Shearer and Levy: Partial Differential Equations – Solutions

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Forward

The following is a collection of my solutions for Michael Shearer and Rachel Levy's text *Partial Differential Equations: An Introduction to Theory and Applications*. These solutions were worked out over the summer of 2017, and will almost certainly contain errors. If you happen to find any, or have suggestions for more elegant/interesting/general approaches to problems, please drop me a line at hstufflebeam@utexas.edu. The figures and diagrams were made with Mathematica.

As an update, I have become to privy to the existence of an errata sheet for the text, which explains some of the funkiness of certain questions. I have adjusted the problems here to represent what is written on the errata sheet, when necessary.

Contents

1 Introduction

1.1

Show that the traveling wave $u(x,t) = f(x-3t)$ satisfies the linear transport equation $u_t + 3u_x = 0$ for any differentiable function $f\colon \mathbb{R} \to \mathbb{R}$.

Solution. This is obvious: if f is differentiable then $u_t = -3f'(x-3t)$ and $u_x = f'(x-3t)$. Hence $u_t + 3u_x = -3f'(x-3t) + 3f'(x-3t) = 0$. $\square$

1.2

Find an equation relating the parameters k, m, n so that the function $u(x,t) = e^{mt}\sin(nx)$ satisfies the heat equation $u_t = ku_{xx}$.

Solution. We have $u_t = mu(x,t)$ and $u_{xx} = -n^2u(x,t)$. Hence, u will satisfy the heat equation above provided that $m = -kn^2$. □

1.3

Find an equation relating the parameters c, m, n so that the function $u(x,t) = \sin(mt)\sin(nx)$ satisfies the wave equation $u_{tt} = c^2u_{xx}$.

Solution. We have $u_{tt} = -m^2u(x,t)$ and $u_{xx} = -n^2u(x,t)$. Hence, u will satisfy the wave equation above provided that $m^2 = c^2n^2$. □

1.4

Find all functions $a, b, c\colon \mathbb{R} \to \mathbb{R}$ such that $u(x,t) = a(t)e^{2x} + b(t)e^x + c(t)$ satisfies the heat equation $u_t = u_{xx}$ for all x, t.

Solution. We have $u_t = a'(t)e^{2x} + b'(t)e^x + c'(t)$ and $u_{xx} = 4a(t)e^{2x} + b(t)e^x$. If u satisfies the heat equation above, then $4a(t) = a'(t)$, $b(t) = b'(t)$, and $c'(t) = 0$. As such, we conclude that a, b, c in general have the forms $a(t) = \lambda_1 e^{4t}$, $b(t) = \lambda_2 e^x$, and $c(t) = \lambda_3$ for $\lambda_i \in \mathbb{R}$. □

1.5

For $m > 1$, define the **conductivity** $k = k(u)$ so that the porous medium equation $u_t = \nabla^2(u^m)$ can be written as the quasilinear heat equation $u_t = \nabla \cdot (k(u)\nabla u)$

Solution. We first recall that, in general, if $f, g\colon \mathbb{R}^m \to \mathbb{R}$ are differentiable, then

$$\nabla \cdot (g\nabla f) = \langle \nabla g, \nabla f\rangle + g\nabla^2 f.$$

Define $k = k(u) = mu^{m-1}$. Then we have

$$\begin{aligned} u_t = \nabla^2(u^m) = \sum_{i=1}^n (u^m)_{x_ix_i} &= \sum_{i=1}^n (mu^{m-1}u_{x_i})_{x_i} \\ &= \sum_{i=1}^n \{m(m-1)u^{m-2}u_{x_i}^2 + mu^{m-1}u_{x_ix_i}\} \\ &= m(m-1)u^{m-2}\|\nabla u\|^2 + mu^{m-1}\nabla^2 u \end{aligned}$$

On the other hand, by the first remark

$$\nabla \cdot (k(u)\nabla u) = \langle \nabla k(u), \nabla u\rangle + k(u)\nabla^2 u$$

and since

$$\langle \nabla k(u), \nabla u\rangle = \langle \nabla mu^{m-1}, \nabla u\rangle = m(m-1)u^{m-2}\langle \nabla u, \nabla u\rangle = m(m-1)u^{m-2}\|\nabla u\|^2$$

we see that indeed $u_t = \nabla \cdot (k(u)\nabla u)$ with the conductivity $k(u) = mu^{m-1}$.

□

1.6

Solve the initial value problem

$$\begin{aligned} u_t + 4u_x &= 1, && -\infty < x < +\infty, \quad t > 0, \\ u(x,0) &= (1+x^2)^{-1}, && -\infty < x < +\infty. \end{aligned}$$

Solution. The general form of the solution is $u(x,t) = f(x-4t)$ for some differentiable $f\colon \mathbb{R} \to \mathbb{R}$. By the initial data, we have $f(x) = \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$. Hence, the particular solution of the IVP is

$$u(x,t) = \frac{1}{1+(x-4t)^2}.$$

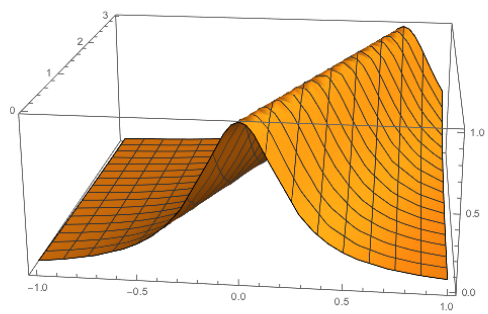


Figure 1: $u(x,t)$

$\square$

1.7

Solve the initial boundary value problem

$$\begin{aligned} u_t + 4u_x &= 0, && 0 < x < +\infty, \quad t > 0, \\ u(x,0) &= 0, && 0 < x < +\infty, \\ u(0,t) &= te^{-t}, && t > 0. \end{aligned}$$

Why is there no solution if the PDE is changed to $u_t - 4u_x = 0$?

Solution. The general solution is again of the form $u(x,t) = f(x-4t)$ for some differentiable function $f\colon \mathbb{R} \to \mathbb{R}$. By the first boundary condition we have that $f(x) = 0$ whenever $t = 0$ and $x > 0$. By the second boundary condition we have $f(-4t) = te^{-t}$ whenever $x = 0$ and $t > 0$. When $x - 4t = 0$, both imply that f vanishes. From the above, we gather that

$$f(\xi) = \begin{cases} 0 & \xi \geqslant 0 \\ -\frac{\xi}{4} e^{\frac{\xi}{4}} & \xi < 0 \end{cases}$$

which implies that the solution u of the BVP is

$$u(x,t) = \begin{cases} 0 & x \geqslant 4y \\ -\frac{x-4t}{4} e^{\frac{x-4t}{4}} & x < 4t \end{cases}$$

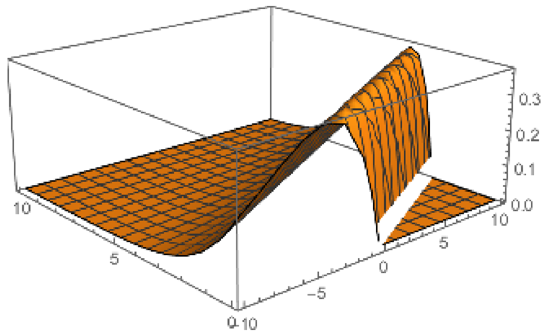


Figure 2: $u(x,t)$ Note that Mathematica is having some trouble on the line $x = 4t$

Suppose, on the other hand, that the PDE were changed to $u_t - 4u_x = 0$. Then the general form of the solution would be $u(x,t) = f(x+4t)$, and the boundary conditions would give contradictory data, e.g. $u(x,0) = f(x) = 0$ for all $x > 0$ at the same time that $u(0,t) = f(4t) = te^{-t} > 0$ when $t > 0$. □

1.8

Consider the linear transport equation $u_t + cu_x = 0$ with initial and boundary conditions

$$\begin{aligned} u(x,0) &= \phi(x), \qquad && \text{if } \ x \geqslant 0, \\ u(0,t) &= \psi(t), \qquad && \text{if } \ t \geqslant 0. \end{aligned}$$

where $\phi, \psi \colon [0, +\infty)$ are differentiable.

(a) Suppose the data ϕ, ψ are differentiable functions. Show that the function $u \colon Q_1 \to \mathbb{R}$ given by

$$u(x,t) = \begin{cases} \phi(x-ct) & x \geqslant ct, \\ \psi(t - \frac{x}{c}) & x \leqslant ct \end{cases}$$

satisfies the PDE away from the line $x = ct$, the boundary condition, and the initial condition.

(b) In the solution above, the line $x = ct$ which emerges from the origin $x = t = 0$ separates the quadrant Q_1 into two regions. On the line, the solution has one-sided limits given by ϕ, ψ. Consequently, the solution will in general have singularities on the line.

(i) Find conditions on the data ϕ, ψ so that the solution is continuous across the line $x = ct$.

(ii) Find conditions on the data ϕ, ψ so that the solution is differentiable across the line $x = ct$.

Solution. (a) We have

$$u_t = \begin{cases} -c\phi(x-ct) & x \geqslant ct \\ \psi(t - \frac{x}{c}) & x \leqslant ct \end{cases}$$

and

$$u_x = \begin{cases} \phi(x-ct) & x \geqslant ct \\ -\frac{1}{c}\psi(t - \frac{x}{c}) & x \leqslant ct \end{cases}$$

hence

$$u_t + cu_x = \begin{cases} -c\phi(x-ct) + c\phi(x-ct) = 0 & x > ct \\ \psi(t - \frac{x}{c}) - \psi(t - \frac{x}{c}) = 0 & x < ct \end{cases}$$

provided that we are also away from the boundary of Q_1.

(b) (i) This seems to always be the case, since by the boundary conditions we get that $\phi(0) = u(0, 0) = \psi(0)$ (and finite!), which is exactly what we need to know to conclude that u is continuous across the line $x = ct$. Indeed, if $((x_k, t_k))$ is any sequence in Q_1 which tends to a point (ξ, τ) on the line $x = ct$, then we can say the following: u takes the value $u_0 = \phi(0) = \psi(0)$ along the line $x = ct$, so it suffices to show that for any $\varepsilon > 0$ we can find a $K \in \mathbb{Z}^{\geq 1}$ such that $u(x_k, t_k) \in \mathcal{B}(u_0; \varepsilon)$ for all $k \geqslant K$. This is true, however, given the fact that both $\phi(x)$ and $\psi(t)$ are continuous. We can simply choose K so large such that all points of the sequence after time K lie in a ball about (ξ, τ) small enough so that $\phi(x_k - ct_k) \in \mathcal{B}(u_0 = \phi(0); \varepsilon)$ (in the case when $x_k \geqslant ct_k$) and $\psi(t_k - \frac{x_k}{c}) \in \mathcal{B}(u_0 = \psi(0); \varepsilon)$ (in the case when $x \leqslant ct_k$) for all $k \geqslant K$. Hence, $u(x_k, t_k) \to u_0$, so u is continuous across the line $x = ct$.

(ii) If we want u to be differentiable across the line $x = ct$, then we must have $-c\phi'(0) = \psi'(0)$. Indeed, this comes from the results above, since if $-c\phi'(0) = \psi'(0)$, we can unambiguously define u_t and u_x on the line $x = ct$. If you look closely at the piecewise definitions of u_t and u_x above, you will see that we actually get this condition for free as well, so as stated the solution is differentiable over the line $x = ct$.

