of each parent function and label some key points.

a. Absolute value $f(x) = |x|$
b. Quadratic $f(x) = x^2$
c. Cubic $f(x) = x^3$
d. Quartic $f(x) = x^4$
e. Natural logarithm $f(x) = \ln x$
f. Linear $f(x) = x$
g. Exponential $f(x) = e^x$
h. Rational $f(x) = \frac{1}{x}$
i. Square root $f(x) = \sqrt{x}$

Recall that transformations of functions can be horizontal or vertical translations, horizontal or vertical stretches or compressions, reflections, or a combination of transformations.

2. Express regularity in repeated reasoning. Match each function with the corresponding transformation.

a. $y = a(f(x))$, $|a| > 1$ ______ **A** Horizontal compression

b. $y = a(f(x))$, $|a| < 1$ ______ **B** Reflection over the x-axis

c. $y = f(bx)$, $|b| > 1$ ______ **C** Vertical translation

d. $y = f(bx)$, $|b| < 1$ ______ **D** Vertical stretch

e. $y = f(x - \mathrm{c})$ ______ **E** Vertical compression

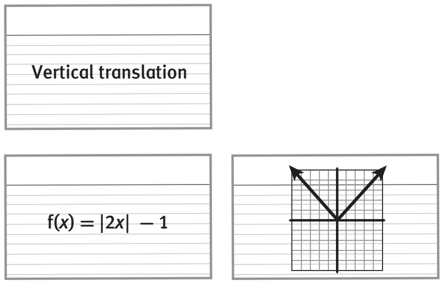
f. $y = f(x) + d$ ______ **F** Horizontal stretch

g. $y = -f(x)$ ______ **G** Reflection over the y-axis

h. $y = f(-x)$ ______ **H** Horizontal translation

My Notes

I Doubt It!



Shuffle the 16 Category cards and place them in the center.

Shuffle the 90 Transformation cards and deal them to the players until no cards remain.

Some players may have more cards than others. The object of the game is to be the first player to discard all of the cards in your hand.

The youngest player goes first, and play proceeds clockwise.

The discard pile will be in the center and starts empty. To begin a turn, the player flips over a Category card. Then, one at a time, each player must discard one or more Transformation cards facedown on the discard pile, calling out the number of cards they discard. Because the transformation cards are discarded facedown, players may discard ones that match the Category card or ones that do not match.

After any person discards Transformation cards, any player who suspects that the card(s) played do not match the current Category card may challenge the play by calling "I doubt it!" The cards played by the challenged player during that turn are flipped over, and one of two things happens.

1. If *all* of the challenged player's discards match the current Category card, the challenger must pick up the entire discard pile, including cards previously played by others.
2. If *any* of the challenged player's discards differ from the current Category card, the person who played the cards must pick up the entire discard pile, including cards previously played by others.

After the challenge is resolved, or if there is no challenge, play continues clockwise. A new Category card is turned over by the next player, and play continues.

The first player to get rid of all of his or her cards and survive any final challenge wins the game. Play "I Doubt It!"

My Notes

When you played “I Doubt It,” you examined transformations graphically and algebraically. You can also look at functions numerically to see the effects of transformations.

3. Create a table of values for each transformation of $f(x)$.

x	$f(x)$
−8	512
−6	216
−4	64
−2	8
1	1
3	27
5	125
7	343

a.

x	$f(x) + 3$
−8	
−6	
−4	
−2	
1	
3	
5	
7	

b.

x	$3f(x)$

4. Create a table of values for each transformation of $g(x)$.

x	$g(x)$
−7	−100
−5	−76
−3	−52
−2	−40
3	−20
4	−32
6	−56
9	−92

a.

x	$g(2x)$

b.

x	$g(x + 2)$

5. Create a table of values for each transformation of $h(x)$.

x	$h(x)$
−4	−4
−2	−7
0	−8
2	−7
6	1
8	8
12	−3
16	56

a.

x	$-h(x)$

b.

x	$h(-x)$

6. How does the transformation $h(-x)$ differ from $-h(x)$?

My Notes

You can use what you know about transformations to help you write the equations of functions and to graph the equations.

Example A

A card game can have between two and six players. The number of cards a player receives at the beginning of the game is given by the function $g(x)$, where x is the number of players. The graph of $g(x)$ is a vertical stretch of the graph of $f(x)=\frac{1}{x}$ by a factor of 60. Write the equation of $g(x)$, and then graph $g(x)$.

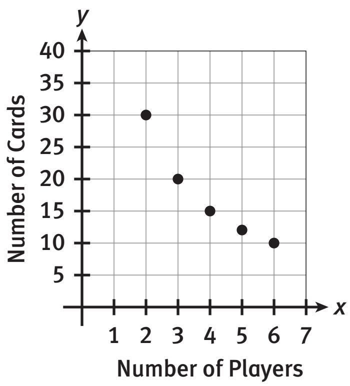
$g(x) = 60f(x)$ — $g(x)$ is a vertical stretch of $f(x)$ by a factor of 60.

$g(x) = 60\left(\frac{1}{x}\right)$ — Substitute $\frac{1}{x}$ for $f(x)$.

$g(x) = \frac{60}{x}$ — Simplify.

To graph $g(x)$, start by making a table of values. Use values of $f(x)$ to find values of $g(x)$.

x	$f(x)$	$g(x) = 60f(x)$
2	$\frac{1}{2}$	30
3	$\frac{1}{3}$	20
4	$\frac{1}{4}$	15
5	$\frac{1}{5}$	12
6	$\frac{1}{6}$	10



MATH TIP

Because the number of players x must be between 2 and 6, the domain of $g(x)$ is set of whole numbers 2, 3, 4, 5, and 6.

Try These A

Use the given information to write the equation of $g(x)$, and then graph $g(x)$.

a. The graph of $g(x)$ is a translation 4 units to the left and 3 units up of the graph of $f(x) = x^3$.

b. The graph of $g(x)$ is a reflection of the graph of $f(x) = \log_2 x$ over the y-axis.

c. A strategy game is played on a square grid. The function $g(x)$ gives the average length of the game in minutes when the side length of the grid is x squares, where $x \geq 2$. The graph of $g(x)$ is a horizontal stretch of the graph of $f(x) = e^x$ by a factor of 2.

My Notes

A function whose graph is symmetric with respect to the y-axis is called an ***even function***. A function whose graph is symmetric with respect to the origin is called an ***odd function***.

MATH TERMS

A function f is **even** if and only if for each x in the domain of f, $f(-x) = f(x)$.

A function f is **odd** if and only if for each x in the domain of f, $f(-x) = -f(x)$.

7. Model with mathematics. Sketch each function shown below to help classify it as even, odd, or neither. Then describe the symmetry of the graph, if any. Use the definitions of even and odd functions to verify your answer algebraically.

a. $f(x) = x^3 + 1$

b. $f(x) = x^2 - 3$

c. $f(x) = x^5 + x$

d. $f(x) = |x|$

e. $f(x) = x^2 - 2x - 8$

f. $f(x) = -\dfrac{1}{x^2}$

My Notes

8. Table 1 shows some values for an even function $f(x)$. Use the definition of an even function to find four more ordered pairs $(x, f(x))$.

Table 1

x	$f(x)$
−2	11
1	2
5	578
9	6402

9. Table 2 shows some values for an odd function $g(x)$. Use the definition of an odd function to find four more ordered pairs $(x, g(x))$.

Table 2

x	$g(x)$
−4	−60
−2	−6
3	24
10	990

10. Reason abstractly. If a function is even, which transformations will always maintain the symmetry of the graph? Explain.

11. If a function is odd, which transformations will always maintain the symmetry of the graph? Explain.

12. You can use a graphing calculator to determine whether a function $f(x)$ is even. First, enter the function rule for $f(x)$ as Y_1. Then enter $Y_1(-X)$ for Y_2. The rule for Y_2 represents $f(-x)$. View the graphs of the two functions on the same screen. What should you expect to see on the screen if $f(x)$ is even?

CONNECT TO TECHNOLOGY

Most graphing calculators can graph a transformation of a function. For example, to translate the graph of $f(x) = x^2 + 4x$ by 3 units to the right, first enter the rule for $f(x)$ as Y_1. Then enter $Y_1(X-3)$ for Y_2. The graph of Y_2 is a translation 3 units to the right of the graph of Y_1.

My Notes

13. Use appropriate tools strategically. Describe how you could use a graphing calculator to determine whether a function $f(x)$ is odd.

14. Use a graphing calculator to classify each function as even, odd, or neither.

a. $f(x) = \frac{2}{x}$

b. $f(x) = 2 \log x$

c. $f(x) = 0.25x^3$

d. $f(x) = -|x| + 2$

Check Your Understanding

Describe each real-world function as a transformation of its parent function. Remember to use complete sentences and words such as *and, or, since, because* to make connections between your thoughts.

15. $g(x) = (2x)^3$

16. $g(x) = |x| - 4$

17. The table represents a function. For each transformation of the function, describe how you would need to change each x-coordinate and each y-coordinate, if at all.

x	−2	−1	0	1	2	3
y	6	3	2	3	2	6

a. a translation 3 units to the right
b. a vertical stretch by a factor of 3
c. a reflection over the x-axis

18. The function $f(x)$ has a y-intercept of -3. The graph of $g(x)$ is a vertical compression of the graph of $f(x)$ by a factor of $\frac{1}{3}$. What is the y-intercept of $g(x)$? Explain how you know.

