CHAPTER 2

Sets, Functions, Relations

2.1. Set Theory

2.1.1. Sets. A *set* is a collection of objects, called *elements* of the set. A set can be represented by listing its elements between braces: $A = \{1, 2, 3, 4, 5\}$. The symbol $\in$ is used to express that an element is (or belongs to) a set, for instance $3 \in A$. Its negation is represented by $\notin$, e.g. $7 \notin A$. If the set is finite, its number of elements is represented $|A|$, e.g. if $A = \{1, 2, 3, 4, 5\}$ then $|A| = 5$.

Some important sets are the following:

1. $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ = the set of natural numbers.[1]
2. $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ = the set of integers.
3. $\mathbb{Q}$ = the set of rational numbers.
4. $\mathbb{R}$ = the set of real numbers.
5. $\mathbb{C}$ = the set of complex numbers.

Is S is one of those sets then we also use the following notations:[2]

1. S^+ = set of positive elements in S, for instance
$$Z^+ = \{1, 2, 3, \cdots\} = \text{ the set of positive integers.}$$
2. S^- = set of negative elements in S, for instance
$$\mathbb{Z}^- = \{-1, -2, -3, \cdots\} = \text{ the set of negative integers.}$$
3. S^* = set of elements in S excluding zero, for instance $\mathbb{R}^*$ = the set of non zero real numbers.

Set-builder notation. An alternative way to define a set, called *set-builder notation*, is by stating a property (predicate) $P(x)$ verified by exactly its elements, for instance $A = \{x \in \mathbb{Z} \mid 1 \le x \le 5\}$ = "set of

[1] Note that $\mathbb{N}$ includes zero—for some authors $\mathbb{N} = \{1, 2, 3, \cdots\}$, without zero.

[2] When working with *strings* we will use a similar notation with a different meaning—be careful not to confuse it.

integers x such that $1 \leq x \leq 5$"—i.e.: $A = \{1, 2, 3, 4, 5\}$. In general: $A = \{x \in \mathcal{U} \mid p(x)\}$, where $\mathcal{U}$ is the universe of discourse in which the predicate $P(x)$ must be interpreted, or $A = \{x \mid P(x)\}$ if the universe of discourse for $P(x)$ is implicitly understood. In set theory the term *universal set* is often used in place of "universe of discourse" for a given predicate.[3]

Principle of Extension. Two sets are *equal* if and only if they have the same elements, i.e.:

$$A = B \equiv \forall x\,(x \in A \leftrightarrow x \in B)\,.$$

Subset. We say that A is a *subset* of set B, or A is *contained* in B, and we represent it "$A \subseteq B$", if all elements of A are in B, e.g., if $A = \{a, b, c\}$ and $B = \{a, b, c, d, e\}$ then $A \subseteq B$.

A is a *proper subset* of B, represented "$A \subset B$", if $A \subseteq B$ but $A \neq B$, i.e., there is some element in B which is not in A.

Empty Set. A set with no elements is called *empty set* (or *null set*, or *void set*), and is represented by $\emptyset$ or $\{\}$.

Note that nothing prevents a set from possibly being an element of another set (which is not the same as being a subset!). For instance if $A = \{1, a, \{3, t\}, \{1, 2, 3\}\}$ and $B = \{3, t\}$, then obviously B is an element of A, i.e., $B \in A$.

Power Set. The collection of all subsets of a set A is called the *power set* of A, and is represented $\mathcal{P}(A)$. For instance, if $A = \{1, 2, 3\}$, then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, A\}\,.$$

Exercise: Prove by induction that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Multisets. Two ordinary sets are identical if they have the same elements, so for instance, $\{a, a, b\}$ and $\{a, b\}$ are the same set because they have exactly the same elements, namely a and b. However, in some applications it might be useful to allow repeated elements in a set. In that case we use *multisets*, which are mathematical entities similar to sets, but with possibly repeated elements. So, as multisets, $\{a, a, b\}$ and $\{a, b\}$ would be considered different, since in the first one the element a occurs twice and in the second one it occurs only once.

[3] Properly speaking, the universe of discourse of set theory is the collection of all sets (which is not a set).

2.1.2. Venn Diagrams. Venn diagrams are graphic representations of sets as enclosed areas in the plane. For instance, in figure 2.1, the rectangle represents the universal set (the set of all elements considered in a given problem) and the shaded region represents a set A. The other figures represent various set operations.

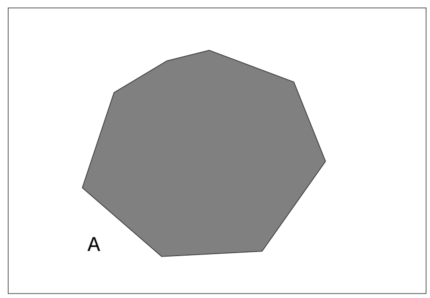

FIGURE 2.1. Venn Diagram.

