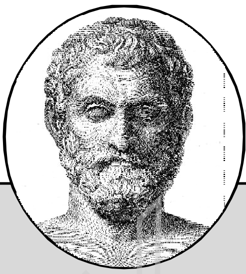
10

Similar Triangles

- Similar polygons and similar triangles.
- Basic proportionality theorem.
- Converse and corollaries of BPT.
- AA and SSS criteria of similar triangles.
- Proof of AA criteria of similar triangles.
- Theorem on areas of similar triangles.



Thales of Miletus

(624-546 B.C., Greece)

Thales was the first known philosopher and mathematician. He is credited with the first use of deductive reasoning in geometry. He discovered many propositions in geometry. He is believed to have found the heights of the pyramids in Egypt, using shadows and the principle of similar triangles

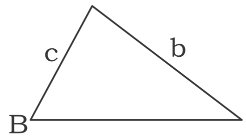
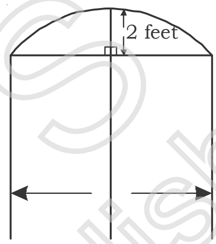
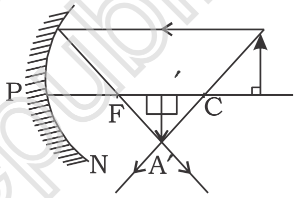
This unit facilitates you in,

- explaining the meaning of similar polygons and similar triangles.
- differentiating between similar triangles and congruent triangles.
- stating and proving basic proportionality theorem (BPT).
- stating and proving converse and corollaries of BPT.
- stating and proving AA criteria for similar triangles.
- stating the SSS and SAS criteria for similar triangles.
- stating and proving theorem related to right angled triangle where perpendicular is drawn from the right angled vertex to the hypotenuse.
- stating and proving the theorem on areas of similar triangles.
- reason deductively and prove riders based on the theorems.
- analyse and solve real life problems based on the theorems.

The universe cannot be read until we have learnt the language in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which it is humanly impossible to comprehend a single word.

- Galileo Galilei

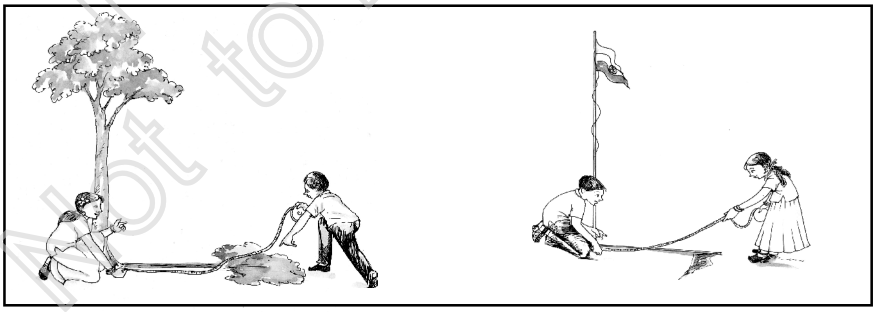
In your previous classes, you have learnt about triangles and their properties. These properties are used to solve many day-to-day life problems. Some of them are applied in studying other subjects like Physics, Chemistry etc. Below are given some examples where triangles and their properties are involved.

1.	$\boxed{A = \sqrt{s(s-a)(s-b)(s-c)}}$ Where 2s = a + b + c. This is the formula to find the area of a triangle. Do you know how this formula is derived?	
2.	AB = 14cm We want to divide AB in the ratio 2 : 5. How can this be done practically?	
3.	There is an arch above the door. The width of the door is 3 feet and the height of the arch is 2 feet. How to find the radius of the arch?	
4.	$\boxed{\frac{1}{u} + \frac{1}{v} = \frac{1}{f}}$ This is called the mirror formula, which you have learnt in physics. Have you ever thought how this formula is derived?	

For all these queries and many more, you will find solutions in this unit.

Consider the following example:

Two groups of students did the following project. One group of students measured the length of the shadow cast by a tree.

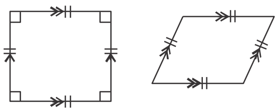
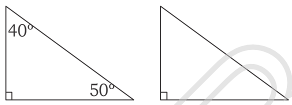
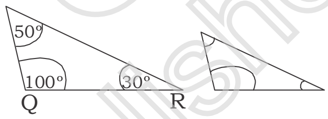


At the same time and next to the tree another group of students measured the length of the shadow cast by a vertical pole of three metres height which was next to the tree. With this information, they wanted to find the height of the tree. Is this possible?

Yes, it is possible to find the height of the tree using an important concept from Geometry called "**Similar Triangles**". In order to understand about similar triangles, let us first learn an important idea about **shape** and **size** of geometrical figures.

> **Know this !**
> Historians tell us that, Thales – Greek Mathematician, about 600 B.C found the height of Pyramid in Egypt on the basis of the length of its shadow.

The geometrical figures on a plane with reference to their shapes and sizes can have three possibilities :

Possibility - 1	**Possibility -2**	**Possibility - 3**
The figures having neither the same shape nor the same size. A square and a rhombus 	The figures having the same shape and same size. Two triangles having same measurements.  These are called congruent figures	The figures having the same shape but not the size. Two equiangular triangles having different measurements  These are called similar figures.

Like congruence, similarity relation also plays an important role in geometry.

Even in our daily life, concept of similarity is used very widely.

For example, the maps used in Geography are based on the concept of similarity.

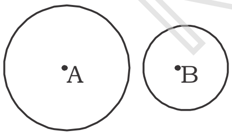
Models are made before constructing big buildings which are similar.

We have already studied about congruent triangles. 'Two triangles having the **same shape** and **same size** are congruent triangles. Now let us study more about the figures which have the same shape and not necessarily the same size, which are called **similar figures**.

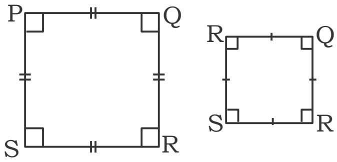
Similar figures

Observe the given photographs. You can at once say that they are the photographs of the same building (Vidhana Soudha) but are in different sizes. Are these photographs similar? Discuss.

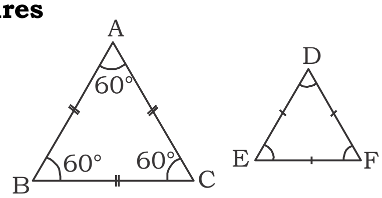
Now, let us consider some similar geometrical figures



Two circles are always similar



Two squares are always similar



Two equiangular triangles are always similar

From the above examples, we can conclude that

"two figures are similar if and only if they have same shape, but not necessarily the same size".

We use the symbol $\sim$ (read as similar to) to indicate the similarities of figures.

For example: Refer to the third set in the above figures.

ΔABC **is similar to** ΔDEF

ΔABC $\sim$ ΔDEF

The other symbol for similarity is $|||$

Know this! Similar, means having same shape.

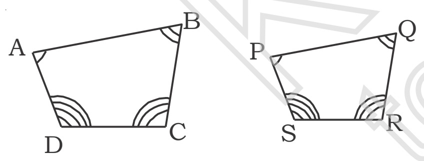
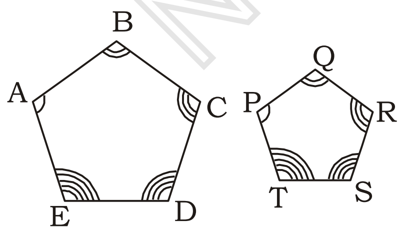
If we have two figures and out of these two, one can be obtained either by diminishing (shrinking) or by enlarging (stretching) the other without any change in their shape, then the two figures are similar. This has happened to the photographs of Vidhana Soudha and hence they are similar.

In all the examples discussed so far, we have assumed the figures to be similar by appearance. But, how to express similarity of figures mathematically. We need some definition of similarity and based on this definition some rules or conditions are evolved to decide whether the two given figures are similar or not.

Now let us study about the similarity of polygons and its definition.

Similar Polygons

Consider the pairs of similar geometrical figures given below. In each case observe the corresponding angles and the ratios of corresponding sides.

<table>
<tr><th>Pairs of similar
Geometrical figures</th><th>Observations</th></tr>
<tr><td>
</td><td>In ⏢ ABCD and ⏢ PQRS
1. $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $\angle D = \angle S$
and
2. $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DA}{SP}$
$\therefore$ ⏢ ABCD $\sim$ ⏢ PQRS</td></tr>
<tr><td>
</td><td>In ⬠ ABCDE and ⬠ PQRST
1. $\angle A = \angle P$, $\angle B = \angle Q$, $\angle C = \angle R$, $\angle D = \angle S$, $\angle E = \angle T$
$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DE}{ST} = \frac{EA}{TP}$
$\therefore$ ⬠ ABCDE $\sim$ ⬠ PQRST</td></tr>
</table>

From the above examples, we can infer that,

"Two polygons of same number of sides are similar,

if condition 1 : all the corresponding angles are equal.

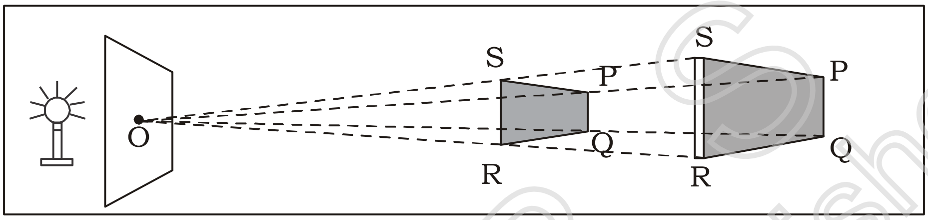
and condition 2 : all the corresponding sides are in the same ratio or in a proportion.

Discuss in groups

Which are the geometrical figures having same number of sides, that :

(a) may or may not be similar? (b) can never be similar?

Activity :



Cut out some geometrical shapes like triangles or quadrilaterals from a piece of card board. Hold these cutouts one by one, between a point source of light and a wall. Look at the shadow cast by each cutout on the wall. **The shadows have the same shape as the original cutouts, but are larger in size.** The shadows are said to be similar to the original cutouts. In the figure PQRS is a quadrilateral and the quadrilateral $P^{l}Q^{l}R^{l}S^{l}$ is its shadow. The quadrilateral $P^{l}Q^{l}R^{l}S^{l}$ is an enlargement (or magnification) of the quadrilateral PQRS.

Note that P^{l} lies on the ray OP, Q^{l} lies on the ray OQ.

R^{l} lies on the ray OR and S^{l} lies on the ray OS.

Also note that Vertex P^{l} corresponds to P, it is written as $P^{l} \leftrightarrow P$

Vertex Q^{l} corresponds to Q, it is written as $Q^{l} \leftrightarrow Q$

Vertex R^{l} corresponds to R, it is written as $R^{l} \leftrightarrow R$

Vertex S^{l} corresponds to S, it is written as $S^{l} \leftrightarrow S$

We can also observe that,

1. $\angle P = \angle P^{l}$ $\angle Q = \angle Q^{l}$ $\angle R = \angle R^{l}$ and $\angle S = \angle S^{l}$

2. $\dfrac{P^{l}Q^{l}}{PQ} = \dfrac{Q^{l}R^{l}}{QR} = \dfrac{R^{l}S^{l}}{RS} = \dfrac{S^{l}P^{l}}{SP}$

Note: This ratio of the corresponding sides is referred to as the **scale factor** for the polygon.

Know this!

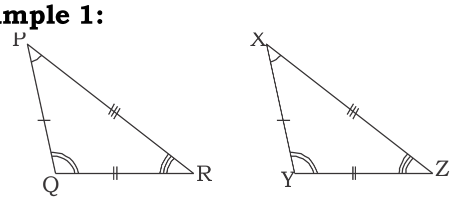
Hills Law: Animals which are geometrically similar, jump to the same height.
Crystalline Silicon is isomorphous with diamond i.e., they have similar structures.

Similarity of Triangles

Recall that any two congruent triangles have **six pairs** of corresponding elements equal, ie., three pairs of corresponding angles and three pairs of corresponding sides. But while determining the congruency of any two triangles we have obtained some criteria involving only **three pairs** of the corresponding elements of the two triangles. They are **SAS, SSS, ASA and RHS Criteria.** We have also found that, when two triangles are equiangular (AAA criteria) they may or may not be congruent triangles.

Observe the following examples.

