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Sequential Round-Robin Tournaments with Multiple Prizes

Abstract

We examine the fairness and intensity of sequential round-robin tournaments with multiple prizes. With three symmetric players and two prizes, the tournament is completely fair if and only if the second prize is valued half of the first prize, regardless of whether matches are organized as Tullock contests or as allpay auctions. For second prizes different from half of the first prize, three-player tournaments with matches organized as Tullock contests are usually fairer than tournaments with matches organized as all-pay auctions. However, unless the second prize is very small, they are less intense in the sense that players exert less ex-ante expected aggregate effort per unit of prize money. Moreover, we specify how the relative size of the second prize influences the extent and the direction of discrimination as well as the intensity of three-player tournaments. Finally, we show that there is no prize structure for which sequential round-robin tournaments with four symmetric players are completely fair in general.

JEL-Codes: C720, D720, Z200.

Keywords: round-robin tournament, multiple prizes, fairness, intensity, Tullock contest, all-pay auction.

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1 Introduction

A round-robin tournament is an all-play-all competition format where each participant meets every other participant in turn. Round-robin tournaments are widely applied to organize sport contests on a large scale, such as the major European football leagues with up to 20 teams like in the English Premier League, as well as on a small scale, such as the first round (group stage) of the FIFA World Cup (since 1950) or the UEFA European Championship (since 1980) with four teams per group. For their World Cups from 2026 on, the FIFA recently announced a transition to an initial group stage with only three teams per group. This modification is particularly remarkable as the natural structure of round-robin tournaments with three teams is sequential.

One reason for the popularity of round-robin tournaments might be the common wisdom that "a round-robin tournament is the fairest way to determine the champion among a known and fixed number of participants" (Wikipedia[1], 2017). For tournaments for which the schedule of matches is sequential, however, this common wisdom is challenged by Krumer et al. (2017a) and Sahm (2017). The authors investigate whether *single-prize* round-robin tournaments with three and four symmetric players are fair with respect to ex-ante winning probabilities and expected payoffs. The different pairwise matches take place one after the other, players are ranked according to the number of matches won, and (only) the player with the most victories receives a prize.

Krumer et al. (2017a) assume that each single match is organized as an all-pay auction and find substantial discrimination by the order of matches in the subgame perfect equilibrium of the sequential game: depending on their position in the sequence of matches, the players have differing ex-ante winning probabilities and expected payoffs.[2] The reason is a *discouragement effect* of trailing players that has been identified in most forms of dynamic contests (Konrad, 2009, Chapter 8).

Sahm (2017) confirms this result for round-robin tournaments where each single match is organized as a Tullock contest. The extent of discrimination is, however, much smaller than with matches organized as all-pay auctions because the discouragement of trailing players in asymmetric intermediate stages is less pronounced due to the non-perfectly discriminating character of the Tullock contest.

In many real world tournaments, such as the group stage of the FIFA World Cup where two teams earn a spot in the next round, the assumption of a single prize seems too narrow because the players have a positive valuation not only for ranking first. In this paper we therefore revisit the analysis of Krumer et al. (2017a) and Sahm (2017) on the fairness of sequential round-robin tournaments with three and four players under the assumption of *multiple prizes*.

The basic idea why the introduction of additional prizes may reduce discrimination in sequential round-robin tournaments is that they induce a second effect which counteracts the discouragement effect and which we call the *lean-back effect*: a second prize weakens the incentives for winners of early matches to provide additional effort in their later matches because this second prize reduces the difference in payoffs from ranking first and second, respectively.

In a first pass, Krumer et al. (2017b) consider three player round-robin tournaments with matches organized as all-pay auctions and show that introducing a second prize

[1] Accessed at https://en.wikipedia.org/wiki/Round-robin_tournament on 02/08/2017

[2] Based on sports data from mega-events, Krumer and Lechner (2017) provide empirical evidence for such discrimination.

which equals the first prize may reduce discrimination but does not lead to a completely fair competition. We extend their analysis in several dimensions.

First, and most importantly, we allow for second prizes which may be smaller than the first prize. In many tournaments, the prize money for the player ranked second is positive but indeed smaller than for the player ranked first; or, as in the initial group stage of the FIFA World Cup where the winning team of each group is paired with a runner-up team of a different group in the next stage, ranking first is more valuable than ranking second. We therefore consider sequential round-robin tournaments with a second prize that equals an arbitrary proportion $a \in [0, 1]$ of the first prize. As our main result, we find that round-robin tournaments with three symmetric players are completely fair if and only if the second prize equals half of the first prize.[3] In this case, the players not only have identical ex-ante ranking probabilities and expected payoffs but exert equal efforts and have equal winning probabilities in each single match of the tournament.

Second, we consider not only tournaments with matches organized as all-pay auctions but also tournaments with matches organized as Tullock contests. Our main result that round-robin tournaments with three players are completely fair if and only if the second prize equals half of the first prize ($a = 1/2$) holds in both cases. For second prizes different from half of the first prize ($a \neq 1/2$), tournaments with matches organized as Tullock contests are usually fairer than tournaments with matches organized as all-pay auctions. However, unless the second prize is very small, they are less intense in the sense that players exert less ex-ante expected aggregate effort per unit of prize money. Qualitatively, the same results apply if we consider three-player round-robin tournaments for which the sequence of matches is not exogenously given but endogenously determined in the sense that the outcome of the first match defines the order of the following two matches.

Third, we analyze not only three-player tournaments but also an instance of a sequential round-robin tournament with four symmetric players. We show that the introduction of a second and a third prize[4] can again mitigate the discrimination induced by the sequential structure but not fully resolve the problem: unlike for three-player tournaments, there is generally no prize structure for which the four-player tournament is completely fair.

Considering tournaments with both, three and four players, also enables us to apply our model for a comparison between the current structure of the FIFA World Cup with four teams per group in the first round and the structure planed from 2026 on with three teams per group in the first round. We illustrate that this structural change is likely to increase both, the fairness and the intensity of the initial group stage.

The remainder of this paper is organized as follows. Section 2 provides the basic analysis and main results for round-robin tournaments with three players and two prizes

[3]In a recent working paper, Dagaev and Zubanov (2017) consider also round-robin tournaments with three symmetric players and two prizes. In contrast to our model, they assume that players have limited resources but face no real effort costs: each player just decides how to split her resources between her two matches. Though the authors find a multiplicity of equilibria which do not allow for unambiguous predictions about the extent and direction of discrimination, the case in which the second prize equals half of the first prize plays a particular role in their model as well and, in line with our results, is likely to entail a fair tournament.

[4]For example, in the initial group stage of the UEFA Champions League, four teams compete in a double round-robin tournament in which the two teams ranked first and second move to the next stage, the team ranked third qualifies for the elimination phase of the UEFA Europa League, and the team ranked fourth drops out.

for an exogenous sequence of matches. Section 3 compares the fairness and intensity of three-player round-robin tournaments with matches organized as Tullock contests and all-pay auctions, respectively. In Section 4, we discuss the robustness of our results considering three-player tournaments with endogenous sequences of matches as well as sequential round-robin tournaments with four players. Section 5 concludes.

2 Tournaments with Three Players

In this section, we consider round-robin tournaments with two prizes and three symmetric, risk-neutral players. Successively, each player is matched one-to-one with each other player in a sequence of three different pairwise matches. The final ranking is determined according to the number of victories: if there is a player with two victories, this player wins the first prize and the player with one victory wins the second prize; if there is a tie because each player has won one match, the first and second prize are assigned randomly with equal probabilities of 1/3 for each player to win the first and second prize, respectively. The value of winning the first prize is identical for all players and normalized to 1. The value of winning the second prize is also identical for all players and expressed as a proportion $a \in [0, 1]$ of the first prize.

Without loss of generality, we consider an exogenous sequence in which player 1 is matched with player 2 in the first match, player 1 is matched with player 3 in the second match, and player 2 is matched with player 3 in the third match.[5] In the following analysis, we distinguish between two types of tournaments that differ in the organization of their matches: in Tullock contest tournaments (TC-tournaments) each match is organized as a Tullock contest and in all-pay auction tournaments (APA-tournaments) each match is organized as an all-pay auction between two players, A and B, with linear costs of effort, see e.g. Konrad (2009, Chapter 2). The structure of the resulting sequential game with its $2^3 = 8$ potential courses is depicted in Figures 1 and 5, respectively. The seven nodes $k \in \{A, \ldots, F\}$ in each figure represent all combinations for which the ranking of the tournament has not yet been determined when the respective match starts.

2.1 Tullock Contest Tournaments

2.1.1 Match behavior

In TC-tournaments, player A's probability of winning match k is

$$p_A^k = \begin{cases} 1/2 & \text{if } x_A^k = x_B^k = 0, \\ \frac{(x_A^k)^r}{(x_A^k)^r + (x_B^k)^r} & \text{else,} \end{cases}$$

where x_i^k denotes the effort of player $i \in \{A, B\}$ in match k, and $r \geq 0$ describes the discrimination power of the contest. In the context of rent-seeking, this type of contest success function was introduced by Tullock (1980) and given an axiomatic foundation by Skaperdas (1996). For $r = 0$ the players' winning probabilities are independent from efforts and equal 1/2. For $r = 1$ the contest is also referred to as lottery contest.

[5] Apart form renaming players, this exogenous sequence is unique. In Section 4.1, we discuss the use of endogenous sequences in which the outcome of the first match determines the order of the two remaining matches.

Player A chooses x_A^k in order to maximize his expected payoff

$$E_A^k = p_A^k(w_A^k - x_A^k) + \left(1 - p_A^k\right)\left(\ell_A^k - x_A^k\right), \tag{1}$$

where w_i^k denotes player i's expected continuation payoff from winning the match and ℓ_i^k denotes his expected continuation payoff from losing the match with $w_i^k \geq \ell_i^k \geq 0$ for $i \in \{A, B\}$. For $w_A^k = \ell_A^k$, the optimal choice is $x_A^k = 0$ for any $x_B^k \geq 0$. If $x_A^k = 0$ and $w_B^k > \ell_B^k$, player B will have no best reply unless there is a smallest monetary unit $\epsilon > 0$. As $\epsilon \to 0$, in the limit, $x_B^k \to 0$ and $p_B^k \to 1$. Otherwise, a unique Nash-equilibrium in pure strategies exists if and only if the discriminating power of the contest is sufficiently small (Nti, 1999); more precisely if and only if for $i, j \in \{A, B\}$ with $i \neq j$ and $w_i^k - l_i^k = \min\{w_A^k - l_A^k, w_B^k - l_B^k\}$ the inequality

$$r \leq 1 + \left(\frac{w_i^k - l_i^k}{w_j^k - l_j^k}\right)^r \tag{2}$$

is satisfied. We will restrict our analysis below to the cases in which this condition is satisfied for each possible match k. Notice that this trivially includes the case of lottery contests where $r = 1$. The equilibrium effort levels can then be derived from the necessary conditions

$$\frac{\partial E_i^k}{\partial x_i^k} = \frac{r(x_i^k)^{r-1}(x_j^k)^r}{[(x_i^k)^r + (x_j^k)^r]^2}(w_i^k - \ell_i^k) - 1 = 0$$

yielding

$$x_i^k = r\frac{(w_i^k - \ell_i^k)^{1+r}(w_j^k - \ell_j^k)^r}{[(w_i^k - \ell_i^k)^r + (w_j^k - \ell_j^k)^r]^2} \tag{3}$$

for $i \neq j \in \{A, B\}$. The resulting equilibrium winning probabilities equal

$$p_i^k = \frac{(w_i^k - \ell_i^k)^r}{(w_i^k - \ell_i^k)^r + (w_j^k - \ell_j^k)^r}. \tag{4}$$

2.1.2 Tournament Equilibria

Figure 1 illustrates the sequential structure of the tournament, where $\Gamma := \frac{1+a}{3}$ denotes the players' expected gross payoff in case of a tie. We solve the game by backward induction for its subgame perfect equilibrium, making repeatedly use of equations (3), (4), and (1). For the cases $a = 1/2$ and $(a, r) = (1, 1)$, the details of this procedure are exemplified in Appendix A.[6] For the case in which all matches are organized as simple lotteries ($r = 1$), Figure 1 also depicts the winning probabilities in each match for $a = 0$, $a = 1/2$ (black box), and $a = 1$ (blue-grey).

Proposition 1. *If the second prize is valued half of the first prize ($a = 1/2$), two-prize round-robin tournaments with three symmetric players and matches organized as Tullock contests are perfectly fair for all $0 \leq r \leq 2$:*

(a) In the unique subgame perfect equilibrium all players have identical ex-ante probabilities of ranking first, second, and third, as well as identical ex-ante expected payoffs.

[6] The details of case $(a, r) = (0, 1)$ are worked out by Sahm (2017).