As an aside, it may be possible that the intended problem involved NOT having data about how ϕ and ψ behave at 0. If this is the case, then the conditions $\phi(0) = \psi(0)$ and $-c\phi'(0) = \psi'(0)$ are not had for free, and must be imposed to ensure continuity and differentiability, respectively, over the line $x = ct$. □

1.9

Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Verify that if $u(x, t)$ is differentiable and satisfies $u = f(x - ut)$, then $u(x, t)$ is a solution of the initial value problem

$$\begin{aligned} u_t + uu_x &= 0, && -\infty < x < +\infty, \quad t > 0 \\ u(x, 0) &= f(x), && -\infty < x < +\infty. \end{aligned}$$

Solution. It is immediately evident that $u(x, 0) = f(x)$ for all x. We then calculate $u_t = -(u + u_t t) f'(x - ut)$ and $u_x = (1 - u_x t) f'(x - ut)$, from which we gather

$$\begin{aligned} u_t + uu_x &= -(u + u_t t) f'(x - ut) + u(1 - u_x t) f'(x - ut) \\ &= -u_t t f'(x - ut) - uu_x t f'(x - ut) \\ &= -(u_t + uu_x) t f'(x - ut) \end{aligned}$$

which gives us the relation

$$(u_t + uu_x)(1 + t f'(x - ut)) = 0.$$

Since $\mathbb{R}$ is a field one of these factors must be zero; we show that in fact $u_t + uu_x$ always vanishes, which proves that u is a solution of the IVP. By inspection, the only thing we need to be worried about is the possibility of having $f'(x - ut) = -\frac{1}{t}$ for some positive time. Recalling that $u_x = (1 - u_x t) f'(x - ut)$, if for some positive time t^* it were true that $f'(x - ut^*) = -\frac{1}{t^*}$, then we would have $u_x|_{t=t^*} = -\frac{1}{t^*} + u_x|_{t=t^*}$ which yields the absurdity $0 = -\frac{1}{t^*}$. Thus, $f'(x - ut) \neq -\frac{1}{t}$ for all positive time, and we conclude that $u_t + uu_x = 0$ for all time $t > 0$. Hence u as described is a solution of the IVP.

□

1.10

Let

$$u_0 = \begin{cases} 1 - x^2 & -1 \leqslant x \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Use the equation $u = u_0(x - ut)$ to find a formula for the solution $u = u(x,t)$ of the inviscid Burgers equation $u_t + uu_x = 0$ with $-1 < x < 1$ and $0 < t < \frac{1}{2}$.

(b) Verify that $u(1,t) = 0$ for all $0 < t < \frac{1}{2}$

(c) Differentiate the formula to find $u_x(1^-,t)$, and deduce that $u_x(1^-,t) \to -\infty$ as $t \to \frac{1}{2}^-$.

Note that $u_x(x,t)$ is discontinuous at $x = \pm 1$.

Solution. (a)

$$u = u_0(x - ut) = \begin{cases} 1 - (x - ut)^2 & -1 \leqslant x - ut \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

Provided that $x - ut \in [-1, 1]$, then, we can write $u = 1 - (x - ut)^2 = 1 - x^2 + 2uxt - u^2t^2$. Rearranging, we find that

$$t^2u^2 + (1 - 2xt)u + x^2 - 1 = 0.$$

By the quadratic formula,

$$u(x,t) = \frac{(2xt - 1) + \sqrt{(1 - 2xt)^2 - 4t^2(x^2 - 1)}}{2t^2}.$$

where we must take the + conjugate root solution, since the − conjugate solution does not satisfy $-1 \leqslant x - ut \leqslant 1$ over the region $(-1, 1) \times (0, \frac{1}{2})$.

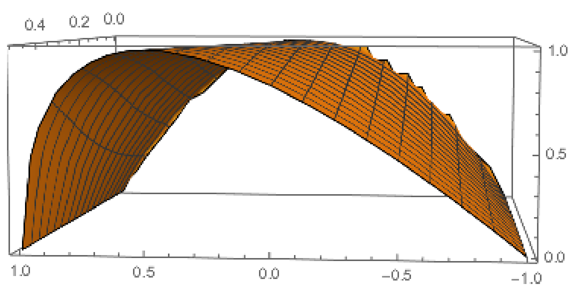


Figure 3: $u(x,t)$

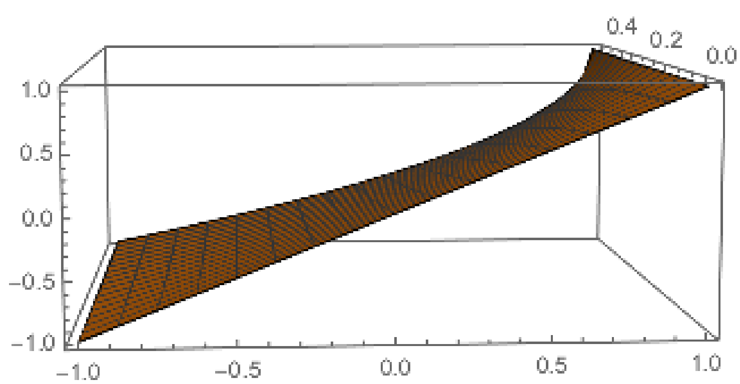


Figure 4: Graph of $x - ut$ corresponding to the + conjugate solution for $u(x,t)$

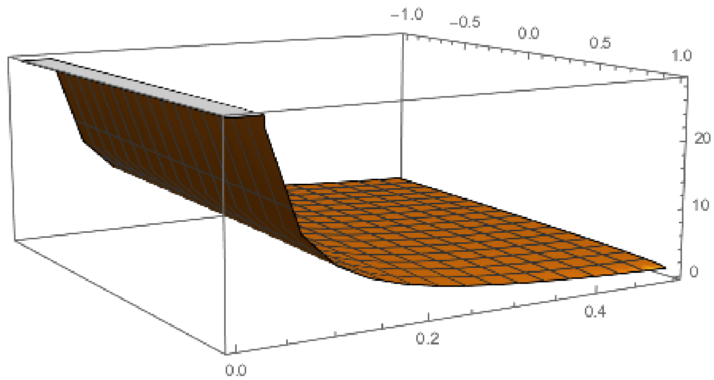


Figure 5: Graph of $x - ut$ corresponding to the $-$ conjugate solution for $u(x,t)$

(b) Note first that $1 - 2t > 0$ for $0 < t < \frac{1}{2}$. Then $u(1,t) = \frac{2t-1+\sqrt{(1-2t)^2}}{2t^2} = \frac{2t-1+(1-2t)}{2t^2} = 0$ for all $0 < t < \frac{1}{2}$.

(c) By direct computation,

$$u_x(x,t) = \frac{2t - 1 - \frac{2t}{\sqrt{(1-2xt)^2-4t^2(x^2-1)}}}{2t^2}$$

so

$$u_x(1^-,t) = \frac{2t-1-\frac{2t}{1-2t}}{2t^2} = \frac{2t-1}{2t^2} - \frac{1}{t(1-2t)}$$

which clearly tends to $-\infty$ as $t \to \frac{1}{2}^-$.

To see what this actually means, consider Figure 3. $u_x(1^-,t)$ is the slope of the wave front at time t, and as time goes to $\frac{1}{2}$, this slope becomes infinitely steep, resulting in the shock.

□

2 Beginnings

2.1

(a) Determine the type of the equation $u_{xx} + u_{xy} + u_x = 0$

(b) Determine the type of the equation $u_{xx} + u_{xy} + \alpha u_{yy} + u_x + u = 0$ for each real value of the parameter α.

(c) Determine the type of the equation $u_{tt} + 2u_{xt} + u_{xx} = 0$. Verify that there are solutions $u(x,t) = f(x-t) + tg(x-t)$ for any twice differentiable functions f, g.

(d) The equation $(1+y)u_{xx} - x^2u_{xy} + xu_{yy} = 0$ is hyperbolic, elliptic, or parabolic depending on the location of (x,y) in the plane. Find a formula to determine where in the $x-y$ plane the equation is hyperbolic. Sketch the $x-y$ plane and label where the equation is hyperbolic, where it is elliptic, and where it is parabolic.

Solution. (a) The principal symbol for the equation is $L^{(p)}[\xi_1, \xi_2] = \xi_1^2 + \xi_1\xi_2$, which informs us that $b^2 > ac$, since $1 > 0$. Hence, the PDE is hyperbolic.

(b) The principal symbol for the equation is $L^{(p)}[\xi_1, \xi_2] = \xi_1^2 + \xi_1\xi_2 + \alpha\xi_2^2$, which informs us that

$$b^2 - ac = 1 - \alpha \text{ is } \begin{cases} > 0 & \alpha \in (-\infty, 1) \\ = 0 & \alpha = 1 \\ < 0 & \alpha \in (1, \infty) \end{cases}.$$

Hence, the PDE is hyperbolic when $\alpha \in (\infty, 1)$, parabolic when $\alpha = 1$, and elliptic when $\alpha \in (1, \infty)$.

(c) The principal symbol for the equation is $L^{(p)}[\xi_1, \xi_2] = \xi_1^2 + 2\xi_1\xi_2 + \xi_2^2$, which informs us that $b^2 > ac$, since $4 > 1$. Hence, the PDE is hyperbolic. Suppose that $f, g \in C^2$, and that $u(x,t) = f(x-t) + tg(x-t)$. Then

$$\begin{aligned} u_{tt} &= f''(x-t) - 2g'(x-t) + tg''(x-t) \\ u_{xt} &= -f''(x-t) + g'(x-t) - tg''(x-t) \\ u_{xx} &= f''(x-t) + tg''(x-t) \end{aligned}$$

hence $u_{tt} + 2u_{xt} + u_{xx} = 0$.

(d) The principal symbol for the equation is $L^{(p)}[\xi_1, \xi_2] = (1+y)\xi_1^2 - x^2\xi_1\xi_2 + x\xi_2^2$, which says that $b^2 - ac = x^4 - x(1+y)$.

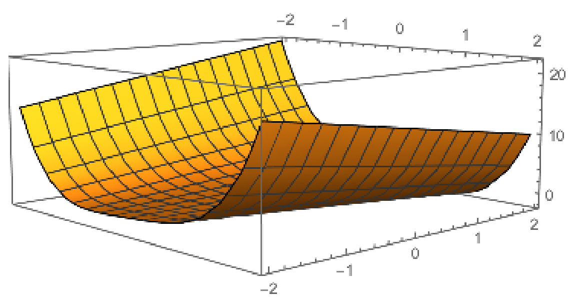


Figure 6: Graph of $b^2 - ac = x^4 - x(1+y)$

The main item of interest is the region of the plane where this function is positive, since that is where the PDE is hyperbolic:

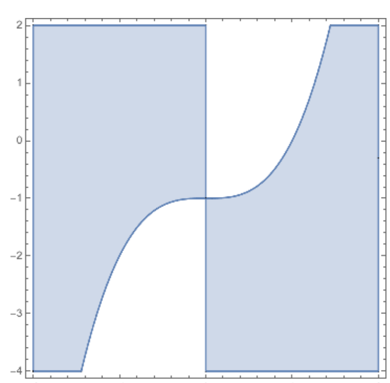


Figure 7: Graph of $b^2 - ac = x^4 - x(1+y) = 0$. The function is positive of the shaded region, negative on the light region, and zero on the boundary.

Thus, we see that the PDE is hyperbolic on the shaded region, elliptic on the light region, and parabolic on the boundary between the two.

□

2.2

Show that with the change of variables $\mathbf{y} = B\mathbf{x}$, the principal symbol of

$$L^{(p)}[\xi] = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j, \qquad \xi = (\xi_1, \dots, \xi_n)^t$$

corresponding to

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} = f(\mathbf{x}, u, u_{x_1}, \dots, u_{x_n})$$

has coefficients c_{ij} given by $C = BAB^T$, where $C = (c_{ij})$.

Solution. With $\mathbf{y} = B\mathbf{x}$, we have that $y_i = b_{ij}x_j$, hence $\frac{\partial y_i}{\partial x_j} = b_{ij}$. We thus write

$$\begin{aligned}
a_{ij}u_{x_i}u_{x_j} &= a_{ij}\left(u_{y_k}\frac{\partial y_k}{\partial x_i}\right)\left(u_{y_k}\frac{\partial y_k}{\partial x_j}\right) \\
&= a_{ij}\left(u_{y_k}b_{ki}\right)\left(u_{y_k}b_{kj}\right) \\
&= a_{ij}u_{y_s}u_{y_t}b_{si}b_{tj} \\
&= b_{si}a_{ij}b^T_{jt}u_{y_s}u_{y_t} \\
&= c_{st}u_{y_s}u_{y_t}
\end{aligned}$$

where c_{st} is the s, t entry of the matrix $C = BAB^T$, as well as the s, t coefficient of the principal symbol after the change of variable. □

2.3

For the series

$$u(x, y) = \sum_{k=0}^{\infty} \frac{1}{k!}u_k(x)y^k$$

write formulas for $u_3(x)$ and $u_4(x)$ in terms of derivative of the functions a, b, c, f, g, h, and G.