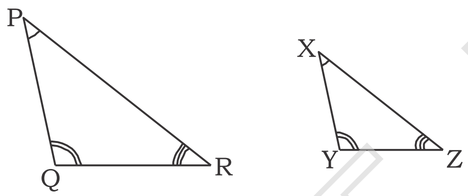
Example 1:



$\angle P = \angle X$, $\angle Q = \angle Y$, $\angle R = \angle Z$

PQ = XY, QR = YZ, PR = XZ

Example 2:



$\angle P = \angle X$, $\angle Q = \angle Y$, $\angle R = \angle Z$;

PQ $\neq$ XY, QR $\neq$ YZ, PR $\neq$ XZ

In example 1 : All the 3 pairs of corresponding angles are equal and all the 3 pairs of corresponding sides are equal.

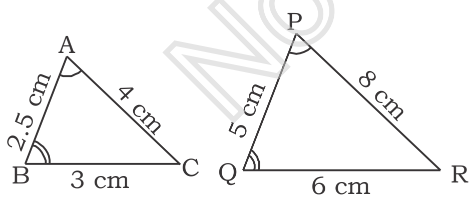
ΔPQR is **congruent** to ΔXYZ   i.e. $\Delta PQR \cong \Delta XYZ$

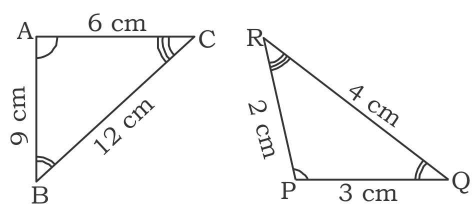
In example 2 : All the 3 pairs of corresponding angles are equal and all the 3 pairs of corresponding sides are not equal.

ΔPQR is **not congruent** to ΔXYZ.

$\therefore$ If two triangles are equiangular and they are not congruent, their corresponding sides may or may not be equal. Then, how are their corresponding sides related to each other?

Observe the pairs of equiangular triangles given below. Write the measures of their corresponding sides in the table. Find their ratios as shown.





Equiangular triangles	Ratio of corresponding sides
1. ΔABC and ΔPQR	$\frac{AB}{PQ} = \frac{2.5}{5}$ $\frac{BC}{QR} =$ ____ $\frac{AC}{PR} =$ ____
2. ΔABC and ΔPQR	$\frac{AB}{PQ} =$ ____ $\frac{BC}{QR} =$ ____ $\frac{AC}{PR} =$ ____

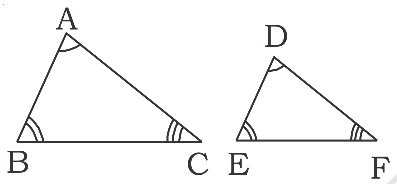
In each of the above examples, what can you conclude about the ratios of the corresponding sides?

We observe that the ratios between corresponding sides are equal.

If the ratios are equal, then the corresponding sides are said to be in proportion.

$\therefore$ We can conclude that, If two triangles are equiangular, then their corresponding sides are in proportion.

This can be symbolically represented as follows.



In ΔABC and ΔDEF

1. $\angle BAC = \angle EDF$ $\angle ABC = \angle DEF$ $\angle BCA = \angle EFD$
2. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

$\therefore$ **$\Delta ABC \sim \Delta DEF$**

Triangles are the only type of polygons, where the two conditions mentioned in the definition of similar figures need not be fulfilled. If any one of these conditions is fulfilled, then the triangles are similar.

Two triangles are said to be similar, if

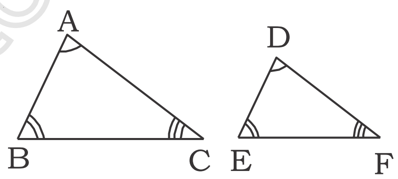
• their corresponding angles are equal. or • their corresponding sides are proportional.

Now let us discuss about the corresponding angles and sides of similar triangles.

Corresponding angles: In two similar triangles, angles which are equal are called corresponding angles.

If, $\Delta ABC \sim \Delta DEF$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$



$\therefore$ $\angle A = \angle D$ as they are opposite to corresponding, sides BC and EF respectively.

Similarly $\angle B = \angle E$ and $\angle C = \angle F$

Corresponding sides: In similar triangles, sides opposite to equal angles are known as corresponding sides and they are in a proportion.

In the figure $\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$

$\therefore$ AB and DE are corresponding sides as they are opposite to the equal angles $\angle C$ and $\angle F$ respectively.

Similarly,

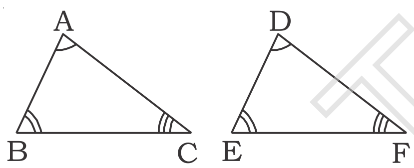
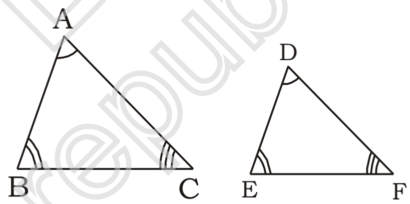
BC and EF are corresponding sides as they are opposite to $\angle A$ and $\angle D$ respectively and

CA and FD are corresponding sides as they are opposite to $\angle B$ and $\angle E$ respectively.

Thus, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ are corresponding ratios between sides of similar triangles which are in a proportion.

Congruency and similarities of triangles

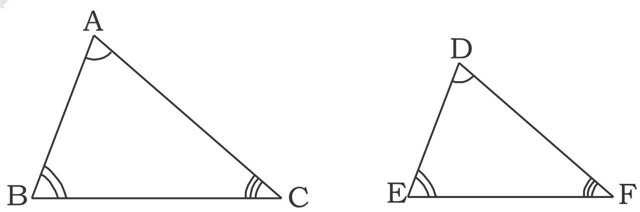
Congruency is a particular case of similarity. In both the cases, three angles of one triangle are equal to the three corresponding angles of the other triangle. But in congruent triangles, the corresponding sides are equal, while in similar triangles, the corresponding sides are proportional.

Congruent triangles	Similar triangles
	
$\Delta ABC \cong \Delta DEF$	$\Delta ABC \sim \Delta DEF$
$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$	$\angle A = \angle D$, $\angle B = \angle E$, $\angle C = \angle F$
AB = DE; BC = EF; CA = FD	$AB \neq DE$; $BC \neq EF$; $CA \neq FD$
$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = 1$	$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} > 1$ or < 1
Same shape and same size	Same shape but not same size.

We can also conclude that, **'Congruent triangles are always similar.'**

but 'Similar triangles are not necessarily congruent'.

Fundamental Properties of similar triangles



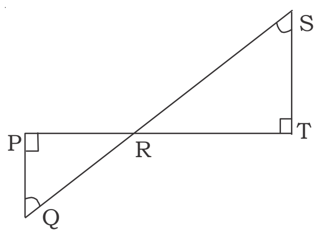
Property 1	**Property 2**
(This is with regard to angles) "If two triangles are similar, then the measure of three angles of one triangle are equal to measure of corresponding three angles of another triangle". If $\Delta ABC \sim \Delta DEF$ then , $\angle BAC = \angle EDF$ $\angle ABC = \angle DEF$ $\angle BCA = \angle EFD$	(This is with regard to sides) "If two triangles are similar, the three sides of one triangle are proportional to the corresponding three sides of another triangle". If $\Delta ABC \sim \Delta DEF$ then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$
Converse of Property 1 and 2	
"If the measure of three angles of one triangle are equal to measure of corresponding three angles of another triangle then, the two triangles are similar". If, in ΔABC and ΔDEF $\angle CAB = \angle FDE$ $\angle ABC = \angle DEF$ $\angle BCA = \angle EFD$	"If three sides of one triangle, are proportional to the corresponding three sides of another triangle then the two triangles are similar". If, in ΔABC and ΔDEF $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ then, $\Delta ABC \sim \Delta DEF$
then, $\Delta ABC \sim \Delta DEF$	
We can combine property 1 and 2 as follows "If two triangles are similar, then * they are equiangular and * the corresponding sides are in proportion.	"If two triangles are similar, then * the corresponding sides are in proportion and * they are equiangular.

ILLUSTRATIVE EXAMPLES

Numerical problems based on similarity of triangles.

Example 1 : In the adjoining figure ⊿PQR $\sim$ ⊿TSR.

Identify the corresponding vertices, corresponding sides and their ratios.



Sol.

Given : ⊿ PQR $\sim$ ⊿TSR

The corresponding Vertices			The corresponding Sides		
P	$\rightarrow$	T	QR	$\rightarrow$	RS
Q	$\rightarrow$	S	PR	$\rightarrow$	RT
R	$\rightarrow$	R	PQ	$\rightarrow$	ST

$\therefore$ The ratios are

$$\frac{PQ}{ST} = \frac{QR}{RS} = \frac{PR}{RT}$$

Example 2 : From the following data, state if $\Delta ABC \sim \Delta DEF$ or not.

(a) $\angle B = 65^0, \angle C = 82^0, \angle D = 33^0, \angle F = 65^0$

(b) AB=5cm, BC=7cm, $\angle B=70^0, \angle E=70^0$, DE=15 cm, EF=28 cm.

Solution :

(a) In $\Delta ABC, \angle B = 65^0, \angle C = 82^0$.

$\therefore \angle A = 180^0 - (65^0 + 82^0) = 180^0 - 147^0 = 33^0$.

In $\Delta DEF, \angle D = 33^0, \angle F = 65^0$

$\therefore \angle E = 180^0 - (33^0 + 65^0) = 180^0 - 98^0 = 82^0$.

$\therefore \angle A = \angle D, \angle B = \angle F$ and $\angle C = \angle E$.

$\Rightarrow \Delta\, ABC \sim \Delta\, DEF$

(b) In Δ ABC and Δ DEF, $\angle B = \angle E = 70^0$.

$$\frac{AB}{DE} = \frac{5}{15} = \frac{1}{3} \qquad \frac{BC}{EF} = \frac{7}{28} = \frac{1}{4} \qquad \therefore \frac{AB}{DE} \neq \frac{BC}{EF}$$

$\Rightarrow \Delta$ ABC is not similar to Δ DEF.

Example 3 : In Δ ABC, XY $\parallel$ BC. If XY = 3cm, AY = 2cm, and AC = 6cm, find the length of BC.

Solution: Given :- XY = 3 cm, AY = 2cm, AC = 6cm and XY $\parallel$ BC.

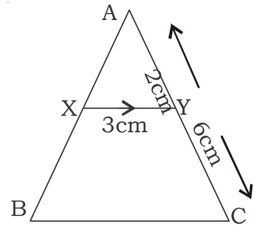
In Δ AXY and Δ ABC,

$\angle AXY = \angle ABC$ ($\because$ Corresponding angles)

$\angle AYX = \angle ACB$ ($\because$ Corresponding angles)

$\angle A$ is common.

$\therefore \Delta AXY \sim \Delta ABC$

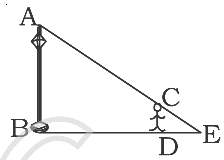


$\Rightarrow$ Corresponding sides are proportional.

$\therefore \frac{AX}{AB} = \frac{XY}{BC} = \frac{AY}{AC} \quad \frac{XY}{BC} = \frac{AY}{AC} \quad \therefore BC = \frac{XY \times AC}{AY} = \frac{3 \times 6}{2} \quad \therefore BC = 9cm.$

Example 4 : A girl of height 90 cm is walking away from the base of a lamp - post at a speed of 1.2 m/s. If the lamp is 3.6m above the ground, find the length of her shadow after 4 seconds.

Sol.



Let, AB denote the lamp - post.

$\therefore$ AB = 3.6m

CD denote the height of the girl.

$\therefore$ CD = 90 cm = 0.9m ($\because$ 90 cm = 0.9 m)

The girl is walking for 4 seconds away from the lamp - post, at a speed of 1.2m/s

$\therefore$ the distance covered is 4×1.2 = 4.8m

$\therefore$ BD = 4.8m

DE denote the shadow cast by the girl.

Let DE = x metres.

In $\triangle$ ABE and $\triangle$ CDE,

$\angle ABE = \angle CDE = 90^0$ ($\because$ lamp-post and the girl are standing vertical to the ground)

$\angle AEB = \angle CED$ ($\because$ Common angle)

$\angle BAE = \angle DCE$ ($\because$ Angle sum property of triangle)

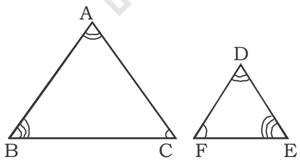
$\therefore \Delta ABE \sim \Delta CDE$

$\Rightarrow \frac{BE}{DE} = \frac{AB}{CD}$; $\frac{4.8+x}{x} = \frac{3.6}{0.9}$; 4.8 + x = 4x ; 3x = 4.8 ; x = 1.6

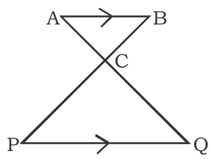
$\therefore$ The shadow of the girl after walking for 4 seconds is 1.6m long.