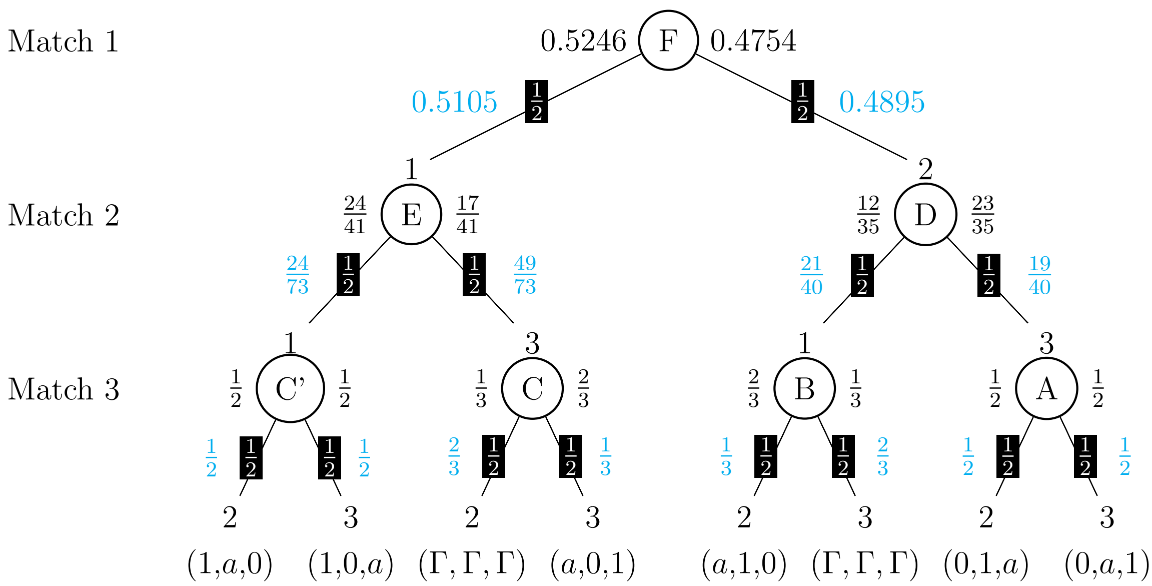


Figure 1: Lottery Contest Game Tree

(b) Each single match of the tournament is fair in the sense that the two matching players always have equal winning probabilities and exert the same effort $x = r/8$.

The proof can be found in Appendix A. We have run a large number of simulations computing the subgame perfect equilibrium of the tournament for all combinations (r, a) on a grid over $[0, \bar{r}(a)] \times [0, 1]$ with a width of 0.01, where $\bar{r}(a)$ denotes the largest value of r for which the tournament has a subgame perfect equilibrium in pure strategies for a given value of a. The results, some of which are depicted in Figure 2, suggest that $a = 1/2$ is not only sufficient but also necessary for the tournament to be fair, i.e. the only prize structure for which the tournament is non-discriminatory. More precisely, we find the following

Simulation Result 1. *If the second prize is not valued half of the first prize* ($a \neq 1/2$)*, two-prize round-robin tournaments with three symmetric players and matches organized as Tullock contests are discriminatory. For each* $r \in (0, \bar{r}(a)]$*, the level of discrimination as measured by the relative standard deviation of the players' ex-ante expected payoffs*

(a) decreases as a function of a *on* $[0, 1/2)$*,*

(b) increases as a function of a *on* $(1/2, 1]$.

Not only the amount but also the direction of discrimination varies with the value of the second prize a. For the sake of concreteness, we will henceforth focus on TC-tournaments with matches organized as lottery contests ($r = 1$) and refer to them as LC-tournaments. Figure 3 depicts the players' weighted qualification probabilities (WQP)[7], probabilities to rank first, and probabilities to rank second as functions of a. Similarly, Figure 4

[7]We define the weighted qualification probability as the aggregate probability to rank first or second, where the probability to rank second is weighted by the second prize as a proportion of the first prize: WQP $= P^{1^{st}} + a \cdot P^{2^{nd}}$. In light of the example of the initial group stage of the FIFA World Cup, the WQP measures the overall probability to qualify for the next round adjusted for the fact that ranking second reduces the winning probability in the next round compared to ranking first.

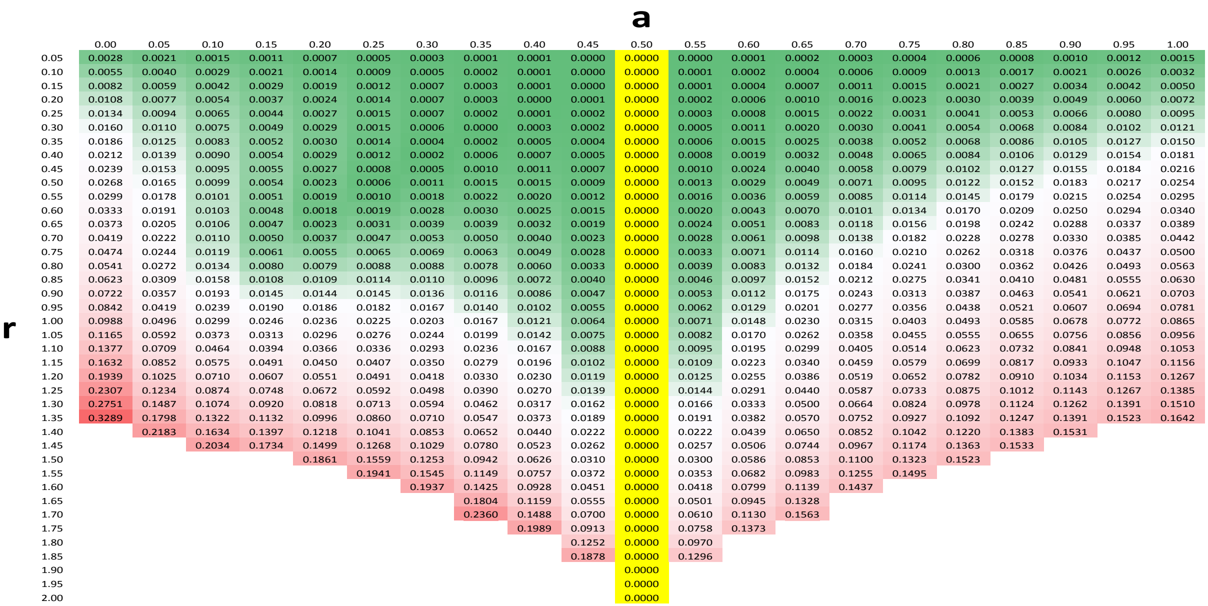


Figure 2: Tullock Contest: relative standard deviations of expected payoffs

shows expected payoffs, expected efforts, and aggregate effort per unit of prize money as functions of a. Table 1 compares the ranking probabilities for all players, and Table 2 compares weighted qualification probabilities, expected payoffs, and expected efforts for LC-tournaments with $a = 0$, $a = 1/2$, and $a = 1$. We observe the following

Simulation Result 2. *LC-tournaments with three symmetric players and two-prizes ($0 < a \leq 1$) are less discriminatory than single-prize tournaments ($a = 0$) as measured by the relative standard deviation of the players' ex-ante expected payoffs (or, alternatively, weighted qualification probabilities).*

(a) If the valuation of the second prize is higher than one half of the first prize ($1/2 < a \leq 1$), the player who competes in the last two matches (player 3) exerts the lowest ex-ante expected effort but has the highest ex-ante expected payoff.

(b) If the valuation of the second prize is lower than one half of the first prize ($0 \leq a < 1/2$), the player who competes in the first two matches (player 1) exerts the highest ex-ante expected effort and has the highest ex-ante expected payoff.

Table 1: LC-Tournaments: Ranking Probabilities

rank	1st			2nd			3rd		
prize	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$
P1	0.3494	0.3333	0.3011	0.2959	0.3333	0.3331	0.3520	0.3333	0.3658
P2	0.3071	0.3333	0.3352	0.3520	0.3333	0.3334	0.3188	0.3333	0.3314
P3	0.3435	0.3333	0.3637	0.3520	0.3333	0.3334	0.3292	0.3333	0.3028
rel. SD	0.0561	0	0.0768	0.0793	0	0.0004	0.0635	0	0.0772

In order to get some intuition for how both, the amount and direction of discrimination vary in the second prize a, we identify two opposing effects: a *discouragement effect* and