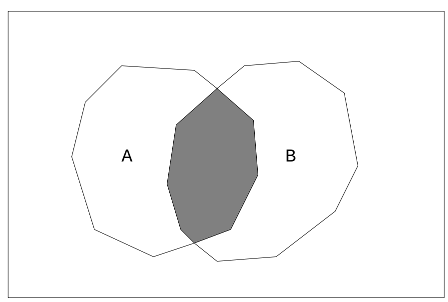

FIGURE 2.2. Intersection $A \cap B$.

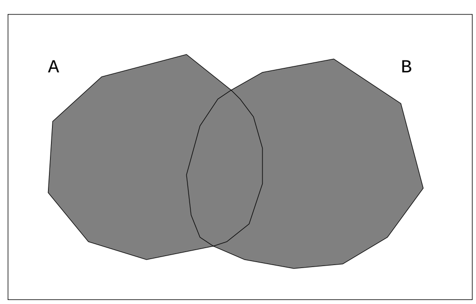

FIGURE 2.3. Union $A \cup B$.

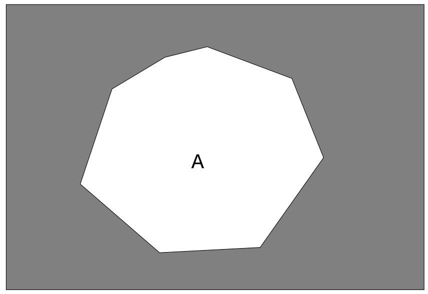

FIGURE 2.4. Complement $\overline{A}$.

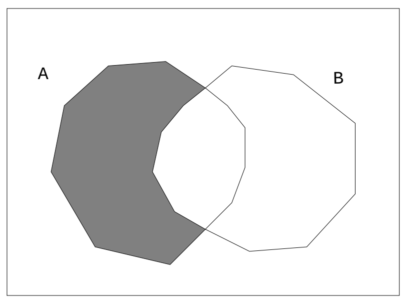

FIGURE 2.5. Difference $A - B$.

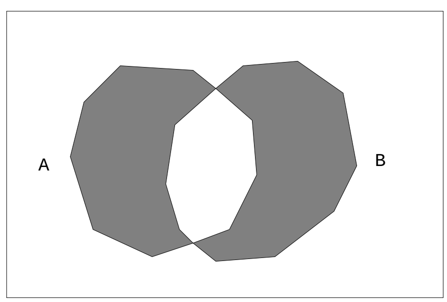

FIGURE 2.6. Symmetric Difference $A \oplus B$.

2.1.3. Set Operations.

1. *Intersection*: The common elements of two sets:
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$
If $A \cap B = \emptyset$, the sets are said to be *disjoint*.

2. *Union*: The set of elements that belong to either of two sets:
$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

3. *Complement*: The set of elements (in the universal set) that do not belong to a given set:
$$\overline{A} = \{x \in \mathcal{U} \mid x \notin A\}\,.$$

4. *Difference* or *Relative Complement*: The set of elements that belong to a set but not to another:
$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\} = A \cap \overline{B}\,.$$

5. *Symmetric Difference*: Given two sets, their symmetric difference is the set of elements that belong to either one or the other set but not both.
$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}\,.$$
It can be expressed also in the following way:
$$A \oplus B = A \cup B - A \cap B = (A - B) \cup (B - A)\,.$$

2.1.4. Counting with Venn Diagrams. A Venn diagram with n sets intersecting in the most general way divides the plane into 2^n regions. If we have information about the number of elements of some portions of the diagram, then we can find the number of elements in each of the regions and use that information for obtaining the number of elements in other portions of the plane.

Example: Let M, P and C be the sets of students taking Mathematics courses, Physics courses and Computer Science courses respectively in a university. Assume $|M| = 300$, $|P| = 350$, $|C| = 450$, $|M \cap P| = 100$, $|M \cap C| = 150$, $|P \cap C| = 75$, $|M \cap P \cap C| = 10$. How many students are taking exactly one of those courses? (fig. 2.7)

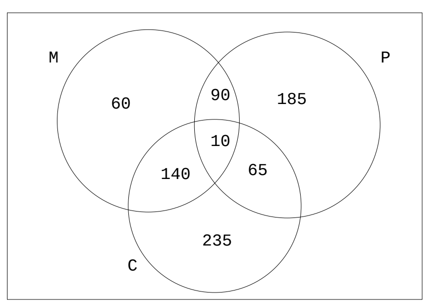

FIGURE 2.7. Counting with Venn diagrams.

We see that $|(M\cap P)-(M\cap P\cap C)| = 100-10 = 90$, $|(M\cap C)-(M\cap P\cap C)| = 150-10 = 140$ and $|(P\cap C)-(M\cap P\cap C)| = 75-10 = 65$.