Solution. It's silly to work these out, since the whole process just comes down to rote calculation. I'll do all the work for $u_3(x)$, but will just remark on how one would go about computing $u_4(x)$.

We have the following three relations from the initial data and the assumption that c is nonzero in some neighborhood including $y = 0$:

$$\begin{aligned}
u_{yy} &= \frac{1}{c(x,y)}\left(f(x, y, u, u_x, u_y) - a(x,y)u_{xx} - 2b(x,y)u_{xy}\right) \\
\partial_x^m u(x, 0) &= u_{\underbrace{x\cdots x}_{\text{m times}}}(x, 0) = g^{(m)}(x) \\
\partial_x^m u_y(x, 0) &= u_{\underbrace{x\cdots x}_{\text{m times}}y}(x, 0) = h^{(m)}(x)
\end{aligned}$$

and we have the following worked out already:

$$\begin{aligned}
u_0(x) &= u(x,0) = g(x) \\
u_1(x) &= u_y(x,0) = h(x) \\
u_2(x) &= u_{yy}(x,0) = \frac{1}{c(x,0)}\left(f(x,0) - a(x,0)g''(x) - 2b(x,0)h'(x)\right)
\end{aligned}$$

To calculate $u_3(x)$, we want to find $u_{yyy}(x,0)$. Differentiating u_{yy} with respect to y, we find that

$$u_{yyy} = -\frac{c_x}{c^2}(f - au_{xx} - 2bu_{xy}) + \frac{1}{c}(f_y - a_y u_{xx} - au_{xxy} - 2b_y u_{xy} - 2bu_{xyy})$$

If you look carefully, you will notice that we know the values of everything in sight at $(x,0)$ except for $u_{xyy}(x,0)$. To find it, we differentiate u_{yy} with respect to x, we find that

$$u_{yyx} = u_{xyy} = -\frac{c_y}{c^2}(f - au_{xx} - 2bu_{xy}) + \frac{1}{c}(f_x - a_x u_{xx} - au_{xxx} - 2b_x u_{xy} - 2bu_{xyx})$$

and here we do know the values of everything in sight at $(x,0)$. Note that everything being smooth in some open neighborhood is also important since it means we can permute the set of variables we are differentiating with respect to. Substituting this expression for u_{xyy} into the one for u_{yyy} and evaluating at $(x,0)$ gives us an expression for $u_3(x)$ in terms of things we know. Out of morbid curiosity, it ends up being

$$\begin{aligned}
u_3(x) &= u_{yyy}(x,0) \\
&= -\frac{c_y(x,0)}{c^2(x,0)}u_2(x) + \frac{1}{c(x,0)}(f_y(x,0) - a_y(x,0)g''(x) - a(x,0)h''(x) - 2b_y(x,0)h'(x) - 2b(x,0)u_{xyy}(x,0)).
\end{aligned}$$

The expression for $u_4(x)$ is undoubtedly worse, but we comment on why we can find it. In the same spirit as above, we first differentiate u_{yyy} with respect to y, but find it has the terms u_{xxyy} and u_{xyyy} we don't yet know. The first can be found by differentiating u_{xyy} with respect to x, which is something we end up knowing, and the second by either differentiating u_{xyy} with respect to y or u_{yyy} with respect to x, and subsequently substituting in our new-found expression for u_{xxyy}. Piecing everything together and evaluating at $(x,0)$, we can assuredly calculate $u_4(x)$. $\square$

2.4

Show that $\zeta \in C^\infty(\mathbb{R})$, where

$$\zeta(x) = \begin{cases} 0 & x \leqslant 0, \\ e^{-\frac{1}{x}} & x > 0 \end{cases}$$

Solution. If $x < 0$, then ζ is clearly C^∞ since it is constant in a sufficiently small neighborhood. If $x > 0$, again ζ is visibly C^∞ because mindless calculation shows $\zeta^{(n)}$ is some Laurent polynomial times ζ, which is differentiable since we are always away from 0. The only thing to check, then, is that ζ is C^∞ at the origin, which we do by showing that the limits $\lim_{t\to 0^-}\zeta^{(n)}(t) = \lim_{t\to 0^+}\zeta^{(n)}(t)$ for all n. The left hand side limit is clearly 0, since ζ vanishes in a sufficiently small neighborhood of every point less than zero. the item of interest is thus the right hand side limit, which is a limit of the derivatives $\zeta^{(n)}$ we said we could always calculate earlier. This limit is indeed zero, which can be proven using a change of variables and L'Hopital's rule. We prove a more general claim, which follows from the next observation. If $p(x)$ is any polynomial under the sun, then $\frac{p(x)}{e^x}$ tends to 0 as $x \to \infty$. Indeed, for each term $\frac{p_n x^n}{e^x}$ we may use L'Hopital's rule. Since the quotient of the n^{th} derivatives of the top and bottom of this term is $\frac{p_n n!}{e^x}$, which tends to 0 as $x \to \infty$, repeated n-fold application of L'Hopital's rule proves that the original term grows arbitrarily small as x grows large. Since this is true for every term of the polynomial (note we could even do this for infinite polynomials using a standard epsilon trick in measure theory, bounding the n^{th} term by $\frac{\varepsilon}{2^n}$), we see that

$\frac{p(x)}{e^x}$ tends to 0 as $x \to \infty$. Now, make the change of variables $x \mapsto \frac{1}{x}$, and conclude that for any polynomial $p(x)$, the quotient $p(\frac{1}{x})e^{-\frac{1}{x}}$ tends to 0 as $x \to 0$.

Now, since every $\zeta^{(n)}$ for $x > 0$ is of the form $p(\frac{1}{x})e^{-\frac{1}{x}}$, we see that the right hand side limit is 0 for every n. Altogether, we have shown that ζ is smooth on all of $\mathbb{R}$.

□

Remark 2.1. *Shearer and Levy are a bit unclear about what phase velocity and group velocity are. Suppose that a PDE has a solution of the form $e^{i\xi x+\sigma t} = e^{i(\xi x+\omega t)}$. The phase velocity is $\frac{\omega}{\xi}$ and the group velocity is $\frac{\mathrm{d}\omega}{\mathrm{d}\xi}$ (sometimes both written with a -). We say a PDE is non-dispersive if the dispersion relation $\sigma(\xi)$ is linear in ξ, which implies that the phase and group velocities are equal. If the dispersion relation is nonlinear, then the PDE is said to be dispersive. The intuition is that a packet of waves containing waves of different spatial frequency (wave number) will spread out in time.*

2.5

Find the dispersion relation $\sigma = \sigma(\xi)$ for the following dispersive equations.

(a) The beam equation $u_{tt} + u_{xxxx} = 0$. Why is the equation dispersive and not dissipative? What makes this equation dispersive, whereas the wave equation is not dispersive?

(b) The linear Benjamin-Bona-Mahoney (BBM) equation $u_t + cu_x + \beta u_{xxt} = 0$. Deduce that the equation is dispersive, and show that the corresponding solutions $u = e^{i\xi x+\sigma(\xi)t}$ are traveling waves. Write a formula for their speed as a function of the wave number ξ. Identify a significant difference between this formula and the wave speeds of KdV traveling waves.

Solution. (a) With the initial data $u(x,0) = e^{i\xi x}$, the function $u(x,t) = e^{i\xi x+\sigma t}$ is a solution to the PDE provided that $\pm i\xi^2 = \sigma$ holds. Indeed, $u_{tt} + u_{xxxx} = \sigma^2 u + \xi^4 u = 0$. Hence, we have $\sigma(\xi) = \pm i\xi^2$ for the dispersion relation. Intuitively, the beam equation is dispersive because the traveling waves $u(x,t)$ travel faster in time as their spatial wave number (frequency) increases. This means that a group of waves of varying frequency will spread out in time.

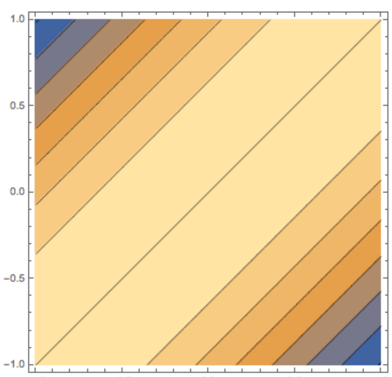


Figure 8: Real part of the solution $u(x,t) = e^{i(x-y)}$

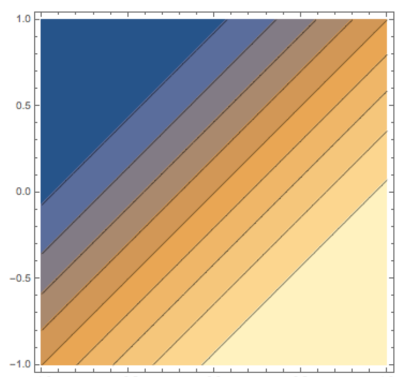


Figure 9: Imaginary part of the solution $u(x,t) = e^{i(x-y)}$

Compare this with

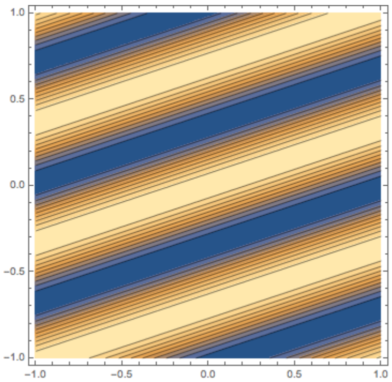


Figure 10: Real part of the solution $u(x,t) = e^{i(3x-9y)}$

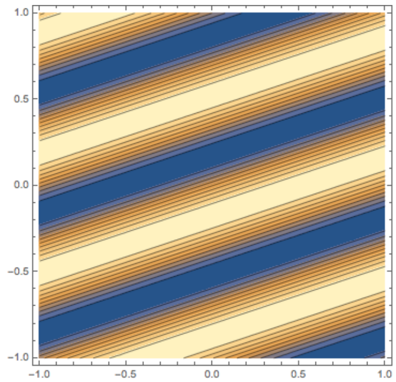


Figure 11: Imaginary part of the solution $u(x,t) = e^{i(3x-9y)}$

From these observations, we conclude that the beam equation $u_{tt} = u_{xxxx}$ is a dispersive equation. On the other hand, the wave equation $u_{tt} - c^2 u_{xx} = 0$ is not dispersive. Let $u(x,0) = e^{i\xi x}$ be initial data for the one dimensional wave equation, the solution of which is thus $u(x,t) = e^{i\xi x + \sigma t}$. This yields the dispersion relation $\sigma = \pm ic\xi$. Indeed, we see that the group velocity $c(\xi) = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} = \pm c$ is the same as the phase velocity, so that the equation is not dispersive. Consider the following contour plots. In each case the phase velocity/group velocity is $c = -7$.

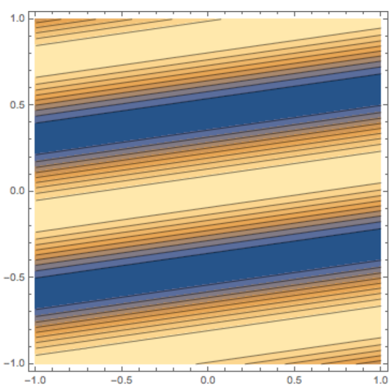


Figure 12: Real part of the solution $u(x,t) = e^{i(x-7y)}$

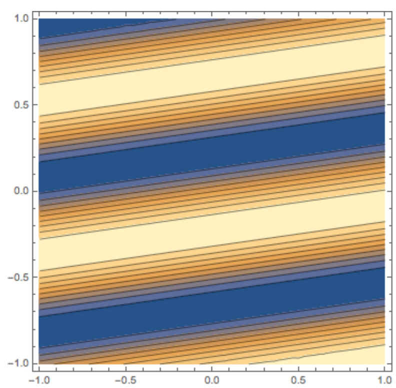


Figure 13: Imaginary part of the solution $u(x,t) = e^{i(x-7y)}$

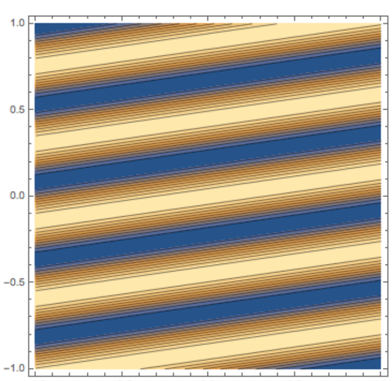


Figure 14: Real part of the solution $u(x,t) = e^{i(2x-14y)}$

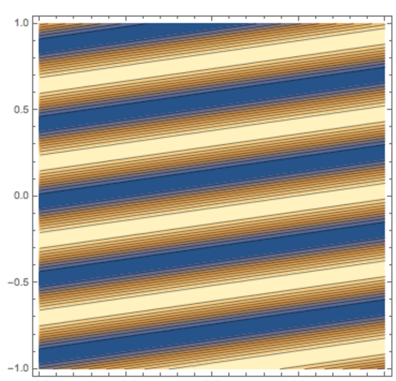


Figure 15: Imaginary part of the solution $u(x,t) = e^{i(2x-14y)}$

Note that the slopes of the contour lines (which encodes the phase velocity) does not increase with an increase in the wave number. This is different from how the beam equation behaves, where we see a change in the slope of the contours.