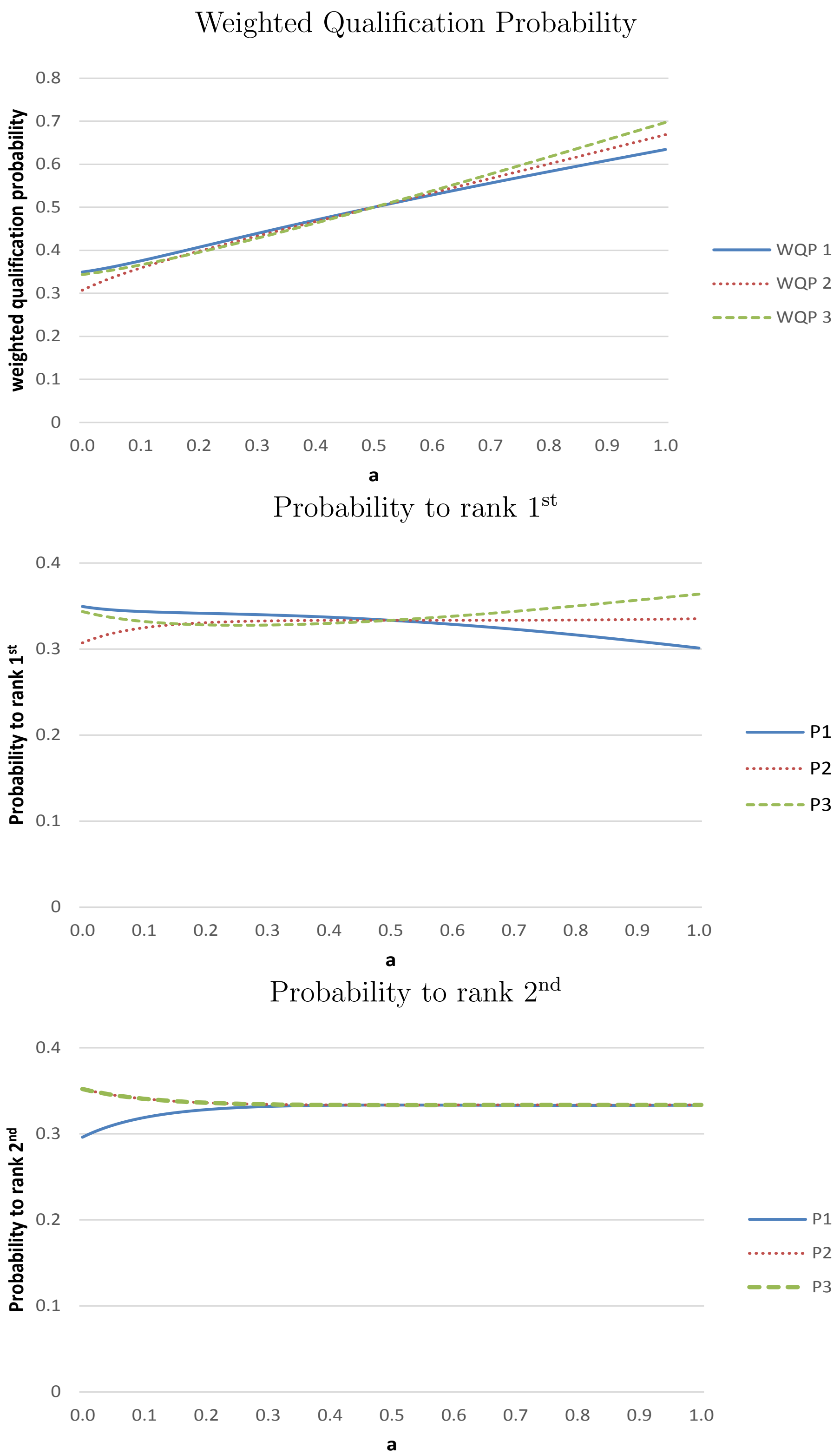


Figure 3: Lottery Contest: Winning Probabilities

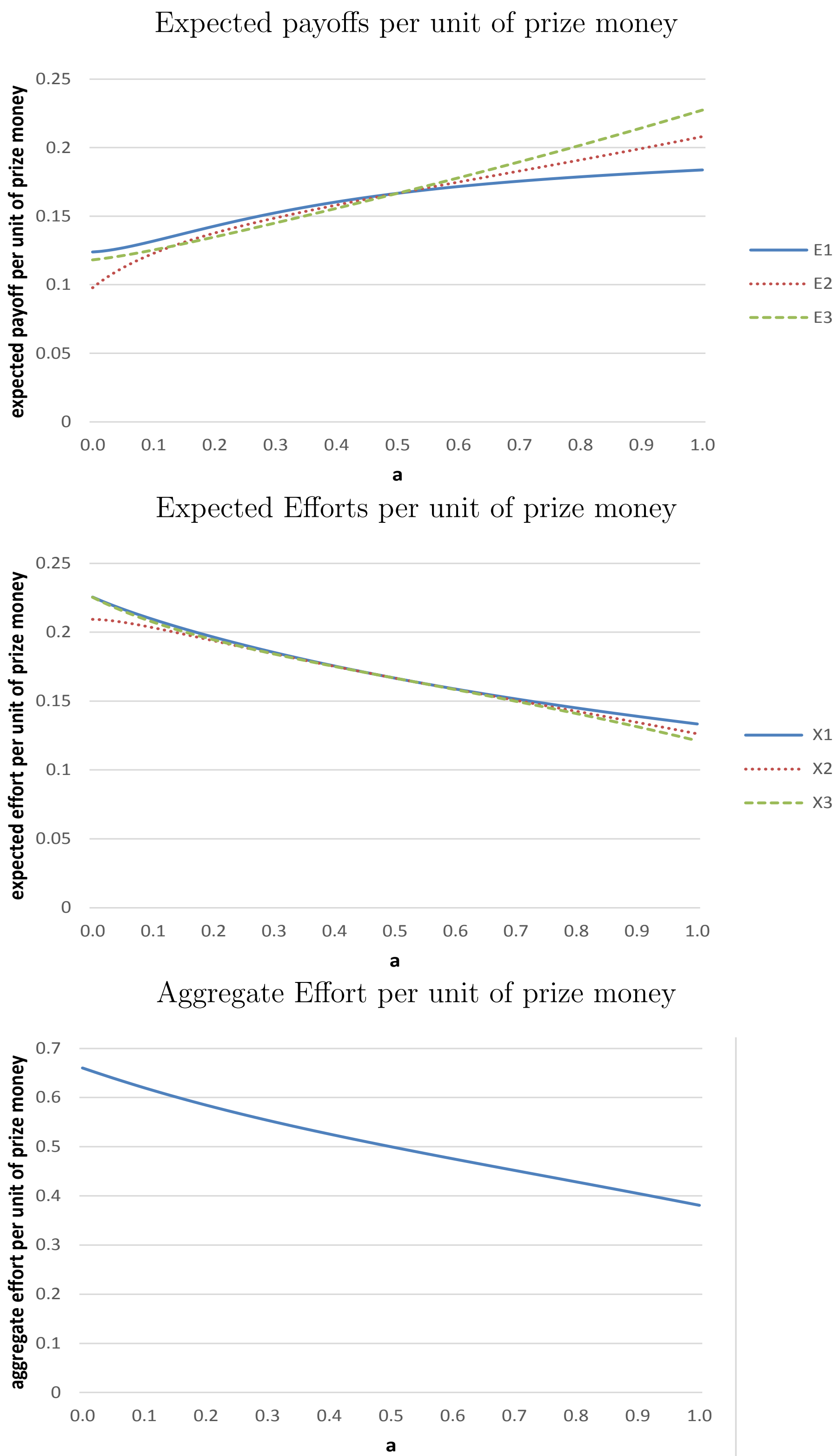


Figure 4: Lottery Contest: Expected Payoffs and Efforts

Table 2: Summary: LC-Tournaments

	WQP			Expected Payoffs			Expected Effort		
prize	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$
P1	0.3494	0.5000	0.6342	0.1239	0.2500	0.3674	0.2262	0.2500	0.2668
P2	0.3071	0.5000	0.6686	0.0978	0.2500	0.4161	0.2070	0.2500	0.2525
P3	0.3435	0.5000	0.6972	0.1181	0.2500	0.4547	0.2253	0.2500	0.2424
rel.SD	0.0561	0	0.0386	0.0988	0	0.0865	0.0387	0	0.0393
$\sum$				0.3398	0.7500	1.2383	0.6585	0.7500	0.7617

a *lean-back effect*. Without a second prize ($a = 0$), the player who skips the first match (player 3) has an (expected) disadvantage when he faces the winner of the first match that discourages him from exerting equivalent effort (Sahm, 2017).[8] Introducing a second prize mitigates this discouragement effect: With a second prize, the winner of the first match has a positive payoff even if he loses his next match and ranks second. This makes him lean back and reduce effort in his second match where he meets player 3. Ceteris paribus, the lean-back effect thus improves the chances of player 3 such that, in equilibrium, player 3 himself responds by a reduction of effort. Because the lean-back effect intensifies as the second prize increases, the discouragement effect dominates for second prizes below a certain threshold while the lean-back effect dominates for second prizes above it. It turns out that the threshold at which the two effects cancel each other equals just half of the first prize.

The designer of a tournament is usually interested not only in fairness but also in aggregate effort because the contest is the more attractive for spectators and sponsors not only the closer it is but also the higher the participants' effort. Another important question is thus how variations of the second prize a affect aggregate effort per unit of prize money ρ.[9] Figure 4 illustrates the following

Simulation Result 3. *An increase in the second prize of LC-tournaments with three symmetric players strictly reduces aggregate expected effort per unit of prize money:* $\partial\rho/\partial a < 0$.

For second prizes below $a = 1/2$, this implies a trade-off between the fairness of the LC-tournament and the induced aggregate effort per unit of prize money: if the contest designer wants to increase the fairness of the tournament by increasing the second prize a, he will have to accept a decrease in aggregate effort per unit of prize money. By contrast, for second prizes above $a = 1/2$, the contest designer can improve both, the fairness and the aggregate effort per unit of prize money, by decreasing the second prize a.

[8] If player 3 meets player 1 in node E, the probability of ranking first after winning this match is 1 for player 1 but smaller than 1 for player 3. Similarly, in node D, the expected probability of ranking first after winning the last match (in node B or A) is 1 for player 2 but smaller than 1 for player 3 because there is a positive probability that player 3 loses match D against player 1 and thus faces a leading player 2 in node B instead of an even player 2 in node A.

[9] In the literature, ρ is also referred to as *rent dissipation rate* as it measures how much of the economic rent is dissipated during the contest in form of the players' efforts.

2.2 All-pay Auction Tournament

2.2.1 Match behavior

In APA-tournaments, player A's probability of winning match k is

$$p_A^k = \begin{cases} 1 & \text{if} \quad x_A^k > x_B^k, \\ 1/2 & \text{if} \quad x_A^k = x_B^k, \\ 0 & \text{if} \quad x_A^k < x_B^k. \end{cases}$$

Again, player A chooses x_A^k in order to maximize his expected payoff as given by equation (1). Here, the unique Nash-equilibrium[10] is in mixed strategies (Krumer et al., 2017a)[11]: for $w_A^k - l_A^k \geq w_B^k - l_B^k$, player A and player B randomize on the interval $[0, w_B^k - l_B^k]$ according to their cumulative distribution functions F_i^k such that

$$E_A^k = w_A^k F_B^k(x_A^k) + l_A^k[1 - F_B^k(x_A^k)] - x_A^k = w_A^k - [w_B^k - l_B^k], \tag{5}$$

$$E_B^k = w_B^k F_A^k(x_B^k) + l_B^k[1 - F_A^k(x_B^k)] - x_B^k = l_B^k, \tag{6}$$

yielding

$$F_A^k(x_A^k) = \frac{x_A^k}{w_B^k - l_B^k},$$

$$F_B^k(x_B^k) = \frac{l_B^k - l_A^k + w_A^k - w_B^k + x_B^k}{w_A^k - l_A^k}.$$

The resulting expected efforts equal

$$E[x_A^k] = \frac{w_B^k - l_B^k}{2}, \tag{7}$$

$$E[x_B^k] = \frac{(w_B^k - l_B^k)^2}{2(w_A^k - l_A^k)}, \tag{8}$$

and the resulting equilibrium winning probabilities equal

$$p_B^k = \frac{w_B^k - l_B^k}{2(w_A^k - l_A^k)}, \tag{9}$$

$$p_A^k = 1 - \frac{w_B^k - l_B^k}{2(w_A^k - l_A^k)}. \tag{10}$$

[10] Baye et al. (1996) provide a full characterization of equilibria in all-pay auctions.

[11] Again, for $w_B^k = \ell_B^k$, the optimal choice is $x_B^k = 0$ for any $x_A^k \geq 0$. If $x_B^k = 0$ and $w_A^k > \ell_A^k$, player A will have no best reply unless there is a smallest monetary unit $\epsilon > 0$. As $\epsilon \to 0$, in the limit, $x_A^k \to 0$ and $p_A^k \to 1$.

2.2.2 Tournament Equilibria

Figure 5 illustrates the sequential structure of the tournament, where $\Gamma := \frac{1+a}{3}$ denotes again the players' expected gross payoff in case of a tie. We solve the game by backward induction for its subgame perfect equilibrium, making repeatedly use of equations (5), (6), (7), (8), (9), and (10). For the cases $a = 1/2$ and $a = 1$, the details of this procedure are exemplified in Appendix B.[12] Figure 5 also depicts the winning probabilities in each match for $a = 0$, $a = 1/2$ (black box), and $a = 1$ (blue-grey).

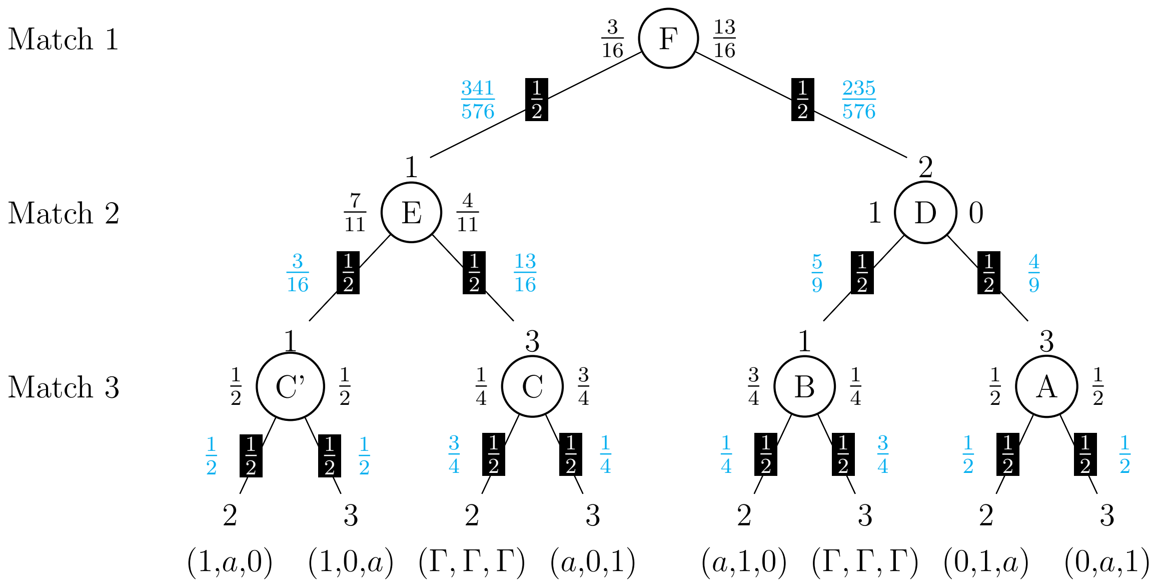


Figure 5: All-pay auction Game Tree

Proposition 2. *If the second prize is valued half of the first prize ($a = 1/2$), two-prize round-robin tournaments with three symmetric players and matches organized as all-pay auctions are perfectly fair:*

(a) In the unique subgame perfect equilibrium all players have identical ex-ante probabilities of ranking first, second, and third, as well as identical ex-ante expected payoffs.

(b) Each single match of the tournament is fair in the sense that the two matching players always have equal winning probabilities and exert the same expected effort $E[x] = 1/4$.

The proof can be found in Appendix B. We have computed the subgame perfect equilibrium of the tournament for all second prizes $a = n/100$ with $n \in \{0, 1, \ldots, 100\}$. The simulation results suggest that $a = 1/2$ is not only sufficient but also necessary for the tournament to be fair, i.e. the only prize structure for which the tournament is non-discriminatory. More precisely, we find the following

Simulation Result 4. *If the second prize is not valued half of the first prize ($a \neq 1/2$), two-prize round-robin tournaments with three symmetric players and matches organized as all-pay auctions are discriminatory. The level of discrimination as measured by the relative standard deviation of the players' ex-ante expected payoffs*

[12]The details of the case $a = 0$ are worked out by Krumer et al. (2017a); for the case $a = 1$, see also Krumer et al. (2017b).

(a) decreases as a function of a *on* $[0, 1/2)$,

(b) increases as a function of a *on* $(1/2, 1]$.

