Simple Harmonic Motion

8.01

Week 12D1

Today's Reading Assignment

MIT 8.01 Course Notes

Chapter 23 Simple Harmonic Motion

Sections 23.1-23.4

Announcements

Sunday Tutoring in 26-152 from 1-5 pm

Problem Set 9 due Nov 19 Tuesday at 9 pm in box outside 26-152

Math Review Nov 19 Tuesday at 9-10:30 pm in 26-152

Exam 3 Tuesday Nov 26 7:30-9:30 pm
Conflict Exam 3 Wednesday Nov 27 8-10 am, 10-12 noon

Nov 20 Drop Date

Mass on a Spring C2: Simple Harmonic Motion

Hooke's Law

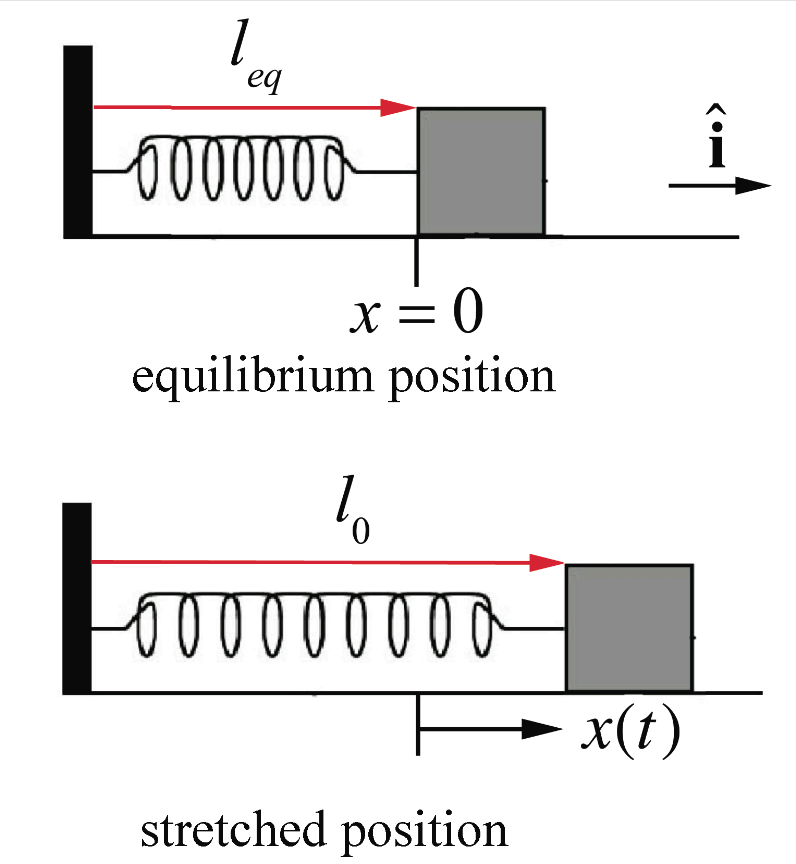
Define system, choose coordinate system.

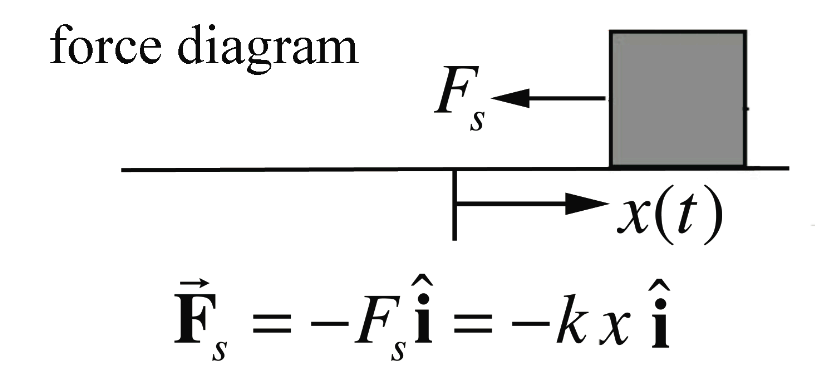
Draw free-body diagram.