19. There is only one function that is both odd and even. What is the equation of this function? ***Justify*** your answer.

ACADEMIC VOCABULARY

When you ***justify*** a statement, you show that it is correct.

My Notes

LESSON 6-1 PRACTICE

Use the given information to write the equation of $g(x)$, and then graph $g(x)$.

20. The graph of g(x) is a reflection of the graph of $f(x) = \log_2 x$ over the x-axis.

21. The graph of g(x) is a vertical compression of the graph of $f(x) = x^2 + 2x$ by a factor of $\frac{1}{2}$.

22. The graph of g(x) is a reflection of the graph of $f(x) = 2^x$ over the y-axis followed by a translation 2 units up.

Determine if the following functions are odd, even, or neither. Then describe the symmetry of the graph of the function, if any.

23. $f(x) = x^4 - 3x^2 + x + 1$

24. $f(x) = x^3 - 3x$

25. $f(x) = x^6 - 4x^4 + 2$

26. $f(x) = \frac{1}{2x}$

27. **Make sense of problems.** Landon kicks a football from a height of 3 ft with an initial vertical velocity of 20 ft/s. The function $f(t) = -16t^2 + 20t + 3$ models the height in feet of the ball t seconds after it is kicked. Next, Mitch kicks a football. The function $g(t)$ models the height in feet of Mitch's football t seconds after it is kicked. The graph of $g(t)$ is a translation 3 units down of the graph of $f(t)$.

a. Write the equation of $g(t)$.

b. From what height does Mitch kick the football? What is its initial vertical velocity? Explain how you know.

CONNECT TO PHYSICS

The height h in feet of a kicked or thrown object after t seconds can be modeled by the equation $h = -16t^2 + v_0 t + h_0$, where v_0 is the initial vertical velocity in feet per second and h_0 is the initial height in feet.

My Notes

Learning Targets:

- Add, subtract, multiply, and divide functions.
- Transform and perform operations with piecewise-defined functions.

SUGGESTED LEARNING STRATEGIES: Note Taking, Create Representations, Summarizing, Discussion Groups, Critique Reasoning

Let *f* and *g* be any two functions. The sum, difference, product, and quotient of *f* and *g* are functions whose domains are the set of all numbers common to the domains of *f* and *g*, and are defined as follows:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \text{ where } g(x) \neq 0$$

Example A

Let $f(x) = 2x + 3$ and $g(x) = 2x^2 + 5x - 12$. Find $f + g$ and $\frac{f}{g}$. State the domain.

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= 2x + 3 + 2x^2 + 5x - 12\\ &= 2x^2 + 7x - 9\end{aligned}$$

The domain of *f* is all real numbers, and the domain of *g* is all real numbers; therefore, the domain of $f + g$ is $\mathbb{R}$.

For $\frac{f}{g}$, the denominator cannot be zero, so its domain is restricted.

$g(x) = 2x^2 + 5x - 12$, or $g(x) = (2x - 3)(x + 4)$, so $x \neq \frac{3}{2}, x \neq -4$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x+3}{2x^2+5x-12}; \text{ domain: } \left\{x \mid x \neq \frac{3}{2}, x \neq -4\right\}$$

Try These A

Find the sum, difference, product, and quotient for the following functions. State the domain of each resulting function.

a. $f(x) = 3x - 4$, $g(x) = 4 - x$

b. $f(x) = x^2 + 6$, $g(x) = \sqrt{1-x}$

c. $f(x) = \sqrt{x^2 - 9}$, $g(x) = \frac{x^2}{x^2+2}$

My Notes

MATH TIP

A piecewise-defined function is called a step function if each rule that defines the function is a constant. For example, the function $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$ is a step function. The graph looks like steps, and would show a domain of all real numbers and a range of $\{-1, 1\}$.

A piecewise-defined function is a function that is defined by two or more rules. Each rule applies to a different part of the domain of the function.

For example, $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ is a piecewise-defined function. The rule $x + 2$ describes the function for the interval $x < 0$, and the rule x^2 describes the function for the interval $x \geq 0$.

When you transform a piecewise-defined function, you must perform the transformation on each rule that defines the function. Similarly, when you perform a function operation involving a piecewise-defined function, you must perform the operation on each rule that defines the function.

Example B

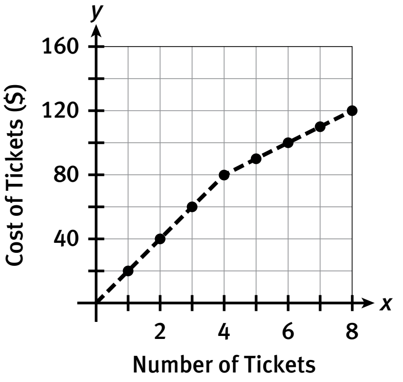
The function $f(x) = \begin{cases} 20x & \text{if } x \leq 4 \\ 10x + 40 & \text{if } x > 4 \end{cases}$ gives the total cost in dollars of x tickets to a board game convention.

a. Graph the function. Determine and analyze any key features.

Apply the rule $20x$ for values of x less than or equal to 4 and the rule $10x + 40$ for values of x greater than 4.

x	$20x$
1	20
2	40
3	60
4	80

x	$10x + 40$
5	90
6	100
7	110
8	120



Key features include the following:

- The domain of $f(x)$ is all the real numbers.
- The range of $f(x)$ is all the real numbers.
- $f(x)$ has a relative maximum at (4, 80).

My Notes

b. The function $g(x) = 10x$ gives the total cost in dollars for x people to buy a board game at the convention. Find $f + g$, and tell what it represents in this situation.

$$(f+g)(x) = \begin{cases} 20x + 10x & \text{if } x \le 4 \\ 10x + 40 + 10x & \text{if } x > 4 \end{cases}$$ Add the rule for $g(x)$ to each rule that defines $f(x)$.

$$= \begin{cases} 30x & \text{if } x \le 4 \\ 20x + 40 & \text{if } x > 4 \end{cases}$$ Then simplify.

$f + g$ represents the cost in dollars for a group of x people to buy one ticket each for the convention, plus a board game.

c. The graph of $h(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of 1.25. Write the equation of $h(x)$. Then tell what $h(x)$ represents in this situation.

$$h(x) = 1.25f(x)$$ To represent the vertical stretch, multiply $f(x)$ by 1.25.

$$= \begin{cases} 1.25(20x) & \text{if } x \le 4 \\ 1.25(10x + 40) & \text{if } x > 4 \end{cases}$$ To find $1.25f(x)$, multiply each rule that defines $f(x)$ by 1.25.

$$= \begin{cases} 25x & \text{if } x \le 4 \\ 12.5x + 50) & \text{if } x > 4 \end{cases}$$ Then simplify.

$h(x)$ represents the total cost in dollars of x tickets to the convention after an increase in ticket prices of 25 percent.

Try These B

a. Reason abstractly. Refer back to the function $f(x)$ in Example B. The graph of $j(x)$ is a translation 2 units up of the graph of $f(x)$. Write the equation of $j(x)$. Then tell what $j(x)$ could represent in this situation.

Consider the functions $p(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ x^2 & \text{if } x \ge 0 \end{cases}$ and $q(x) = x + 3$.

b. Graph $p(x)$ on a coordinate plane. Determine and analyze any key features such as domain, range, relative extrema, and zeros.

c. Find the sum, difference, product, and quotient of the functions. State the domain of each.

d. The graph of $r(x)$ is a translation 3 units to the left of the graph of $p(x)$. Write the equation of $r(x)$.

MATH TIP

Some transformations of piecewise-defined functions affect not only the rules that define the function but also the interval for each rule. Transformations that affect the intervals include horizontal translations, horizontal stretches, horizontal compressions, and reflections across the *y*-axis.

My Notes

Check Your Understanding

1. **Express regularity in repeated reasoning.** Is addition of functions commutative? Give an example to support your answer.
2. Explain how to determine the domain of the quotient of two functions $f(x)$ and $g(x)$.
3. Explain why $y = \begin{cases} 3 & \text{if } x \leq 5 \\ 6 & \text{if } x \geq 5 \end{cases}$ is *not* a piecewise-defined function.
4. **Critique the reasoning of others.** A student claims that the graph of $g(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ (x-2)^3 & \text{if } x \geq 0 \end{cases}$ is a translation 2 units to the right of the graph of $f(x) = \begin{cases} x + 4 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0 \end{cases}$. Is the student correct? Explain.

LESSON 6-2 PRACTICE

Find the sum, difference, product, and quotient of these functions. State the domain.

5. $f(x) = x^2 + 1$, $g(x) = 2x^2 - 3x + 5$
6. $f(x) = 2^x$, $g(x) = 2^{x-1}$
7. $f(x) = \begin{cases} 2x + 1 & \text{if } x \leq 2 \\ 4x - 3 & \text{if } x > 2 \end{cases}$, $g(x) = 2x + 1$

Use the function $f(x) = \begin{cases} x^2 - 1 & \text{if } x < 1 \\ -2x & \text{if } x \geq 1 \end{cases}$ and the given information to write the equation of $g(x)$, and then graph $g(x)$.