Then the region corresponding to students taking Mathematics courses only has cardinality $300-(90+10+140) = 60$. Analogously we compute the number of students taking Physics courses only (185) and taking Computer Science courses only (235). The sum $60 + 185 + 235 = 480$ is the number of students taking exactly one of those courses.

2.1.5. Properties of Sets. The set operations verify the following properties:

1. *Associative Laws:*
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$
2. *Commutative Laws:*
$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$
3. *Distributive Laws:*
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
4. *Identity Laws:*
$$A \cup \emptyset = A$$
$$A \cap \mathfrak{U} = A$$
5. *Complement Laws:*
$$A \cup \overline{A} = \mathfrak{U}$$
$$A \cap \overline{A} = \emptyset$$
6. *Idempotent Laws:*
$$A \cup A = A$$
$$A \cap A = A$$
7. *Bound Laws:*
$$A \cup \mathfrak{U} = \mathfrak{U}$$
$$A \cap \emptyset = \emptyset$$
8. *Absorption Laws:*
$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$
9. *Involution Law:*
$$\overline{\overline{A}} = A$$

10. 0/1 *Laws:*

$$\overline{\emptyset} = \mathcal{U}$$
$$\overline{\mathcal{U}} = \emptyset$$

11. *DeMorgan's Laws:*

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

2.1.6. Generalized Union and Intersection. Given a collection of sets $A_1, A_2, \dots, A_N$, their union is defined as the set of elements that belong to at least one of the sets (here n represents an integer in the range from 1 to N):

$$\bigcup_{n=1}^{N} A_n = A_1 \cup A_2 \cup \cdots \cup A_N = \{x \mid \exists n\,(x \in A_n)\}\,.$$

Analogously, their intersection is the set of elements that belong to all the sets simultaneously:

$$\bigcap_{n=1}^{N} A_n = A_1 \cap A_2 \cap \cdots \cap A_N = \{x \mid \forall n\,(x \in A_n)\}\,.$$

These definitions can be applied to infinite collections of sets as well. For instance assume that $S_n = \{kn \mid k = 2, 3, 4, \dots\}$ = set of multiples of n greater than n. Then

$$\begin{aligned}\bigcup_{n=2}^{\infty} S_n = S_2 \cup S_3 \cup S_4 \cup \cdots &= \{4, 6, 8, 9, 10, 12, 14, 15, \dots\} \\ &= \text{set of composite positive integers}\,.\end{aligned}$$

2.1.7. Partitions. A *partition* of a set X is a collection $\mathcal{S}$ of non overlapping non empty subsets of X whose union is the whole X. For instance a partition of $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ could be

$$\mathcal{S} = \{\{1, 2, 4, 8\}, \{3, 6\}, \{5, 7, 9, 10\}\}\,.$$

Given a partition $\mathcal{S}$ of a set X, every element of X belongs to exactly one member of $\mathcal{S}$.

Example: The division of the integers $\mathbb{Z}$ into even and odd numbers is a partition: $\mathcal{S} = \{\mathbb{E}, \mathbb{O}\}$, where $\mathbb{E} = \{2n \mid n \in \mathbb{Z}\}$, $\mathbb{O} = \{2n + 1 \mid n \in \mathbb{Z}\}$.

Example: The divisions of $\mathbb{Z}$ in negative integers, positive integers and zero is a partition: $\mathcal{S} = \{\mathbb{Z}^+, Z^-, \{0\}\}$.

2.1.8. Ordered Pairs, Cartesian Product. An ordinary pair $\{a, b\}$ is a set with two elements. In a set the order of the elements is irrelevant, so $\{a, b\} = \{b, a\}$. If the order of the elements is relevant, then we use a different object called *ordered pair*, represented (a, b). Now $(a, b) \neq (b, a)$ (unless $a = b$). In general $(a, b) = (a', b')$ iff $a = a'$ and $b = b'$.

Given two sets A, B, their *Cartesian product* $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) \mid (a \in A) \wedge (b \in B)\} .$$

Analogously we can define triples or 3-tuples (a, b, c), 4-tuples (a, b, c, d), $\dots$, n-tuples $(a_1, a_2, \dots, a_n)$, and the corresponding 3-fold, 4-fold,$\dots$, n-fold Cartesian products:

$$A_1 \times A_2 \times \cdots \times A_n = \\ \{(a_1, a_2, \dots, a_n) \mid (a_1 \in A_1) \wedge (a_2 \in A_2) \wedge \cdots \wedge (a_n \in A_n)\} .$$

If all the sets in a Cartesian product are the same, then we can use an exponent: $A^2 = A \times A$, $A^3 = A \times A \times A$, etc. In general:

$$A^n = A \times A \times \overset{(n \text{ times})}{\cdots} \times A .$$

An example of Cartesian product is the *real plane* $\mathbb{R}^2$, where $\mathbb{R}$ is the set of real numbers ($\mathbb{R}$ is sometimes called *real line*).