EXERCISE 10.1

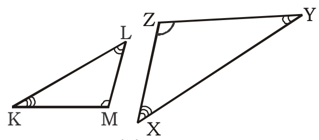
1. In the given pairs of similar triangles, write the corresponding vertices, corresponding sides and their ratios.



(a)

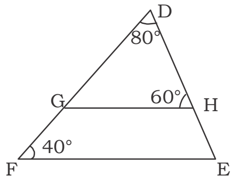


(b)

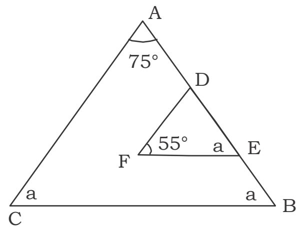


(c)

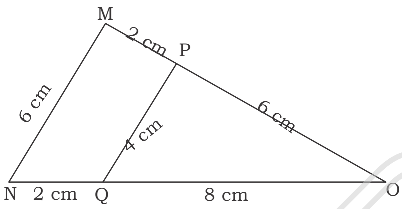
2. Study the following figures and find out in each case whether the triangles are similar. Give reason.



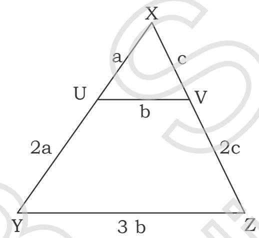
(i)



(ii)

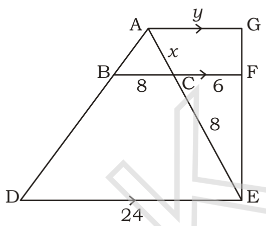


(iii)

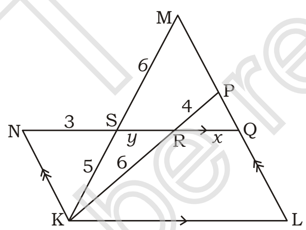


(iv)

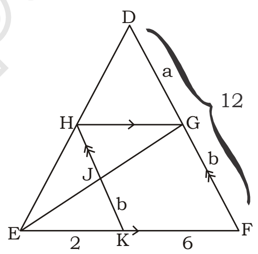
3. Find the unknown values in each of the following figures. All lengths are given in centimetres. (Measures are not to scale)



(i)

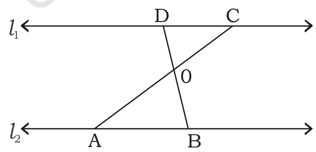


(ii)



(iii)

4. In the given figure, $l_1 \| l_2$ show that $\Delta AOB \sim \Delta COD$.



5. From the follwing data, state whether ΔABC is similar to ΔDEF or not.

(a) $\angle A = 70^0$, $\angle B = 80^0$, $\angle D = 70^0$, $\angle F = 30^0$

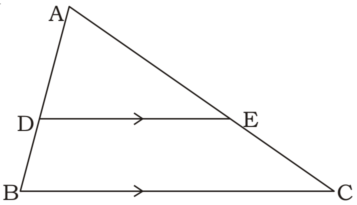
(b) $\angle B=50^0$, $\angle E=50^0$, AB=5cm, BC=6cm, DE=2.5cm, EF=3cm.

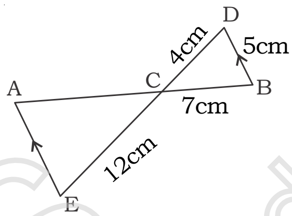
(c) AB=8cm, BC=9cm, CA=15 cm, DE=4cm, EF=3cm, FD=5cm.

(d) AB=16cm, AC=24cm, DE=4cm, DF=6cm, $\angle A = 62^0$, $\underline{|D} = 62°$

(e) $\angle C = 25^0$, $\angle F = 25^0$, AC = 25cm, BC = 30 cm, DF = 5cm, EF = 7cm.

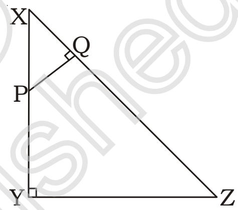
6. In ΔABC, $DE \parallel BC$
(a) If DE = 6cm and BC = 18cm, find AB:AD.
(b) If BC = 20cm, DE=4cm and AE = 3cm, find AC.
(c) If AD = 3cm, DB = 15cm, AE =4cm, find AC.
(d) If AE = 3.5cm, EC = 7cm, DB = 8cm, find AD.
(e) If AB = 12cm, AD = 3cm, BC = 16cm, find DE.





7. In the given figure, $AE \parallel DB$, BC = 7cm, BD = 5cm, DC = 4cm. If CE = 12cm, find AE and AC.

8. In ⊾XYZ, P is any point on XY and $PQ \perp XZ$ If XP = 4cm, XY = 16cm and XZ = 24cm, find XQ.



9. Select the set of numbers in the following, which can form similar triangles.
(i) 3,4,6 (ii) 9, 12, 18 (iii) 8, 6, 12 (iv) 8, 4, 9 (v) 2, 4½, 4

10. A vertical pole of 10m casts a shadow of 8m at certain time of the day. What will be the length of the shadow cast by the tower standing next to the pole, if its height is 110 m?

11. A ladder resting against a vertical wall has its foot on the ground at a distance of 6ft. from the wall. A man on the ground climbs two - thirds of the ladder. What will be his distance from the wall now?

Fundamental geometrical result on proportionality:

Activity: Study the following triangles carefully. In ΔABC, DE is drawn parallel to BC. Observe the measures of the line segments AD, DB, AE and CE. Now consider the ratios of the two sides which are divded by DE.

What can you conclude about the ratios? Discuss in groups.

In ΔABC, $DE \parallel BC$

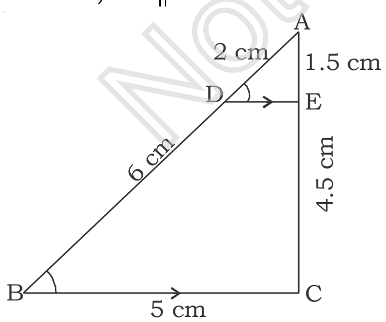


Fig. 1

In ΔABC, $DE \parallel BC$

$$\frac{AD}{DB} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{AE}{EC} = \frac{1.5}{4.5} = \frac{1}{3}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} = \frac{1}{3}$$

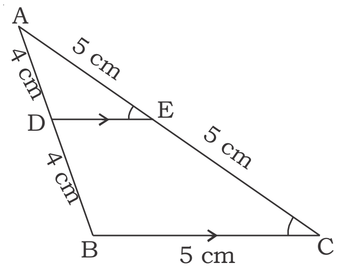


Fig. 2

In ΔABC, $DE \parallel BC$

$$\frac{AD}{DB} = \frac{4}{4} = 1$$

$$\frac{AE}{EC} = \frac{5}{5} = 1$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} = 1$$

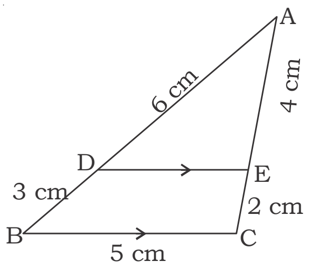


Fig. 3

In ΔABC, $DE \parallel BC$

$$\frac{AD}{DB} = \frac{6}{3} = 2$$

$$\frac{AE}{EC} = \frac{4}{2} = 2$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} = 2$$

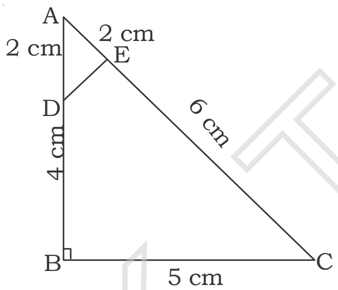


Fig. 4

In ◺ABC, DE is not parallel to the side BC.

$$\frac{AD}{DB} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{AE}{EC} = \frac{2}{6} = \frac{1}{3}$$

$$\therefore \frac{AD}{DB} \neq \frac{AE}{EC}$$

In the first three triangles, each of the ratios are equal but in the fourth triangle, the ratios are not equal. So in a triangles, what is the important condition / criteria that makes the ratios equal?

The only important condition is that in ΔABC, if DE is parallel to BC, then the ratios $\frac{AD}{DB}$ and $\frac{AE}{EC}$ will be equal.

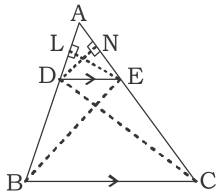
The above result can be generalized and it is called **Basic Proportionality Theorem (B.P.T) or Thales Theorem.** It can be stated as,

"If a straight line is drawn parallel to one side of a triangle, then it divides the other two sides proportionally".

Now let us logically prove the Thales theorem.

Thales Theorem : [Basic proportionality theorem]

If a straight line is drawn parallel to a side of a triangle, then it divides the other two sides proportionally".



Data: In ΔABC

$DE \parallel BC$

To prove: $\frac{AD}{DB} = \frac{AE}{EC}$

Construction: 1. Join D,C and E,B 2. Draw $EL \perp AB$ and $DN \perp AC$

Proof:

Statement	Reason
$\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta BDE} = \frac{\cancel{\frac{1}{2}} \times AD \times \cancel{EL}}{\cancel{\frac{1}{2}} \times DB \times \cancel{EL}}$	$\left(\because A = \frac{1}{2} \times b \times h\right)$
$\therefore \frac{\Delta ADE}{\Delta BDE} = \frac{AD}{DB}$	
$\frac{\text{Area of } \Delta ADE}{\text{Area of } \Delta CDE} = \frac{\cancel{\frac{1}{2}} \times AE \times \cancel{DN}}{\cancel{\frac{1}{2}} \times EC \times \cancel{DN}}$	$\left(\because A = \frac{1}{2} \times b \times h\right)$
$\therefore \frac{\Delta ADE}{\Delta CDE} = \frac{AE}{EC}$	
$\Rightarrow \boxed{\frac{AD}{DB} = \frac{AE}{CE}}$	$\left[\because \Delta BDE = \Delta CDE \text{ and Axiom -1}\right]$
	QED

While proving Thales Theorem, the figure can be drawn in two more ways. This is because, the straight line drawn parallel to a side of a triangle can be outside the triangle also. (above the triangle or below the triangle). Study the following figures and with the help of the same constructions try to prove them individually.

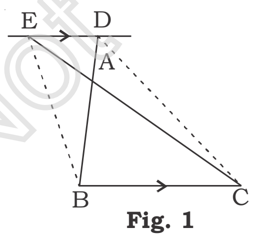


Fig. 1

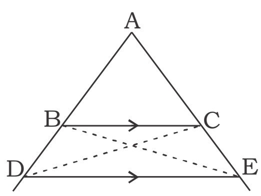
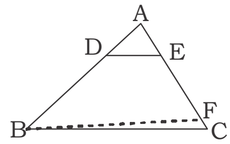


Fig. 2

In figure 1, the line DE is above the vertex A, where as in figure 2 the line DE is below the base line BC. Note that 'D' is the point where the parallel line cuts AB, or BA produced or AB produced. Similarly the point E is the point where the parallell line cuts AC or CA prduced or AC produced.

Converse of Thales Theorem

"If a straight line divides two sides of a triangle proportionally, then the straight line is parallel to the third side".



Data: In ΔABC, $\frac{AD}{DB} = \frac{AE}{EC}$

To prove: $DE \parallel BC$

Construction: Draw $DE \parallel BF$

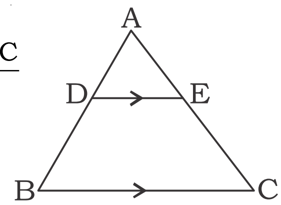
Proof:

Statement	Reason
If BC is not parallel to DE, then let BF be parallel to DE	
$\frac{AD}{DB} = \frac{AE}{EF}$	[∵ Thales Theorem]
But $\frac{AD}{DB} = \frac{AE}{EC}$	[∵ Data]
$\therefore \frac{AE}{EF} = \frac{AE}{EC}$	(∵ Axiom1)
EC = EF	
Point E coincides point C	
Hence, $DE \parallel BC$	QED

Now, let us study about some corollaries of Thales theorem.

