## Session 27: Fractals - Handout


"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

Benoit Mandelbrot (1924-2010)

Geometry literally means earth measure. (Geo is from the Greek $\gamma \eta$ meaning earth, and metry is from the Greek $\mu \eta \tau \rho o \nu$.) Euclid's famous book The Elements contains a logical development of geometry, and is said to be second only to the Bible in number of editions published. It was written in about 300 BC and, until well into the twentieth century (at which time its contents were incorporated into standard textbooks) it was considered something that all well-educated people read.

If we look around us we see the shapes of Euclidean geometry in hexagonal honeycombs, in polygonal rock formations, in our spherical planet, and also its elliptical orbit.


No wonder geometry, earth measure, was so named! In this section, however, we'll be looking at fractal geometry rather than the smooth, regular shapes of Euclidean geometry, with its circles, spheres, squares, and polygons. The very name fractal means fragmented and jagged.


The shapes in the image above are fractals. They don't look much like standard Euclidean geometrical objects. When these shapes were first being discovered and explored many mathematicians referred to them as pathological monsters because of their counter-intuitive properties. Some of the ways in which these shapes are different from classical geometrical shapes is that they can have dimensions that are between whole numbers, they can have finite area but infinite perimeter, and they can be continuous everywhere but nowhere differentiable (a concern in calculus).

Given that these fractal shapes are so different from classical geometrical shapes, given that they have such unusual properties, and given that they don't model the natural shapes we saw on the previous page, how can they be classified as part of 'geometry' (earth measure) at all? Along with that question we need to consider another question, however. Are the smooth images of polygons, a sphere and ellipses on the previous page truly representative of most natural shapes we encounter?

Is earth and the universe truly made up of circles, polygons, spheres, and ellipses? What about trees and mountain peaks; clouds and seaweed; peacock feathers and seahorse tails; forests, arteries and coastlines? (Consider the images on the next page.) It turns out that the rough, jagged shapes of fractals are precisely what we need to model most shapes in the natural world and for modeling natural process as well including such things as weather systems and fluid flow.

So what are fractals? Mathematical fractals are shapes that display self-similarity, shapes that continue to display detail at all levels of magnification, shapes that are the result of an infinite process of iteration, and shapes that can have dimensionality that is not a whole number. Nothing in the natural world displays the infinite detail that a fractal does, but then again nothing in the natural world is a perfect circle either. Just as we can use circles to model such things as ripples in a still pond, we can use fractals to model shapes in the
natural world as well.


How are fractal images created? They are the result of infinite iteration. Consider the image below; it started out at stage 0 as an equilateral triangle. Then a central inverted triangle ${ }^{1}$ was removed. This leaves 3 new equilateral triangles, and the process is repeated on each of them - leaving, then, 9 new smaller triangles on which to repeat the process.


The above image is known as the Sierpinski Gasket. Actually, what we see is a pseudofractal because we don't have the actual fractal until this process has been repeated infinitely many times. Although we cannot physically carry that out, mathematics is powerful enough to give us information about the actual shape as if we were able to take it to an infinite number of iterations.

Another famous fractal is the Hilbert Curve, which is shown on the next page. It is slightly more complicated to generate than the Sierpinski Gasket. For the Hilbert Curve we start with three line segments as shown in stage 0 . Then we shrink ${ }^{2}$ this shape to half its original size, and we copy it four times. Two of the copies, the two shown at the top of

[^0]stage 1, are left in their original orientation, and the other two copies are both rotated $90^{\circ}$ as shown in the figure. These four smaller copies of the original are then connected by the three line segments shown in grey.


Notice that in stage 0 , if we think of the square grid as being cut into four quadrants, each of the four vertices of the object would be at the midpoint of one of the quadrants. Consider in stage 1 the sixteen small squares that make up the grid, and notice that each vertex of the curve is at the midpoint of one of these squares. So one way to think of the Hilbert Curve is as a path that is traveling through all the midpoints within a square grid and letting the squares become smaller and smaller until every point in the square is a midpoint of an infinitely fine grid.

Below is an image of further iterations of the Hilbert Curve. Can you see that each stage contains 4 half-sized copies of the image at the previous stage, two of which are in their original orientation and two of which have been rotated $90^{\circ}$ ?


The Hilbert Curve is just that, a mathematical curve. It is made of line segments, and line segments are one-dimensional. But if we iterate this infinitely many times, then every midpoint of every grid square, no matter how fine the grid becomes, is covered. In other words the entire two-dimensional square becomes filled in. So, what is the dimensionality of the Hilbert Curve? It is 1 or 2 or somewhere in between, and is that even possible? ${ }^{3}$

Another curve that is interesting to consider in terms of dimension is the Cantor Set. It begins as a straight line. The middle third of this line is removed, leaving two lines. Each of these lines has its middle third removed, and the process continues. The completed Cantor Set has no length. It is made up of a fine dusting of infinitely many points. It begins as a one-dimensional object and ends as infinitely many zero-dimensional objects. So what is its dimension? 0? 1? Something between 0 and 1?