8. The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the x-axis.
9. The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the y-axis.
10. **Model with mathematics.** The charges at an airport parking garage are \$4 per hour for the first 5 hours and a maximum of \$20 per day.
    a. Write the equation of a piecewise-defined function $f(x)$ that represents the cost in dollars of parking in the garage for x hours in a single day.
    b. The airport plans to add a security charge of \$1 per day regardless of the number of hours parked in the garage. Write the equation of a function $g(x)$ that represents the new cost in dollars of parking in the garage. Describe the graph of $g(x)$ as a transformation of the graph of $f(x)$.

ACTIVITY 6 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 6-1

Identify the transformation of $f(x)$ that each function represents.

1. $g(x) = f(x - 5)$
2. $g(x) = 3f(x)$

Describe each function as one or more transformations of its parent function.

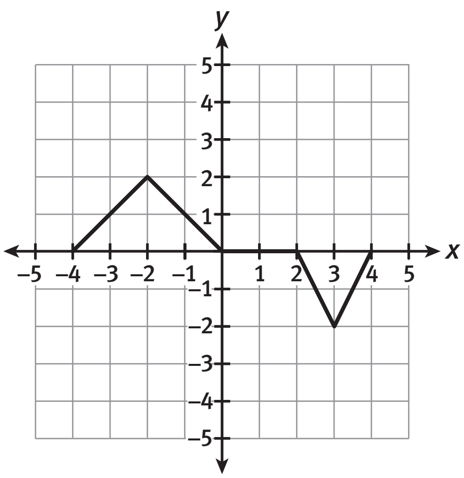
3. $g(x) = \log(x + 4)$
4. $g(x) = -x^2 - 3$
5. $g(x) = \dfrac{2}{x+5}$
6. $g(x) = e^{-4x}$

Use the table below to help you create a table of values for each transformation of $f(x)$.

x	−8	−5	−1	0	4	9
$f(x)$	17	14	10	9	5	10

7. $y = -2f(x)$
8. $y = f(x - 4)$

Use the graph of $f(x)$ for Items 9–12. Sketch each transformation of $f(x)$.



9. $y = \frac{1}{2}f(x)$
10. $y = f(x - 3)$
11. $y = -f(x)$
12. $y = f(x) - 2$

Determine if the following functions are odd, even, or neither. Then describe the symmetry of the graph of the function, if any.

13. $f(x) = |x^5|$
14. $f(x) = e^{x-5}$
15. $f(x) = \sqrt[3]{x}$
16. $f(x) = \dfrac{x}{x^2+1}$
17. Explain how the real-world models of functions $p(x) = 4r(x)$ and $q(x) = r(4x)$ differ.

Use the given information to write the equation of $g(x)$, and then graph $g(x)$.

18. The graph of $g(x)$ is a translation 2 units to the right of the graph of $f(x) = e^x$ followed by a reflection over the x-axis.
19. The graph of $g(x)$ is vertical stretch of the graph of $f(x) = x^2 + 4x$ by a factor of 2 followed by a translation 4 units down.
20. The function $f(x) = -(x - 25)^2 + 625$ gives the area in square feet of a rectangular pen that can be enclosed by a small roll of fencing, where x is the length of the pen. The function $g(x)$ gives the area in square feet of a rectangular pen that can be enclosed by a large roll of fencing. The graph of $g(x)$ is a translation 25 units to the right and 1875 units up of the graph of $f(x)$.
    a. Write the equation of $g(x)$.
    b. Graph $f(x)$ and $g(x)$ on the same coordinate grid.
    c. What is the perimeter of a rectangular pen enclosed with a small roll of fencing? What is the perimeter of a rectangular pen enclosed with a large roll? Explain how you know.
21. The function $f(x) = 2000e^{0.03x}$ models the amount in dollars in an investment account after x years for an initial investment of \$2000. The function $g(x)$ models a second investment account. The graph of $g(x)$ is a horizontal compression of the graph of $f(x)$ by a factor of $\frac{3}{5}$.
    a. Write the equation of $g(x)$.
    b. In which account would you rather invest your money, the one modeled by $f(x)$ or the one modeled by $g(x)$? Justify your choice.

22. The points (2, −1) and (4, −8) lie on the graph of $f(x)$. Assuming that $f(x)$ is odd, which of the following points must also lie on the graph of $f(x)$?

A. (−1, 2) **B.** (−4, −8)
C. (−2, 1) **D.** (8, −4)

Lesson 6-2

Find the sum, difference, product, and quotient of the functions in each item below. State the domain.

23. $f(x) = 3x - 7$, $g(x) = -8x - 9$

24. $f(x) = \frac{1}{x^2}$, $g(x) = \sqrt{x - 4}$

25. $f(x) = x^2 - x - 6$, $g(x) = x + 2$

26. $f(x) = \begin{cases} 3x - 1 & \text{if } x \le 2 \\ -4x & \text{if } x > 2 \end{cases}$, $g(x) = 2x - 5$

Graph each piecewise-defined function. Determine and analyze any key features such as domain, range, relative extrema, and zeros.

27. $f(x) = \begin{cases} -2 & \text{if } x < -1 \\ 2 & \text{if } -1 \le x \le 1 \\ 4 & \text{if } x > 1 \end{cases}$

28. $f(x) = \begin{cases} x^2 - 3 & \text{if } x \le 0 \\ 2x - 3 & \text{if } 0 < x < 5 \\ 7 & \text{if } x \ge 5 \end{cases}$

Use the function $f(x) = \begin{cases} -x + 5 & \text{if } x \le 2 \\ (x - 4)^2 & \text{if } x > 2 \end{cases}$ and the given information to write the equation of $g(x)$, and then graph $g(x)$.

29. The graph of $g(x)$ is a translation 5 units left of the graph of $f(x)$.

30. The graph of $g(x)$ is a vertical stretch of the graph of $f(x)$ by a factor of 3.

31. The graph of $g(x)$ is a reflection of the graph of $f(x)$ over the x-axis.

32. What is the domain of $\left(\frac{f}{g}\right)(x)$ given that $f(x) = \sqrt{x - 3}$ and $g(x) = (x + 2)^2$?

A. $[0, \infty)$ **B.** $[3, \infty)$
C. $\{x \mid x \ne -2\}$ **D.** $\mathbb{R}$

33. The cost in dollars of ordering x football jerseys from a company is given by

$$f(x) = \begin{cases} 35x & \text{if } 0 < x \le 10 \\ 32x & \text{if } x > 10 \end{cases}.$$

a. The function $g(x)$ is a vertical compression of $f(x)$ by a factor of 0.9. Write the equation of $g(x)$. Then tell what $g(x)$ represents in this situation.
b. The function $h(x) = 2x$ gives the discount in dollars that a returning customer receives when buying x jerseys. Find $f - h$, and tell what it represents in this situation.

34. The function $f(x) = 3.6x$ gives the cost in dollars of buying x gallons of gasoline. The function $g(x) = 30x$ gives the distance in miles Anuja can drive her car on x gallons of gasoline. Find $\frac{f}{g}$, and tell what it represents in this situation.

MATHEMATICAL PRACTICES
Look For and Express Regularity in Repeated Reasoning

35. Use the functions listed below to make conjectures about whether each product is even, odd, or neither. Give at least two examples to support each of your conjectures.

Even: $f(x) = x^2$, $g(x) = x^2 + 1$, $h(x) = x^4$

Odd: $p(x) = x^3$, $q(x) = \frac{1}{x}$, $r(x) = \frac{1}{2}x^3$

a. the product of two even functions
b. the product of two odd functions
c. the product of an even function and an odd function

Modeling with Power Functions

ACTIVITY 7

Highway Safety

Lesson 7-1 Finding a Regression Line

My Notes

Learning Targets:

- Write an equation that models a data set.
- Transform data to determine whether a power function is a good model for a data set.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Discussion Groups, Quickwrite, Group Presentation

The braking distance of a car is the distance the car travels between the time the driver hits the brakes and the time the car stops. The braking distance is dependent on the car's speed.

The Federal Highway Administration is one agency interested in determining safe stopping distances under various conditions. Below is a table of data collected by the agency under test conditions in which the road was dry and the driver traveling at a constant speed was confronted with a situation requiring a sudden stop.

Observed Braking Distances

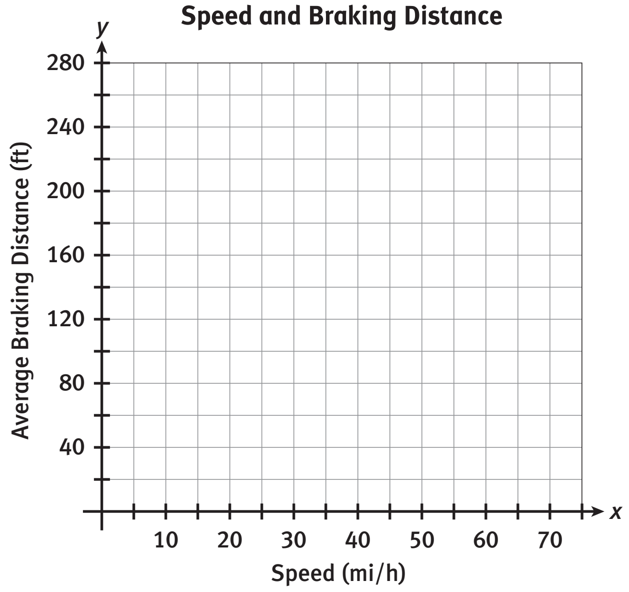
Speed (mi/h)	Average Braking Distance (ft)
20	20
25	28
30	40.5
35	52.5
40	72
45	92.5
50	118
55	148.5
60	182
65	220.5

DISCUSSION GROUP TIPS

As you listen to the group discussion, take notes to aid comprehension and to help you describe your own ideas to others in your group. Ask questions to clarify ideas and to gain further understanding of key concepts.