(b) For any of this to make sense, we want to take $\beta < 0$ (or at least that's what it looks like to me). Suppose we have initial data $u(x,0) = e^{i\xi x}$ for the BBM equation, yielding solution $u(x,t) = e^{i\xi x + \sigma t}$. Then from the PDE we have the equation $\sigma + ci\xi - \beta\xi^2\sigma = 0$, which gives us the dispersion relation $\sigma = -\frac{ci\xi}{1-\beta\xi^2}$. Note that σ is bounded by the assumption $\beta < 0$. The equation is dispersive, since the phase velocity $\frac{\omega}{\xi} = -\frac{c}{1-\beta\xi^2}$ is nonlinear. The fact that the corresponding solutions are traveling waves is evident from inspection.

Lastly, we consider how the phase velocities of wave solutions to the BBM equation relate to those of the linearity KdV equation, which is $u_t + cu_x + \beta u_{xxx} = 0$. As noted in the text, the dispersion relation for $lKdV$ is $\sigma = -i(c\xi - \beta\xi^3)$. The phase velocity is thus $\frac{\omega}{\xi} = c - \beta\xi^2$ which, unlike the phase velocity of the traveling waves solving BBM, is unbounded.

□

2.6

Suppose in the traffic flow model discussed in section 2.4 that the speed v of cars is taken to be a positive monotonic differentiable function of density $v = v(u)$.

(a) Should v be increasing or decreasing?

(b) How would you characterize the maximum velocity $v_{\max}$ and the maximum density $u_{\max}$?

(c) Let $Q(u) = uv(u)$. Prove that Q has a maximum at some density u^* in the interval $(0, u_{\max})$.

(d) Can there be two local maxima of the flux?

Solution. (a) v should be (strictly) monotonic decreasing with density in a continuously differentiable manner. This is because we expect traffic velocity to decrease as traffic density increases.

(b) It makes sense to let $v_{\max}$ be the limit of velocity as $u \to 0$, and to let $u_{\max}$ be the sup over all u such that $v > 0$. Lastly, it also behooves us to ensure that $v_{\max}$ and $u_{\max}$ are finite, since otherwise we don't have a realistic model. In particular, for this model we could let $v_{\max}$ be the velocity of cars assuming no congestion (where density is taken to be zero), and $u_{\max}$ be the maximum density of cars physically possible, i.e., the greatest density of cars (assuming they are all the same length) without overlap.

(c) With $Q(u) = uv(u)$, by the assumptions above, we have $Q(u_{\max}) = Q(0) = 0$. Since Q is continuous on the interval $[0, u_{\max}]$ and differentiable on $(0, u_{\max})$, by the mean value theorem $\frac{\mathrm{d}Q}{\mathrm{d}u}$ vanishes somewhere on $(0, u_{\max})$. Since Q is positive on $(0, u_{\max})$, it follows that this critical point is a local maximum.

For example, with $Q(u) = v_{\max}u(1 - \frac{1}{u_{\max}}u)$, we have $\frac{\mathrm{d}Q}{\mathrm{d}u} = v_{\max}(1 - \frac{2}{u_{\max}}u)$. Q thus has a critical point at $u = \frac{1}{2}u_{\max}$ in the interval $(0, u_{\max})$. Since $\frac{\mathrm{d}^2Q}{\mathrm{d}u^2} = -2\frac{v_{\max}}{u_{\max}} < 0$, it follows that $\frac{1}{2}u_{\max}$ is a local maximum for the flux Q.

(d) Yes it is possible, depending on the relation $v(u)$. In the event that v is cubic in u, Q is quartic, and thus might have two local maxima.

□

3 First Order PDE

3.1

Use the substitution $v = u_y$ to solve for $u = u(x, y)$:

$$u_{xy} = 5u_y, \quad u(x, x) = 0, \quad u_y(x, x) = 2.$$

Solution. Letting $v = u_y$, the PDE becomes $v_x = 5v$ with initial condition $v(x, x) = 2$. This yields the solution $v(x, y) = A(y)e^{5x}$ for some $A(y)$. The initial data then informs us that $A(y) = 2e^{-5y}$, so that the solution of the new PDE is $u_y(x, y) = v(x, y) = 2e^{5(x-y)}$. Hence, we find that $u(x, y) = -\frac{2}{5}e^{5(x-y)} + B(x)$ for some $B(x)$. The initial data gives $u(x, x) = -\frac{2}{5} + B(x) = 0$, so that $B(x) = \frac{2}{5}$. Thus the solution of the IVP is

$$u(x, t) = -\frac{2}{5}e^{5(x-y)} + \frac{2}{5}.$$

□

3.2

Solve for $u = u(x, y)$:

$$(1 + t^2)u_t + u_x = 0, \quad u(x, 0) = \sin x.$$

Solution. Since $1 + t^2 > 0$, we can rewrite the PDE as

$$u_t + \frac{1}{1 + t^2}u_x = 0.$$

The corresponding characteristic equations are thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1 + t^2}, \quad x(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = 0, \quad \hat{u}(0) = \hat{u}(0; x_0) = \sin x_0$$

which yield

$$x(t) = \arctan t + x_0 \quad \text{and} \quad \hat{u}(t) = \hat{u}(t; x_0) = \sin x_0.$$

Hence, the solution of the IVP is

$$u(x, t) = \hat{u}(t; x_0(x, t)) = \sin(x - \arctan t).$$

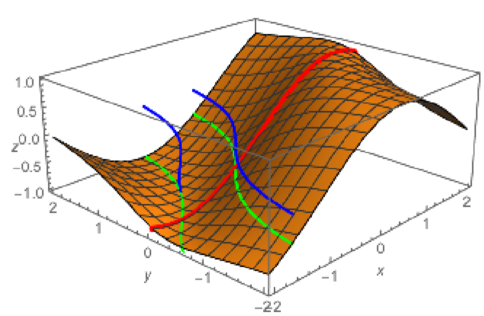


Figure 16: Graph of u is orange; graph of the initial curve is red; graphs of characteristics corresponding to $x_0 = -.5$ and $x_0 = -1.5$ are shown in blue; graphs of $\hat{u}$ on each characteristic are in green.

□

3.3

Solve for $u = u(x, y)$:

$$u_t + u_x + 3u = e^{2x+t}, \quad u(x, 0) = x.$$

Solution. We reorganize a smidgen to get $u_t + u_x = e^{2x+t} - 3u$, which has the corresponding characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1, \quad x(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = e^{2x+t} - 3\hat{u}, \quad \hat{u}(0) = \hat{u}(0; x_0) = x_0$$

which yield

$$x(t) = t + x_0 \quad \text{and thus} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = e^{3t+2x_0} - 3\hat{u}, \quad \hat{u}(0) = \hat{u}(0; x_0) = x_0$$

so that

$$\hat{u} = \hat{u}(t; x_0) = e^{-3t} \int e^{6t+2x_0} \mathrm{d}t = e^{-3t} \left\{ \frac{1}{6} e^{6t+2x_0} + A(x_0) \right\}$$

for some constant $A(x_0)$. The initial condition informs us that $A(x_0) = x_0 - \frac{1}{6} e^{2x_0}$. Hence, the solution of the IVP is

$$u(x, t) = \hat{u}(t; x_0(x, t)) = e^{-3t} \left\{ \frac{1}{6} e^{2(2t+x)} + x - t - \frac{1}{6} e^{2(x-t)} \right\}.$$

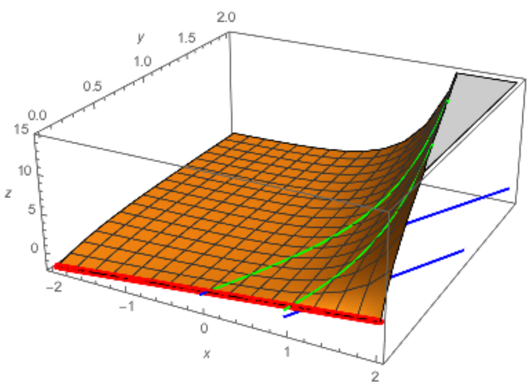


Figure 17: Graph of u is orange; graph of the initial curve is red; graphs of characteristics corresponding to $x_0 = 0$ and $x_0 = 1$ are shown in blue; graphs of $\hat{u}$ on each characteristic are in green.

□

3.4

Solve

$$u_x + u_y = u, \quad u(x, 0) = \cos x$$

using the general method of characteristics. Show that the initial curve Γ is noncharacteristic.

Solution. We parametrize the initial curve Γ by $\gamma(s) = [s, 0]$. Thus, $u(x, y) = \cos x$ for $[x, y] \in \Gamma$. Examining the PDE in a general setting, we see that with $a(\mathbf{x}, u) = [1, 1], c(\mathbf{x}, u) = u$, where $\mathbf{x} \in \mathbb{R}^2$, the IVP can be written as

$$\langle a(\mathbf{x}, u), \nabla u \rangle = c(\mathbf{x}, u), \quad \text{and} \quad u(\mathbf{x}) = \cos x_1 \quad \mathbf{x} \in \Gamma.$$

This set up yields the characteristic equations

$$a = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x}(s, 0) = \mathbf{x}_0(s) = \begin{pmatrix} s \\ 0 \end{pmatrix} \quad \text{and} \quad c = \frac{\mathrm{d}z}{\mathrm{d}\tau} = z, \quad z(s, 0) = \cos s.$$

Solving these equations gives us

$$\mathbf{x}(s, \tau) = \begin{pmatrix} \tau \\ \tau \end{pmatrix} + \begin{pmatrix} \xi_1(s) \\ \xi_2(s) \end{pmatrix} \quad \text{and} \quad z(s, \tau) = A(s) e^{\tau},$$

for some functions ξ_1, ξ_2, A, which after applying initial conditions yields

$$\mathbf{x}(s,\tau) = \begin{pmatrix} \tau + s \\ \tau \end{pmatrix} \quad \text{and} \quad z(s,\tau) = \cos(s)e^{\tau}.$$

Writing $\mathbf{x} = [x, y]$, we solve the first equation for s, τ as functions of x, y, giving $s = x - y$ and $\tau = y$. Hence, the solution of the Cauchy problem is

$$u(x,y) = z(s(x,y), \tau(x,y)) = \cos(x-y)e^{y}.$$

To confirm that the initial curve Γ is noncharacteristic, we compute the jacobian of the transformation $\mathbf{x}(s,\tau)$:

$$\begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial \tau} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which is invertible on Γ.

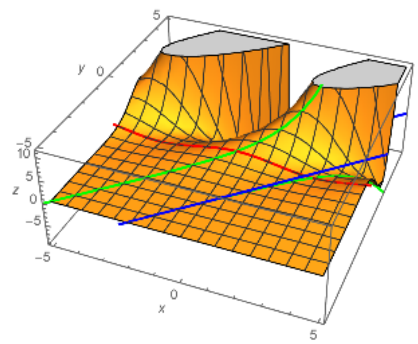


Figure 18: Graph of u is orange; graph of Γ is red; graphs of characteristics corresponding to $x_0 = 0$ and $x_0 = 3$ are shown in blue; graphs of z on each characteristic are in green.