[^1]

What is dimension? It is not as easy a concept as it may seem. In fact, there are many different definitions of dimension, each with its own area of application. What typically comes to mind is the type of dimension learned in high school, Euclidean dimension, which has to do with degrees of freedom or how many coordinates are necessary to locate a point in the given shape. There is also topological dimension, Hausdorff dimension, MinkowskiBouligand dimension, and, what we'll be using in considering fractals, self-similarity dimension.
$\qquad$


$$
3^{1}=3
$$

$$
3^{2}=9
$$



$$
3^{3}=27
$$

The equations to the right of the segment, square and cube represent the number of pieces an edge was cut into ( 3 in each case) and the number of smaller copies of the original that resulted from this cutting (3, 9 and 27). Notice that the exponents needed to make these equations true happen to be the dimension of each of these shapes. The equation for finding dimension is as follows:

$$
E^{D}=C \text { or alternatively } D=\frac{\log C}{\log D}
$$

Where $E$ is the number of pieces each edge has been cut into, $C$ is the number of smaller copies of the original that result from this cutting, and $D$ is the dimension of the object.


Consider again the Sierpinski Gasket. It begins as a 2-dimensional triangle, but once the process of removal is iterated infinitely many times there are only line segments left, no area is left at all, and yet the shape still has width and height to it. Though it has no area, its perimeter is infinite. What is its dimension? Notice that each edge has been cut into 2 pieces and that the result (at each stage) is one triangle become three smaller triangles. Using the formula above we get

$$
2^{D}=3 \text { which is } D=\frac{\log 3}{\log 2} \approx 1.584962501
$$

The self-similarity dimension falls between 1 (a line segment) and 2 (a polygon). If you haven't worked with fractal dimension before this may seem strange, but self-similarity dimensions tells us something different from what Euclidean dimension tells us. Self-similarity dimension tells us how a shape fills space. For instance, consider the coastlines of South Africa and of Norway. South Africa's coastline is very smooth, but the


South Africa


Norway
coastline of Norway, with all its fjords, is very jagged. The self-similarity dimension of South Africa's smooth coast is 1.02 , and the self-similarity dimension of Norway's coast is 1.52; the higher the number, the rougher the shape. The fractal shapes we have been working with have been perfectly self-similar, that is a small piece can be scaled up to exactly match the original shape. Coastlines are self-similar, but not perfectly so. The process for finding the dimension of coastlines and other natural objects is somewhat more complicated than what we are doing with our fractal shapes, but not a lot more complicated.

It is possible to generate shapes that look more natural than the fractals we have considered so far. We have only looked at one method of constructing fractals - that of using an initiator and generator (also called MRCM), but fractals can be generated in many ways. The image below is a Randomized Sierpinski Gasket. The only difference between the generation of this image and the Sierpinski Gasket we saw earlier is that rather than taking midpoints of triangles, points that will form the new vertices are chosen randomly in a given radius around the midpoint.


The fractal known as the Koch Curve, shown below, can be randomized (by flipping a coin

to determine which direction each new triangle faces) to give a shape that models a coastline. Its dimension is about 1.26, somewhere between the dimensions of the coastlines

for South Africa and Norway, and very close to the dimension of the coastline of Great Britain.

There are also non-linear fractals. The Mandelbrot Set, part of which is seen below, is a non-linear fractal. This fractal is generated using an equation rather than a geometrical shape. It is graphed on the complex plane.


There is much to be said about fractal geometry. Hopefully this gives you a good idea of some of the basics. A couple of excellent and very accessible books on the topic are $A n$ Eye for Fractals: A Graphic and Photographic Essay by Michael McGuire and Fractals: The Patterns of Chaos by John Briggs. Another book that contains an excellent introduction to fractal geometry is The Mathematical Tourist: Snapshots of Modern Mathematics by Ivars Peterson (chapters 5 and 6). If you want to dig a lot deeper check out Fractals, Chaos, and Power Laws: Minutes from an Infinite Paradise by Manfred Schroeder or Chaos and Fractals: New Frontiers of Science by Peitgen, Jürgens and Saupe.


[^0]:    ${ }^{1}$ This central triangle was created by finding the mid-points of each side of the original triangle and connecting those mid-points.
    ${ }^{2}$ Notice that we can think of the Sierpinski Gasket as being created by a process of shrinking and copying also. We can see the triangle in stage 0 as having each of its sides shrunk by a factor of two and then copying the smaller triangle three times and placing the copies in the manner shown in stage 1 . This way of thinking of the process is known as MRCM: Multiple Reduction Copy Machine.

[^1]:    ${ }^{3}$ While in the process of writing this section I visited a friend who was finishing up mowing her huge lawn on her riding lawn-mower. She curves back and forth as she works her away across the lawn. I asked how long it takes to complete the job, 3 hours! That's as long as it takes me to drive to and from my favorite campground from home! It just hit me what an interesting thing it is to cover area with a curve!