Hooke's Law

$$\vec{\mathbf{F}}_{\text{spring}} = -kx\,\hat{\mathbf{i}}$$

$$-kx = m\frac{d^2x}{dt^2}$$





Concept Q.: Simple Harmonic Motion

Which of the following functions $x(t)$ has a second derivative which is proportional to the negative of the function

$$\frac{d^2x}{dt^2} \propto -x?$$

1. $x(t) = \frac{1}{2}at^2$
2. $x(t) = Ae^{t/T}$
3. $x(t) = Ae^{-t/T}$
4. $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$

SHM: Angular Frequency

Newton's Second Law $F_x = -kx \Rightarrow \text{SHM}$

Simple Harmonic Oscillator Differential Equation (SHO)
$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x$$

Particular Solution:
$$x(t) = C\cos((2\pi / T)t)$$

Required Condition:
$$\frac{d^2x}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 x$$

$$2\pi / T = \sqrt{k / m}$$

Angular Frequency:
$$\omega_0 = 2\pi f = 2\pi / T = \sqrt{k / m}$$

Notation for particular solution: $x(t) = C\cos(\omega_0 t)$

Mass on a Spring C2: Demonstrate Initial Conditions

Summary: SHO

Equation of Motion:

$$-kx = m\frac{d^2x}{dt^2}$$

Solution: Oscillatory with Period

$$T = 2\pi / \omega_0 = 2\pi\sqrt{m/k}$$

Position:

$$x = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

Velocity:

$$v_x = \frac{dx}{dt} = -\omega_0 C\sin(\omega_0 t) + \omega_0 D\cos(\omega_0 t)$$

Initial Position at *t* = 0:

$$x_0 \equiv x(t=0) = C$$

Initial Velocity at *t* = 0:

$$v_{x,0} \equiv v_x(t=0) = \omega_0 D$$

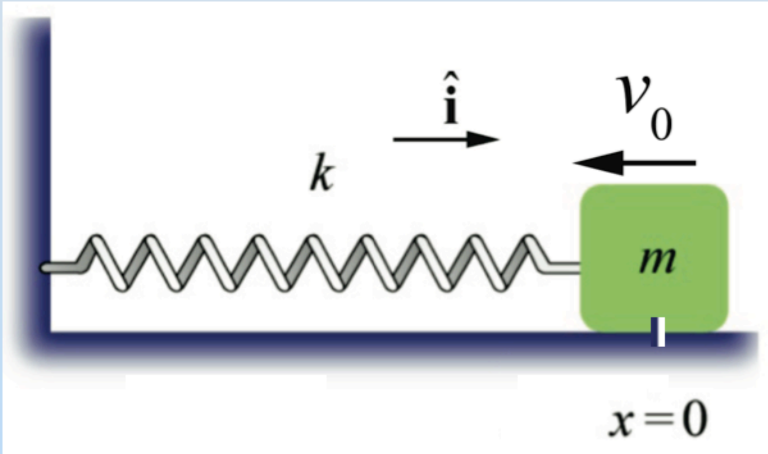
General Solution:

$$x = x_0\cos(\omega_0 t) + \frac{v_{x,0}}{\omega_0}\sin(\omega_0 t)$$

Table Problem: Simple Harmonic Motion Block-Spring

A block of mass m , attached to a spring with spring constant k, is free to slide along a horizontal frictionless surface. At $t = 0$ the block-spring system is released from the equilibrium position $x_0 = 0$ and with speed v_0 in the negative x-direction.

a) What is the position as a function of time? b) What is the x-component of the velocity as a function of time?



Demo: Spray Paint Oscillator C4

Illustrating choice of alternative representations for position as a function of time (amplitude and phase or sum of sin and cos)