□

3.5

Verify that $u(x,t) = \hat{u}(t; \tilde{x_0}(x,t))$ constructed in general in Section 3.1 is indeed a solution of

$$u_t + c(x,t,u)u_x = r(x,t,u), \quad t > 0, \quad u(x,0) = f(x)$$

Solution. In particular, we have $x = x(t; x_0)$ and $u = \hat{u}(t; x_0)$ the solutions of the characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c, \quad x(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = r, \quad \hat{u}(0) = \hat{u}(0; x_0) = f(x_0).$$

Inverting the relation $x(t; x_0)$ to get $x_0 = \tilde{x_0}(x,t)$, we can compute

$$\begin{aligned} u_x &= D_1(u)D_1(\tilde{x_0}) \\ u_t &= D_2(u) = D_1(\hat{u}) + D_2(\hat{u})D_2(\tilde{x_0}) \end{aligned}$$

$$\begin{aligned} u_t + cu_x = u_t + \frac{\mathrm{d}x}{\mathrm{d}t}u_x &= D_1(\hat{u}) + D_2(\hat{u})D_2(\tilde{x_0}) + D_1(u)D_1(\tilde{x_0})\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \partial_t \hat{u}(t, \tilde{x_0}(x(t), t)) \\ &= \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} \\ &= r \end{aligned}$$

□

3.6

With characteristics described by $y = x + k$, solve

$$u_x + u_y = u, \quad u(x,0) = \cos x.$$

Solution. The characteristic equations are

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1, \quad y(0) = k, \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}x} = \hat{u}, \quad \hat{u}(y^{-1}(0)) = \hat{u}(y^{-1}(0); 0) = \cos(y^{-1}(0))$$

which yield, as advertised,

$$y(x) = x + k, \quad \hat{u}(x;k) = A(k)e^x.$$

Since $\cos(-k) = \hat{u}(y^{-1}(0)) = A(-k)e^{-k}$, we see that $A(k) = \cos(k)e^k$ and thus

$$u(x,y) = \hat{u}(x; k(x,y)) = \cos(y-x)e^y.$$

□

3.7

For the avalanche flow equation

$$u_t + yu_x + S(u(u-1))_y = 0, \quad S > 0$$

suppose an initial distribution of particles is given by

$$u(x,y,0) = u_0(x,y) = x + y, \quad (x,y) \in \mathbb{R}^2.$$

Find the solution $u(x,y,t), (x,y,t) \in \mathbb{R}^2 \times \mathbb{R}^{>0}$ by the method of characteristics.

Solution. We write the PDE out as

$$u_t + yu_x + S(2u-1)u_y = 0$$

which has corresponding characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = S(2\hat{u}-1), \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = 0$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \hat{u}(0) = \hat{u}(0; x_0, y_0) = u_0(x_0, y_0) = x_0 + y_0.$$

We thus find that

$$\begin{aligned}
\hat{u}(t) &= \hat{u}(t; x_0, y_0) = u_0(x_0, y_0) = x_0 + y_0 \\
y(t) &= S(2\hat{u}-1)t + y_0 \\
x(t) &= \frac{1}{2}S(2\hat{u}-1)t^2 + y_0 t + x_0
\end{aligned}$$

Hence we conclude that implicitly the solution to the PDE is given by

$$u(x,y,t) = \hat{u}(t; x_0(x,y,t), y_0(x,y,t)) = \frac{1}{2}S(2u-1)t^2 + x - yt + y - S(2u-1)t.$$

□

Amended in the Errata

Use the method of characteristics to solve the initial value problem for $u = u(x, y, t)$ on the domain $x, y \in \mathbb{R}$, and for small $t > 0$:

$$u_t + yu_x + uu_y = 0, \quad u(x, y, 0) = x + y.$$

Show that the solution has a singularity as $t \to t^*$ for some $t^* > 0$, and find the value of t^*.

Solution. The characteristic equations are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \hat{u}, \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = 0$$

with initial conditions

$$x(0) = x_0, \quad y(0) = y_0, \quad \hat{u}(0) = \hat{u}(0; x_0, y_0) = u_0(x_0, y_0) = x_0 + y_0.$$

We thus find that

$$\begin{aligned}
\hat{u}(t) &= \hat{u}(t; x_0, y_0) = u_0(x_0, y_0) = x_0 + y_0 \\
y(t) &= \hat{u}t + y_0 \\
x(t) &= \frac{1}{2}\hat{u}t^2 + y_0 t + x_0
\end{aligned}$$

Hence we conclude that implicitly the solution to the PDE is given by

$$u(x, y, t) = \hat{u}(t; x_0(x, y, t), y_0(x, y, t)) = x + \frac{1}{2}ut^2 - (y + u)t + y$$

which can be easily solved for u:

$$u(x, y, t) = 2\frac{x + y - yt}{-t^2 + 2t + 2}.$$

The denominator vanishes at $t = 1 \pm \sqrt{3}$, so with $t > 0$ the blowup occurs at $t^* = 1 + \sqrt{3}$.

Below the graphs of u are shown for various values of t:

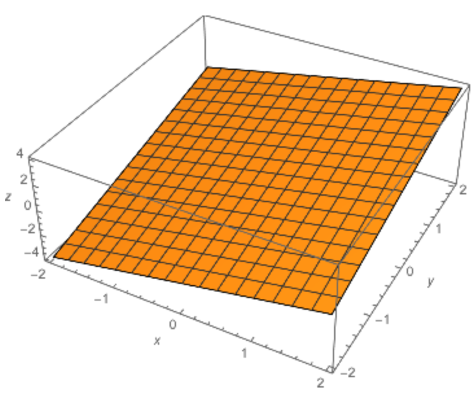


Figure 19: $t = 0$

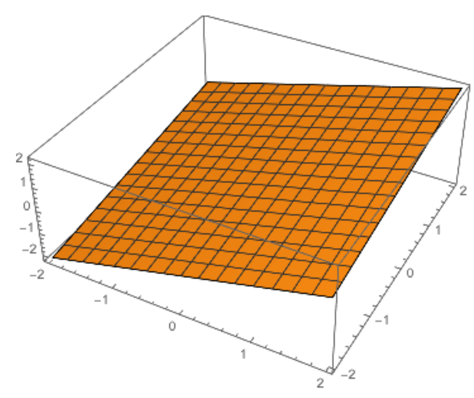


Figure 20: $t = 1$

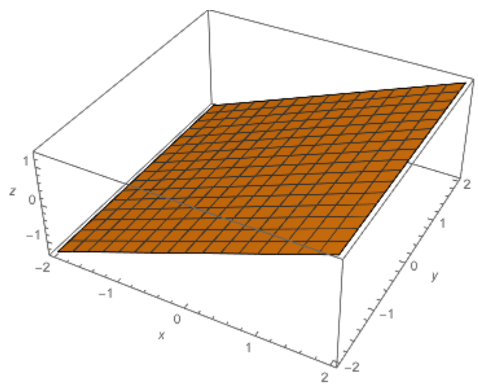


Figure 21: $t = 2$

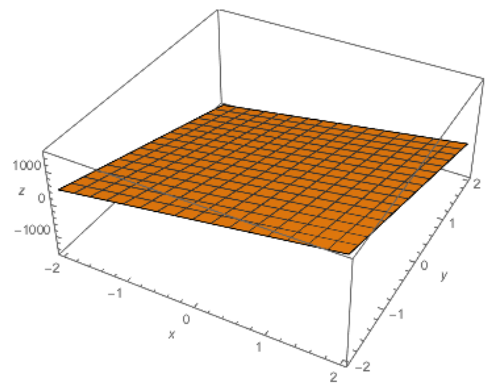


Figure 22: $t = 2.73$

□

3.8

(a) Use the method of characteristics to solve the initial value problem

$$u_t + tu_x = u^2, \quad -\infty < x < \infty, \quad 0 < t < 1; \quad u(x,0) = \frac{1}{1+x^2}, \quad -\infty < x < \infty.$$

(b) Show that the solution blows up as $t \to 1$, that is, $\lim_{t\to 1^-} \max_x u(x,t) = \infty$.

Solution. (a) We have the characteristics

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t, \quad x(0) = x_0, \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = \hat{u}^2, \quad \hat{u}(0) = \hat{u}(0;x_0) = \frac{1}{1+x_0^2}$$

which yield the solutions

$$x(t) = \frac{1}{2}t^2 + x_0, \quad \text{and} \quad \hat{u}(t;x_0) = \frac{1}{1+x_0^2 - t}.$$

Thus, the explicit solution is

$$u(x,t) = \hat{u}(t;x_0(x,t)) = \frac{1}{1+(x-\frac{1}{2}t^2)^2 - t}.$$

(b) Now, we calculate

$$\lim_{t\to 1^-} \max_x u(x,t) = \lim_{t\to 1^-} \max_x \frac{1}{1+(x-\frac{1}{2}t^2)^2 - t} = \lim_{t\to 1^-} \frac{1}{1-t} = +\infty.$$

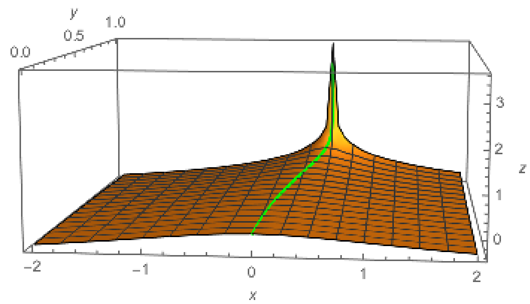


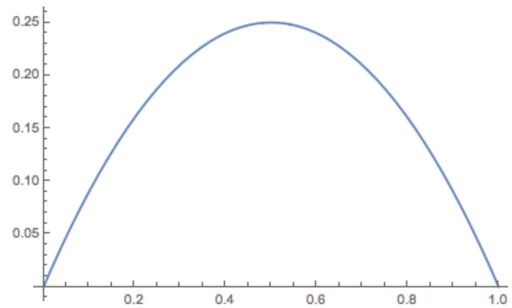
Figure 23: Graph of $\log_{100} u$; The green curve is the one parametrized by $(\frac{1}{2}t^2, t, \log_{100}\frac{1}{1-t})$, which is along the "ridgeline" of $\log_{100} u$.

□

3.9

Sketch the graph of the traffic flow flux Q as a function of density u. Explain each zero of Q in terms of the physical model.

Solution. Earlier, we chose the constitutive law $Q(u) = \beta u(1 - \frac{u}{\alpha})$, where α is the maximum density and β is the maximum velocity. Everyone knows what a parabola looks like, so I think the following sketch with constants normalized is sufficient:



There are of course two zeros, provided that $\beta > 0$ meaning we do have car movement. The least zero is the one corresponding to when there are no cars on the road, that is, when the density is zero. In this case, there is clearly no flux. The greatest zero occurs when maximum density is reached, which is where the velocity of cars vanishes corresponding to a stand-still traffic jam.

□

3.10

Formulate constitutive laws for the traffic flux Q as a function of density assuming that traffic speed $v(p)$ is a quadratic decreasing function of density ρ. How many parameters are in the model? Is it possible to make the flux nonconcave as a function of density?

Solution. We begin with the relation $Q(\rho) = \rho v(\rho)$ which gives correct units for flux. We could take, for instance,

$$Q(\rho) = \rho v(\rho) = \beta\rho\left(1 - \left(\frac{\rho}{\alpha}\right)^2\right)$$

with α and β as before.