Again, not only the amount but also the direction of discrimination varies with the value of the second prize a. Figure 6 depicts the players' weighted qualification probabilities, the probabilities to rank first, and the probabilities to rank second as functions of a. Similarly, Figure 7 shows expected payoffs, expected efforts, and aggregate effort per unit of prize money as functions of a. Table 3 compares the ranking probabilities for all players, and Table 4 compares weighted qualification probabilities, expected payoffs, and expected efforts for APA-tournaments with $a = 0$, $a = 1/2$, and $a = 1$. We observe the following

Simulation Result 5. *APA-tournaments with three symmetric players and two-prizes* $(0 < a \leq 1)$ *are less discriminatory than single-prize tournaments* $(a = 0)$ *as measured by the relative standard deviation of the players' ex-ante expected payoffs (or, alternatively, weighted qualification probabilities).*

(a) If the valuation of the second prize is higher than one half of the first prize $(1/2 < a \leq 1)$*, the player who competes in the last two matches (player 3) exerts the lowest ex-ante expected effort but has the highest ex-ante expected payoff.*

(b) If the valuation of the second prize is lower than one half of the first prize $(0 \leq a < 1/2)$*, the player who competes in the first and last match (player 2) exerts the highest ex-ante expected effort and has the highest ex-ante expected payoff.*

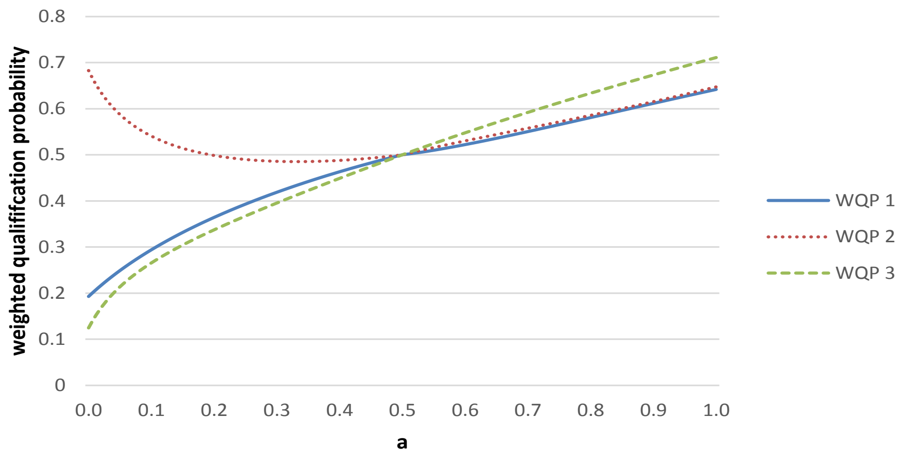
Table 3: All-pay auction tournaments: Winning Probabilities

rank	1^{st}			2^{nd}			3^{rd}		
prize	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$
P1	0.1927	0.3333	0.2879	0.7339	0.3333	0.3538	0.0734	0.3333	0.3582
P2	0.6828	0.3333	0.3242	0.1330	0.3333	0.3231	0.1842	0.3333	0.3527
P3	0.1245	0.3333	0.3878	0.1330	0.3333	0.3231	0.7424	0.3333	0.2891
rel. SD	0.7459	0.0000	0.1239	0.8497	0.0000	0.0439	0.8784	0.0000	0.0941

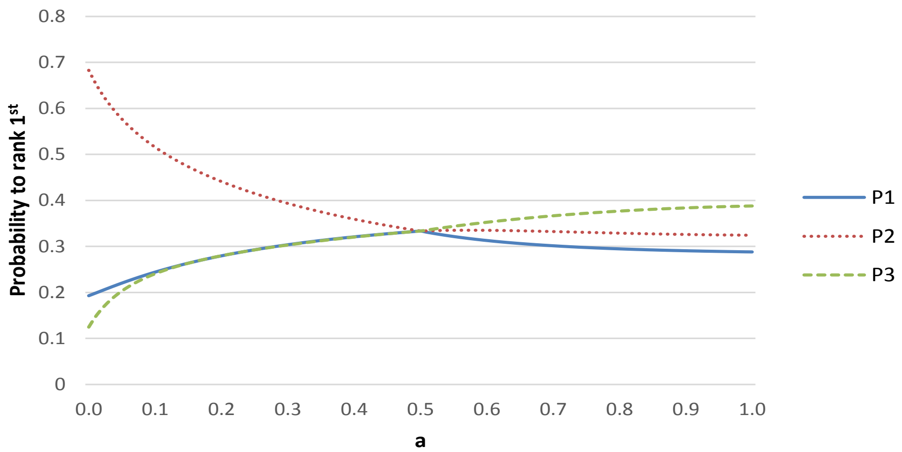
Table 4: Summary: all-pay auction tournaments

	WQP			Expected Payoffs			Expected Effort		
prize	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$	$a = 0$	$a = \frac{1}{2}$	$a = 1$
P1	0.1927	0.5000	0.6418	0.0833	0.0000	0.2060	0.1094	0.5000	0.4357
P2	0.6828	0.5000	0.6473	0.4167	0.0000	0.2708	0.2661	0.5000	0.3765
P3	0.1245	0.5000	0.7109	0.0000	0.0000	0.3827	0.1245	0.5000	0.3282
rel.SD	0.7459	0.0000	0.0471	1.0801	0.0000	0.2547	0.4235	0.0000	0.1156
$\sum$				0.5000	0.0000	0.8595	0.5000	1.5000	1.1404

Weighted Qualification Probability



Probability to rank 1st



Probability to rank 2nd

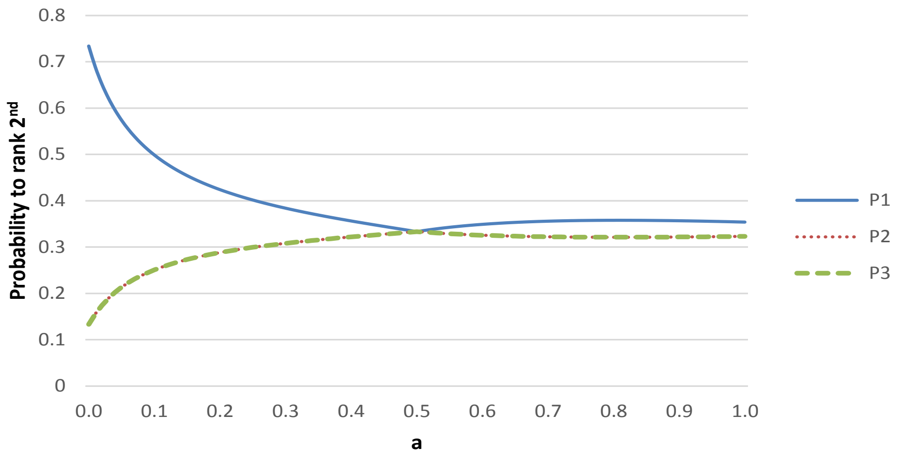


Figure 6: All-Pay Auction: Winning Probabilities

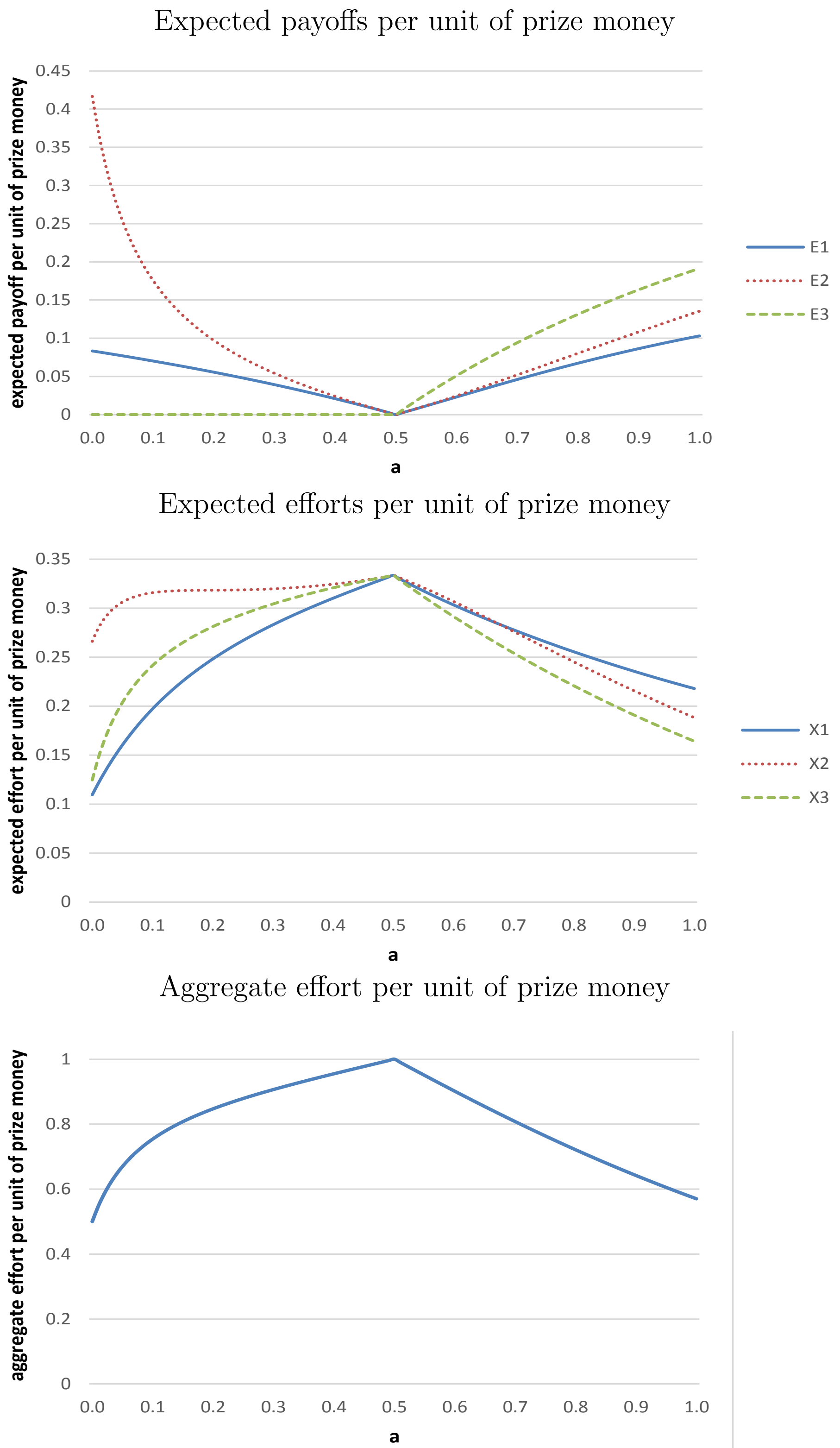


Figure 7: All-Pay Auction: Expected Payoffs and Efforts

As above, the intuition for these results lies in the concurrence of the two opposing effects: while the discouragement effect discriminates against player 3, the lean-back effect increasingly favors player 3 as the second prize rises.

Finally, the question how variations of the second prize a affect aggregate effort per unit of prize money ρ is now answered as follows.

Simulation Result 6. *An increase in the second prize of APA-tournaments with three symmetric players*

(a) strictly increases aggregate expected effort per unit of prize money ($\partial\rho/\partial a > 0$) if $0 \leq a < 1/2$,

(b) strictly decreases aggregate expected effort per unit of prize money ($\partial\rho/\partial a < 0$) if $1/2 < a \leq 1$.

Hence, for APA-tournaments there is no trade-off between fairness and aggregate effort per unit of prize money: the contest designer can always improve both by moving the second prize a towards half of the first prize.

3 Comparison

The previous section has shown that two-prize round-robin tournaments with three symmetric players will be completely fair if the second prize is valued half the first prize, regardless of whether matches are organized as Tullock contests or as all-pay auctions. However, even for $a = 1/2$, aggregate expected effort per unit of prize money is usually distinct. In APA-tournaments aggregate effort peaks at $a = 1/2$ and is equal to the total prize value. In contrast, in Tullock contests with $0 < r < 2$, aggregate effort per unit of prize money will be smaller than 1 if $a = 1/2$ and can be increased by a reduction of the second prize.

In this section, we compare TC-tournaments and APA-tournaments with respect to fairness and aggregate effort provision also for second prizes different from half of the first prize. For the sake of concreteness, we focus again on TC-tournaments with matches organized as simple lotteries, i.e., $r = 1$. We measure the level of discrimination by the relative standard deviation of the players' ex-ante weighted qualification probabilities. Considering identical levels of second prizes $a \neq 1/2$, we find that tournaments with matches organized as lottery contests are less discriminatory than tournaments with matches organized as all-pay auctions.