Corollary 1:

In ΔABC; $DE \parallel BC$ $\therefore \frac{AD}{DB} = \frac{AE}{EC}$ [∵ B.P.T Theorem]



Add 1 to both the sides, $\frac{AD}{DB} + 1 = \frac{AE}{EC} + 1$; $\frac{AD + DB}{DB} = \frac{AE + EC}{EC}$

$\therefore \frac{AB}{DB} = \frac{AC}{EC}$ This is called componendo

Corollary 2 :

In ΔABC, $DE \parallel BC$ $\therefore \frac{AD}{DB} = \frac{AE}{EC}$ [∵ BPT theorem]

Taking reciprocals, we get $\frac{DB}{AD} = \frac{EC}{AE}$

Add 1 to both the sides. $\frac{DB}{AD} + 1 = \frac{EC}{AE} + 1$ $\therefore \frac{DB + AD}{AD} = \frac{EC + AE}{AE}$

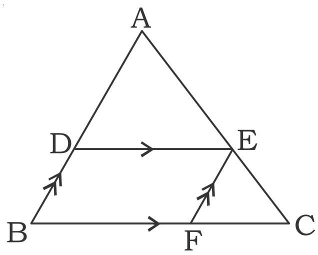
$\frac{AB}{AD} = \frac{AC}{AE}$ This is also called componendo

Corollary 3:

In ΔABC, $DE \parallel BC$ and $EF \parallel AB$

$\therefore$ DEFB is a parallelogram [$\because$ definition of parallelogram]

$\therefore$ $BF = DE$ $\left(\begin{array}{l}\because \text{ In a parallelogram} \\ \text{opposite sides are equal.}\end{array}\right)$



In ΔABC, $DE \parallel BC$ $\therefore \dfrac{AD}{AB} = \dfrac{AE}{AC}$ [$\because$ corollary 2]

In ΔABC, $EF \parallel AB$ $\therefore \dfrac{BF}{BC} = \dfrac{AE}{AC}$ [$\because$ corollary 1]

$$\therefore \frac{DE}{BC} = \frac{AE}{AC} \quad [\because BF = DE]$$

$$\therefore \boxed{\frac{AD}{AB} = \frac{DE}{BC} = \frac{AE}{AC}}$$

Note : Converse for all the three corollaries exists and are true.

Corollary 3 can be stated as follows :-

If a straight line is drawn parallel to a side of a triangle then the sides of intercepted triangle will be proportional to the sides of the given triangle.

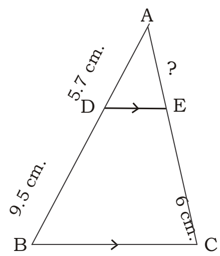
Numerical problems based on Thales theorem

ILLUSTRATIVE EXAMPLES

1. In ΔABC, $DE \parallel BC$, AD = 5.7 cm, BD = 9.5cm, EC = 6cm. Find AE.

Sol. Given: In ΔABC, $DE \parallel BC$ $\therefore \dfrac{AD}{DB} = \dfrac{AE}{EC}$ ($\because$ BPT)

$\dfrac{5.7}{9.5} = \dfrac{AE}{6}$ $\therefore AE = \dfrac{5.7 \times 6}{9.5}$ AE = 3.6cm



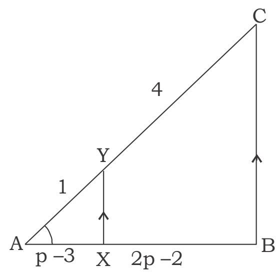
2. In the adjoining figure $XY \parallel BC$.

AX = p-3, BX =2p-2 and $\dfrac{AY}{CY} = \dfrac{1}{4}$. Find 'p'.

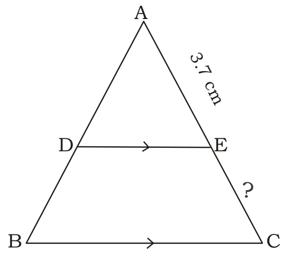
Sol. Given : In ΔABC, $XY \parallel BC$ $\therefore \dfrac{AX}{XB} = \dfrac{AY}{YC}$ ($\because$ Thales theorem)

$$\frac{p-3}{2p-2} = \frac{1}{4}$$

4 (p-3) = 1(2p-2), 4p-12 = 2p-2, 2p = 10 $\therefore$ p = 5



3. In ΔABC, DE || BC and $\frac{AD}{DB} = \frac{2}{3}$. If AE = 3.7cm, find EC.

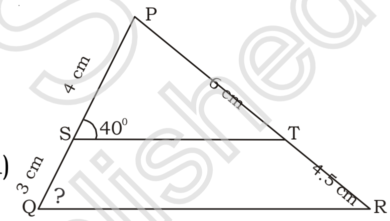


Sol. Given: In ΔABC, DE||BC.

$$\therefore \frac{AD}{DB} = \frac{AE}{EC} \quad (\because \text{ Thales theorem})$$

$$\therefore \; EC = \frac{AE \times DB}{AD} = \frac{3.7 \times 3}{2} \quad \therefore EC = 5.5 \text{ cm.}$$

4. In ΔPQR, S and T are two points on PQ and PR respectively such that PS = 4cm, SQ = 3cm, PT = 6cm, TR = 4.5cm and $\angle PST = 40^0$. Find $\angle PQR$.



Sol. In Δ PQR, $\frac{PS}{SQ} = \frac{4}{3} \;\therefore\; \frac{PT}{TR} = \frac{6}{4.5} = \frac{4}{3}$

$$\therefore \frac{PS}{SQ} = \frac{PT}{TR} \Rightarrow ST \parallel QR \quad (\because \text{ Converse of Thales theorem})$$

$\therefore \angle PST = \angle PQR = 40^0$

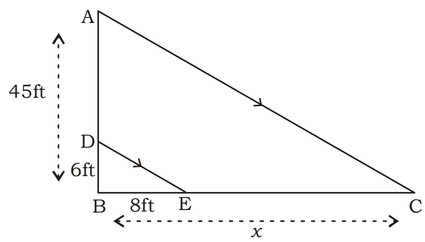
5. At a certain time of the day, a man 6 feet tall, casts his shadow 8feet long. Find the length of the shadow cast by a building 45 feet high, at the same time.

Solution : Let the length of the shadow of the building = x.

Since DE || AC, $\Delta DBE \sim \Delta ABC$.

$$\text{i.e., } \frac{\text{height of the man}}{\text{height of the building}} = \frac{\text{length of shadow cast by the man}}{\text{length of shadow cast by the building}}$$

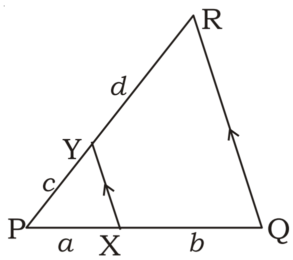
$$\frac{6\,ft}{45\,ft} = \frac{8\,ft}{x} \qquad \therefore x = \frac{8 \times 45}{6} = 60\,ft.$$



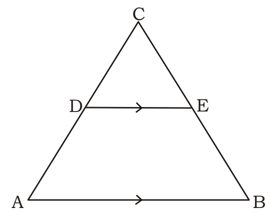
$\therefore$ Length of shadow cast by the building is 60 ft.

Exercise 10.2

1. Study the adjoining figure. Write the ratios in relation to basic proportionality theorem and its corollories, in terms of a, b, c and d.

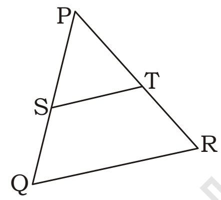


2. In the adjoining figure, DE||AB, AD = 7cm, CD = 5cm and BC = 18cm. Find BE and CE.



3. In ΔPQR, S is a point on PQ such that ST||QR and $\frac{PS}{SQ} = \frac{3}{5}$.

   If PR = 5.6cm, find PT.



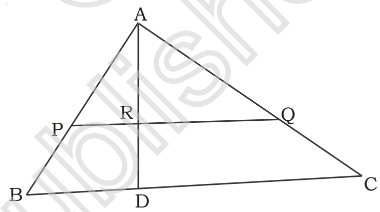
4. In Δ ABC, D and E are points on the sides AB and AC respectively such that DE||BC.

(i) If AD = 6cm, DB = 9cm, and AE = 8cm, find AC.

(ii) If AD = 8cm, AB = 12cm, AE = 12cm, find CE.

(iii) If AD = 4x-3, BD = 3x-1, AE = $8x{-}7$, and CE = $5x{-}3$ find the value x.

5. In the figure, PQ||BC AP = 3cm, AR = 4.5 cm and AB = 5 cm. Find the length of AD.



6. In ΔPQR, E and F are points on the sides PQ and PR respectively. For each of the following cases, verify EF||QR.

(i) PE = 3.9cm, EQ = 3cm, PF = 3.6cm, FR = 2.4 cm.

(ii) PE = 4cm, QE = 4.5cm, PF = 8cm, FR = 9cm.

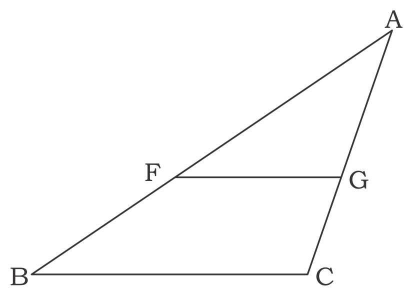
(iii) PQ = 1.28cm, PR = 2.56cm, PE = 0.18cm, PF = 0.36cm.

7. Which of the following sets of data make FG||BC?

(i) AB = 14cm, AF = 6cm, AC = 7cm, AG = 3cm.

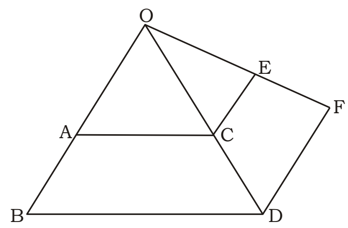
(ii) AB = 12cm, FB = 3cm, AC = 8cm, AG = 6cm.

(iii) AF = 6cm, FB = 5cm, AG = 9cm, GC = 8cm.

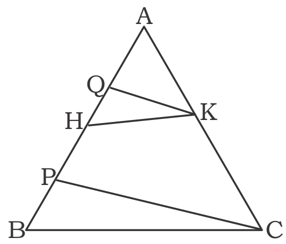


8. In the adjoining figure, AC||BD and CE||DF.

   If OA = 12cm, AB = 9cm, OC = 8cm and EF = 4.5cm, find OE.



9. In the figure, PC||QK and BC||HK. If AQ = 6cm, QH = 4cm, HP = 5cm and KC = 18cm, find AK and PB.



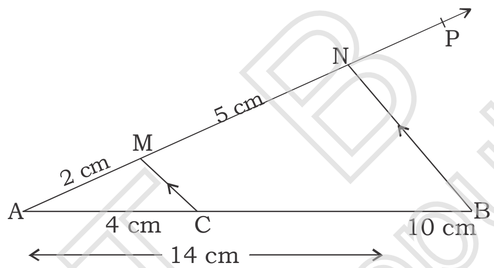
10. At a certain time of the day a tree casts its shadow 12.5 feet long. If the height of the tree is 5 feet, find the height of another tree that casts its shadow 20 feet long at the same time.

Practical application of Thales Theorem :

1. Let us consider the problem discussed in Page 230.

To divide a line in a given ratio.

Divide the line segment AB = 14 cm in the ratio 2 : 5



Steps :
1. Draw AB = 14 cm
2. Draw AP to make an angle of any measure with AB
   [For convenience let $\angle PAB < 60^0$]
3. Mark points M and N on AP such that AM = 2 cm, MN = 5cm.
4. Join N and B
5. Draw MC || NB using set squares
6. Measure AC and CB. AC = ___ cm CB = ___ cm.

The above result can be proved in the following few steps using Thale's theorem.

Data: In ΔABN, $\frac{AM}{MN} = \frac{2}{5}$, and MC || NB

To prove: $\frac{AC}{CB} = \frac{2}{5}$

Proof: In ΔABN, MC || NB

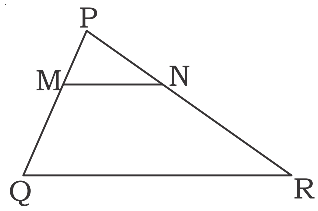
$\therefore \frac{AM}{MN} = \frac{AC}{CB} = \frac{2}{5}$ [$\because$ Thales Theorem]

Now let us solve some riders based on Thales theorem.