My Notes

1. Make a scatter plot showing the relationship between a car's speed and the average braking distance.



2. Perform a linear regression using the speed data and the braking distance data.
   a. Give the equation of the regression line and graph the line on the scatter plot. Round numerical values to three decimal places.
   b. Give the meaning of the slope and y-intercept of the model in terms of the context of the situation.
   c. What is the correlation coefficient for the data? Interpret the meaning of this value.
   d. Do you think a linear model is necessarily the best model for the data in the scatter plot? Explain your answer.

WRITING MATH

There is not a universal rule for rounding numerical values from a regression equation. In this activity, round the values to three decimal places.

MATH TIP

The correlation coefficient r is a measure of how linear the relationship between two variables is. The value of r can range between -1 and 1, with values close to -1 indicating a strong negative correlation, values close to 0 indicating little or no correlation, and values close to 1 indicating a strong positive correlation.

My Notes

MATH TIP

When you transform a data set, you perform the same operation or set of operations on each value in the data set.

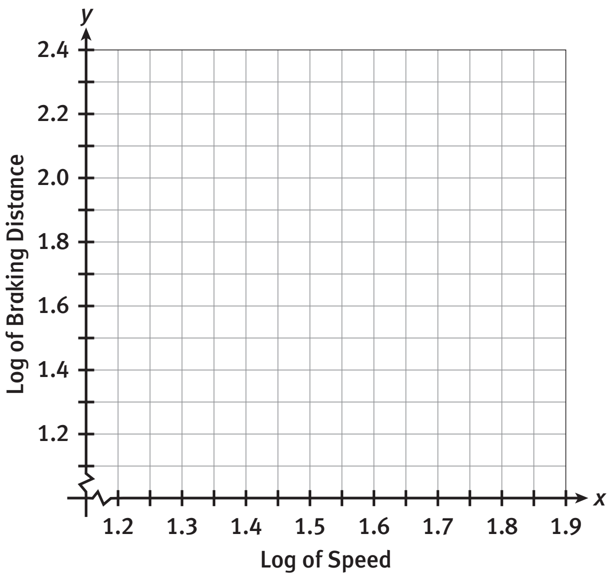
3. Transform the *x*-values and *y*-values of the original scatter plot.

a. Take the common logarithm of the speeds and the common logarithm of the stopping distances to complete the table. Round values to the nearest thousandth.

Speed (mi/h)	Average Braking Distance (ft)
20	20
25	28
30	40.5
35	52.5
40	72
45	92.5
50	118
55	148.5
60	182
65	220.5

Log of Speed	Log of Braking Distance

b. Make a new scatter plot showing the relationship between the common logarithm of the speed and the common logarithm of the average stopping distance.



c. What pattern do you notice in the new scatter plot?

My Notes

4. Perform a linear regression using the transformed data.

a. Give the equation of the regression line and graph the line on the scatter plot.

b. What is the correlation coefficient for the transformed data? What does the value indicate about the transformed data?

c. Is a linear model a better fit for the original data or the transformed data? Use the correlation coefficient to explain how you know.

MATH TIP

When a data set is transformed by taking the logarithm of both the *x*-values and the *y*-values, the transformation is called a log-log transformation.

You can use the equation of the regression line of the log-log transformed data to write the equation of a function that models the original data set.

5. To do so, start with the equation for the regression line of the transformed data.

a. Substitute log x for x and log y for y in the equation.

b. Use the properties of logarithms to solve the equation from Part a for y. List a property or give an explanation for each step in your work.

MATH TIP

In Item 5b, start by rewriting the equation so that it has a single logarithm on one side of the equation and a constant on the other side. Then write the equation in exponential form. Finally, solve for *y*.

A ***power function*** has the form $f(x) = ax^b$, where a and b are nonzero real numbers.

6. Look at the final equation from Item 5b that models the braking distance of a car. Is this equation a power function? Explain.

7. **Use appropriate tools strategically.** Use a graphing calculator to perform a power regression using the speed and braking distance data.
   a. Give the equation of the regression.

   b. How does the regression confirm that a power function is an appropriate model for the speed and braking distance data?

   c. How does the regression equation confirm that you solved the equation in Item 5b correctly?

8. A car is traveling on a highway at a speed of 70 mi/h. The driver sees a tree ahead that has fallen across the road.
   a. Use your braking distance model to determine how far the car will travel from the time the driver hits the brakes to the time the car comes to a stop, assuming it does not hit the tree first. Consider the precision of the original data set when rounding your answer.

   b. Attend to precision. Explain how you used the precision of the original data set to decide how to round your answer.

My Notes

CONNECT TO TECHNOLOGY

After you enter the speed and braking distance data into a graphing calculator, you can perform a power regression by pressing STAT. Move the cursor to the right to highlight the CALC menu. Then scroll down and select A:PwrReg.

Check Your Understanding

9. **Construct viable arguments.** Use properties of logarithms to show that if $\log y = \log a + b \log x$ where a and b are nonzero real numbers, then $y = ax^b$.
10. The graph of the equation $\log y = \log 4 + 3 \log x$ is linear. Explain how you can write a power function that models the relationship between x and y.
11. Explain how you can transform a data set to determine whether a power function is a good model for the data.
12. Is a power function a type of exponential function? Explain.
13. The correlation coefficient for a power regression for a data set is 0.542. Based on this information, is a power function a good model for the data set? Explain.

My Notes

LESSON 7-1 PRACTICE

The table shows the relationship between the body mass and heart rate of several mammals.

Mass and Heart Rate of Mammals

Animal	Mass (kg)	Heart Rate (beats/min)
Mouse	0.027	723
Hamster	0.09	400
Rat	0.26	250
Japanese monkey	6.6	147
Roe deer	20	104
Human	66	80
Horse	494	40

14. **Attend to precision.** Make a scatter plot of the data.

15. Do you think a linear model is necessarily the best model for the data in the scatter plot? Support your answer.

16. What transformation can you perform on the data to check whether a power function is a good model for the data?

17. Transform the data using the transformation you chose.
a. Find a linear regression equation for the transformed data.
b. Find the correlation coefficient. What does this indicate about the transformed data?

18. **Model with mathematics.** What power function models the original data set?

19. Use your model to predict the heart rate of an elephant with a mass of 3400 kg.

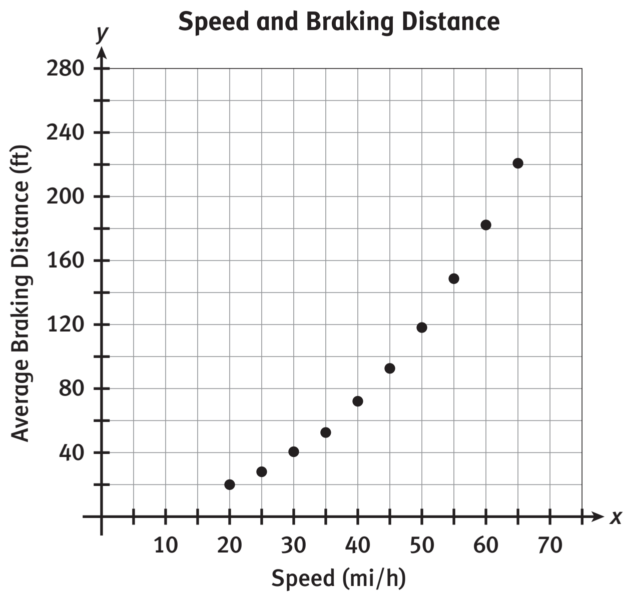
My Notes

Learning Targets:

- Graph power functions.
- Identify and analyze key features of the graphs of power functions.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Think-Pair-Share, Self-Revision/Peer Revision, Graphic Organizer

1. The scatter plot below shows the braking distance and speed data from the previous lesson.



a. Graph the power function from Item 7a in Lesson 7-1 on the scatter plot.

b. Explain how the graph confirms that a power function is a good model for the data set.

2. Analyze the key features of the graph of the power function.

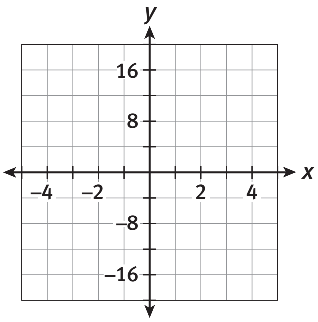
a. What are the domain and range of the function?

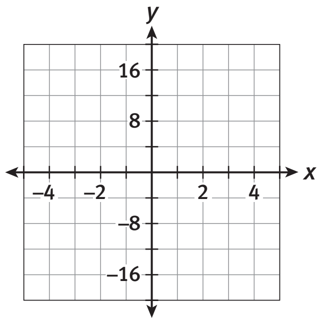
b. What is the minimum value of the function? Why does it make sense in the context of the problem situation that the function has this minimum value?

c. Does the value of the function increase or decrease over its domain? Why does it make sense in the context of the problem situation that the function behaves in this way?