Simulation Result 7. *For identical values of the second prize $a \neq 1/2$, two-prize round-robin tournaments with three symmetric players and an exogenous sequence of matches are less discriminatory if matches are organized as lottery contests than as all-pay auctions.*

The result follows from our simulations described in the previous section and is illustrated by Figure 8. Figure 9 shows that APA-tournaments generate more aggregate effort per unit of prize money unless the second prize is very small.

Simulation Result 8. *In two-prize round-robin tournaments with three symmetric players, ex-ante expected aggregate effort per unit of prize money is higher in APA-tournaments than LC-tournaments if the second prize exceeds a certain threshold (about 4 percent of the first prize).*

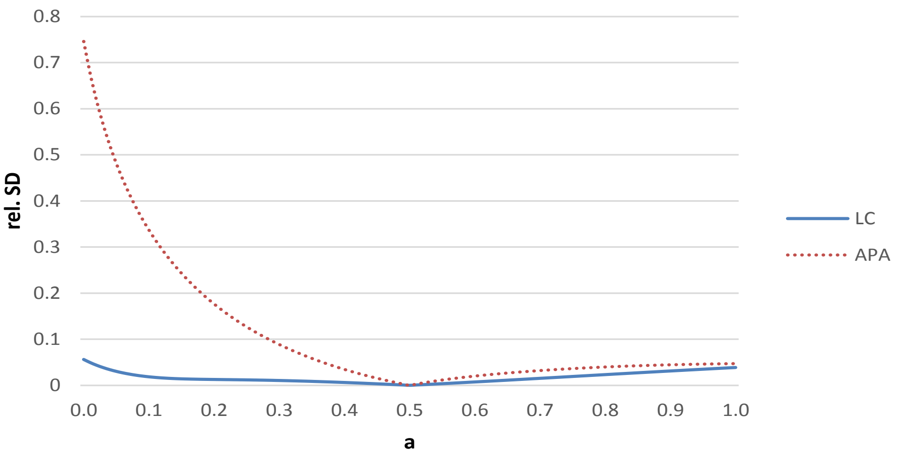


Figure 8: relative standard deviation of WQP

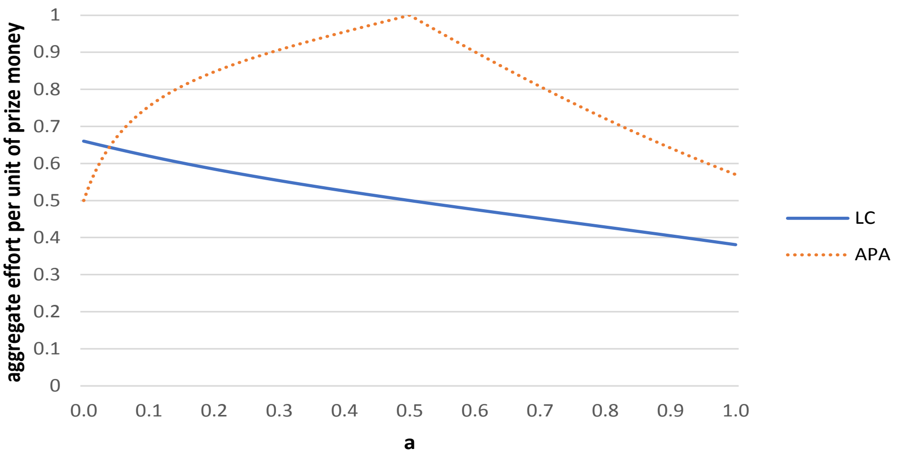


Figure 9: aggregate effort per unit of prize money

4 Discussion

In this section, we discuss two extensions of our model: tournaments with endogenous sequences of matches and tournaments with four instead of three players.

4.1 Endogenous sequence of matches

The idea behind making the tournament fairer by introducing a second prize is to countervail the discouragement effect by an opposing lean-back effect. Sahm (2017) proposes an alternative way of mitigating the discouragement effect by relying on an endogenous sequence of matches. He shows that in single-prize round-robin tournaments with three players fairness increases if player 3 is always matched first with the winner of the first match (WF). Krumer et al. (2017b) consider APA-tournaments with three players and discuss a combination of endogenous sequences and a second prize which equals the first prize ($a = 1$). Besides the WF-structure, they also consider an endogenous sequence in which player 3 is always matched first with the loser of the first match (LF).

In Appendix C, we extend their analysis to arbitrary values of the second prize $0 \leq a \leq 1$. Figures 10 and 11 illustrate our simulation results[13] showing the players' ranking

[13] Again, we have computed the subgame perfect equilibria of the respective tournaments for all second

probabilities and expected payoffs per unit of prize money in both, the WF-structure and the LF-structure, for the LC-tournament and the APA-tournament, respectively. The striking result is that a second prize valued half of the first prize always results in a completely fair tournament. Thus, Propositions 1 and 2 are robust under endogenous sequences of matches as well. Moreover, the simulations suggest that also the tournaments with endogenous sequences of matches are completely fair *only* if the second prize equals half of the first prize ($a = 1/2$).[14]

4.2 Tournaments with four players

As Krumer et al. (2017a) show for APA-tournaments and Sahm (2017) for TC-tournaments, sequential round-robin tournaments with a single prize and *four* players are unfair as well. Therefore, the question whether the fairness of four-player tournaments can also be improved by introducing a second (and maybe third) prize suggests itself.

To shed some light on this question, in Appendix D we consider a round-robin tournament with four players, three prizes and the following exogenous sequence of matches:[15] first player 1 is matched with player 2, second player 3 with player 4, third player 1 with player 3, fourth player 2 with player 4, fifth player 1 with player 4, and sixth player 2 with player 3. The valuation of prizes is common among all players and equals 1 for the first rank, a for the second rank, and b for the third rank with $0 \leq b \leq a \leq 1$. Figure 13 depicts the respective structure where $\Delta := \frac{1+a}{2}$, $\Theta := \frac{a+b}{3}$, and $\Omega := \frac{1+a+b}{3}$.

We have run a large number of simulations (i.e., computed the subgame prefect equilibrium of the tournament for a large number of a-b combinations) suggesting that, in contrast to three-player tournaments, in four-player tournaments there is no prize-structure for which the tournament is completely fair, regardless of whether matches are organized as all-pay auctions or lottery contests.

Figure 12 depicts the players' ex-ante probabilities of ranking first or second as well as their ex-ante expected payoff per unit of prize money for the case with only two prizes ($b = 0$).[16] It exemplifies that there does not exist a totally fair tournament for any value of the second prize a, neither if matches are organized as lottery contests nor as all-pay auctions. However, consistent with our previous findings, for any given value of a the lottery contest is less discriminatory than the all-pay auction. Moreover, the figure illustrates that the extent and direction of discrimination in APA-tournaments with four players is very sensitive to variations of the second prize a.

4.3 Comparison between tournaments with three and four players

To assess whether the FIFA's transition to a three team group stage leads to an improvement in terms of more fairness and intensified competition, we compare the current

prizes $a = n/100$ with $n \in \{0, 1, \ldots, 100\}$.

[14]In the LF-structure of LC-tournaments, there exist $a_1 < a_2 < 1/2$ such that for a second price equal to a_1 (a_2) all players have identical ex-ante expected payoffs (ranking probabilities) but differing ranking probabilities (expected payoffs); moreover, these prize structures fail to provide fairness in each single match of the tournament.

[15]With four players, there are 30 different exogenous sequences of matches (Sahm, 2017). The exogenous sequence we consider here has been referred to as *case A* by Krumer et al. (2017a) and Sahm (2017).

[16]This may seem reasonable for many sports tournaments like the current group stage of the FIFA World Cup where only two teams advance to the next round.

organization of FIFA World Cups and the structure planned from 2026 on.[17] We use the relative standard deviation of the weighted qualification probability as a measure of fairness and the aggregate effort per unit of prize money per match as a measure of intensity.[18]

Figure 14 plots the relative standard deviation of WQP and the aggregate effort per unit of prize money per match as a function of a in LC- and APA-tournaments for three and four players, respectively. Two empirically relevant values of second prizes are highlighted by the vertical lines. The left line corresponds to the average second prize for FIFA World Cups $a_{\text{FIFA}} \approx 0.29$, and the right line corresponds to the combined average second prize of FIFA World Cups and UEFA European Championships $a_{\text{FIFA+UEFA}} \approx 0.47$.[19] The graphs reveal that implications for LC- and APA-tournaments differ only in quantity but not in quality.

In LC-tournaments for both, a_{FIFA} and $a_{\text{FIFA+UEFA}}$, three-player tournaments are fairer compared to the 4-team counterpart as their relative standard deviations of WQP are (at least slightly) smaller. Additionally, match intensity in three-player LC-tournaments is considerably higher than in 4-team tournaments for each value of a: aggregate effort per unit of prize money per match for 4 teams is less than two thirds the one for 3 teams.

Regarding APA-tournaments, 3-team tournaments are substantially fairer in absolute terms than their four-player equivalents for both, a_{FIFA} and $a_{\text{FIFA+UEFA}}$: the relative standard deviation of the WQP is about 5 percentage points higher for both values of a. Moreover, the 3-team APA-tournament is at least double as intense per match as the 4-team APA-tournament for both, a_{FIFA} and $a_{\text{FIFA+UEFA}}$.

To summarize, by a transition to the 3-team group format, the FIFA increases both, fairness and intensity per match, regardless of whether APA-tournaments or LC-tournaments are a better real-world description.

5 Conclusion

We have examined the fairness and intensity of sequential round-robin tournaments with multiple prizes. With three symmetric players and two prizes, the tournament is completely fair if and only if the second prize is valued half of the first prize, regardless of whether matches are organized as Tullock contests or as all-pay auctions. For second prizes different from half of the first prize, three-player tournaments with matches organized as Tullock contests are usually fairer than tournaments with matches organized as all-pay auctions. However, unless the second prize is very small, they are less intense in the sense that players exert less ex-ante expected aggregate effort per unit of prize money.

[17]The reform affects the organization of the first round of the tournament and consists in switching from eight groups with four teams each to 16 groups with three teams each. Moreover, the FIFA seriously discusses the avoidance of draws by penalty shoot-out already in the group stage of the World Cup from 2026 on.

[18]Given that the total number of matches in the group stage is the same before and after the reform ($3 \cdot 16 = 6 \cdot 8$), it seems reasonable that the organizer's objective is to maximize the attractiveness of each single match.

[19]In FIFA World Cups since 1998 and in UEFA European Championships from 1992 to 2012, the winner of each group in the first round has always been paired with the runner-up of another group in the second round. Across all these tournaments, a group winner wins his second round (elimination) match with an average probability of 68.1%. Thus the chances of a runner-up to win his second round (elimination) match are approximately one half the chances of a group winner ($a_{\text{FIFA+UEFA}} = \frac{1-0.682}{0.682} \approx 0.47$). Considering only the respective World Cups, the average winning probability in the second round for a runner-up is even lower with only 22.5% which corresponds to a second prize of about 29% of the first prize.

Moreover, we have specified how the relative size of the second prize influences the extent and the direction of discrimination as well as the intensity of three-player tournaments. For tournaments with matches organized as lottery contests and a second prize below half of the first price, there is a trade-off between fairness and intensity. Such a trade-off does not exist for tournaments with matches organized as all-pay auctions because a second prize that equals half of the first prize then maximizes not only fairness but also ex-ante expected aggregate effort per unit of prize money.

Finally, we have shown that there is no prize structure for which sequential round-robin tournaments with four symmetric players are completely fair in general, though the introduction of a second (and third) prize may reduce the extent of discrimination.

Appendix

A Backward Induction of the Tullock Contest Tournament

A.1 Case $a = 1/2$ (Proof of Proposition 1)

Notice that $a = 1/2$ implies $\Gamma = 1/2$. Let $r \in (0, 2]$.

3rd stage: player 2 vs player 3

In node A, player 2 has won the first match and player 3 has won the second match. Thus $w_2^A = w_3^A = 1$, $\ell_2^A = \ell_3^A = 1/2$, which yields

$$x_2^A = x_3^A = \frac{(1-\frac{1}{2})^{r+1} \cdot (1-\frac{1}{2})^r}{[(1-\frac{1}{2})^r + (1-\frac{1}{2})^r]^2} = \frac{1}{8}r,$$

$p_2^A = p_3^A = \frac{1}{2}$, and $E_2^A = E_3^A = \frac{1}{2} \cdot (1 + \frac{1}{2}) - \frac{1}{8}r = \frac{3}{4} - \frac{1}{8}r$.

In node B, player 2 has won the first match and player 1 has won the second match. Thus $w_2^B = 1$, $w_3^B = 1/2$, $\ell_2^B = 1/2$ and $\ell_3^B = 0$, which yields $x_2^B = x_3^B = r/8$, $p_2^B = p_3^B = 1/2$, $E_2^B = \frac{3}{4} - \frac{1}{8}r$, and $E_3^B = \frac{1}{4} - \frac{1}{8}r$.