Illustrative Examples

1. In ΔPQR $2PM = 3PN$ and $2PQ = 3PR$

Prove that MQRN is a trapezium



Proof : (i) $\frac{2PM}{2PQ} = \frac{3PN}{3PR}$ [$\because$ Data]

$\frac{PM}{PQ} = \frac{PN}{PR}$

$\therefore \overline{MN} \parallel \overline{QR}$ [$\because$ Converse of corollary 2 of B.P.T]

(ii) Since $\overline{QP}$ and $\overline{RP}$ intersects at 'P',

$\overline{QP}$ cannot be parallel to $\overline{RP}$

$\Rightarrow$ MQRN is a trapezium

[$\because$ If only one pair of opposite sides of a quadrilateral is parallel then it is a trapezium]

2. Prove that "In a trapezium, the line joining the mid points of non-parallel sides is (i) parallel to the parallel sides and

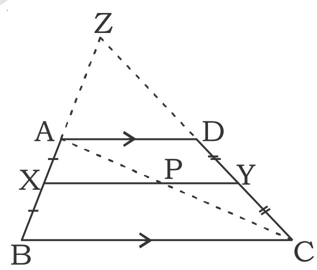
(ii) Half of the sum of the parallel sides"

Data : In the trapezium ABCD

(i) $AD \parallel BC$

(ii) $AX = XB$

(iii) $DY = YC$



To Prove: (i) $XY \parallel AD$

OR $XY \parallel BC$

(ii) $XY = \frac{1}{2}(AD + BC)$

Constrution : 1. Extend BA and CD to meet at Z.

2. Join A and C. Let it cut XY at P

Proof: Step 1: In ΔZBC,

$AD \parallel BC$ [$\because$ Data]

$\therefore \frac{ZA}{AB} = \frac{ZD}{DC}$ [$\because$ BPT]

$\therefore \frac{ZA}{2AX} = \frac{ZD}{2DY}$ [$\because$ X & Y are mid points of AB and DC]

$\therefore \frac{ZA}{AX} = \frac{ZD}{DY}$

$\Rightarrow XY \parallel AD$ ---(i) [$\because$ Converse of B.P.T]

Step 2: In ΔABC,

1. $AX = XB$ [$\because$ Data]

2. $XP \parallel BC$ [$\because$ Proved]

$\therefore AP = PC$ [$\because$ Converse of mid point theorem]

$\therefore XP = \frac{1}{2}BC$ [$\because$ M.P Theorem]

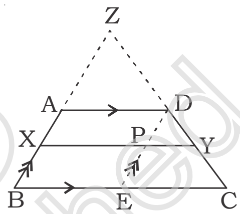
$|||^{ly}$ In ΔADC, $PY = \frac{1}{2}AD$

By adding, we get $XP + PY = \frac{1}{2}BC + \frac{1}{2}AD$

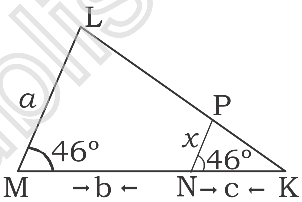
$\therefore XY = \frac{1}{2}(BC + AD)$ --- (ii) QED

This problem can also be solved by doing alternate construction.
i.e. instead of joining AC, draw DE $\parallel$ AB.
Let it cut XY at P.

Now, discuss in groups and write the proof.



3. In Δ LMK, Prove that $x = \frac{ac}{b+c}$



Data:- In Δ LMN, $\angle LMN = \angle PNK = 46°$

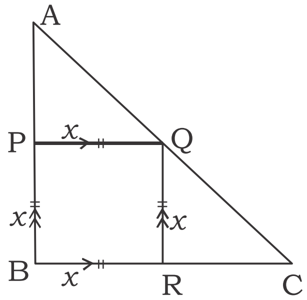
$LM = a$, $PN = x$, $MN = b$, $NK = c$

To prove:- $x = \frac{ac}{b+c}$

	Statement	Reason
Proof:-	In Δ LMK, $LM \parallel PN$	If corresponding angles are equal then lines are parallel.
	$\therefore \frac{PN}{LM} = \frac{NK}{MK}$	Corollary to Thales.
	$\frac{x}{a} = \frac{c}{b+c}$	
	$\therefore x = \frac{ac}{b+c}$.	

4. Rhombus PQRB is inscribed in Δ ABC such that $\angle B$ is one of its angle. P,Q and R lie on AB, AC and BC respectively. If AB = 12 cm and BC = 6cm find the sides of rhombus PQRB.

	Statement	Reason
Solution:-	In Δ ABC, $PQ \parallel BC$	In a rhombus opposite sides are parallel.
	$\therefore \frac{AP}{AB} = \frac{PQ}{BC}$	Corollary to Thales.



$$\frac{12-x}{12} = \frac{x}{6}$$

$12x = 72 - 6x$

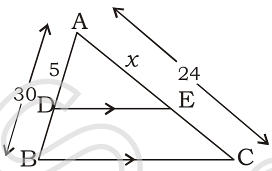
$18x = 72$

$x = 4$ cm

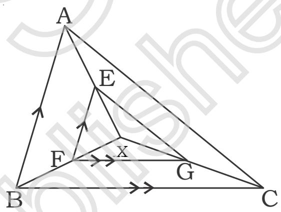
The sides of the rhombus is 4cm each.

Exercise 10.3

Riders based on Thales theorem.

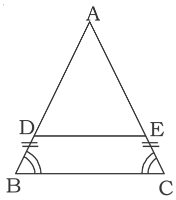


1. In ΔABC, $DE \parallel BC$. calculate 'x' or for what value of 'x', $\overline{DE}$ will be parallel to $\overline{BC}$.

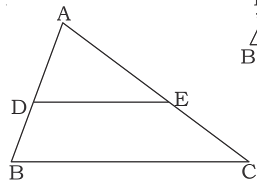


2. X is any point inside ΔABC. XA, XB and XC are joined. 'E' is any point on $\overline{AX}$. If $EF \parallel AB$, $FG \parallel BC$. Prove that $\overline{EG} \parallel \overline{AC}$. (Hint: B.P.T & Axiom 1).

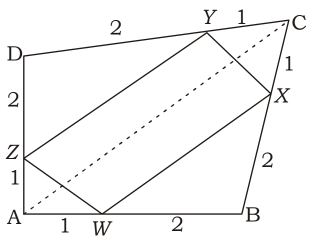
3. State 'Mid point Theorem'. Prove the theorem using 'Converse of Thale's Theorem.



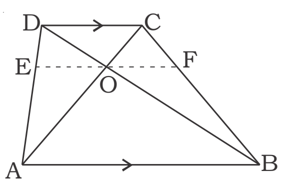
4. In ΔABC, $\angle B = \angle C$, D and E are the points on $\overline{AB}$ and $\overline{AC}$ such that $\overline{BD} = \overline{CE}$, prove that $\overline{DE} \parallel \overline{BC}$.



5. In ΔABC, D & E are the points on AB and $\overline{AC}$ such that $AD \times EC = AE \times DB$. Prove that $DE \parallel BC$



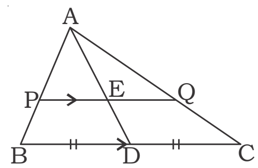
6. ABCD is a quadrilateral in which W, X, Y and Z are the points of trisection of sides AB, BC, CD and DA respectively. Prove that WXYZ is a parallelogram. (Hint: Join A, C)



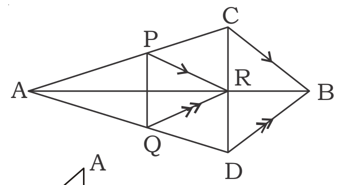
7. ABCD is a trapezium in which $AB \parallel DC$ Diagonals intersect at 'O'. Show that $\frac{AO}{BO} = \frac{CO}{OD}$ (Use BPT to prove it)

   (Hint: Draw $EF \parallel AB \parallel CD$ through 'O' and apply BPT to ΔABD and ΔACD

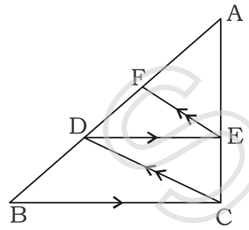
8. In ΔABC, $PQ \parallel BC$ and $BD = DC$ Prove that $PE = EQ$



9. In the figure, $PR \parallel BC$ and $QR \parallel BD$ Prove that $PQ \parallel CD$



10. In ΔABC, $DE \parallel BC$ and $CD \parallel EF$ Prove that $AD^2 = AF \times AB$



Criteria for similarity of triangles

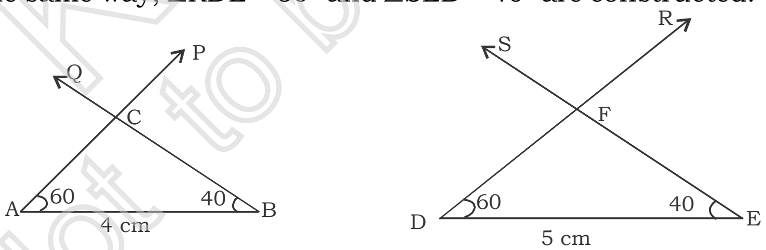
Recall that we have certain criteria for determining the congruency of triangles. In the same way what is the criteria for determining the similarity of triangles? Is it necessary to check all the six pairs of corresponding elements of two triangles? let us study the criteria for similarity of two triangles. Let us discover these criteria through activities.

Angle-Angle (AA) similarity criterion for two triangles

You are familiar with construction of triangles for given measurments. Consider the following activity.

> Criteria means a standard which is established so that judgement or decision, especially a scientific one can be made.

Two line segments AB and DE of lengths 4 cm and 5 cm respectively are drawn. At A and B angles $\angle PAB = 60°$ and $\angle QBA = 40°$ are constructed. In the same way, $\angle RDE = 60°$ and $\angle SED = 40°$ are constructed.



Let the rays AP and BQ intersect at C and rays DR and ES intersect at F. Now, we have the triangles ABC and DEF.

Measure $\angle ACB$ and $\angle DFE$. What do you observe? In ΔABC and ΔDEF, you can see that $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$. That is, the corresponding angles of the two triangles are equal.

What can you say about their corresponding sides? Note that, the ratio of the corresponding 2 sides AB and DE, i.e., $\frac{AB}{DE} = \frac{4}{5} = 0.8$

Now, measure the other sides of the two triangles and find the ratios of the corresponding sides. What do you find? You will find that $\frac{BC}{EF}$ and $\frac{AC}{DF}$ are also equal to 0.8.

Repeat this activity by constructing several pairs of equiangular triangles. In each case, you will find that the corresponding sides are in proportion. This activity leads us to the criterion for similarity of triangles which is referred to as AA criteria (Angle-Angle Criteria).

It is stated as "In two triangles, if the corresponding angles are equal, then their corresponding sides will be in proportion and hence the two triangles are similar".

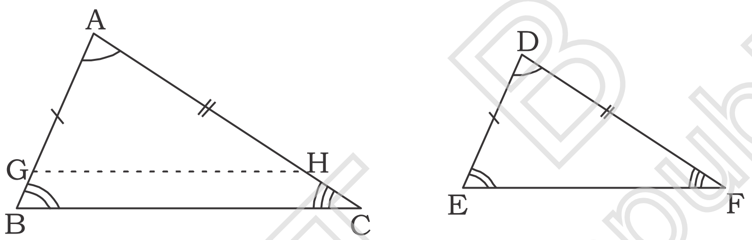
OR

If two triangles are equiangular, then their corresponding sides are proportional".

Now, let us prove this statement logically.

Theorem (AA similarity Criterion)

"If two triangles are equiangular, then their corresponding sides are proportional".



Data : In ΔABC and ΔDEF

(i) $\angle BAC = \angle EDF$ (ii) $\angle ABC = \angle DEF$

To prove: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

Construction: Mark points 'G' and 'H' on AB and AC such that

(i) AG = DE and (ii) AH = DF Join G and H

Proof:

Statement	Reason
compare ΔAGH and ΔDEF,	
$AG = DE$	[$\because$ Construction]
$\angle GAH = \angle EDF$	[$\because$ Data]
$AH = DF$	[$\because$ Construction]
$\therefore \Delta AGH \cong \Delta DEF$	[$\because$ SAS]
$\angle AGH = \angle DEF$	[$\because$ CPCT]
But $\angle ABC = \angle DEF$	[$\because$ Data]

$\Rightarrow \angle AGH = \angle ABC$ [$\because$ Axiom-1]

$\therefore GH \parallel BC$ [$\because$ If corresponding. angles are equal then lines are $\parallel$.]

$\therefore$ in ΔABC $\frac{AB}{AG} = \frac{BC}{GH} = \frac{CA}{HA}$ [$\because$ third corollary to Thales theorem]

Hence $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$ [$\because \Delta AGH \cong \Delta DEF$]

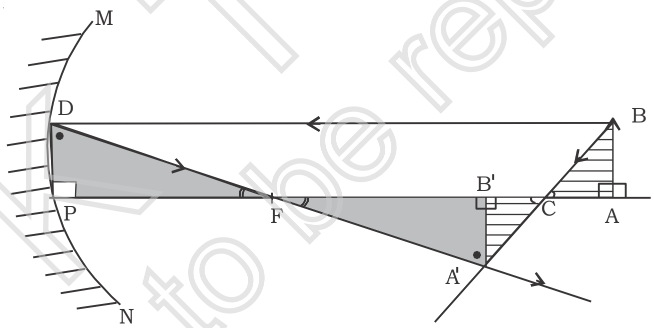
QED

We know that, if two angles of a triangle are equal to corresponding two angles of another triangle, then by the angle sum property of a triangle their thrid angles will also be equal. Therefore, AAA similarity criterion can also be stated as follows: "If two angles of one triangle are equal to corresponding two angles of another triangle, then the two triangles are similar. This is reffered to as the **AA similarity criterion** for two triangles.