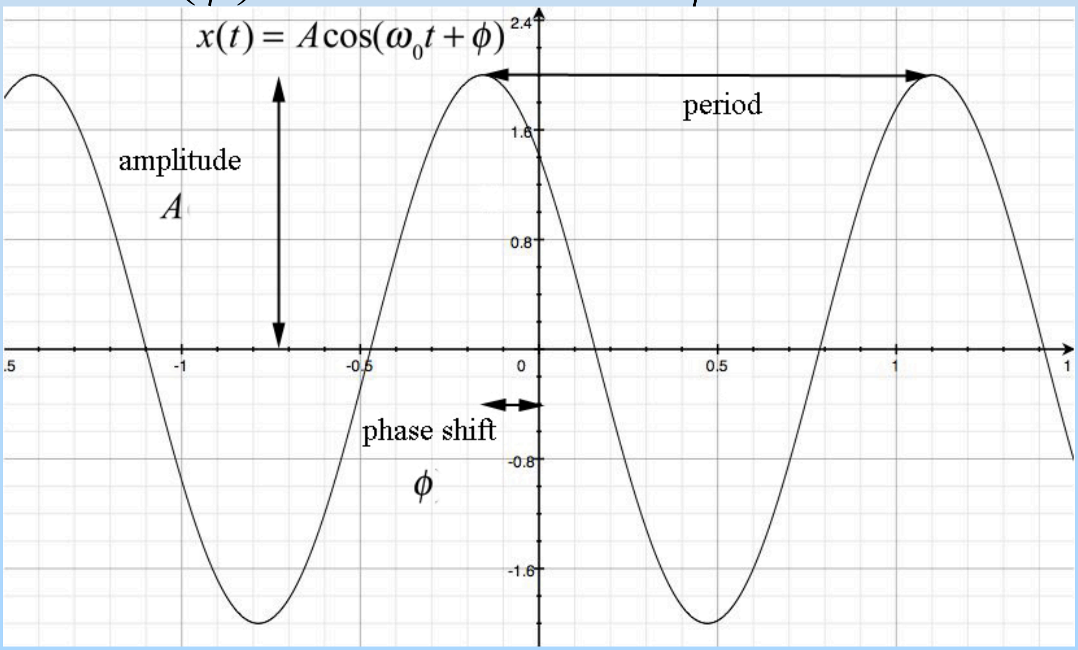
Phase and Amplitude

$$x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

$$x(t) = A\cos(\omega_0 t + \phi) \ \Rightarrow$$

$$C = A\cos(\phi) \qquad A = \sqrt{C^2 + D^2}$$

$$D = -A\sin(\phi) \qquad \tan\phi = -D / C$$

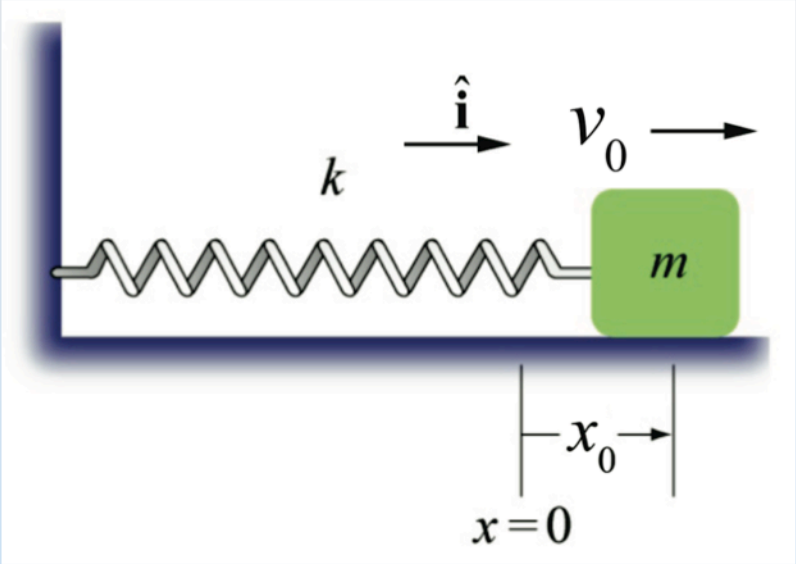


Mass on a Spring: Energy

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$v_x(t) = -\omega_0 A\sin(\omega_0 t + \phi)$$

$$\omega_0 = \sqrt{k/m}, \quad A = \left({x_0}^2 + \frac{v_{x,0}^2}{{\omega_0}^2} \right)^{1/2}$$