My Notes

3. Graph the power functions $y = x^2$ and $y = x^4$ on the coordinate plane on the left and the power functions $y = -x^2$ and $y = -x^4$ on the coordinate plane on the right.





4. **Express regularity in repeated reasoning.** Use the graphs from Item 3 to analyze key features of power functions for which b is a positive even integer.
   a. What is the domain of this type of power function?

   b. Describe the symmetry of the graph of this type of power function.

   c. What are the real zeros of this type of function, if any?

   d. How does the sign of a affect the range of this type of power function?

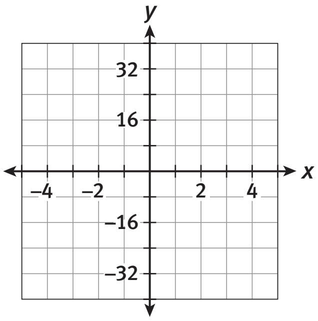
   e. How does the sign of a affect the end behavior of the function's graph?

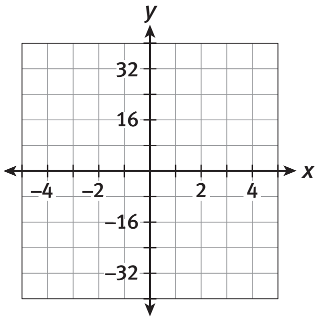
   f. How does the sign of a affect the intervals over which the function is increasing and decreasing?

MATH TIP

The end behavior of a graph is a description of what happens to the value of the function as x approaches negative infinity and as x approaches positive infinity.

5. Graph the power functions $y = x^3$ and $y = x^5$ on the coordinate plane on the left and the power functions $y = -x^3$ and $y = -x^5$ on the coordinate plane on the right.





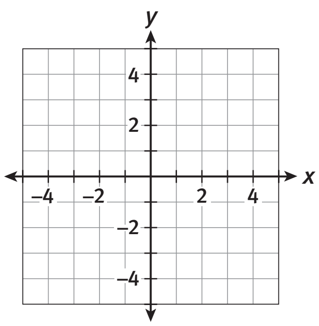
6. **Express regularity in repeated reasoning.** Use the graphs from Item 5 to analyze key features of power functions for which b is a positive odd integer.
   a. What are the domain and range of this type of power function?
   b. Describe the symmetry of the graph of this type of power function.
   c. What are the real zeros of this type of function, if any?
   d. How does the sign of a affect the end behavior of the function's graph?
   e. How does the sign of a affect the intervals over which the function is increasing and decreasing?

7. Graph the power functions $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$ on the same graphing calculator screen.
   a. What is the domain of $y = x^{\frac{1}{2}}$?
   b. What is the domain of $y = x^{\frac{1}{3}}$?
   c. Write the equation of each function in radical form.
   d. **Make use of structure.** How do the radical forms of the functions help to explain the difference in the domains of the power functions $y = x^{\frac{1}{2}}$ and $y = x^{\frac{1}{3}}$?

MATH TIP

For a natural number n, the expression $x^{\frac{1}{n}}$ is equivalent to $\sqrt[n]{x}$.

8. a. Graph the power function $y = x^{-1}$ on the coordinate plane below.



My Notes

b. How does the graph of this function differ from the graphs of the other power functions in this lesson?

c. Describe the end behavior of the function's graph.

d. Describe the behavior of the function as x approaches 0.

e. What are the asymptotes of the function's graph?

f. **Make use of structure.** Write the equation of the function without using an exponent. How does this form of the equation explain why the graph has a vertical asymptote?

Check Your Understanding

9. A power function of the form $y = ax^b$ has a positive integer exponent and a minimum value of 0.
a. Is the value of b even or odd? How do you know?
b. Is the value of a positive or negative? How do you know?

10. Make a conjecture about the domain of a power function of the form $y = x^{\frac{1}{n}}$, where n is a positive even integer. Explain the reasoning you used to make your conjecture.

11. **Critique the reasoning of others.** A student claims that the graphs of all power functions pass through the point (0, 0). Is the student correct? Explain.

ACADEMIC VOCABULARY

A *conjecture* is a mathematical statement that is based on evidence but has not yet been proven.

LESSON 7-2 PRACTICE

Make a table that describes key features of the graphs of the following power functions. The key features should include domain, range, symmetry, maximum or minimum, end behavior, and intervals over which the function is increasing or decreasing.

12. $y = -3x^7$ **13.** $y = x^{\frac{1}{4}}$ **14.** $y = 2x^6$

15. $y = -x^{\frac{1}{5}}$ **16.** $y = -\frac{2}{3}x^4$ **17.** $y = 16x^5$

18. **Reason quantitatively.** The function $f(x) = 0.0008x^{3.91}$ models the number of people infected with a virus, where x is the time in months.
a. Explain why the function is a power function.
b. Graph the function.
c. Find $f(24)$ and explain what it represents in this situation.

ACTIVITY 7 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 7-1

Students in a physics class collected the following data about a ball dropped from a height.

Distance Fallen by Dropped Ball

Time (s)	Distance Fallen (m)
0.10	0.04
0.16	0.11
0.22	0.22
0.28	0.36
0.34	0.54
0.40	0.75
0.46	0.99
0.52	1.28

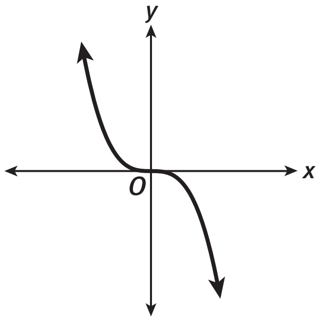
1. Make a scatter plot of the data.
2. Perform a linear regression using the time and distance data.
   a. Give the equation of the regression line.
   b. Give the meaning of the slope and y-intercept of the model in terms of the context of the situation.
   c. What is the correlation coefficient? Interpret the meaning of this value.
3. Perform a transformation on the data to check whether a power function is a good model for the data.
   a. Make a table showing the results of the transformation.
   b. Tell how you transformed the data.
4. Perform a linear regression using the transformed data.
   a. Give the equation of the regression line.
   b. What is the correlation coefficient? Interpret the meaning of this value.
5. Is a linear function or a power function a better model for the original data set? Explain how you know.
6. Use the equation of the regression line of the transformed data to write the equation of a power function that models the original data set. List a property or give an explanation for each step in your work.
7. Graph the power function on the scatter plot of the original data.
8. The students in the physics class drop a ball from a height of 10 m.
   a. Use your model from Item 6 to determine how many seconds it will take for the ball to hit the ground.
   b. Explain how you used the precision of the original data set to decide how to round your answer in Part a.
9. In the absence of air resistance, the formula $d = \frac{1}{2}gt^2$ gives the distance d in meters an object falls in t seconds, where g is the acceleration due to gravity in m/s^2.
   a. Use your model from Item 6 to approximate the acceleration due to gravity. Explain how you determined your answer.
   b. The actual acceleration due to gravity is about 9.8 m/s^2. Did the students' data lead to a good approximation of the acceleration due to gravity? Explain.

Lesson 7-2

Graph each power function.

10. $f(x) = \frac{1}{2}x^3$
11. $f(x) = 3x^{\frac{1}{3}}$
12. $f(x) = -2x^4$
13. $f(x) = 4x^{-1}$

14. Which could be the equation of the power function graphed below?



A. $y = -\frac{1}{3}x^4$ **B.** $y = -\frac{1}{4}x^3$

C. $y = \frac{1}{4}x^3$ **D.** $y = \frac{1}{3}x^4$

15. Explain how you determined your answer to Item 14.

16. The graph of which of these power functions is symmetric with respect to the y-axis?
A. $y = -3x^5$ **B.** $y = -3x^4$
C. $y = x^{\frac{1}{3}}$ **D.** $y = x^{\frac{1}{2}}$

17. Which of these power functions has a minimum of 0?
A. $f(x) = -2x^7$ **B.** $f(x) = -4x^2$
C. $f(x) = 3x^6$ **D.** $f(x) = 5x^3$

18. How does the sign of a affect the range of the power function $y = ax^{\frac{1}{4}}$?

State whether or not each function is a power function. Then explain your reasoning.

19. $f(x) = -4x^{\frac{2}{3}}$ **20.** $f(x) = -\frac{5}{x}$

21. $f(x) = x^3 - 2x^2$ **22.** $f(x) = 0.6x^{2.5}$

23. $f(x) = 3\sqrt{2x}$ **24.** $f(x) = 4(3^x)$

25. Which description matches the power function $y = 6x^6$?
A. decreasing throughout its domain
B. increasing throughout its domain
C. decreasing for $x < 0$ and increasing for $x > 0$
D. increasing for $x < 0$ and decreasing for $x > 0$

Describe the end behavior of each real-world model of a power function without first graphing the function.

26. $f(x) = -2x^7$ **27.** $f(x) = -\frac{2}{3}x^8$

28. $f(x) = 0.3x^6$ **29.** $f(x) = x^9$

30. Which of these power functions has no real zeros?
A. $f(x) = -4x^{-1}$ **B.** $f(x) = x^{\frac{1}{6}}$
C. $f(x) = 3x^{0.5}$ **D.** $f(x) = 6x^7$

The Beaufort scale is a numerical scale that indicates wind speed. On this scale, 0 represents calm or no wind, and 12 represents a hurricane. The function $v = 1.87B^{1.5}$ can be used to determine the wind speed v in mi/h, where B is the Beaufort scale number.