Now, let us study some applications of similar triangles.

1. Application of A A criteria to derive mirror formula.

Let AB be the object kept beyond 'C'. Its image will be A'B' formed between F and C. Nature of image will be inverted, diminished and real.



Compare △ DPF and △ B^IA^IF

$\angle DPF = \angle A^IB^IF = 90^0$

$D\hat{F}P = A^I\hat{F}B^I$ [$\because$ Vertically opposite angles]

$\Rightarrow$ △ DPF $\sim$ △ A^IB^IF [$\because$ Equiangular triangles]

$\frac{DP}{A^IB^I} = \frac{PF}{B^IF} = \frac{DF}{A^IF}$ [$\because$ A A criteria]

$$\boxed{\frac{DP}{A'B'}} = \frac{f}{v-f} = \frac{DF}{A'F} \qquad \text{....... (i)}$$

Compare $\triangle$ ABC and $\triangle$ B'A'C

$\angle BAC = \angle A'B'C = 90^0$

$\angle BCA = \angle ACB$ [$\because$ Vertically opposite angles]

$\therefore \angle ABC = B'A'C$

$\Rightarrow \triangle ABC \sim \triangle B'A'C$ [$\because$ Equiangular triangles]

$$\frac{AB}{B'A'} = \frac{AC}{B'C} = \frac{BC}{A'C} \qquad [\because \text{ AA criteria}]$$

$$\frac{AB}{B'A'} = \frac{u-2f}{2f-v} \qquad \text{...... (ii)}$$

compare (i) and (ii), Since DP = AB, we get

$$\frac{f}{v-f} = \frac{u-2f}{2f-v}$$

$uv - uf - 2vf + \cancel{2f^2} = \cancel{2f^2} - vf$

$uv - uf = 2vf - vf$

$uv - uf = vf$

Divide the equation by (uvf)

$(uv - uf = vf) \div uvf$

$$\Rightarrow \frac{1}{f} - \frac{1}{v} = \frac{1}{u}$$

$$\Rightarrow \frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

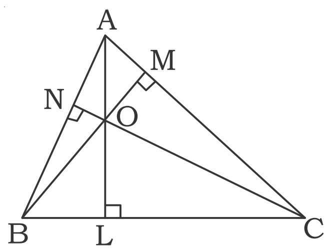
Illustrative Examples

Example 1: In $\triangle$ABC, AL, BM and CN are the altitudes which concurr at 'O'.

Prove that (i) $\triangle AMB \sim \triangle ANC$ **(ii) $\frac{AN}{BN} \cdot \frac{BL}{CL} \cdot \frac{CM}{AM} = 1$**

Sol. Data : In $\triangle$ABC, AL, BM and CN are the altitudes.

To prove : (i) $\triangle AMB \sim \triangle ANC$ (ii) $\frac{AN}{BN} \cdot \frac{BL}{CL} \cdot \frac{CM}{AM} = 1$



Proof : In $\triangle$AMB and $\triangle$ANC

$\angle AMB = \angle ANC = 90°$ ($\because$ Data)

$\angle BAM = \angle NAC$ ($\because$ common angle)

$\therefore \triangle AMB \sim \triangle ANC$ ($\because$ AA criteria)

|||ly $\triangle BLA \sim \triangle BNC$

and $\triangle BMC \sim \triangle ALC$ can be proved

$\triangle AMB \sim \triangle ANC$ $\therefore \dfrac{AN}{AM} = \dfrac{AC}{AB}$

$\triangle BLA \sim \triangle BNC$ $\therefore \dfrac{BL}{BN} = \dfrac{AB}{BC}$

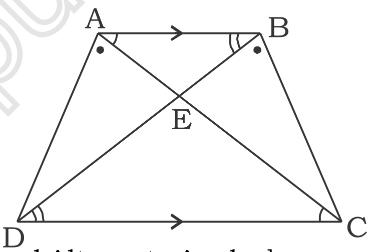
$\triangle BMC \sim \triangle ALC$ $\therefore \dfrac{CM}{CL} = \dfrac{BC}{AC}$

$$\frac{AN}{AM} \times \frac{BL}{BN} \times \frac{CM}{CL} = \frac{AC}{AB} \times \frac{AB}{BC} \times \frac{BC}{AC}$$

$$\therefore \frac{AN}{BN} \times \frac{BL}{LC} \times \frac{CM}{AM} = 1$$

or AN × BL × CM = BN × LC × AM

Example 2: In the trapezium ABCD, (i) AB || DC (ii) $\Delta AED \sim \Delta BEC$. Prove that AD = BC



Data: In the ⏢ABCD

(i) AB || DC

(ii) $\Delta AED \sim \Delta BEC$

To Prove: AD = BC

Proof: Compare ΔEDC and ΔEBA

$\angle EDC = \angle EBA$ [$\because$ AB || DC and Alternate Angles]

$\angle ECD = \angle EAB$

$\therefore \Delta EDC \sim \Delta EBA$ [$\because$ Equiangular triangles]

$\Rightarrow \dfrac{ED}{EB} = \dfrac{EC}{EA}$ [$\because$ AA]

$\Rightarrow \dfrac{ED}{EC} = \dfrac{EB}{EA}$

But, $\Delta AED \sim \Delta BEC$ [$\because$ Data]

$\Rightarrow \dfrac{ED}{EC} = \dfrac{EA}{EB} = \dfrac{AD}{BC}$ [$\because$ AA]

$\Rightarrow \dfrac{EB}{EA} = \dfrac{EA}{EB}$ [$\because$ Axiom 1]

$EA^2 = EB^2$

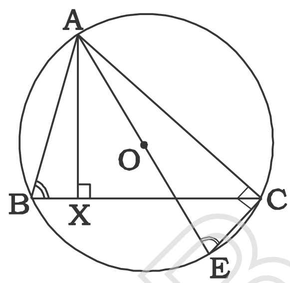
$EA = EB$

$\Rightarrow \quad \frac{EA}{EB} = \frac{AD}{BC} = 1$

$\therefore \quad AD = BC$

Example 3: Brahma Gupta's theorem (628 A.D.) prove that,

"The rectangle contained by any two sides of a triangle, is equal to rectangle contained by altitude drawn to the third side and the circum diameter."



Data: In ΔABC, $AX \perp BC$

'O' is the circumcenter of ΔABC, AE is the diameter

To prove: AB.AC = AX.AE

Construction : Join E and C

Proof: Compare ◺AXB and ◺ACE

$\angle AXB = \angle ACE = 90°$ [$\because$ angle in a semicircle is a right angle]

$\angle ABX = \angle AEC$ [$\because$ angles in the same segment]

$\therefore$ ◺AXB ~ ◺ACE [$\because$ Equiangular triangles]

$\Rightarrow \quad \frac{AX}{AC} = \frac{AB}{AE}$ [$\because$ AA criteria]

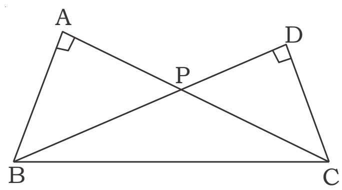
$\Rightarrow$ AB.AC = AX.AE QED

EXERCISE 10.4

Riders based on AA similarity criteria.

1. ◺BAC and ◺BDC are two right angled triangles with common hypotenuse BC. The sides AC and BD intersect at 'P'

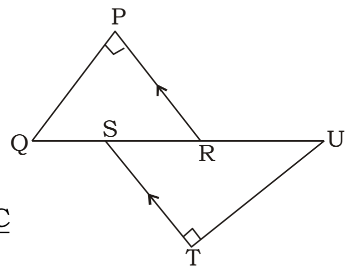
   Prove that AP. PC = DP. PB



2. In the figure

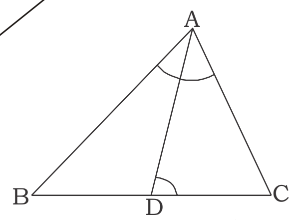
   $\angle QPR = \angle UTS = 90^0$ and $PR \parallel TS$

   Prove that $\triangle PQR \sim \triangle TUS$



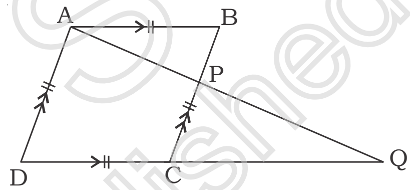
3. 'D' is a point on BC such that $\angle ADC = \angle BAC$

   Prove that $\frac{CA}{CD} = \frac{CB}{CA}$



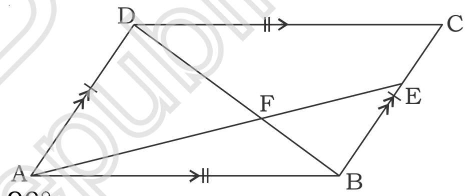
4. If the diagonals of a quadrilateral divide each other proportionally, then prove that the quadrilateral is a trapezium.

5. ABCD is a rhombus. 'P' is any point on BC, AP is joined and produced to meet $\overline{DC}$ produced at Q. Prove that $\frac{1}{BC} = \frac{1}{PC} - \frac{1}{CQ}$



6. The diagonal BD of a $\parallel^{gm}$ ABCD intersects AE at 'F'. 'E' is any point on BC.

   Prove that DF.EF = FB. FA

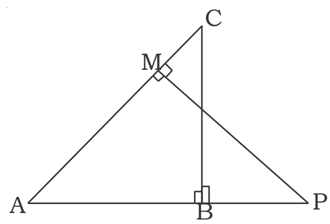


7. In the adjoining figure $\angle ABC = 90^\circ$ and $\angle AMP = 90^\circ$

   Prove that

   (i) $\triangle ABC \sim \triangle AMP$

   (ii) $\frac{CA}{PA} = \frac{BC}{MP}$

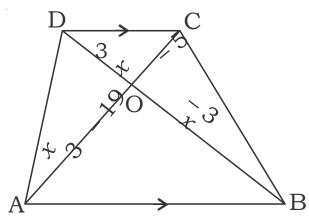


8. In the ⏢ABCD,

   AO = $3x - 19$; OC = $x - 5$

   BO = $x - 3$; OD = 3

   Find 'x'



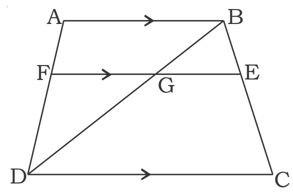
9. In the trapezium ABCD

   $AB \parallel DC$

   $EF \parallel AB$ and $DC = 2AB$,

   $\frac{BE}{EC} = \frac{3}{4}$

   Prove that $7EF = 10AB$



10. If the mid points of three sides of a triangle are joined in an order, then prove that the four triangles so formed are similar to each other and to the original triangle.

Side-Side-Side (SSS) similarity criterion for two triangles

You have seen in the previous theorem that, if the three angles (AAA) of one triangle are equal to the corresponding three angles (AAA) of another triangle, then their corresponding sides (SSS) are proportional.

Let us state the converse of this statement.

"If the three sides (SSS) of a triangle are proportional to the corresponding three sides (SSS) of another triangle, then their corresponding angles (AAA) are equal".

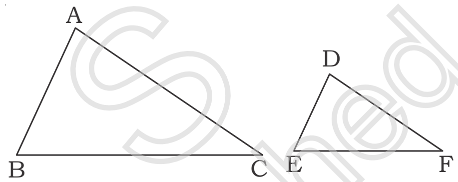
In ΔABC and ΔDEF,

if $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$

then $\angle BAC = \angle EDF$

$\angle ABC = \angle DEF$

$\angle BCA = \angle EDF$



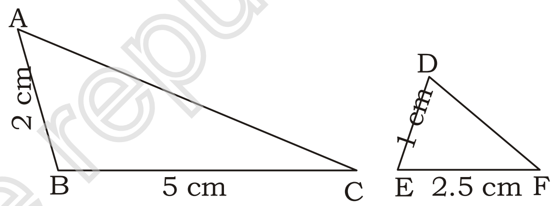
i.e., ΔBAC and ΔDEF are equiangular and hence similar.