Constant energy oscillates between kinetic and potential energies

$$K(t) = (1/2)m(v_x(t))^2 = (1/2)m{\omega_0}^2 A^2 \sin^2(\omega_0 t + \phi)$$

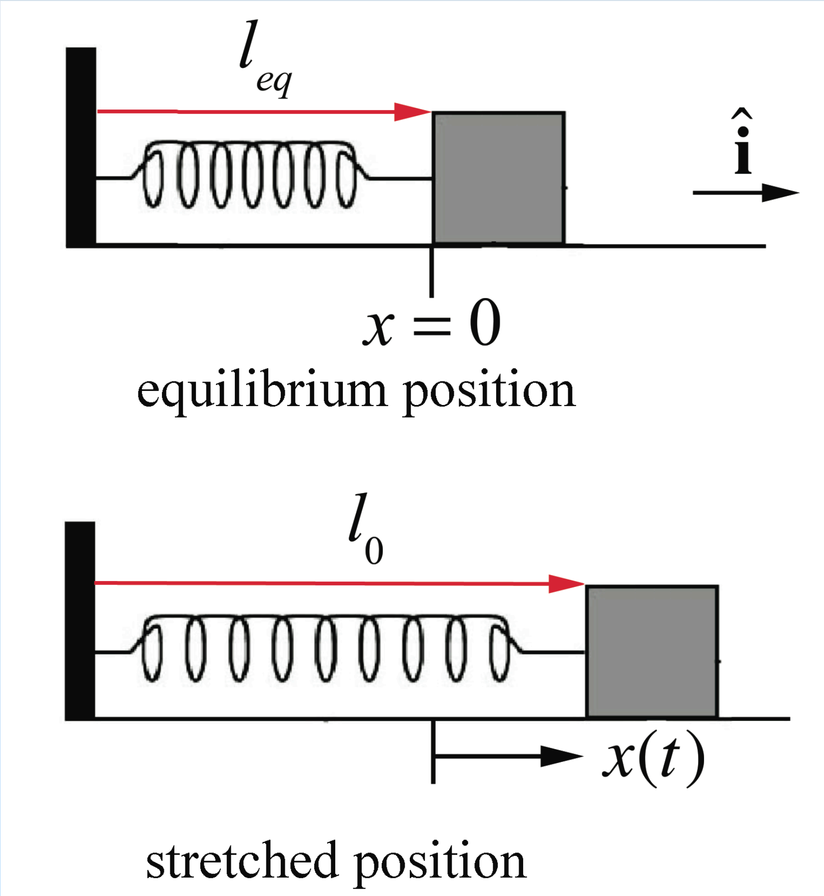
$$K(t) = (1/2)kA^2 \sin^2(\omega_0 t + \phi)$$

$$U(t) = (1/2)kx^2 = (1/2)kA^2 \cos^2(\omega_0 t + \phi)$$

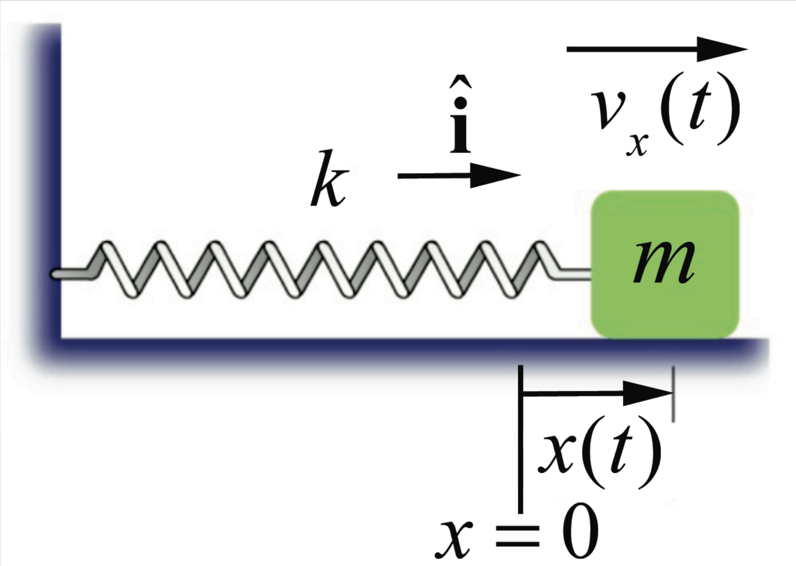
$$E = K(t) + U(t) = (1/2)kA^2 = (1/2)mv_{x,0}^2 + (1/2)k{x_0}^2 = \text{constant}$$

Worked Example: Block-Spring Energy Method

A block of mass is attached to spring with spring constant *k* . The block slides on a frictionless surface. Use the energy method to find the equation of motion for the spring-block system.



Energy and Simple Harmonic Motion



$$E = K(t) + U(t) = (1/2)k(x(t))^2 + (1/2)m(v_x(t))^2$$

Apply chain rule:

$$dE/dt = 0 \;\; \Rightarrow \;\; 0 = kx\frac{dx}{dt} + mv_x\frac{dv_x}{dt}$$

$$0 = kx + m_x\frac{d^2x}{dt^2}$$

Concept Question: SHM Velocity

A block of mass m is attached to a spring with spring constant k is free to slide along a horizontal frictionless surface.
At t = 0 the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest.
What is the x -component of the velocity of the block when it first comes back to the equilibrium?

