Set Theory: A First Course

Set theory is a rich and beautiful subject whose fundamental concepts permeate virtually every branch of mathematics. One could say that set theory is a unifying theory for mathematics, since nearly all mathematical concepts and results can be formalized within set theory.

This textbook is meant for an upper undergraduate course in set theory. In this text, the fundamentals of abstract sets, including relations, functions, the natural numbers, order, cardinality, transfinite recursion, the axiom of choice, ordinal numbers, and cardinal numbers, are developed within the framework of axiomatic set theory.

The reader will need to be comfortable reading and writing mathematical proofs. The proofs in this textbook are rigorous, clear, and complete, while remaining accessible to undergraduates who are new to upper-level mathematics. Exercises are included at the end of each section in a chapter, with useful suggestions for the more challenging exercises.

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Set Theory

A FIRST COURSE

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PREFACE

Set theory is a rich and beautiful subject whose fundamental concepts permeate virtually every branch of mathematics. Yet, most mathematics students receive only a cursory overview of the theory of sets in their lower division courses. Set theory is a subject that is sufficiently important and interesting to merit its own undergraduate course. The book is intended to offer the reader, or department, just such a course.

This introductory textbook delivers a one-semester course in undergraduate set theory at the upper-division level. Sets, of course, are the central objects that shall be investigated. Since essentially every significant concept in modern mathematics can be defined in terms of sets, students who possess a fairly deep understanding of set theory will find a first course in abstract algebra or real analysis much less formidable and, perhaps, much more accessible. My first undergraduate course in set theory, for example, gave me a definite advantage over other students in my ensuing mathematics courses.

In this book, the fundamental facts about abstract sets–relations, functions, natural numbers, order, cardinality, transfinite recursion, the axiom of choice, ordinal numbers, and cardinal numbers–are covered and developed within the framework of axiomatic set theory. Mathematicians have shown that virtually all mathematical concepts and results can be formalized within set theory. This has been recognized as one of the greatest achievements of modern mathematics, and, consequently, one can now say that "set theory is a unifying theory for mathematics."

The textbook is suitable for a broad range of readers, from undergraduate to graduate students, who desire a better understanding of the fundamental topics in set theory that may have been, or will be, overlooked in their other mathematics courses. I have made an effort to write clear and complete proofs throughout the text.

Many modern books in undergraduate set theory are written for a reader who is well versed in mathematical argument and proof. My primary goal was to produce a book that would be accessible to a relatively unsophisticated reader. Nevertheless, I have composed completely rigorous proofs. In addition, these proofs favor detail over brevity. Another goal was to write a book that is focused on those topics that a student is likely to see in advanced courses (including graduate courses). Thus, the book is comparatively concise and can be covered in one semester. Most other undergraduate set theory texts cannot possibly be covered in a semester.

On the Origins of Set Theory

In the nineteenth century, mathematicians were investigating problems that concerned functions in real analysis. Some of these problems involved the representation of a function by a trigonometric series and the convergence of such series. The young Georg Cantor then proved a uniqueness theorem for trigonometric series. Cantor, in his proof, introduced a process for constructing collections of real numbers that involved an infinite iteration of the limit operation. Cantor's novel proof led him to a deeper investigation of sets of real numbers and then to the concept of transfinite numbers.

It has been said that set theory, as a subject in mathematics, was created on the day that Georg Cantor proved that the set of real numbers is uncountable, that is, there is no one-to-one correspondence between the real numbers and the natural numbers. In the two decades following this proof, Cantor developed a fascinating theory of ordinal and cardinal numbers.

Cantor's revolutionary research raised many difficult issues and paradoxes. As the axiomatic method had assumed an important role in mathematics, Ernst Zermelo developed an axiomatic system for set theory and published the first axiomatization of set theory in 1908. Zermelo's axioms resolved the difficulties introduced by Cantor's development of set theory. After receiving some proposed revisions from Abraham Fraenkel, in 1930 Zermelo presented his final axiomatization of set theory, now known as the Zermelo–Fraenkel axioms.

Topics Covered

The book presents the axioms of Zermelo–Fraenkel set theory and then uses these axioms to derive a variety of theorems concerning functions, relations, ordering, the natural numbers, and other core topics in mathematics. These axioms shall also be used to investigate infinite sets and to generalize the concepts of induction and recursion.

The Zermelo–Fraenkel axioms are now generally accepted as the standard foundation for set theory and for mathematics. On the other hand, there are textbooks that introduce set theory using a naive point of view. As naive set theory[1] is known to be inconsistent, such a theory cannot seriously be offered as a foundation for mathematics or set theory.