Observe: The following triangles In ΔABC and ΔDEF

$\frac{AB}{DE} = \frac{2}{1} = 2;$

$\frac{BC}{EF} = \frac{5}{2.5} = 2$

$\therefore \frac{AB}{DE} = \frac{BC}{EF}$



But the two triangles ABC and DEF are **not** equiangular (one is obtuse angled and another is an acute angled and hence they are not similar.)

From the above example we can say that if two sides of one triangle are proportional to two sides of another triangle then those two triangles **need not** be equiangular and hence they are not similar.

We can conclude that, : For two triangles to be similar, the fundamental criteria is that all the three sides of one triangle must be proportional to all the three sides of another triangle.

Note: Discuss the proof for SSS criteria in groups

Side Angle Side (SAS) similarity criterion for two triangles.

If two sides of one triangle are proportional to two sides of another triangle and the angles formed by those sides are equal, then the two triangles are equiangular and therefore they are similar.

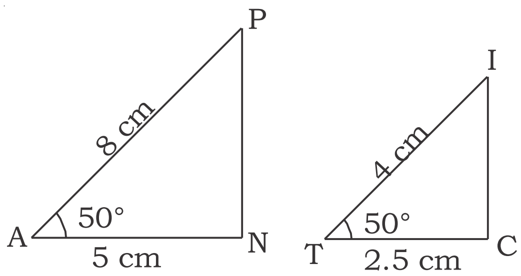
Data : In ΔPAN & ΔITC

1. $\frac{PA}{IT} = \frac{AN}{TC}$
2. $\angle PAN = \angle ITC$

then, $\angle APN = \angle TIC$

$\angle ANP = \angle TCI$

Hence $\Delta PAN \sim \Delta ITC$



Exercise 10.5

1. In which of the following cases the pairs of triangles are similar? Write the similarity criterion used by you for answering the question and also write the pairs of similar triangles in the symbolic form.

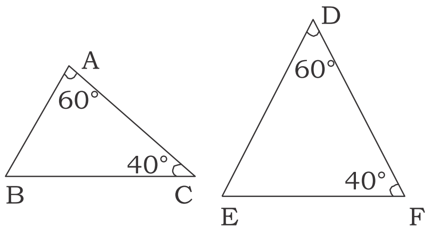


Fig 1

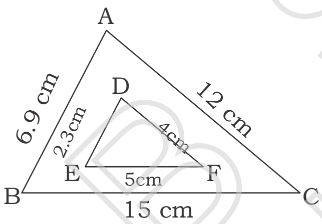


Fig 2

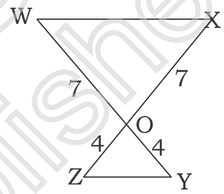


Fig 3

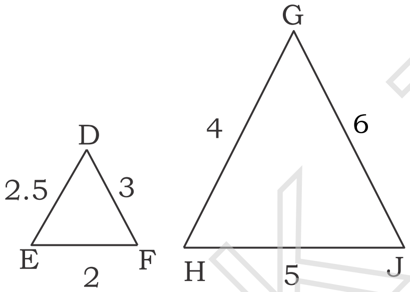


Fig 4

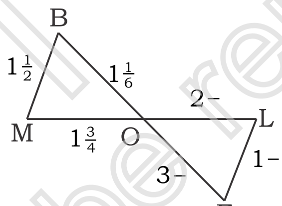


Fig 5

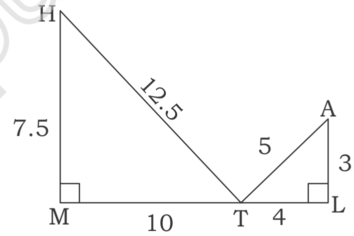


Fig 6

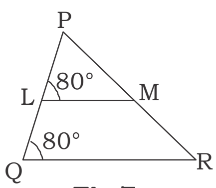


Fig 7

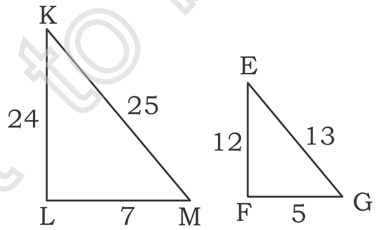


Fig 8

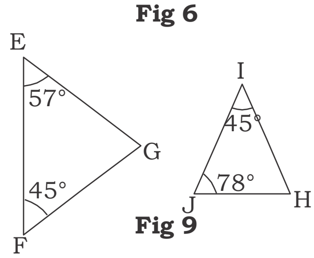


Fig 9

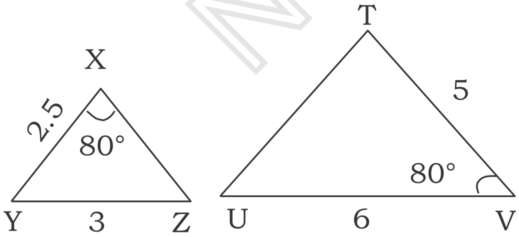
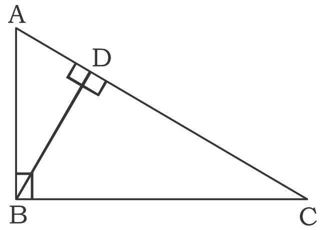


Fig 10

Theorem:-

"In a right angled triangle, the perpendicular to the hypotenuse from the right angled vertex, divides the original triangle into two right angled triangles, each of which is similar to the original triangle."



Data:- In $\triangle$ ABC, (i) $\angle ABC = 90°$.

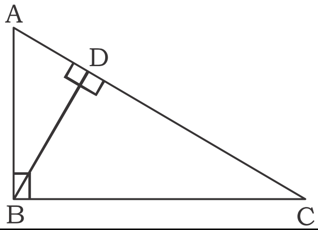
(ii) $BD \perp AC$.

To Prove:- $\triangle ABD \sim \triangle BDC \sim \triangle ABC$

	Statement	Reason
Proof:- Compare,	$\triangle$ADB and $\triangle$ABC	
	(i) $\angle ADB = \angle ABC = 90°$	Data
	(ii) $\angle BAD = \angle CAB$	Common angle
	$\therefore$ (iii) $\angle ABD = \angle ACB$	Sum of 3 angles of a Δ is 180°
	$\therefore$ $\triangle ADB \sim \triangle ABC$...(1)	Equiangular triangles
Compare	$\triangle$BDC and $\triangle$ABC	
	(i) $\angle BDC = \angle ABC = 90°$	Data
	(ii) $\angle BCD = \angle ACB$	Common angle
	$\therefore$ (iii) $\angle DBC = \angle BAC$	Sum of 3 angles of a Δ is 180°
	$\therefore$ $\triangle BDC \sim \triangle ABC$...(2)	Equiangular triangles
Now, we have	$\triangle ABD \sim \triangle BDC \sim \triangle ABC$	(from 1 and 2)
		QED

Note:- In the third step to prove $\triangle ADB \sim \triangle BDC$ we have used the results of first two steps and axiom 1. Alternatively without all these we can show that they are similar by comporing the angles and show that they are equiangular and hence similar. Discuss in groups and the proof separately.

COROLLARIES

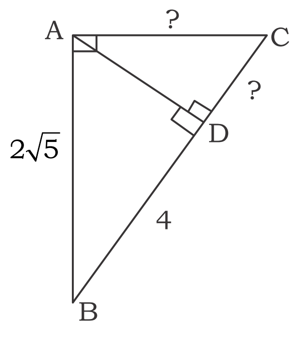


Corollary - 1	Corollary - 2	Corollary - 3
$\triangle ADB \sim \triangle ABC$ [$\because$ Equiangular]	$\triangle BDC \sim \triangle ABC$ [$\because$ Equiangular]	$\triangle ADB \sim \triangle BDC$ [$\because$ Equiangular]
$\therefore \frac{AD}{AB}=\frac{DB}{BC}=\frac{AB}{AC}$ [$\because$ AA criteria]	$\because \frac{BD}{AB}=\frac{DC}{BC}=\frac{BC}{AC}$ [$\because$ AA criteria]	$\because \frac{AD}{BD}=\frac{DB}{DC}=\frac{AB}{BC}$ [$\because$ AA criteria]
$\Rightarrow AB^2 = AC.AD$	$\Rightarrow BC^2 = AC.DC$	$\Rightarrow BD^2 = AD.DC$
Now recall the definition of gemetric mean (G.M) From the above equation, we can say that AB is the geometric mean of AC and AD.	By the definition of geometric mean (G.M), we can say that BC is the geometric mean of AC and DC.	Again by the definition of geometric mean (G.M) we can say that BD is the geometric mean of AD.DC.

Note:-

The converse for the above three corollaries exists and are true.
So discuss in groups, the proof for each and write them separately.

Example 1: In $\triangle$ ABC, $\angle BAC = 90°$ and AD is the altitude. If $AB = 2\sqrt{5}$, $BD = 4$. Then find BC and AC.



Sol. $AB^2 = BC.BD$ [$\therefore$ Corollary]

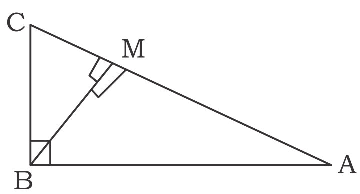
$(2\sqrt{5})^2 = BC.4 \quad \therefore \frac{20}{4} = BC \quad \therefore 5 = BC$

$CD = BC - BD = 5 - 4 = 1$

$AC^2 = BC.CD \quad AC^2 = 5 \times 1 \quad \therefore AC = \sqrt{5}$

Example 2: In $\triangle$ ABC, $\angle ABC = 90°$ and BM is the altitude.

If AM = 16MC Prove that AB = 4 BC



Sol. $AB^2 = AC.AM$ [$\because$ Corollary]

$AB^2 = AC.\ 16\ MC \quad \therefore AB = 4\sqrt{AC.\ MC}$

$BC^2 = AC.MC$ [$\because$ Corollary]

$\therefore BC = \sqrt{AC.\ MC}$

But $AB = 4\sqrt{AC.\ MC}$

$\Rightarrow AB = 4\ BC \quad [\because \sqrt{AC.\ MC} = BC]$

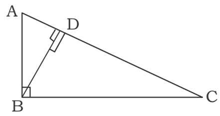
EXERCISE 10.6

1. In $\triangle$ ABC, $\angle ABC = 90°$, $BD \perp AC$

   (a) If BD = 8cm, AD = 4cm, find CD

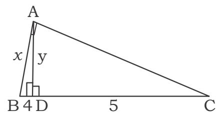
   (b) If AB = 5.7cm, BD = 3.8cm, CD = 5.4cm, find BC

   (c) If AB = 75cm, BC = 1m, AC = 1.25m, find BD



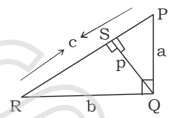
2. In $\triangle$ ABC, $\angle BAC = 90°$, $AD \perp BC$,

   BD = 4cm, DC = 5cm, Find x and y.



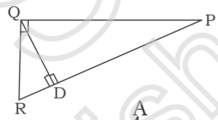
3. In $\triangle$ PQR, $\angle PQR = 90°$, $QS \perp PR$.

   If PQ = a, QR = b, RP = c and QS = p, show that pc = ab.



4. In $\triangle$ PQR, $\angle PQR = 90°$, $QD \perp PR$.

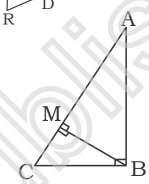
   If PD = 4DR. Prove that PQ = 2QR.



5. In $\triangle$ ABC, $\angle ABC = 90°$, $BM \perp AC$

   (a) BM = $x + 2$, AM = $x + 7$, CM = x, find x.

   (b) AM = $8x^2$, MC = $2x^2$, then find BM and AB.



Areas of similar Triangles

So far we have studied the two fundamental properties regarding angles and sides of similar triangles. You may now be very eager to know the relationship between the areas of similar triangles.

To understand the relation, conduct the following activitiy. Consider two similar triangles ABC and DEF, with the following measurments.

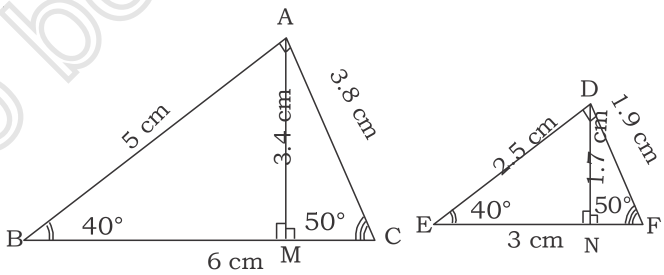
In ΔABC and ΔDEF,

$\angle BAC = \angle EDF = 90°$

$\angle ABC = \angle DEF = 40°$

$\angle BCA = \angle EFD = 50°$

$\therefore \Delta ABC \sim \Delta DEF$



$$\frac{AB}{DE} = \frac{5}{2.5} = 2 \quad \frac{BC}{EF} = \frac{6}{3} = 2 \quad \frac{CA}{FD} = \frac{3.8}{1.9} = 2 \quad \therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD} = 2$$

Now let us consider the ratio between their areas.