Now, suppose in general that $v(\rho) = \alpha\rho^2 + \beta\rho + \xi$ is quadratic and decreasing so that $v'(\rho) = 2\alpha\rho + \beta < 0$ for all ρ in the interval $[0, \rho_{\max}]$. Then we can interpret ξ as the maximum velocity assuming no cars (so

$\rho = 0$), and we take $\alpha = -\frac{\beta\rho_{\max}+\xi}{\rho_{\max}^2}$ to ensure that $v(\rho_{\max}) = 0$. It is indeed possible to make the flux convex on some interval: indeed, we can just choose β satisfying

$$3\frac{\beta\rho_{\max} + \xi}{\rho_{\max}^2} > \beta$$

since this will make $Q''(\rho) > 0$ over some subinterval of $[0, \rho_{\max}]$. □

3.11

Write the details of how to use the Implicit Function Theorem on $u = u_0(x - ut)$ to prove: If u_0 is smooth and bounded on $(-\infty, \infty)$ then for each $x_0 \in \mathbb{R}$ there is an interval $I \subset \mathbb{R}$ containing x_0 such that the solution $u(x, t)$ exists, is C^1, and is unique for all $x \in I$ and all small enough t. Recall we are trying to solve the IVP

$$u_t + uu_x = 0 \quad \text{for} \quad (x, t) \in \mathbb{R} \times \mathbb{R}^{>0}, \quad u(x, 0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R}$$

Solution. Consider the function $F\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ defined by $F(x, y, \hat{u}) = \hat{u} - u_0(x - \hat{u}t)$. We want to prove the existence of a unique C^1 function $u(x, t)$ such that $\hat{u} = u(x, t)$ for all $(x, t) \in \mathbb{R} \times \mathbb{R}^{>0}$ in some neighborhood of $(x_0, 0)$. Note that $F(x_0, 0, u_0(x_0)) = u_0(x_0) - u_0(x_0) = 0$ for any $x_0 \in \mathbb{R}$ and that F is a C^1 function on all of $\mathbb{R}$, which is all we need for the implicit function theorem to guarantee that u is C^1. We check

$$(D_3F(x_0, 0, u_0(x_0))) = (1)$$

which is nonsingular. Hence, there exists an open set $A \times B \subset \mathbb{R}^2$ containing $(x_0, 0)$, an open set $V \subset \mathbb{R}$ containing $u_0(x_0)$, and a unique C^1 function $u\colon A \times B \to V$ such that $u(x_0, 0) = u_0(x_0)$ and $u(x, t) = \hat{u}$ satisfies $F(x, t, \hat{u}) = 0$ for all $(x, t) \in A \times B$. Since B is open, there is some open neighborhood of positive time on which $u(x, t) = u_0(x - ut) = \hat{u}$ solves the IVP.

I am a little confused as to why u_0 is assumed to be smooth and bounded on $\mathbb{R}$, since it seems we were able to prove the result using only continuous differentiability and pointwise boundedness on $\mathbb{R}$. Clearly, pointwise boundedness on $\mathbb{R}$ is necessary for F to even make sense, but I fail to see why boundedness on all of $\mathbb{R}$ should be necessary. □

3.12

Let $u_0(x) = H(x)x^2$, where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x \geqslant 0$ is the Heaviside step function. Write the solution $u(x, t)$ of

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0; \quad u(x, 0) = u_0(x), \quad -\infty < x < \infty$$

as an explicit formula for $t > 0$.

Solution. It is visibly true that $u_0(x)$ is C^1 on $\mathbb{R}$, so let's go ahead with solving the IVP. By the method of characteristics, the solution of the IVP is, in implicit form,

$$u(x, t) = H(x - ut)(x - ut)^2.$$

We have two cases we are concerned with, the first where $x - ut < 0$, where $u(x, t) = 0$, and the second where $x - ut \geqslant 0$, where $u(x, t) = x^2 - 2xtu + u^2t^2$. Since $u(x, t) \geqslant 0$ and $t > 0$, we see that $u(x, t) = 0$ when $x - ut \leqslant x < 0$.

Suppose then that $x - ut \geqslant 0$, so that $u = x^2 - 2xtu + u^2t^2$. Solving for u, we obtain the two possible solutions

$$u(x, t) = \frac{2xt + 1 \pm \sqrt{4xt + 1}}{2t^2}.$$

To see which solution we need to take, note that (with $t > 0$) $x - ut \geqslant 0$ implies that $\frac{x}{t} - u \geqslant 0$. Then we subtract

$$\frac{x}{t} - u_+ = \frac{x}{t} - \frac{2xt + 1 + \sqrt{4xt + 1}}{2t^2} = \frac{-1 - \sqrt{4xt + 1}}{2t^2} < 0$$

and

$$\frac{x}{t} - u_- = \frac{x}{t} - \frac{2xt+1-\sqrt{4xt+1}}{2t^2} = \frac{-1+\sqrt{4xt+1}}{2t^2} \geqslant 0.$$

Hence, we need to take the negative conjugate solution u_-. Hence, the full solution is

$$u(x,t) = \begin{cases} 0 & x \leqslant 0 \\ \frac{2xt+1-\sqrt{4xt+1}}{2t^2} & x > 0 \end{cases}$$

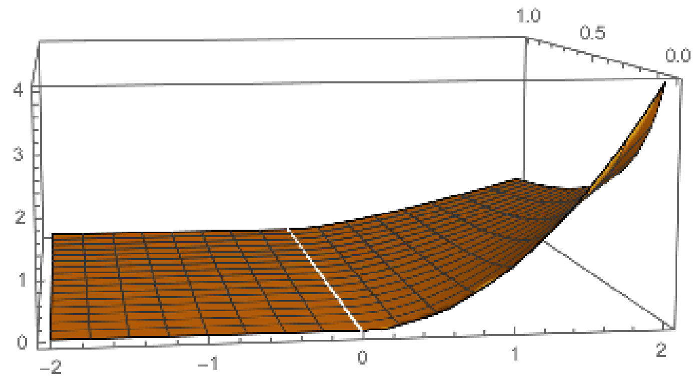


Figure 24: Graph of u

□

3.13

Get the answer

$$v = \frac{u_0'(x_0)}{1+u_0'(x_0)t}$$

by differentiating the implicit solution $u = u_0(x - ut)$ with respect to x.

Solution. We compute $\partial_x u(x,t) = \partial_x u_0(x-ut) = u_0'(x-ut)(1-u_x t)$. Hence,

$$\frac{u_x}{1-u_x t} = u_0'(x-ut)$$

so with $v = u_x$ and $x = ut + x_0$ we solve

$$\frac{v}{1-vt} = u_0'(x_0)$$

to find

$$v = \frac{u_0'(x_0)}{1+u_0'(x_0)t}$$

□

3.14

Use the method of characteristics to prove global ($t > 0$) existence of a smooth solution of

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = u_0(x), \quad -\infty < x < \infty$$

when the initial data are given by a strictly increasing but bounded C^1 function u_0.

Solution. I think we need u_0 to be smooth and pointwise bounded. If this is so, then the method of characteristics gives us the implicit solution $u(x,t) = u_0(x-ut)$. Then

$$u_x = \frac{u_0'(x-ut)}{1+u_0'(x-ut)t} = \frac{u_0'(x_0)}{1+u_0'(x_0)t} \quad \text{and} \quad u_t = \frac{-uu_0'(x-ut)}{1+u_0'(x-ut)t} = \frac{-uu_0'(x_0)}{1+u_0'(x_0)t}$$

which both exist and are smooth on $\mathbb{R} \times \mathbb{R}^{>0}$ since u_0 is smooth and $u_0' > 0$. By repeatedly differentiating, we obtain all the derivatives of u in terms of previous ones, which we know are smooth. Hence, u is smooth on $\mathbb{R} \times \mathbb{R}^{>0}$. $\square$

3.15

Carry through the analysis presented in Section 3.4 for a general scalar conservation law

$$u_t + f(u)_x = 0$$

where $f \colon \mathbb{R} \to \mathbb{R}$ is a given C^2 function. Derive an implicit solution $u(x,t)$ of the Cauchy problem, and formulate a condition for the solution to remain smooth for all time. Likewise, if the condition is violated, find an expression for the time at which the solution first breaks down.

Solution. Suppose that $u(x,0) = u_0(x)$ for some smooth u_0. We write $u_t + f(u)_x = u_t + f'(u)u_x = 0$, which has characteristic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f'(\hat{u}), \quad x(0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = 0, \quad \hat{u}(0) = \hat{u}(0; x_0) = u_0(x_0)$$

yielding the solutions

$$x(t) = f'(\hat{u})t + x_0 \quad \text{and} \quad \hat{u}(t; x_0) = u_0(x_0)$$

which gives the solution of the Cauchy problem

$$u(x,t) = \hat{u}(t; x_0(x,t)) = u_0(x - f'(u)t).$$

Now, suppose that either f is convex (so that $f'' \geqslant 0$) and that u_0 is monotonic increasing, or that f is concave (so $f'' \leqslant 0$) and u_0 is monotonic decreasing. Then u_x and u_t are defined for all $t > 0$, since as above

$$u_x = \frac{u_0'(x - ut)}{1 + u_0'(x - ut)f''(u)t} = \frac{u_0'(x_0)}{1 + u_0'(x_0)f''(u)t} \quad \text{and} \quad u_t = \frac{-f'(u)u_0'(x - ut)}{1 + u_0'(x - ut)f''(u)t} = \frac{-f'(u)u_0'(x_0)}{1 + u_0'(x_0)f''(u)t}.$$

As above, we can obtain every derivative of u by repeatedly differentiating, and so we conclude that in either of the above cases u is smooth.

On the other hand, suppose that $f''(u_0(x_0)) > 0$ and $u_0'(x_0) < 0$ at some point x_0, or that $f''(u_0(x_0)) < 0$ and $u_0'(x_0) > 0$. Then the break down time is

$$t^* = \inf\left\{ -\frac{1}{f''(u_0(x_0))u_0'(x_0)} \middle| f''(u_0(x_0)) > 0 \ \text{ and } \ u_0'(x_0) < 0 \quad \text{or} \quad f''(u_0(x_0)) < 0 \ \text{ and } \ u_0'(x_0) > 0 \right\}.$$

$\square$

4 The Wave Equation

4.1

Consider the IVP

$$\begin{aligned} u_{tt} &= u_{xx}, \quad x \in \mathbb{R},\ t > 0 \\ u(x,0) &= \phi(x), \quad x \in \mathbb{R} \\ u_t(x,0) &= \psi(x), \quad x \in \mathbb{R}. \end{aligned}$$

where ϕ is the function defined by

$$\phi(x) = \begin{cases} 0 & x < 1 \\ x - 1 & 1 \leqslant x < 2 \\ 3 - x & 2 \leqslant x < 3 \\ 0 & 3 \leqslant x \end{cases}$$

and $\psi \equiv 0$. In the $x-t$ plane representation of the solution in Fig. 4.5, we find that $u \equiv 0$ in the middle section with $t > \frac{1}{2}$. Show that if we keep the same ϕ but make ψ nonzero, with $\operatorname{supp}\phi = [1,3]$, then u will still be constant in this middle section. Find a condition on ψ that is necessary and sufficient to make this constant 0.

Solution. We consult d'Alembert's solution

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} \psi(\xi)\mathrm{d}\xi\right].$$

If (x_0, t_0) is in this middle region as pictured in Fig 4.5, then $x_0 - ct_0 < 1$ and $x_0 + ct_0 > 3$, hence the first two terms in d'Alembert's solution vanish. Thus the solution of the IVP is just the last term:

$$\frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\mathrm{d}\xi = \frac{1}{2c}\int_{[1,3]} \psi(\xi)\mathrm{d}\xi$$

which is just some constant. This holds for every (x_0, t_0) in this region, hence we conclude that u is constant there. To ensure that this constant is zero, it is necessary and sufficient to require this integral to vanish. □

4.2

Consider C^3 solutions of the wave equation

$$u_{tt} = c^2 u_{xx}.$$

For $c = 1$, define the energy density $e = \frac{1}{2}(u_t^2 + u_x^2)$, and let $p = u_t u_x$ (the momentum density).

(a) Show that $e_t = p_x$ and $e_x = p_t$.

(b) Conclude that both e and p satisfy the wave equation.

Solution. Notice that u being C^3 implies that u satisfies the Schwarz-Clairaut Theorem for equality of mixed partials of order up to 3, although it seems we only require this equality to hold up to order 2. Now, with this fact and the assumption that $u_{xx} = u_{tt}$ we calculate

$$\begin{aligned} e_t &= u_t u_{tt} + u_x u_{xt} = u_t u_{xx} + u_x u_{xt} = p_x \\ e_x &= u_t u_{tx} + u_x u_{xx} = u_t u_{xt} + u_x u_{tt} = p_t \end{aligned}$$

Hence it follows that

$$\begin{aligned} e_{tt} &= p_{xt} = p_{tx} = e_{xx} \\ p_{tt} &= e_{xt} = e_{tx} = p_{xx} \end{aligned}$$

□

4.3

Suppose $u(x,t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$. Show that

(a) For each $y \in \mathbb{R}$, the function $u(x-y,t)$ also satisfies the wave equation.