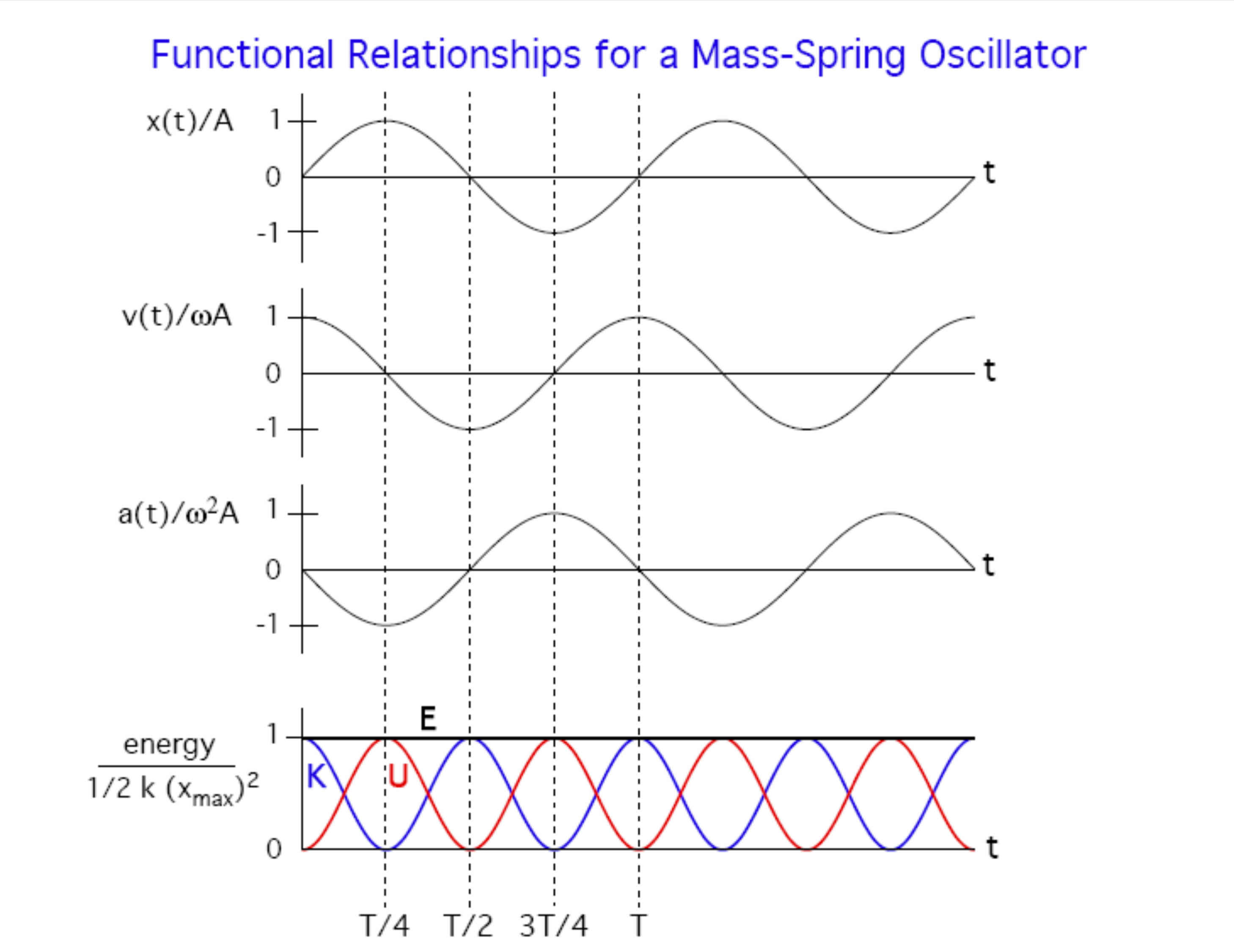
1. $v_x = -x_0 \dfrac{4}{T}$

2. $v_x = x_0 \dfrac{4}{T}$

3. $v_x = -\sqrt{\dfrac{k}{m}} x_0$

4. $v_x = \sqrt{\dfrac{k}{m}} x_0$

Graphical Representations



SHM: Oscillating Systems

$$\frac{d^2x}{dt^2} = -bx$$

$$x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

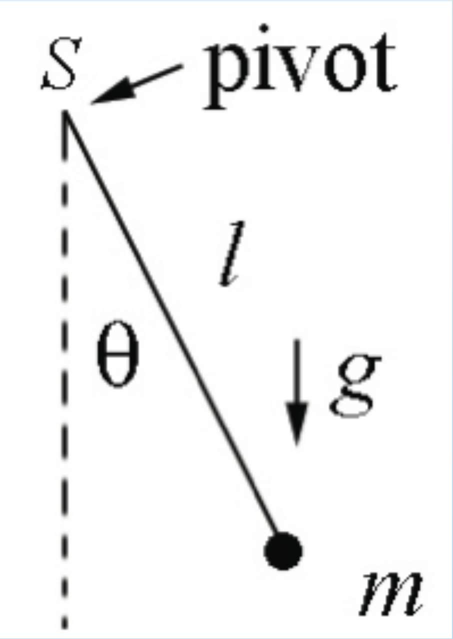
$$v_x = \frac{dx}{dt} = -\omega_0 C\sin(\omega_0 t) + \omega_0 D\cos(\omega_0 t)$$

$$\omega_0 = \sqrt{b}$$

Table Problem: Simple Pendulum by the Energy Method

1. Find an expression for the mechanical energy when the pendulum is in motion in terms of $\theta(t)$ and its derivatives, *m*, *l*, and *g* as needed.

2. Find an equation of motion for $\theta(t)$ using the energy method.