31. Explain why the function that describes the Beaufort scale is a power function.

32. Graph the function that describes the Beaufort scale.

33. What is the wind speed during a strong gale that measures 9 on the Beaufort scale?

MATHEMATICAL PRACTICES
Look For and Express Regularity in Repeated Reasoning

34. For a power function of the form $y = ax^b$, the coefficient a is negative and the exponent b is a positive even integer. Based on this information, draw four conclusions about the graph of the power function.

Compositions of Functions and Inverses

Search and Rescue

Lesson 8-1 Composition of Functions

Learning Targets:

- Determine the composition of two functions.
- Determine the inverse of a function.

SUGGESTED LEARNING STRATEGIES: Think Aloud, Create Representations, Look for a Pattern, Group Presentation, Work Backward

A hiker who was last seen at a visitor center in a national park has been reported missing. A search-and-rescue (SAR) team estimates that the hiker would have been walking at a rate of 2 mi/h. The SAR team needs to determine the possible search area in which to look for the hiker.

1. The theoretical search area is circular. Write the equation of a function $f(x)$ that gives the number of square miles in the search area when its radius is x miles.

2. Write the equation of a function $g(x)$ that gives the distance in miles the hiker could have walked in x hours at a rate of 2 mi/h.

You have seen that the operations of addition, subtraction, multiplication, and division can be applied to functions. Another operation with functions is the ***composition*** of one function with another. The composition of the function f with the function g is written $f \circ g$ or $(f \circ g)(x) = f(g(x))$. The domain of $(f \circ g)$ is the set of all x in the domain of g so that the values of $g(x)$ are in the domain of f.

3. Use the equations you wrote in Items 1 and 2 to find the composition $(f \circ g)(x)$.

4. **Reason quantitatively.** What does the input of the composition $(f \circ g)(x)$ represent in this situation? How do you know?

My Notes

MATH TERMS

The **composition** of functions is an operation that combines functions in such a way that the output of one function is used as the input for the other.

READING MATH

$f \circ g$ is read "f of g." $(f \circ g)(x)$ and $f(g(x))$ are read "f of g of x."

My Notes

CONNECT TO STATISTICS

SAR teams rely on statistics to help them focus on the portions of the possible search area where the probability of finding a missing person is greatest. For example, data collected about the behavior of other lost hikers can help rescuers make predictions about where a hiker is most likely to be found.

MATH TIP

An operation on functions is commutative if the order of the functions within the operation does not matter.

5. **Reason quantitatively.** What does the output of the composition $(f \circ g)(x)$ represent in this situation? How do you know?

6. What is the reasonable domain of $(f \circ g)(x)$ in this situation? Explain.

7. **Model with mathematics.** The SAR team will use the distance in miles that the hiker could have walked as the radius of the theoretical search area. Explain why the composition $(f \circ g)(x)$ is a good model for the SAR team to use for calculating the number of square miles in the theoretical search area.

8. The hiker was last seen walking away from a visitor center at the park 6 hours ago. What is the theoretical search area in square miles? Explain how you used the composition $(f \circ g)(x)$ to find your answer.

9. Use the equations you wrote in Items 1 and 2 to find the composition $(g \circ f)(x)$.

10. Is $(f \circ g)(x)$ equivalent to $(g \circ f)(x)$? What does this tell you about whether composition of functions is commutative?

My Notes

One way to transform a function involves composition with the absolute-value function.

11. Express regularity in repeated reasoning. For each function listed below, use graphs to compare $f(x)$ with $g(f(x))$, where $g(x) = |x|$. Describe the effects of this absolute-value transformation.

a. $f(x) = x^3$ **b.** $f(x) = x^2 - 4$

c. $f(x) = x^5 + 2x^3$ **d.** $f(x) = \frac{1}{x}$

e. $f(x) = \sin x$ **f.** $f(x) = \ln x$

TECHNOLOGY TIP

Graph *f*(*x*) as Y1 and abs(Y1) as Y2. Turn Y1 on and off or trace each function to see ordered pairs as well as where the graphs differ or coincide. Use ZOOM Trig for the window for part e. It is also helpful to use the calculator's TABLE function.

12. Express regularity in repeated reasoning. For each function listed below, use graphs to compare $f(x)$ with $f(g(x))$, where $g(x) = |x|$. Describe the effects of this absolute-value transformation.

a. $f(x) = x^3$ **b.** $f(x) = \ln x$

c. $f(x) = x^5 + 2x^3$ **d.** $f(x) = e^x$

e. $f(x) = \sin x$ **f.** $f(x) = \sqrt{x}$

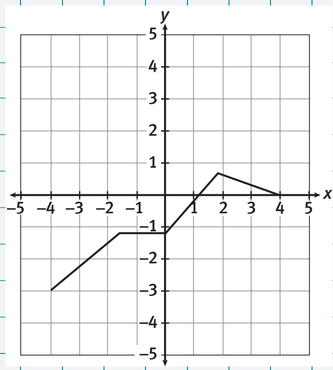
TECHNOLOGY TIP

Graph *f*(*x*) as Y1 and Y1(abs(*x*)) as Y2. Turn Y1 off and on as needed.

13. A function *f* is defined on the closed interval from −4 to 4 and has the graph shown.

a. Sketch the graph of $y = g(f(x))$, where $g(x) = |x|$.

b. Sketch the graph of $y = f(g(x))$, where $g(x) = |x|$.



My Notes

c. Sketch the graph of $y = g(f(x))$, where $g(x) = -x$.

d. Sketch the graph of $y = f(g(x))$, where $g(x) = -x$.

You can often write the equation of a given function as a composition of two other simpler functions.

Example A

Given $h(x) = \log(3x - 4)$, write the equations of two functions $f(x)$ and $g(x)$ so that $f(g(x)) = h(x)$.

Step 1: Analyze $h(x)$. To evaluate $h(x)$, you start with x and find the value of $3x - 4$. Then you find the common logarithm of the result.

Step 2: Write the equation for $g(x)$. $g(x)$ is evaluated first in the composition $f(g(x))$, so let $g(x) = 3x - 4$.

Step 3: Write the equation for $f(x)$. Let $f(x) = \log x$.

Step 4: Check by finding $f(g(x))$.

$$f(g(x)) = f(3x - 4) = \log(3x - 4) = h(x)$$

Solution: $f(x) = \log x$ and $g(x) = 3x - 4$

Try These A

Write the equations of two functions $f(x)$ and $g(x)$ so that $f(g(x)) = h(x)$.

a. $h(x) = \frac{1}{2}x - 5$

b. $h(x) = \frac{1}{x^2 + 3}$

c. $h(x) = 2^{x+1}$

d. $h(x) = x^2 + 6x + 9$

MATH TIP

Writing a function as the composition of two or more functions is called *decomposing* the function. There is often more than one way to decompose a function.

CONNECT TO AP

Being able to decompose functions will be helpful when solving some types of problems in calculus.

My Notes

Check Your Understanding

14. Critique the reasoning of others. Given that $h(x) = 3x + 5$, a student claims that $(h \circ h)(x) = (3x + 5)^2$. What mistake did the student make? Write the correct equation for $(h \circ h)(x)$.

15. Explain how to determine the range of $f(g(x))$ given $f(x)$ and $g(x)$.

16. Given that $q(x) = 2x + 4$ and $p(q(x)) = \sqrt{2x + 5}$, what is the equation of $p(x)$? Explain how you determined your answer.

17. a. Given the functions $f(x) = \ln x$ and $g(x) = -|x|$, find the composition $f(g(x))$.

b. What is the domain of the composition? What does the domain indicate about the composition?

MATH TIP

You can compose a function with itself. So, for example, $(f \circ f)(x) = f(f(x))$.

LESSON 8-1 PRACTICE

Find $f(g(x))$ and $g(f(x))$ for each pair of functions. State the domain.

18. $f(x) = x^2 - 4$, $g(x) = x - 2$

19. $f(x) = \frac{1}{x}$, $g(x) = x^2 + 3x - 4$

20. $f(x) = x^2 + 3$, $g(x) = \sqrt{x - 5}$

21. $f(x) = e^{x-3}$, $g(x) = \ln x + \ln 3$

Write the equations of two functions $f(x)$ and $g(x)$ so that $f(g(x)) = h(x)$.

22. $h(x) = |x - 2| + 4$

23. $h(x) = (x + 1)^4 + (x + 1)$

Gold has a density of 19.3 g/cm^3. On a certain date, gold is valued at \$55.62 per gram. Use this information for Items 24–27.

24. Write the equation of a function $f(x)$ that gives the value in dollars of an object made of gold with a mass of x grams.

25. Write the equation of a function $g(x)$ that gives the mass in grams of a cube of gold with a side length of x cm.

26. Reason quantitatively. Find the composition $f(g(x))$. State its reasonable domain. What does the composition represent in this situation?

27. What is the value of a cube of gold having a side length of 20 cm? Explain how you used the composition $f(g(x))$ to find your answer.

My Notes

MATH TERMS

If $f(x) = y$, then the function f^{-1} is the **inverse function** of f if $f^{-1}(y) = x$. The domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.

READING MATH

The notation f^{-1} is read as "the inverse of f." Note that the -1 in this notation does not represent an exponent.