In node C, player 1 has won the first match and player 3 has won the second match. Thus $w_2^C = 1/2$, $w_3^C = 1$, $\ell_2^C = 0$, and $\ell_3^C = 1/2$, which yields $x_2^C = x_3^C = r/8$, $p_2^C = p_3^C = 1/2$, $E_2^C = \frac{1}{4} - \frac{1}{8}r$, and $E_3^C = \frac{3}{4} - \frac{1}{8}r$.

In node C', player 1 has won the first match and the second match. Thus $w_2^{C'} = w_3^{C'} = 1/2$ and $\ell_2^{C'} = \ell_3^{C'} = 0$, which yields $x_2^{C'} = x_3^{C'} = r/8$, $p_2^{C'} = p_3^{C'} = 1/2$, and $E_2^{C'} = E_3^{C'} = \frac{1}{4} - \frac{1}{8}r$.

2nd stage: player 1 vs player 3

In node D, player 2 has won the first match. Thus $w_1^D = \frac{1}{2}p_2^B + \frac{1}{2}p_3^B = 1/2$, $w_3^D = E_3^A = \frac{3}{4} - \frac{1}{8}r$, $\ell_1^D = 0$, and $\ell_3^D = E_3^B = \frac{1}{4} - \frac{1}{8}r$, which yields $x_1^D = x_3^D = r/8$, $p_1^D = p_3^D = 1/2$, $E_1^D = \frac{1}{4} - \frac{1}{8}r$, and $E_3^D = \frac{1}{2} - \frac{1}{4}r$.

In node E, player 1 has won the first match. Thus $w_1^E = 1$, $w_3^E = E_3^C = \frac{3}{4} - \frac{1}{8}r$, $\ell_1^E = \frac{1}{2}p_2^C + \frac{1}{2}p_3^C = 1/2$, and $\ell_3^E = E_3^{C'} = \frac{1}{4} - \frac{1}{8}r$, which yields $x_1^E = x_3^E = r/8$, $p_1^E = p_3^E = 1/2$, $E_1^E = \frac{3}{4} - \frac{1}{8}r$, and $E_3^E = \frac{1}{2} - \frac{1}{4}r$.

1st stage: player 1 vs player 2

In node F, $w_1^F = E_1^E = \frac{3}{4} - \frac{1}{8}r$, $w_2^F = p_1^D E_2^B + p_3^D E_2^A = \frac{3}{4} - \frac{1}{8}r$, $\ell_1^F = E_1^D = \frac{1}{4} - \frac{1}{8}r$, and $\ell_2^F = p_1^E E_2^{C'} + p_3^E E_2^C = \frac{1}{4} - \frac{1}{8}r$, which yields $x_1^F = x_2^F = r/8$, $p_1^F = p_2^F = 1/2$, $E_1^F = \frac{1}{2} - \frac{1}{4}r$, and $E_2^F = \frac{1}{2} - \frac{1}{4}r$.

Moreover, player 3's expected payoff equals $E_3^F = p_1^F E_3^E + p_2^F E_3^D = \frac{1}{2} - \frac{1}{4}r$ and the players' ex-ante probabilities of ranking first or second, respectively, are given by

$$P_1^{1^{st}} = p_1^F \big(p_1^E + \frac{1}{3} p_3^E p_2^C\big) + \frac{1}{3} p_2^F p_1^D p_3^B = 1/3$$

and similarly

$$P_2^{1^{st}} = P_3^{1^{st}} = 1/3 \quad \text{as well as} \quad P_1^{2^{nd}} = P_2^{2^{nd}} = P_3^{2^{nd}} = 1/3.$$

Finally, the players' expected total effort is calculated as $E[x_1] = E[x_2] = E[x_3] = r/4$.

A.2 Case $(a, r) = (1, 1)$

Notice that $a = 1$ implies $\Gamma = 2/3$.

3rd stage: player 2 vs player 3

In node A, player 2 has won the first match and player 3 has won the second match. Thus $w_2^A = w_3^A = 1$ and $\ell_2^A = \ell_3^A = 1$. Hence, $x_2^A = x_3^A = 0$, $p_2^A = p_3^A = \frac{1}{2}$, and $E_2^A = E_3^A = 1$.

In node B, player 2 has won the first match and player 1 has won the second match. Thus $w_2^B = 1$, $w_3^B = 2/3$, $\ell_2^B = 2/3$ and $\ell_3^B = 0$, which yields $x_2^B = 2/27$, $x_3^B = 4/27$, $p_2^B = 1/3$, $p_3^B = 2/3$,$E_2^B = 19/27$ and $E_3^B = 8/27$.

In node C, player 1 has won the first match and player 3 has won the second match. Thus $w_2^C = 2/3$, $w_3^C = 1$, $\ell_2^C = 0$, and $\ell_3^C = 2/3$, which yields $x_2^C = 4/27$, $x_3^C = 2/27$, $p_2^C = 2/3$, $p_3^C = 1/3$, $E_2^C = 8/27$, and $E_3^C = 19/27$.

In node C', player 1 has won the first match and player 1 has won the second match. Thus $w_2^{C'} = w_3^{C'} = 1$ and $\ell_2^{C'} = \ell_3^{C'} = 0$, which yields $x_2^{C'} = x_3^{C'} = 1/4$, $p_2^{C'} = p_3^{C'} = 1/2$, and $E_2^{C'} = E_3^{C'} = 1/4$.

2nd stage: player 1 vs player 3

In node D, player 2 has won the first match. Thus $w_1^D = 1p_2^B + \frac{2}{3}p_3^B = 7/9$, $w_3^D = 1$, $\ell_1^D = 0$, and $\ell_3^D = E_3^B = 8/27$, which yields $x_1^D = 931/4800 \approx 0.1940$, $x_3^D = 2527/14400 \approx 0.1755$, $p_1^D = 21/40$, $p_3^D = 19/40$, $E_1^D = 343/1600 \approx 0.2144$, and $E_3^D = 6553/14400 \approx 0.4551$.

In node E, player 1 has won the first match. Thus $w_1^E = 1$, $w_3^E = E_3^C = 19/27$, $\ell_1^E = \frac{2}{3}p_2^C + 1p_3^C = 7/9$, and $\ell_3^E = E_3^{C'} = 1/4$, which yields $x_1^E = 784/15987 \approx 0.0490$, $x_3^E = 4802/47961 \approx 0.1001$, $p_1^E = 24/73$, $p_3^E = 49/73$, $E_1^E = 38455/47961 \approx 0.8018$, and $E_3^E = 65383/143883 \approx 0.4544$.

1st stage: player 1 vs player 2

In node F, $w_1^F = E_1^E = 38455/47961$, $w_2^E = p_1^D E_2^B + p_3^D E_2^A = 38/45$, $\ell_1^F = E_1^D = 343/1600$, and $\ell_2^F = p_1^E E_2^{C'} + p_3^E E_2^C = 554/1971$, which yields

$$\begin{aligned} x_1^F &= \frac{11281496980295116208}{76853928836949940095} \approx 0.1468, \\ x_2^F &= \frac{88927795399770112}{631676127426985809} \approx 0.1408, \\ p_1^F &= \frac{135232131}{264926851} \approx 0.5105, \\ p_2^F &= \frac{129694720}{264926851} \approx 0.4895, \\ E_1^F &= \frac{687142423813067612659}{1870112268365781875645} \approx 0.3674, \\ E_2^F &= \frac{19186967627511130238}{46112357302169964057} \approx 0.4161. \end{aligned}$$

Moreover, player 3's expected payoff equals

$$E_3^F = p_1^F E_3^E + p_2^F E_3^D = \frac{28889786784307}{63530783504055} \approx 0.4547$$

and the players' ex-ante probabilities of ranking first or second, respectively, are given by

$$\begin{aligned} P_1^{1^{st}} &= p_1^F \big(p_1^E + \frac{1}{3} p_3^E p_2^C\big) + \frac{1}{3} p_2^F p_1^D p_3^B \approx 0.3011, \\ P_2^{1^{st}} &= \frac{1}{3} p_1^F p_3^E p_2^C + p_2^F p_1^D \big(p_2^B + \frac{1}{3} p_3^B\big) + p_2^F p_3^D p_2^A \approx 0.3352, \\ P_3^{1^{st}} &= p_1^F p_3^E \big(\frac{1}{3} p_2^C + p_2^B\big) + p_2^F \big(\frac{1}{3} p_1^D p_3^B + p_3^D p_2^A\big) \approx 0.3637. \end{aligned}$$

$$\begin{aligned} P_1^{2^{nd}} &= p_1^F p_3^E \big(\frac{1}{3} p_2^C + p_3^C\big) + p_2^F p_1^D \big(p_2^B + \frac{1}{3} p_3^B\big) \approx 0.3331, \\ P_2^{2^{nd}} &= \frac{1}{3} p_1^F p_3^E p_2^C + p_2^F p_1^D \big(p_2^B + \frac{1}{3} p_3^B\big) + p_2^F p_3^D p_2^A \approx 0.3334, \\ P_3^{2^{nd}} &= p_1^F p_3^E \big(\frac{1}{3} p_2^C + p_2^B\big) + p_2^F \big(\frac{1}{3} p_1^D p_3^B + p_3^D p_2^A\big) \approx 0.3334. \end{aligned}$$

Finally, the players' expected total effort is calculated as

$$\begin{aligned}
E[x_1] &= \left(P_1^{1^{st}} + P_1^{2^{nd}}\right) - E_1 \approx 0.2668, \\
E[x_2] &= \left(P_2^{1^{st}} + P_2^{2^{nd}}\right) - E_2 \approx 0.2525, \\
E[x_3] &= \left(P_3^{1^{st}} + P_3^{2^{nd}}\right) - E_3 \approx 0.2425.
\end{aligned}$$

B Backward Induction of the All-pay Auction Tournament

B.1 Case $a = 1/2$ (Proof of Proposition 2)

Notice that $a = 1/2$ implies $\Gamma = 1/2$.

3rd stage: player 2 vs player 3

In node A, $w_2^A = w_3^A = 1$, $\ell_2^A = \ell_3^A = 1/2$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_2^A = E_3^A = 1/2$, $E[x_2^A] = E[x_3^A] = 1/4$, and $p_2^A = p_3^A = 1/2$.

In node B, $w_2^B = 1$, $w_3^B = 1/2$, $\ell_2^B = 1/2$ and $\ell_3^B = 0$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_2^B = 1/2$, $E_3^B = 0$, $E[x_2^B] = E[x_3^B] = 1/4$, and $p_2^B = p_3^B = 1/2$.

In node C, $w_2^C = 1/2$, $w_3^C = 1$, $\ell_2^C = 0$, and $\ell_3^C = 1/2$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_2^C = 0$, $E_3^C = 1/2$, $E[x_2^C] = E[x_3^C] = 1/4$, and $p_2^C = p_3^C = 1/2$, .

In node C', $w_2^{C'} = w_3^{C'} = 1/2$ and $\ell_2^{C'} = \ell_3^{C'} = 0$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_2^{C'} = E_3^{C'} = 0$, $E[x_2^{C'}] = E[x_3^{C'}] = 1/4$, and $p_2^{C'} = p_3^{C'} = 1/2$.

2nd stage: player 1 vs player 3

In node D, $w_1^D = w_3^D = 1/2$ and $\ell_1^D = \ell_3^D = 0$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_1^D = E_3^D = 0$, $E[x_1^D] = E[x_3^D] = 1/4$, and $p_1^D = p_3^D = 1/2$.

In node E, $w_1^E = 1$, $w_3^E = 1/2$, $\ell_1^E = 1/2$, and $\ell_3^E = 0$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_1^E = 1/2$, $E_3^E = 0$, $E[x_1^E] = E[x_3^E] = 1/4$, and $p_1^E = p_3^E = 1/2$.

1st stage: player 1 vs player 3

In node F, $w_1^F = w_2^F = 1/2$ and $\ell_1^F = \ell_2^F = 0$, which implies that the players randomize on the interval $[0, 1/2]$. This yields $E_1^F = E_2^F = 0$, $E[x_1^F] = E[x_2^F] = 1/4$, and $p_1^F = p_2^F = 1/2$. Moreover, player 3's expected payoff equals $E_3^F = p_1^F E_3^E + p_2^F E_3^D = 0$ and the players' ex-ante ranking probabilities are given by

$$P_1^{1^{st}} = p_1^F\left(p_1^E + \frac{1}{3}p_3^E p_2^C\right) + \frac{1}{3}p_2^F p_1^D p_3^B = 1/3$$

and similarly

$$P_2^{1^{st}} = P_3^{1^{st}} = 1/3 \quad \text{as well as} \quad P_1^{2^{nd}} = P_2^{2^{nd}} = P_3^{2^{nd}} = 1/3.$$

Finally, the players' expected total effort is calculated as $E[x_1] = E[x_2] = E[x_3] = 1/2$.