(b) Both u_x and u_t satisfy the wave equation.

(c) For any $a > 0$, the function $u(ax, at)$ satisfies the wave equation. Note that the restriction $a > 0$ is not necessary.

Solution. We need u to be C^3! Indeed, if this is true then we can go ahead and calculate:

(a) Letting $\tilde{u} = u(x - y, t)$, we have $\tilde{u}_{tt} = u_{tt} = c^2 u_{xx} = c^2 \tilde{u}_{xx}$.

(b)

$$(u_x)_{tt} = u_{xtt} = u_{ttx} = (u_{tt})_x = (c^2 u_{xx})_x = c^2 u_{xxx} = c^2 (u_x)_{xx}$$
$$(u_t)_{tt} = u_{ttt} = (u_{tt})_t = (c^2 u_{xx})_t = c^2 u_{xxt} = c^2 u_{txx} = c^2 (u_t)_{xx}$$

(c) Letting $\tilde{u} = u(ax, at)$, we have $\tilde{u}_{tt} = a^2 u_{tt} = a^2 c^2 u_{xx} = c^2 \tilde{u}_{xx}$.

□

4.4

(a) Let $u(x, t)$ be a solution of the wave equation with $c = 1$, valid for all x, t. Prove that for all x, t, h, k

$$u(x + h, t + k) + u(x - h, t - k) = u(x + k, t + h) + u(x - k, t - h).$$

(b) Write a corresponding identity if u satisfies the wave equation with $c = 2$.

Solution. (a) We appeal to the general form of the solution, which is of the form

$$u(x, t) = F(x - t) + G(x + t)$$

for C^2 functions F, G. We can just go ahead and compute

$$u(x+h, t+k) + u(x-h, t-k) = F(x-t+h-k) + F(x-t-h+k) + G(x+t+h+k) + G(x+t-h-k)$$
$$u(x+k, t+h) + u(x-k, t-h) = F(x-t+k-h) + F(x-t-k+h) + G(x+t+k+h) + G(x+t-k-h)$$

So that the equality clearly holds.

(b) With $c = 2$, it is readily seen that the following holds:

$$u(x + 2h, t + k) + u(x - 2h, t - k) = u(x + 2k, t + h) + u(x - 2k, t - h),$$

since the general solution is now $u(x, t) = F(x - 2t) + G(x + 2t)$.

□

4.5

Consider the quarter-plane problem

$$
\begin{aligned}
u_{tt} &= 4u_{xx}, \quad & x > 0,\ t > 0, \\
u(0,t) &= 0, & t > 0 \\
u(x,0) &= \phi(x), & x > 0 \\
u_t(x,0) &= \psi(x), & x > 0
\end{aligned}
$$

Let $\phi(x)$ be the function described in 4.1, and let $\psi(x) \equiv 0$. Sketch the solution $u(x,t)$ as a function of x for $t = \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, 1, 2$.

Solution. Before we plot the solution, lets actually say what it is. Since the initial data is only defined for positive values, our solution splits into two parts depending on whether x is less than or greater than $4t$ (it doesn't actually matter which part we go with on the line $x = 4t$, since the two parts agree). We have

$$
u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+4t) + \phi(x-4t)\right] & x > 4t \\ \frac{1}{2}\left[\phi(x+4t) - \phi(4t-x)\right] & x \leqslant 4t \end{cases}
$$

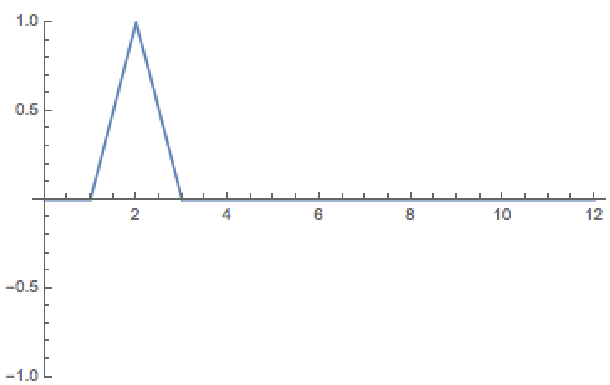


Figure 25: $u(x,t)$ at $t = 0$

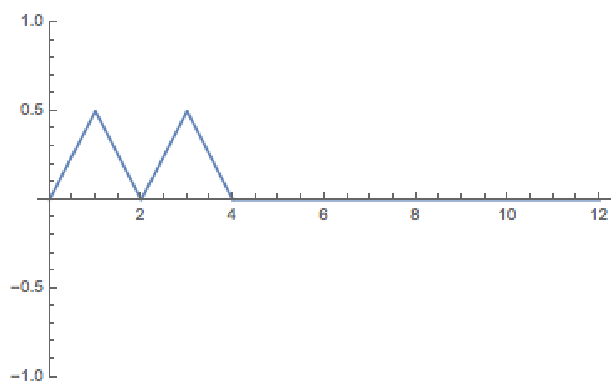


Figure 26: $u(x,t)$ at $t = \frac{1}{4}$

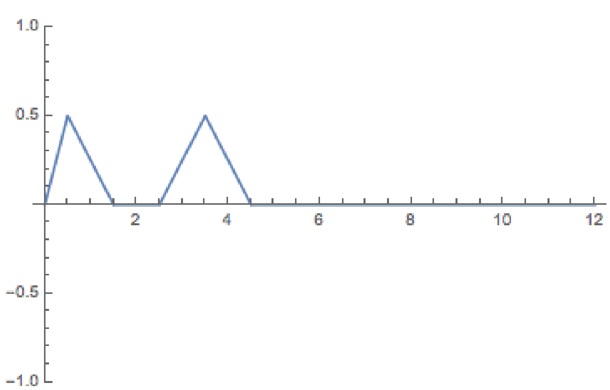


Figure 27: $u(x,t)$ at $t = \frac{3}{8}$

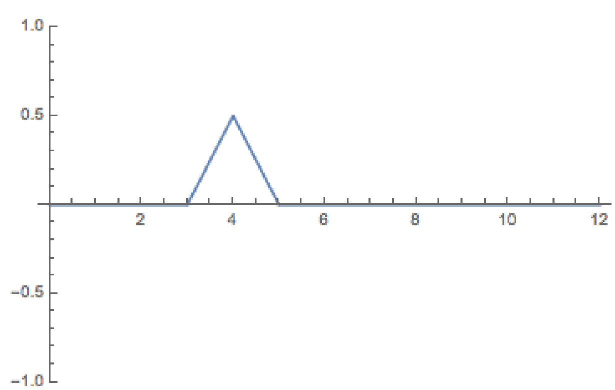


Figure 28: $u(x,t)$ at $t = \frac{1}{2}$

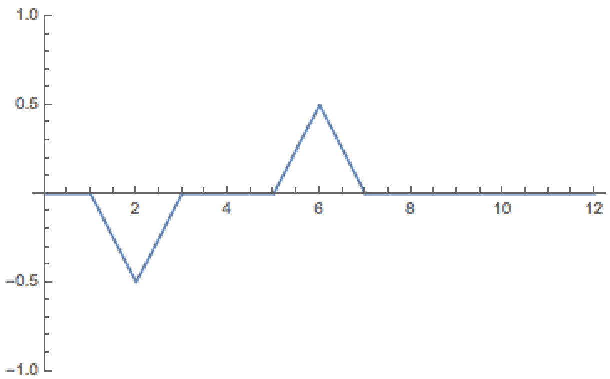


Figure 29: $u(x,t)$ at $t = 1$

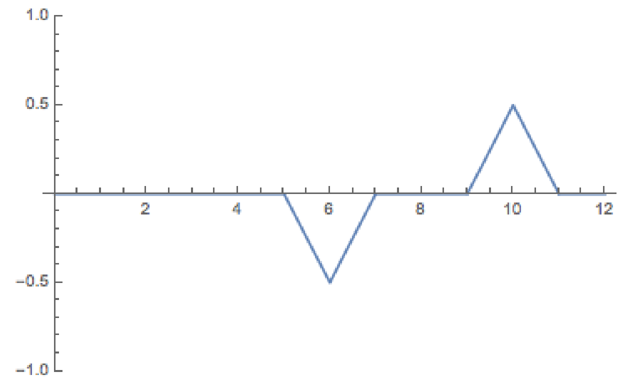


Figure 30: $u(x,t)$ at $t = 2$

□

4.6

Consider the quarter-plane problem with a homogeneous Neumann boundary condition

$$\begin{aligned} u_{tt} &= u_{xx}, \quad && x > 0,\ t > 0, \\ u_x(0,t) &= 0, && t > 0 \\ u(x,0) &= \phi(x), && x > 0 \\ u_t(x,0) &= \psi(x), && x > 0 \end{aligned}$$

Suppose that $\operatorname{supp} \phi = [1,2] = \operatorname{supp} \psi$.

(a) Solve for $u(x,t),\ x \geqslant 0,\ t > 0$.

(b) Where can you guarantee $u = 0$ in the first quadrant of the $x - t$ plane?

(c) Consider $\phi \equiv 0$; write a formula for u.

(d) If 0 is in the support of ϕ or ψ (e.g. if $\lim_{x\to 0^+} \phi(x) \neq 0$), write conditions that guarantee u is (a) continuous and (b) C^1. Explain your answers in terms of the behavior of the data around the boundary of the domain. Any compatibility condition will be effectively at the origin, but you will need to match u, u_x, and u_t across $x = t$.

Solution. (a) Let's first solve the more general problem where $u_x(0,t) = h(t)$ and h is not necessarily identically 0. If $x > t$, then we just take d'Alembert's solution. On the other hand, if $x < t$ we have the following to work with:

$$\begin{aligned} u(x,0) &= F(x) + G(x) = \phi(x) \\ u_t(x,0) &= -F'(x) + G'(x) = \psi(x) \\ u_x(0,t) &= F'(-t) + G'(t) = h(t) \end{aligned}$$

Since $\operatorname{supp} \phi = [1,2] = \operatorname{supp} \psi$, we can define without cause for concern $\psi(0) = \phi(0) = 0$, from which we surmise that $-F(x) + G(x) = \int_0^x \psi(\xi)\mathrm{d}\xi + A$ and $-F(-x) + G(x) = \int_0^x h(\xi)\mathrm{d}\xi + B$ for constants A and

B. These two relations also tell us that $A = B$, which is found by evaluating both expressions at $x = 0$. Now, notice that $F(-x) = G(x) - \int_0^x h(\xi)\mathrm{d}\xi - B$, and $G(x) = \frac{1}{2}\left[\phi(x) + \int_0^x \psi(\xi)\mathrm{d}\xi + A\right]$, which allow us to calculate (remember we are dealing with the case $x < t$)

$$\begin{aligned}
u(x,t) &= F(x-t) + G(x+t) \\
&= G(t-x) - \int_0^{t-x} h(\xi)\mathrm{d}\xi - B + G(x+t) \\
&= \frac{1}{2}\left[\phi(t-x) + \int_0^{t-x} \psi(\xi)\mathrm{d}\xi + A\right] - \int_0^{t-x} h(\xi)\mathrm{d}\xi - B + \frac{1}{2}\left[\phi(x+t) + \int_0^{x+t} \psi(\xi)\mathrm{d}\xi + A\right] \\
&= \frac{1}{2}\left[\phi(t-x) + \phi(x+t) + \int_0^{x+t} \psi(\xi)\mathrm{d}\xi + \int_0^{t-x} \psi(\xi)\mathrm{d}\xi\right] - \int_0^{t-x} h(\xi)\mathrm{d}\xi - B + A \\
&= \frac{1}{2}\left[\phi(t-x) + \phi(x+t) + \int_{t-x}^{x+t} \psi(\xi)\mathrm{d}\xi\right] + \int_0^{t-x} \psi(\xi)\mathrm{d}\xi - \int_0^{t-x} h(\xi)\mathrm{d}\xi
\end{aligned}$$

If we now specialize to $h \equiv 0$, we recover the solution of our original IBVP when $x < t$. Altogether,

$$u(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x-t) + \phi(x+t) + \int_{x-t}^{x+t} \psi(\xi)\mathrm{d}\xi\right] & t < x \\ \frac{1}{2}\left[\phi(t-x) + \phi(x+t) + \int_{t-x}^{x+t} \psi(\xi)\mathrm{d}\xi\right] + \int_0^{t-x} \psi(\xi)\mathrm{d}\xi & x \leqslant t \end{cases}$$

(b) u must vanish in the region of the first quadrant defined by $x + t < 1$, the region defined by $x - t > 2$, and the region defined by $t - x > 2$, since in these regions both backward characteristics cross the $t = 0$ line outside the support of ϕ and ψ. Hence the region of dependence of a point in either of these regions contains no nonzero initial data.