The basics of logic and elementary set theory are first discussed in Chapter 1. Since students, typically, easily grasp the topics covered in this chapter, it can be covered at a fairly quick pace. The chapter ends with a brief overview of the Zermelo–Fraenkel axioms.

Chapter 2 examines the first six axioms of Zermelo–Fraenkel set theory and begins proving theorems, justified by the axioms, about sets. It is also shown that the set operations discussed in Chapter 1 can be derived from these six axioms.

A relation is defined as a set of ordered pairs in Chapter 3, where equivalence relations and induced partitions are also discussed. Then a precise set-theoretic definition of a function in terms of ordered pairs is presented. The chapter ends with a section on order relations and a section on the concepts of congruence and preorder.

In Chapter 4, after representing the natural numbers as sets, the fundamental principles of number theory (for example, proof by mathematical induction) are derived from a few very basic notions involving sets.

Cantor's early work on the "size of infinite sets" is the subject of Chapter 5. In particular, we explore the notion of a one-to-one correspondence between two sets and the concept of a countable set.

Well-ordered sets and transfinite recursion are examined in Chapter 6. In Chapter 7, it is shown that the axiom of choice implies Zorn's Lemma, the Ultrafilter Theorem, and the Well-Ordering Theorem. The theory of ordinals is carefully developed in Chapter 8, and the theory of cardinals is presented in Chapter 9. The last section of Chapter 9 discusses closed unbounded sets and stationary sets.

How to Use the Book

It is strongly recommended that the reader be acquainted with the basics of sets, functions, relations, logic, and mathematical induction. These topics are typically introduced in a "techniques of proof" course (for example, see [1]), and they will be more seriously discussed in this book. As the emphasis will be on theorems and their proofs, the reader should be comfortable reading and writing mathematical proofs. For example, one should know how to prove that a function is one-to-one and to prove that one set is a subset of another set.

In the book, naturally, there is a progressive increase in complexity. The first five chapters cover the important topics that every mathematics undergraduate should know about the theory of sets, including the Recursion Theorem and the Schröder–Bernstein Theorem. The last four chapters present more challenging material, beginning in Chapter 6 with the principle of transfinite recursion. When applying definition by transfinite recursion there are two functional forms that may be used: a set function or a class function. In both of these cases, we prove that one can construct a new function by recursion. The two proofs are similar, except that the class form requires the replacement axiom.

Moreover, the class version implies the set version. Students will likely find these technical proofs relatively difficult to follow or appreciate; however, they should be assured that such recursive definitions are valid and that one can read the rest of the book without possessing a deep understanding of these proofs.

If time is short, certain topics can be overlooked. For example, the following sections can be skipped without loss of continuity:

- 3.5 Congruence and Preorder
- 4.2.1 The Peano Postulates
- 7.2 Filters and Ultrafilters
- 8.4 The Cumulative Hierarchy
- 9.3 Closed Unbounded and Stationary Sets.

Furthermore, the proofs presented in Chapter 7 of Zorn's Lemma 7.1.1 and of the Well-Ordering Theorem 7.3.1 do not appeal to transfinite recursion on the ordinals; however, such proofs are carefully outlined in Exercises 14 and 15, respectively, beginning on page 193 of Chapter 8. Therefore, in Chapter 7, one could just highlight the proofs of 7.1.1 and 7.3.1 and then, after completing Section 8.2, assign these shorter "ordinal recursion" proofs as exercises.

The symbol Ⓢ marks the end of a solution, and the symbol □ identifies the end of a proof. Exercises are given at the end of each section in a chapter. An exercise marked with an asterisk $*$ is one that is cited, or referenced, elsewhere in the book. Suggestions are also provided for those exercises that a newcomer to upper-division mathematics may find more challenging.

Acknowledgments

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THE GREEK ALPHABET

A	α		alpha
B	β		beta
Γ	γ		gamma
Δ	δ		delta
E	ϵ	ε	epsilon
Z	ζ		zeta
H	η		eta
Θ	θ	ϑ	theta
I	ι		iota
K	κ		kappa
Λ	λ		lambda
M	μ		mu
N	ν		nu
O	o		omicron
Ξ	ξ		xi
Π	π		pi
P	ρ		rho
Σ	σ		sigma
T	τ		tau
Υ	υ		upsilon
Φ	ϕ	φ	phi
X	χ		chi
Ψ	ψ		psi
Ω	ω		omega