B.2 Case $a = 1$ (see also Krumer et al. (2017b))

Notice that $a = 1$ implies $\Gamma = 2/3$.

3rd stage: player 2 vs player 3

In node A, $w_2^A = 1$, $w_3^A = 1$, $\ell_2^A = 1$ and $\ell_3^A = 1$, which implies that the players choose $x_2^A = x_3^A = 0$. This yields $E_2^A = 1$, $E_3^A = 1$, $p_2^A = 1/2$, and $p_3^A = 1/2$.

In node B, $w_2^B = 1$, $w_3^B = 2/3$, $\ell_2^B = 2/3$ and $\ell_3^B = 0$, which implies that the players randomize on the interval $[0, 1/3]$. This yields $E_2^B = 2/3$, $E_3^B = 1/3$, $E[x_2^B] = 1/12$, $E[x_3^B] = 1/6$, $p_2^B = 1/4$, and $p_3^B = 3/4$.

In node C, $w_2^C = 2/3$, $w_3^C = 1$, $\ell_2^C = 0$, and $\ell_3^C = 2/3$, which implies that the players randomize on the interval $[0, 1/3]$. This yields $E_2^C = 1/3$, $E_3^C = 2/3$, $E[x_2^C] = 1/6$, $E[x_3^C] = 1/12$, $p_2^C = 3/4$, and $p_3^C = 1/4$.

In node C', $w_2^{C'} = w_3^{C'} = 1$ and $\ell_2^{C'} = \ell_3^{C'} = 0$, which implies that the players randomize on the interval $[0, 1]$. This yields $E_2^{C'} = E_3^{C'} = 0$, $E[x_2^{C'}] = E[x_3^{C'}] = 1/2$, and $p_2^{C'} = p_3^{C'} = 1/2$.

2nd stage: player 1 vs player 3

In node D, $w_1^D = 1p_2^B + \frac{2}{3}p_3^B = 3/4$, $w_3^D = 1$, $\ell_1^D = 0$, and $\ell_3^D = E_3^B = 1/3$, which implies that the players randomize on the interval $[0, 2/3]$. This yields $E_1^D = 1/12$, $E_3^D = 1/3$, $E[x_1^D] = 1/3$, $E[x_3^D] = 8/27$, $p_1^D = 5/9$, and $p_3^D = 4/9$.

In node E, $w_1^E = 1$, $w_3^E = E_3^C = 2/3$, $\ell_1^E = \frac{2}{3}p_2^C + 1p_3^C = 3/4$, and $\ell_3^E = E_3^{C'} = 0$, which implies that the players randomize on the interval $[0, 1/4]$. This yields $E_1^E = 3/4$, $E_3^E = 5/12$, $E[x_1^E] = 3/64$, $E[x_3^E] = 1/8$, $p_1^E = 3/16$, and $p_3^E = 13/16$.

1st stage: player 1 vs player 3

In node F, $w_1^F = E_1^E = 3/4$, $w_2^E = p_1^D E_2^B + p_3^D E_2^A = 22/27$, $\ell_1^F = E_1^D = 1/12$, and $\ell_2^F = p_1^E E_2^{C'} + p_3^E E_2^C = 13/48$, which implies that the players randomize on the interval $[0, 235/432]$. This yields $E_1^F = 89/432 \approx 0.2060$, $E_2^F = 13/48 \approx 0.2708$, $E[x_1^F] = 235/864 \approx 0.2720$, $E[x_2^F] = 55225/248832 \approx 0.2219$, $p_1^F = 341/576 \approx 0.5920$, and $p_2^F = 235/576 \approx 0.4080$. Moreover, player 3's expected payoff equals $E_3^F = p_1^F E_3^E + p_2^F E_3^D = 2645/6912 \approx 0.3785$ and the players' ex-ante ranking probabilities are given by

$$
\begin{aligned}
P_1^{1^{st}} &= p_1^F \big(p_1^E + \frac{1}{3}p_3^E p_2^C\big) + \frac{1}{3}p_2^F p_1^D p_3^B \approx 0.2879, \\
P_2^{1^{st}} &= \frac{1}{3}p_1^F p_3^E p_2^C + p_2^F p_1^D \big(p_2^B + \frac{1}{3}p_3^B\big) + p_2^F p_3^D p_2^A \approx 0.3242, \\
P_3^{1^{st}} &= p_1^F p_3^E \big(\frac{1}{3}p_2^C + p_2^B\big) + p_2^F \big(\frac{1}{3}p_1^D p_3^B + p_3^D p_2^A\big) \approx 0.3878.
\end{aligned}
$$

$$
\begin{aligned}
P_1^{2^{nd}} &= p_1^F p_3^E \big(\frac{1}{3} p_2^C + p_3^C\big) + p_2^F p_1^D \big(p_2^B + \frac{1}{3} p_3^B\big) \approx 0.3538, \\
P_2^{2^{nd}} &= \frac{1}{3} p_1^F p_3^E p_2^C + p_2^F p_1^D \big(p_2^B + \frac{1}{3} p_3^B\big) + p_2^F p_3^D p_2^A \approx 0.3231, \\
P_3^{2^{nd}} &= p_1^F p_3^E \big(\frac{1}{3} p_2^C + p_2^B\big) + p_2^F \big(\frac{1}{3} p_1^D p_3^B + p_3^D p_2^A\big) \approx 0.3231.
\end{aligned}
$$

Finally, the players' expected total effort is calculated as

$$
\begin{aligned}
E[x_1] &= \big(P_1^{1^{st}} + P_1^{2^{nd}}\big) - E_1 \approx 0.4358, \\
E[x_2] &= \big(P_2^{1^{st}} + P_2^{2^{nd}}\big) - E_2 \approx 0.3765, \\
E[x_3] &= \big(P_3^{1^{st}} + P_3^{2^{nd}}\big) - E_3 \approx 0.3324.
\end{aligned}
$$

C Three-player-Tournaments with Endogenous Sequences

C.1 The winner-first structure (WF)

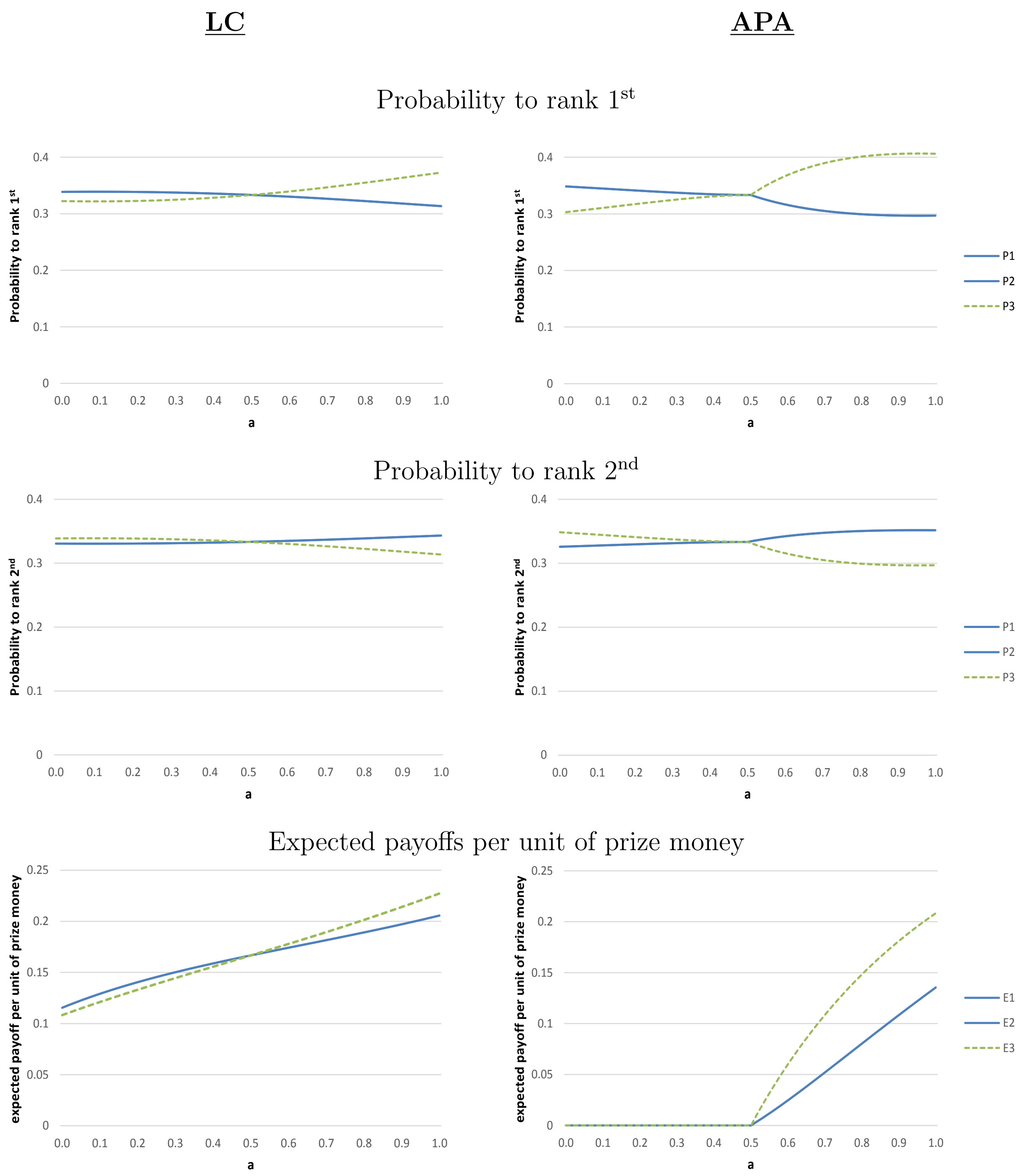


Figure 10: Winner First: LC vs APA

C.2 The loser-first structure (LF)