Learning Targets:

- Find the inverse of a function.
- Restrict the domain of a function so that its inverse is also a function.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Discussion Groups, Create Representations, Look for a Pattern, Quickwrite

The function $f(x) = 2.5(x - 5)$ gives the distance in miles that a search-and-rescue helicopter can travel within x minutes of receiving an emergency call.

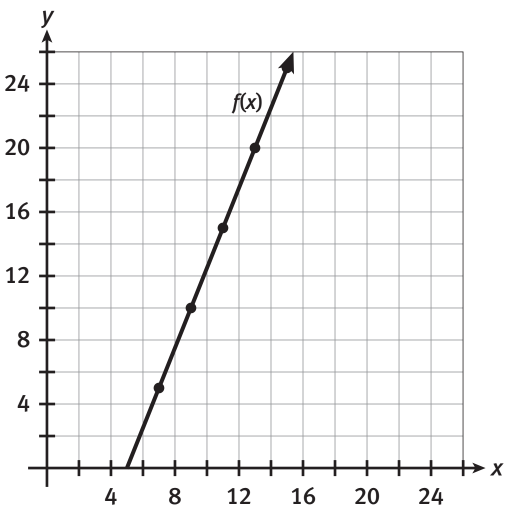
A search-and-rescue operator wants to develop the equation of a new function that will give the time in minutes from receiving an emergency call that it will take the helicopter to travel x miles. This new function will be the ***inverse function*** of f.

1. Complete the table of values for $f(x)$. Then use it to complete the table of values for the inverse function $f^{-1}(x)$.

Time (min), x	7	9	11	13	15
Distance (mi), $f(x)$					

Distance (mi), x					
Time (min), $f^{-1}(x)$	7	9	11	13	15

2. The coordinate grid below shows the graph of $f(x)$. Use the table of values from Item 28 to help you graph $f^{-1}(x)$ on the same coordinate grid.



My Notes

A reflection of the graph of a function over the line $y = x$ causes the x- and y-coordinates of the points on the graph to be switched.

3. Use a dashed line to graph $y = x$ on the coordinate grid on the previous page. Does the graph of $f^{-1}(x)$ appear to be a reflection of the graph of $f(x)$ over the line $y = x$? Explain.

4. Why does it make sense that the graph of the inverse of a function is a reflection of the graph of the function over the line $y = x$?

5. a. Reason abstractly. The search-and-rescue operator determines that $f(53) = 120$. Based on this information, what is $f^{-1}(120)$? Explain how you know.

b. Reason quantitatively. Explain the meanings of $f(53)$ and $f^{-1}(120)$ in the context of the situation.

6. Write the equation of $f^{-1}(x)$. Explain how you determined your answer.

My Notes

You can test whether two functions are inverses by using compositions. Because inverse functions undo each other, their compositions should equal the original input variable. In other words, if $f(g(x)) = x$ and $g(f(x)) = x$, then f and g are inverse functions.

7. Use compositions to check whether the equation you wrote in Item 6 is the inverse of the function $f(x) = 2.5x(x - 5)$.

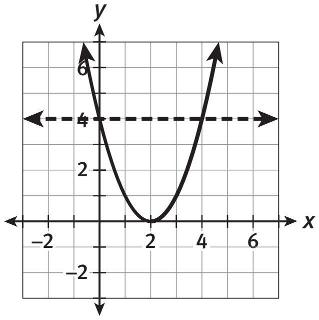
Sometimes the inverse of a function is not itself a function. In such cases, you may be able to restrict the domain of the original function so that its inverse will be a function.

Example A

Find the inverse function of $f(x) = (x - 2)^2$. Restrict the domain of f if needed, and describe the restriction. Then state the domain and range of the inverse function.

Step 1: Determine whether the inverse of f is a function.

Graph $f(x) = (x - 2)^2$.



Notice that for any y-value greater than 0, a horizontal line passes through more than one point on the graph. Therefore, the inverse of f is not a function.

Step 2: Restrict the domain of f if needed.

If you restrict the domain to include only the right half of the parabola (shown in bold), the inverse of f will be a function. So, restrict the domain of f to $\{x \mid x \geq 2\}$.

Step 3: Find the inverse function.

$f(x) = (x - 2)^2$	
$y = (x - 2)^2$	Replace $f(x)$ with y.
$x = (y - 2)^2$	Switch y and x to find the inverse function.
$\sqrt{x} = y - 2$	Take the positive square root of both sides.
$\sqrt{x} + 2 = y$	Add 2 to both sides.
$\sqrt{x} + 2 = f^{-1}(x)$	Replace y with $f^{-1}(x)$.

MATH TIP

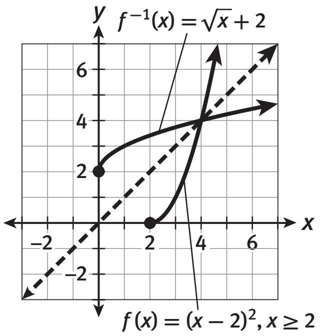
If a horizontal line passes through more than one point on the graph of a function, then the function has more than one x-value for the same y-value. As a result, the inverse of the function will have more than one y-value for the same x-value, which means that the inverse is not a function.

MATH TIP

When you restrict the domain of f to $x \geq 2$, you also restrict the range of its inverse to $y \geq 2$. So, when you are finding the equation of the inverse, you know that the quantity $y - 2$ will always be nonnegative. As a result, you can ignore the negative square root in the calculations in Step 3.

Step 4: Check by graphing.

Each graph represents a function, and the graphs are reflections of each other across the line $y = x$. So, f^{-1} is the inverse function of f.



Solution: $f^{-1}(x) = \sqrt{x} + 2$ is the inverse function of $f(x) = (x - 2)^2$, where the domain of f is $\{x \mid x \geq 2\}$. The domain of f^{-1} is $\{x \mid x \geq 0\}$, and the range is $\{y \mid y \geq 2\}$.

Try These A

Find the inverse function. Restrict the domain of f if needed, and describe the restriction. Then state the domain and range of the inverse function.

a. $f(x) = \frac{1}{2}x^3$ **b.** $f(x) = (x + 4)^4 - 1$

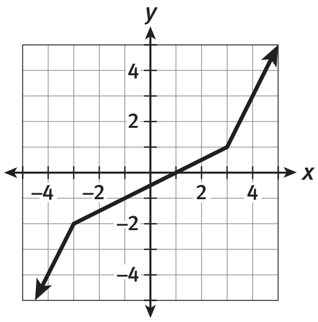
c. $f(x) = \log_2 x$ **d.** $f(x) = \sqrt{x + 3}$

Check Your Understanding

8. Explain how you can use a graph of the function $f(x) = 5x - 6$ to determine whether its inverse is a function.

9. **Make use of structure.** How would you restrict the domain of $f(x) = |x| + 4$ so that its inverse is a function? Explain how you decided on the domain restriction.

10. The graph shows a function f. Sketch the graph of its inverse function f^{-1}, and explain how you determined the coordinates of points on the graph of f^{-1}.



11. Find the inverse function of $f(x) = \frac{1}{x}$. What relationship do you notice between f and f^{-1} in this case?

My Notes

TECHNOLOGY TIP

To graph a function with a restricted domain on a graphing calculator, enter the function rule in parentheses followed by the domain restriction in parentheses. For example, to graph $f(x) = (x - 2)^2$ with the domain restricted to $x \geq 2$, you would enter ((X − 2)^2) (X ≥ 2) as Y1. To enter inequality symbols in a function rule, press 2nd TEST.

My Notes

LESSON 8-2 PRACTICE

Make use of structure. Find the inverse function. Restrict the domain of f if needed, and describe the restriction. Then state the domain and range of the inverse function.

12. $f(x) = -2(x+1)^2 - 4$

13. $f(x) = e^{2x}$

14. $f(x) = \frac{2}{x} + 4$

15. $f(x) = |x - 3| + 2$

Use composition to determine whether each pair of functions are inverses.

16. $f(x) = 2x - 3$ and $g(x) = \frac{1}{2}x + 3$

17. $f(x) = (x-4)^2 + 5$ for $x \geq 4$ and $g(x) = \sqrt{x-5} + 4$

The function $f(x) = \frac{4}{3}\pi x^3$ gives the volume in cm^3 of a sphere with a radius of x cm. Use this information for Items 18–20.

18. Write the equation of the inverse of f, and explain what the inverse represents in this situation.

19. Graph the inverse function.

20. **Reason quantitatively.** Find $f(12)$ and $f^{-1}(12)$, and explain what each represents in this situation.

ACTIVITY 8 PRACTICE

Write your answers on notebook paper.
Show your work.

Lesson 8-1

Yvonne earns $9 per hour for the first 40 hours she works in a week. She earns $13.50 per hour after that. Each week she deposits 15 percent of her earnings into her college savings account.

1. Write the equation of a function $f(x)$ that can be used to determine Yvonne's weekly earnings when she works x hours and $x \geq 40$.
2. Write the equation of a function $g(x)$ that can be used to determine the amount Yvonne deposits in her savings account when her weekly earnings are x dollars.
3. Find the composition $g(f(x))$. What does the composition represent in this situation?
4. Last week, Yvonne deposited $60.08 in her savings account. How many hours did she work that week? Explain how you used the composition $g(f(x))$ to find your answer.