(c) We have, based on our results from the first part,

$$u(x,t) = \frac{1}{2}\left[\phi(t-x) + \phi(x+t)\right].$$

(d) We first handle the task of ensuring that u is continuous. All we need here is that ϕ be continuous at 0, and that ψ be Riemann integrable. Then everything on the right hand side of the definition of u in part (a) is continuous on the whole of the quarter plane. Indeed, taking the limit of u along a sequence $(x,t) \to (x_0, t_0)$ of the quarter plane tending to some point (x_0, t_0) therein is evidently convergent to the value $u(x_0, t_0)$ if the preceding conditions hold.

For u to be guaranteed C^1, we need to require a little more. Let's first calculate the derivatives of u:

$$u_x(x,t) = \begin{cases} \frac{1}{2}\left[\phi'(x-t) + \phi'(x+t) + \psi(x+t) - \psi(x-t)\right] & x > t \\ \frac{1}{2}\left[-\phi'(x-t) + \phi'(x+t) + \psi(x+t) - \psi(t-x)\right] & x \leqslant t \end{cases}$$

$$u_t(x,t) = \begin{cases} \frac{1}{2}\left[-\phi'(x-t) + \phi'(x+t) + \psi(x+t) + \psi(x-t)\right] & x > t \\ \frac{1}{2}\left[\phi'(t-x) + \phi'(x+t) + \psi(x+t) + \psi(t-x)\right] & x \leqslant t \end{cases}$$

If we want u to be C^1, then, we need to have $\phi \in C^1$, and we need $\phi'(0) = 0$. This can be seen by taking the limits of u_t along sequences which tend to a point (x_0, t_0) on the line $x = t$ from each side of the line.

Looking at the origin in particular, we can see why the condition $\phi'(0) = 0$ is necessary. Indeed, the boundary data has $u_x(0,t) = 0$ for all $t > 0$, while $u(x,0) = \phi(x)$ implies $u_x(x,0) = \phi'(x)$ for all $x > 0$.

Taking the limit as $t \to 0$ and $x \to 0$ in these respective relations should yield the same value if we assume that u is C^1.

□

4.7

Consider the more general case above where $u_x(0,t) = h(t)$ for h not identically 0. Derive the solution for $x < t$. Also derive a suitable compatibility condition at the origin that ensures the solution is continuous when the data are continuous. What about the first derivatives across $x = t$?

Solution. Oh boy! We already took care of the first part in the preceding problem, although for some reason my answer does not match with Shearer and Levy's since we've got that rogue integral of ψ floating around.

$$u(x,t) = \frac{1}{2}\left[\phi(t-x) + \phi(x+t) + \int_{t-x}^{x+t} \psi(\xi)\mathrm{d}\xi\right] + \int_0^{t-x} \psi(\xi)\mathrm{d}\xi - \int_0^{t-x} h(\xi)\mathrm{d}\xi.$$

This solution appears to be correct, however, since we can directly calculate

$$u_x(x,t) = \frac{1}{2}\left[-\phi'(t-x) + \phi'(x+t) + \psi(x+t) + \psi(t-x)\right] - \psi(t-x) + h(t-x)$$

which gives $u_x(0,t) = h(t)$ as we want.

As suggested by the analysis at the end of the last problem, we should take $\phi'(0) = h(0)$ for u to be C^1. Otherwise, continuity of the initial data guarantees continuity of u. □

4.8

Consider the wave equation that includes frictional damping:

$$u_{tt} + \mu u_t = c^2 u_{xx},$$

in which $\mu > 0$ is a damping constant. Show that if u is a C^2 solution with $u_x \to 0$ as $x \to \infty$, then the total energy

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2 + c^2 u_x^2)\mathrm{d}x$$

is a decreasing function.

Devise a C^2 function $f(x)$ with the property that f approaches some constant as $x \to \pm\infty$, but f' does not simultaneously approach 0.

Solution. As outlined in the text, we really want u_t and u_x to be in $\mathcal{L}^2$. That way u_t and u_x both approach zero as $x \to \pm\infty$ (otherwise we could find a sufficiently large ball about the origin and a positive $\delta > 0$ such that $u_t > \delta$ and $u_x > \delta$ in the complement of said ball, the integral over which of u_t^2 or u_x^2 would be infinite). If this is the case, then we can multiply the PDE by u_t, integrate by parts, and find that

$$E'(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \left(u_t^2 + c^2 u_x^2\right)\mathrm{d}x = -\int_{-\infty}^{\infty} \mu u_t^2 \mathrm{d}x$$

which says that the energy is decreasing.

For the added exercise at the end, typical examples of such functions include $f(x) = \frac{1}{x^m} sin(|x|^n)$ where $m > 1+n$ (this is actually related to an exercise in Baby Rudin). One that I just pulled out of my rear end, which I happen to like, but am too tired to write out explicitly uses bump functions. We define a function f on the positive real line which, we extend to an even function on the whole line, as follows: centered at each positive integer n take a ball of diameter $\frac{1}{n}$ and define on this ball a bump function reaching a height of $\frac{1}{n}$, such that it attains a slope of n. Then let f be zero everywhere else, and extend as an even function to the whole real line. Then clearly f is smooth, tends to 0 as $|x| \to = \infty$, and has a derivative with no limit as $|x| \to \infty$. □

4.9

Consider the quarter-plane problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) &= 0, \qquad t > 0 \\ u(x,0) &= \phi(x), \quad x > 0 \\ u_t(x,0) &= \psi(x), \quad x > 0 \end{aligned}$$

.

(a) Formulate the mechanical energy $E(t)$ for solutions, and show that it is conserved. Specify any assumptions you need on the initial data.

(b) For the nonzero boundary condition $u(0,t) = h(t)$, evaluate $E'(t)$ in terms of the data ϕ, ψ, h.

Solution. (a) Following the analysis in the text, we multiply the PDE by u_t, and integrate by parts, assuming u_x and u_t are in $\mathcal{L}^2$. We find that

$$\frac{1}{2}\int_0^\infty \frac{\partial}{\partial t}(u_t)^2 \mathrm{d}x = \int_0^\infty u_{tt}u_t \mathrm{d}x = \int_0^\infty c^2 u_{xx}u_t \mathrm{d}x = c^2 u_x u_t|_{x=0}^{x=\infty} - \frac{1}{2}\int_0^\infty c^2 \frac{\partial}{\partial t}(u_x)^2 \mathrm{d}x$$

which implies that

$$E'(t) = \frac{1}{2}\int_0^\infty (u_t^2 + c^2 u_x^2)\mathrm{d}x = c^2 u_x u_t|_0^\infty.$$

Thus, energy will be conserved provided that either of $u_x(0,t) = 0$ or $u_t(0,t) = \psi(0) = 0$ holds (as u_t and u_x being integrable implies that they tend to 0 as x tends to ∞).

(b) We simply substitute into the above result:

$$E'(t) = \frac{1}{2}\int_0^\infty (u_t^2 + c^2 u_x^2)\mathrm{d}x = c^2 u_x u_t|_{x=0}^{x=\infty} = -c^2 u_x(0,t)u_t(0,t) = -c^2 u_x(0,t)h'(t).$$

I'm not sure what else you can do.

□

4.10

Let $f(x,t)$ be a continuous function, and let $\Delta(x,t)$ denote the domain of dependence of the point (x,t) for $u_{tt} = c^2 u_{xx}$. Use the Fundamental Theorem of Calculus to show directly that

$$u(x,t) = \frac{1}{2c}\int\int_{\Delta(x,t)} f(y,s)\mathrm{d}y\mathrm{d}s$$

satisfies

$$u_{tt} = c^2 u_{xx} + f(x,t), \quad u(x,0) = u_t(x,0) = 0.$$

Solution. We have that

$$u(x,t) = \frac{1}{2c}\int\int_{\Delta(x,t)} f(y,s)\mathrm{d}y\mathrm{d}s = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \frac{1}{2c} f(y,s)\mathrm{d}y\mathrm{d}s = \int_0^t \tilde{u}(x,t,s)\mathrm{d}s$$

where $\tilde{u}(x,t,s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\mathrm{d}y$. Recall that $\tilde{u}$ solves the IVP

$$\begin{aligned}
\tilde{u}_{tt}(x,t;s) &= c^2\tilde{u}_{xx}(x,t;s), \quad t > s,\ x \in \mathbb{R}\\
\tilde{u}(x,s;s) &= 0,\\
\tilde{u}_t(x,s;s) &= f(x,s).
\end{aligned}$$

We then calculate directly

$$\begin{aligned}
u_t(x,t) &= \int_0^t \tilde{u}_t(x,t,s)\mathrm{d}s + \tilde{u}(x,t,t) = \int_0^t \tilde{u}_t(x,t,s)\mathrm{d}s\\
u_{tt}(x,t) &= \int_0^t \tilde{u}_{tt}(x,t,s)\mathrm{d}s + \tilde{u}_t(x,t,t) = \int_0^t \tilde{u}_{tt}(x,t,s)\mathrm{d}s + f(x,t)\\
u_x(x,t) &= \int_0^t \tilde{u}_x(x,t,s)\mathrm{d}s\\
u_{xx}(x,t) &= \int_0^t \tilde{u}_{xx}(x,t,s)\mathrm{d}s
\end{aligned}$$

Since we have $\tilde{u}_{tt} - c^2\tilde{u}_{xx} = 0$, we verify that

$$u_{tt} - c^2 u_{xx} = \int_0^t \left(\tilde{u}_{tt} - c^2\tilde{u}_{xx}\right)\mathrm{d}s + f(x,t) = f(x,t).$$

□

4.11

Consider the wave equation in three dimensions, with initial conditions in which $\phi(x) = f(|x|)$ is rotationally symmetric, the function f satisfies $f(r) = 0$, $r \geqslant \varepsilon$, and $\psi \equiv 0$. Show that the solution $u(x,t)$ is (a) rotationally symmetric, and (b) zero outside a circular strip centered at the origin and having width ε.

Solution. To be more clear, the IVP we are considering is:

$$\begin{aligned}
u_{tt} &= c^2\nabla^2 u, \quad \mathbf{x} \in \mathbb{R}^3,\ t \geqslant 0\\
u(\mathbf{x},0) &= \phi(\mathbf{x}) = f(|x|), \quad f(r) = 0,\ r \geqslant \varepsilon\\
u_t(\mathbf{x},0) &= \psi(\mathbf{x}) \equiv 0.
\end{aligned}$$

(a) From the method of spherical means, the solution is

$$u(\mathbf{x},t) = t\fint_{S(\mathbf{x},ct)} \psi(\mathbf{y})\mathrm{d}S + \frac{\partial}{\partial t}\left\{t\fint_{S(\mathbf{x},ct)} \phi(\mathbf{y})\mathrm{d}S\right\} = \frac{\partial}{\partial t}\left\{t\fint_{S(\mathbf{x},ct)} f(|\mathbf{y}|)\mathrm{d}S\right\}.$$

Since this last integral does not depend upon x itself, only $|x|$, it follows that u is rotationally symmetric.

(b) Suppose that $\mathbf{x}$ and t are such that $||\mathbf{x}| - ct| \geqslant \varepsilon$. Then $S(\mathbf{x},ct) \cap \mathcal{B}(\mathcal{O};\varepsilon) = \emptyset$. Hence, the last integral vanishes, meaning $u(\mathbf{x},t) = 0$.

□

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