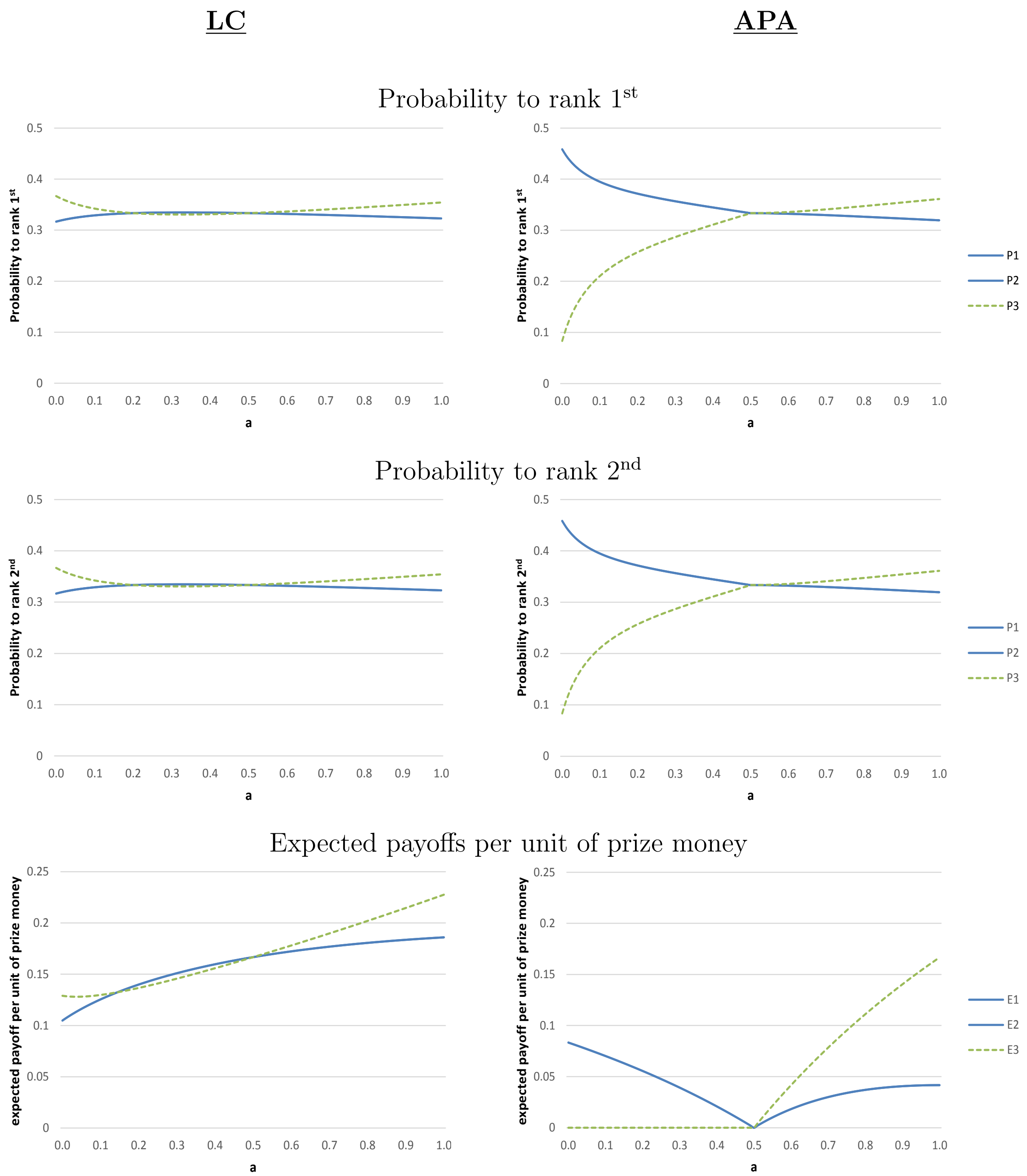


Figure 11: Loser First: LC vs APA

D Four-Player-Tournaments

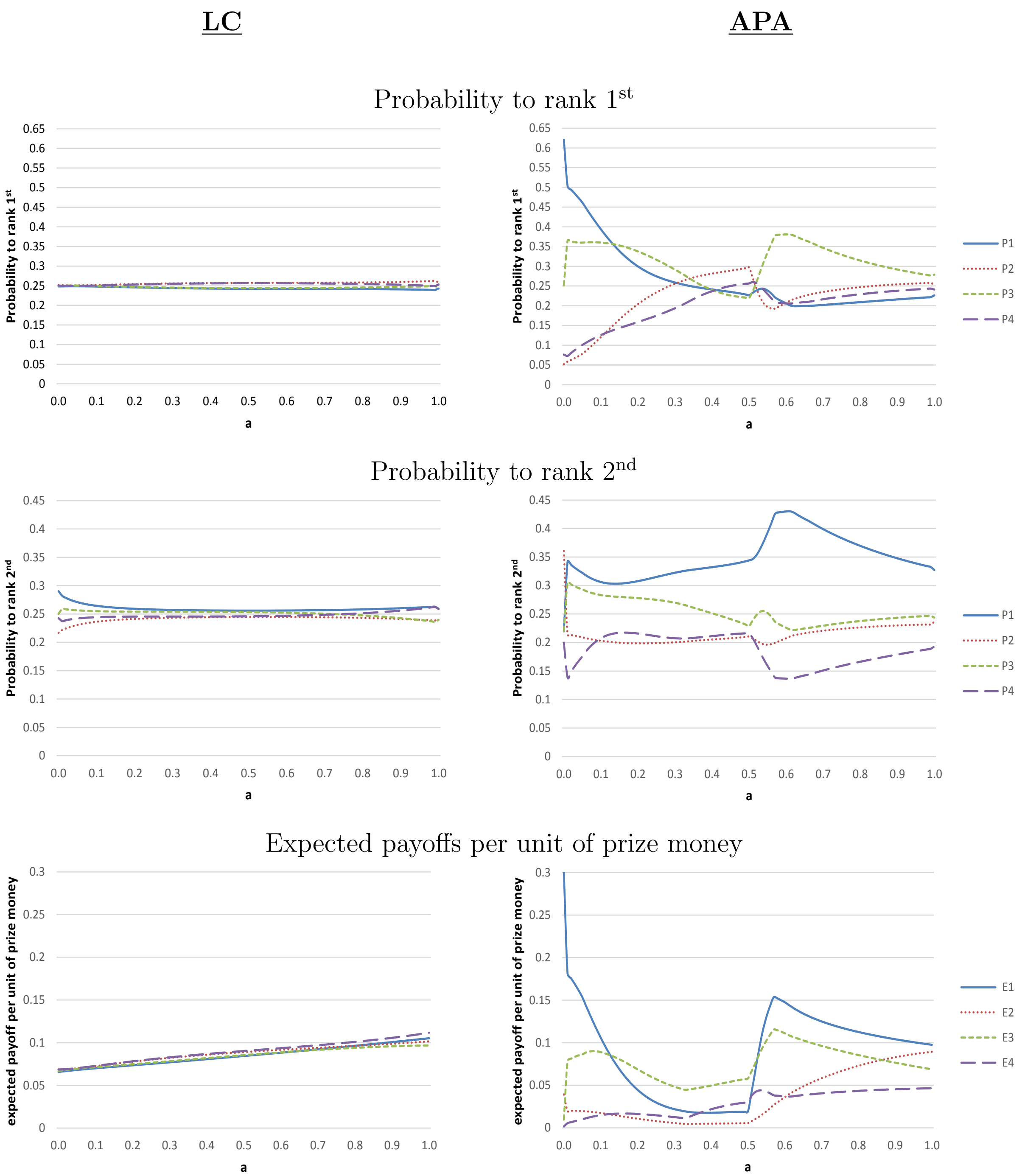


Figure 12: Four-player round-robin tournament with $b = 0$

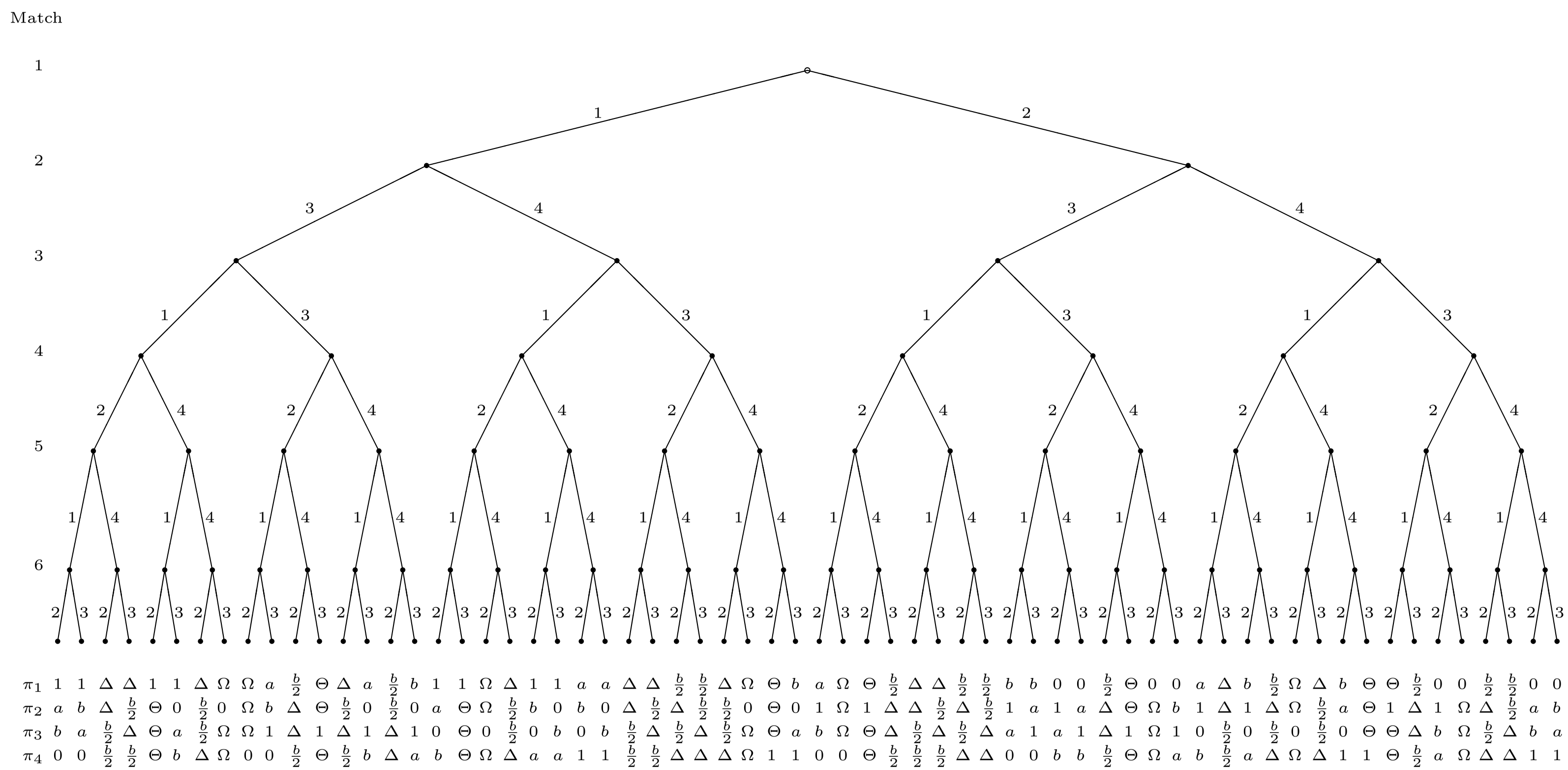


Figure 13: Four-player round-robin tournament with exogenous sequence: 1-2,3-4,1-3,2-4,1-4,2-3

E Comparison: Three- vs. Four-Player-Tournaments

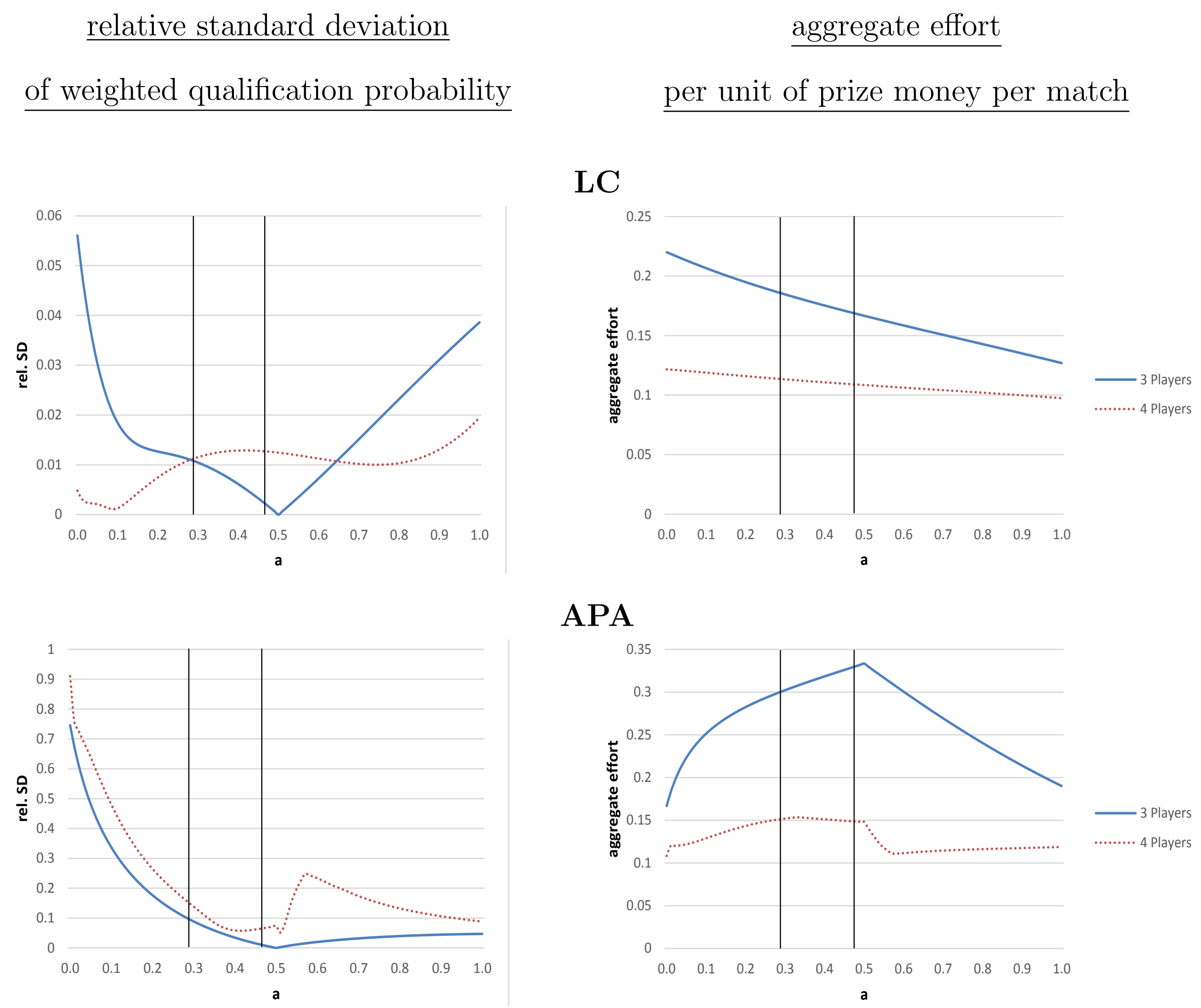


Figure 14: Fairness and Intensity in Three- vs. Four-Player Tournaments

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