The function $f(x) = 331.4 + 0.6x$ gives the speed of sound in m/s in dry air, where x is the temperature in °C. The function $g(x) = \frac{5}{9}(x - 32)$ converts a temperature x in °F to °C.

5. Find $f(g(x))$, and tell what this composition represents in this situation.
6. What is the speed of sound in dry air when the temperature is 98°F? Explain how you determined your answer.

Find $f(g(x))$ and $g(f(x))$ for each pair of functions. State the domain.

7. $f(x) = 3x - 7$, $g(x) = -8x - 9$
8. $f(x) = \frac{1}{x^2}$, $g(x) = \sqrt{x - 4}$
9. $f(x) = \ln x$, $g(x) = 2x - 3$
10. $f(x) = x^2 + 3x - 1$, $g(x) = x + 4$

Write the equations of two functions $f(x)$ and $g(x)$ so that $f(g(x)) = h(x)$.

11. $h(x) = (x - 3)^2 - 5$
12. $h(x) = 16x^4$
13. $h(x) = 2\sqrt{x + 8}$
14. $h(x) = \frac{x + 2}{x + 5}$
15. The function $t(d)$ models the high temperature in °F in a certain city on day d of the year. The function $n(t)$ models the number of visitors to a pool in the city on a day when the high temperature is t °F. What does the composition $n(t(d))$ represent in this situation?

Use the tables to find each value.

x	1	3	5	7
$f(x)$	2	4	6	8

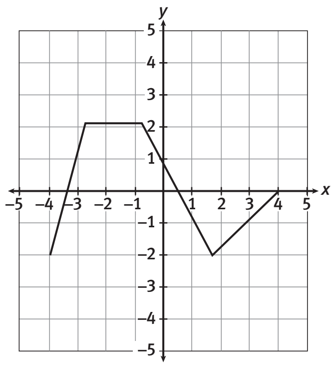
x	2	6	8	10
$g(x)$	3	7	11	15

16. $f(g(2))$
17. $g(f(5))$
18. $f(g(6))$

Given that $f(x) = x^2$ and $g(x) = 3x + 1$, find each value.

19. $f(g(4))$
20. $g(f(-2))$
21. $f(g(0))$

A function f has the graph shown. Use the graph for Items 22–23.



22. Sketch the graph of $y = g(f(x))$, where $g(x) = |x|$.
23. Sketch the graph of $y = f(g(x))$, where $g(x) = |x|$.

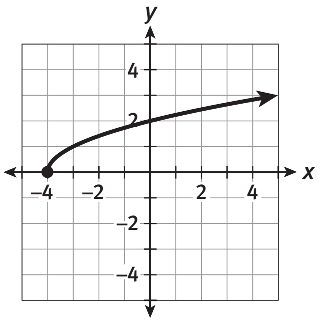
Lesson 8-2

24. Based on the table below, what is $f^{-1}(10)$?

x	0	5	10	20
$f(x)$	0	10	20	30

A. 0 **B.** 5
C. 10 **D.** 20

The graph represents a function $f(x)$. Use the graph for Items 25 and 26.



25. For what value of x is $f^{-1}(x) = -4$? Explain how you know.

26. Sketch a graph of $f^{-1}(x)$.

Find the inverse function. Restrict the domain of f if needed, and describe the restriction. Then state the domain and range of the inverse function.

27. $f(x) = 2x^2 + 4$

28. $f(x) = -3 \log (x + 2)$

29. $f(x) = \sqrt[3]{x-6}$

30. $f(x) = (x + 1)^4 - 1$

31. Given that $f(x) = \frac{2x-1}{4}$, what is $f^{-1}(3)$?

A. $\frac{11}{2}$ **B.** $\frac{13}{2}$
C. 8 **D.** 26

32. Explain why the inverse of $f(x) = (x + 1)(x - 3)$ is not a function. Explain how you could restrict the domain of $f(x)$ so that its inverse would be a function.

33. The table represents the function $g(x)$. Make a table that represents its inverse, $g^{-1}(x)$.

x	−5	−2	7	12
$g(x)$	−4	1	6	−10

Use composition to determine whether each pair of functions are inverses.

34. $f(x) = \log_3 (x - 1)$ and $g(x) = 3^{x+1}$

35. $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$

36. $f(x) = -4(x - 6)$ and $g(x) = -\frac{1}{4}(x - 24)$

The function $f(x) = \left(\frac{1}{2}\right)^x$ gives the probability of getting heads x times in a row when tossing a coin. Use this information for Items 37 and 38.

37. Write the equation of the inverse of f, and explain what the inverse represents in this situation.

38. Find $f^{-1}(0.01)$, and explain what it represents in this situation.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

39. Recall that the formula $A = Pe^{rt}$ gives the balance A in an account that earns continuously compounded interest, where P is the principal, r is the annual interest rate expressed as a decimal, and t is the time in years. Gilberto deposits \$1000 in an account that earns continuously compounded interest. After 1 year, the balance in the account is \$1030.45. How many years will it take for the balance in the account to reach \$1500? Show your work, and include in your answer both the equation of a function that gives the account balance after x years and the equation of a function that gives the number of years for the balance in the account to reach x dollars.

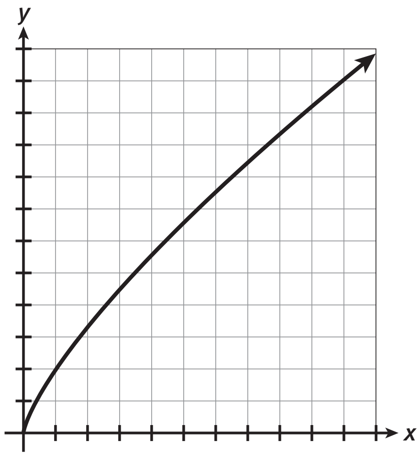
Transformations, Compositions, and Inverses

FEEDING FRENZY

Even when an animal is sitting still, its body is using energy. An animal's basal metabolic rate, or BMR, is a measure of the amount of energy the animal uses per day when at rest. An animal's BMR is related to its weight. In general, the more an animal weighs, the higher its BMR and the more food it needs to eat each day.

1. The function $f(x) = 70x^{\frac{3}{4}}$ can be used to estimate the BMR of mammals and many other types of animals. In this function, x is the mass of the animal in kilograms, and $f(x)$ is its BMR in kilocalories (kcal) per day. Note that 1 kcal is equal to 1 food Calorie.

a. The graph of the function $f(x) = 70x^{\frac{3}{4}}$ is given below. Write appropriate scales and labels for the axes.



b. What are the domain and range of the function? Explain mathematically why the domain is not all real numbers.

c. Does the value of the function increase or decrease over its domain? Why does it make sense in the context of the problem situation that the function behaves in this way?

d. Describe the graph of $f(x)$ as a transformation of the graph of $h(x) = x^{\frac{3}{4}}$. Explain how you identified the transformation.

2. A male Bengal tiger at a zoo has an estimated BMR of 4000 kcal/day, and a female Bengal tiger has an estimated BMR of 2850 kcal/day.

a. Find the inverse of $f(x) = 70x^{\frac{3}{4}}$, and explain what the inverse represents in this situation.

b. Predict how much greater the mass of the male tiger is than that of the female tiger. Explain how you determined your answer.

3. The howler monkeys at a zoo are fed a diet that contains 3.35 kcal/gram.

a. Write a function $g(x)$ that gives the mass of food in grams that a howler monkey must eat to obtain x kilocalories of energy.

b. Find $g(f(x))$, and tell what it represents in this situation.

c. One of the howler monkeys has a mass of 9 kg. How many grams of food must it eat each day just to meet the needs of its basal metabolic rate?

Transformations, Compositions, and Inverses

FEEDING FRENZY

Scoring Guide	Exemplary	Proficient	Emerging	Incomplete
	The solution demonstrates these characteristics:			
Mathematics Knowledge and Thinking (Item 2)	• Clear and accurate understanding of a monomial with a rational exponent and a clear understanding of inverse functions in this context	• A functional understanding of undoing a function to find a solution of a monomial with a rational exponent and also understanding of inverse operations to solve the word problem	• Partial understanding of functions using rational exponents and/or partial understanding of solving them	• Little or no understanding of functions with rational exponents or techniques to solve
Problem Solving (Items 1, 2, 3)	• An appropriate and efficient strategy that results in a correct answer	• A strategy that may include unnecessary steps but results in a correct answer	• A strategy that results in some incorrect answers	• No clear strategy when solving problems
Mathematical Modeling / Representations (Items 1, 2)	• Clear and accurate understanding of representations of graphs of monomials with rational exponents • Clear and accurate understanding of finding an inverse function and using it	• A functional understanding of representations of graphs of monomials with rational exponents • Mostly accurate understanding of finding an inverse function and using it	• Partial understanding of representations of graphs of monomials with rational exponents • Partial understanding of finding an inverse function and using it	• Little or no understanding of representations of graphs of monomials with rational exponents • Inaccurate or incomplete understanding of inverses
Reasoning and Communication (Item 3)	• Precise use of appropriate math terms and language to describe the composite function and its meaning in the context of the problem with appropriate units	• Correct characterization of the solution of the composite function (may not necessarily understand it completely in the context of the problem)	• Misleading or confusing characterization of the composite function	• Incomplete or inaccurate characterization of the composite function