Let $AM \perp BC$ and $DN \perp EF$.

AM = 3.4cm, DN = 1.7 cm.

$$\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{\frac{1}{2} \times BC \times AM}{\frac{1}{2} \times EF \times DN} = \frac{\frac{1}{2} \times 6 \times 3.4}{\frac{1}{2} \times 3 \times 1.7} = 4$$

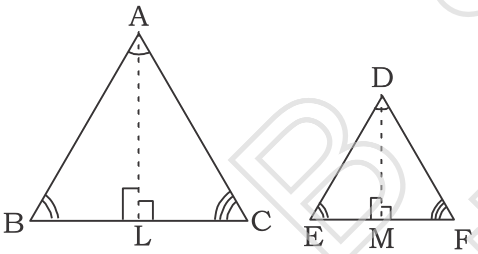
This means that area of ΔABC is four times the area of ΔDEF.

Repeat this activity for two right angled triangles and two obtuse angled triangles which are similar. What is your inference? Discuss in groups.

We can generalize this in the following way :

"The areas of similar triangles are proportional to squares on the corresponding sides."

Theorem : "The areas of similar triangles are proportional to the squares of the corresponding sides"



Data: ΔABC ~ ΔDEF

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$$

To prove: $\frac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \frac{BC^2}{EF^2}$

Const.: Draw AL ⊥ BC and DM ⊥ EF

Proof:

Statement	Reason
Compare $\triangle$ALB and $\triangle$DME	
$\angle ABL = \angle DEM$	[∵ Data]
$\angle ALB = \angle DME = 90^0$	[∵ construction]
∴ $\triangle$ALB ~ $\triangle$DME	[∵ Equiangular]
⇒ $\frac{AL}{DM} = \frac{AB}{DE}$	[∵ AA - criteria]
but $\frac{BC}{EF} = \frac{AB}{DE}$	[∵ Data]

$\therefore \quad \dfrac{AL}{DM} = \dfrac{BC}{EF}$ $\qquad$ [$\because$ Transitive property]

$\therefore \quad \dfrac{\text{Area of } \Delta ABC}{\text{Area of } \Delta DEF} = \dfrac{\frac{1}{2} \times BC \times AL}{\frac{1}{2} \times EF \times DM}$ $\qquad$ $\left(\because \text{Area of } \Delta = \dfrac{1}{2} \times b \times h\right)$

Now, $\dfrac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \dfrac{BC \times AL}{EF \times DM}$

$= \left(\dfrac{BC}{EF}\right) \times \left(\dfrac{AL}{DM}\right)$

$= \dfrac{BC}{EF} \times \dfrac{BC}{EF}$ $\qquad$ $\left[\because \dfrac{AL}{DM} = \dfrac{BC}{EF} \text{ is proved}\right]$

$= \dfrac{BC^2}{EF^2}$

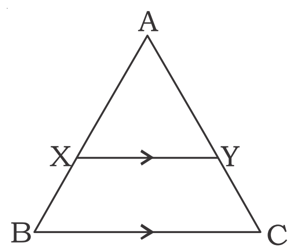
$\therefore \dfrac{ar\ (\Delta ABC)}{ar\ (\Delta ABC)} = \dfrac{BC^2}{EF^2}$

From data, $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$

$\therefore \dfrac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \dfrac{AB^2}{DE^2} = \dfrac{BC^2}{EF^2} = \dfrac{AC^2}{DF^2}$

Illustrative Examples

1. In ΔABC, $\overline{XY} \parallel \overline{BC}$ and XY divides the triangle into two parts of equal areas. Find $\left(\dfrac{BX}{AB}\right)$



Data: In ΔABC

(i) $XY \parallel BC$

(ii) $\Delta ABC = 2\Delta AXY$

To find: $\dfrac{BX}{AB}$

Solution: $\Delta ABC \sim \Delta AXY$ $\qquad$ [$\because$ $XY \parallel BC$]

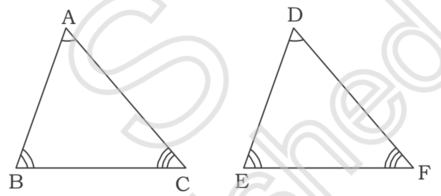
$\dfrac{ar(\Delta ABC)}{ar(\Delta AXY)} = \dfrac{AB^2}{AX^2}$ $\qquad$ [$\because$ Thales theorem]

$\frac{2}{1} = \frac{AB^2}{AX^2}$ $\frac{\sqrt{2}}{1} = \frac{AB}{AX}$ $\frac{1}{\sqrt{2}} = \frac{AX}{AB}$ $1 - \frac{1}{\sqrt{2}} = 1 - \frac{AX}{AB}$ $\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{AB - AX}{AB}$

$$\boxed{\frac{\sqrt{2}-1}{\sqrt{2}} = \frac{BX}{AB}}$$

Important results : 1. $\frac{AX}{AB} = \frac{1}{\sqrt{2}}$ 2. $\frac{XB}{AB} = \frac{2-\sqrt{2}}{2}$ 3. $\frac{AX}{XB} = \frac{1}{\sqrt{2}-1}$

2. "If the area of two similar triangles are equal, then they are congruent"- Prove.



Data: * $\Delta ABC \sim \Delta DEF$

* ar (ΔABC) = ar (ΔDEF)

To Prove: $\Delta ABC \cong \Delta DEF$

Proof: $\frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2}$ [$\because$ Theorem]

$1 = \frac{AB^2}{DE^2} = \frac{BC^2}{EF^2} = \frac{CA^2}{FD^2}$ [$\because$ areas are equal by data]

$\Rightarrow$ $AB^2 = DE^2$ $\therefore AB = DE$

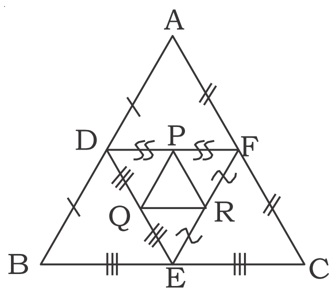
$BC^2 = EF^2$ $\therefore BC = EF$

$CA^2 = FD^2$ $\therefore CA = FD$

$\therefore \Delta ABC \cong \Delta DEF$ [$\because$ SSS]

3. D, E and F are the mid points of sides of ΔABC. P,Q and R are the mid points of sides of ΔDEF. This process of marking the midpoints and forming a new triangle is continued.

How are the areas of triangles so formed related? Investigate and draw inference.



We know that

$\Delta ADF \sim \Delta DBE \sim \Delta CEF \sim \Delta ABC$

$$\therefore \frac{ar\ (\Delta ABC)}{ar\ (\Delta DEF)} = \frac{BC^2}{DF^2} = \frac{BC^2}{\frac{1}{4}BC^2} = 4$$

$$\therefore \Delta DEF = \frac{1}{4}\Delta ABC$$

$$\therefore \frac{ar\ (\Delta ABC)}{ar\ (\Delta PQR)} = \frac{BC^2}{QR^2} = \frac{BC^2}{\frac{1}{16}BC^2} = 16$$

$$\therefore \Delta PQR = \frac{1}{16}\Delta ABC$$

So the areas are $1, \frac{1}{4}, \frac{1}{16}$ ....

These are in G.P with r = $\frac{1}{4}$

Know this!

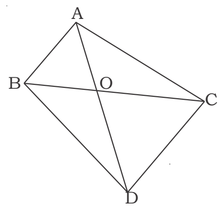
Areas of similar triangles are proportional to

1. Squares on their corresponding sides.
2. Squares on their corresponding altitudes.
3. Squares on their corresponding medians.
4. Squares on their corresponding circum-radii.
5. Squares on their corresponding angular bisectors.
6. Squares on their corresponding in-radii.

Note : The converse of the above six statements also holds good.

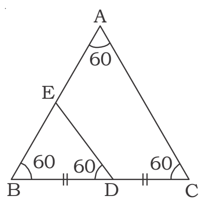
Exercise 10.7

Riders based on areas of similar triangles.



1. ΔABC and ΔBDC are on the same base BC

   Prove that $\frac{ar\ (\Delta ABC)}{ar\ (\Delta DBC)} = \frac{AO}{DO}$

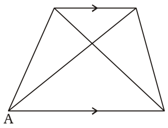


2. ΔABC and ΔBDE are two equilateral triangles and BD = DC. Find the ratio between areas of ΔABC and ΔBDE.

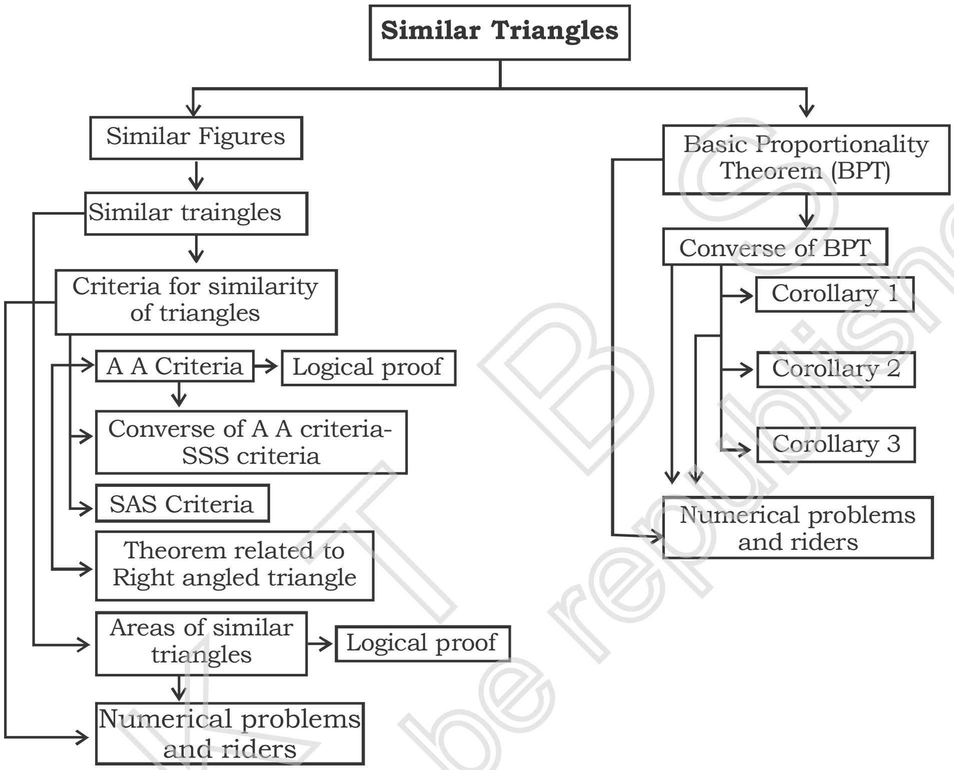
3. Two isosceles triangles are having equal vertical angles and their areas are in the ratio 9 : 16. Find the ratio of their corresponding altitudes.

4. The corresponding altitudes of two similar triangles are 3cm and 5cm respectively. Find the ratio between their areas.

5. In the trapezium ABCD, AB || CD, AB = 2CD and ar (ΔAOB) = 84cm^2, find the area of ΔCOD



6. In the above figure find the ratios between areas of ΔAOB and ΔCOD, if AB = 3CD.
7. Prove that the ratio of the areas of two similar triangles is equal to the square of the ratio of their corresponding medians.



ANSWERS

EXERCISE 10.1

3] (i) $x = 4$, $y = 9$ (ii) $x = 2.64$, $y = 0.96$ (iii) $a = 9$, b = 3 6] (i) 3:1 (ii) 15cm (iii) 24cm (iv) 4cm (v) 4cm 7] 15cm, 21cm 8] 2.6cm 10] 88m 11] 2 ft

EXERCISE 10.2

2] BE = 10.5cm, CE = 7.5cm 3] 2.1cm 4] (i) 20cm (ii) 6cm (iii) $x = 1$
5] 7.5cm 8] 6cm 9] AK = 12cm, PB = 10cm 10] 8 ft

EXERCISE 10.6

1] (a) 16cm (b) 6.6 cm (c) 0.6m 2] x = 6cm, y = $2\sqrt{5}$ cm 5] (a) $\frac{4}{3}$ (b) $4x^2, 4x^2\sqrt{5}$

EXERCISE 10.7

2] 4:1 3] 3:4 4] 9:25 5] 21 cm^2 6] 9:1