Worked Example: Simple Pendulum Small Angle Approximation

Equation of motion $$-mg\sin\theta = ml\frac{d^2\theta}{dt^2}$$

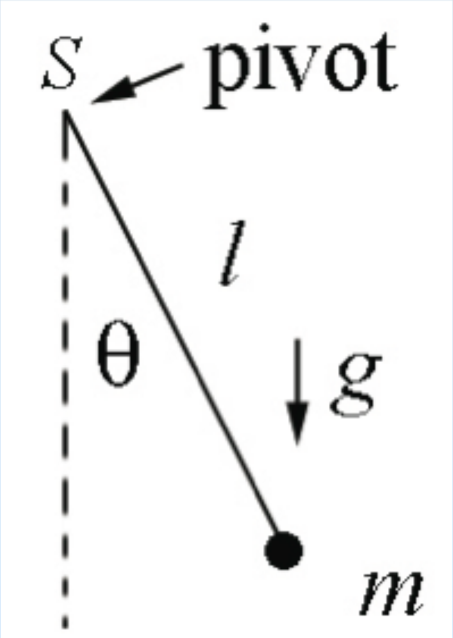
Angle of oscillation is small $$\sin\theta \cong \theta$$

Simple harmonic oscillator $$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta$$

Analogy to spring equation $$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Angular frequency of oscillation $$\omega_0 \cong \sqrt{g/l}$$

Period $$T_0 = \frac{2\pi}{\omega_0} \cong 2\pi\sqrt{l/g}$$

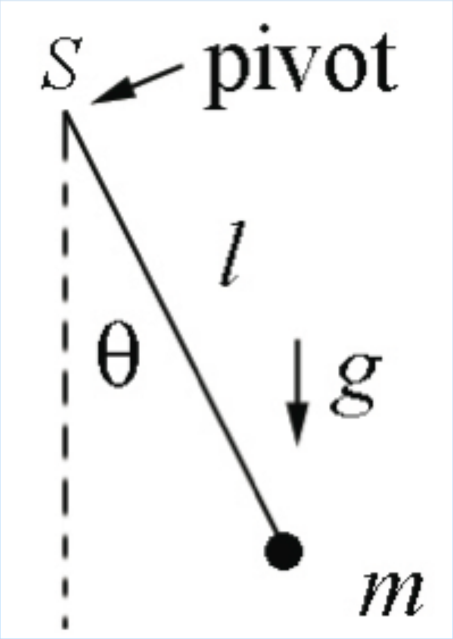


Periodic vs. Harmonic

Equation of motion $$-\frac{g}{l}\sin\theta = \frac{d^2\theta}{dt^2} \Rightarrow periodic$$

Angle of oscillation is small, linear restoring torque

$$\sin\theta \cong \theta$$



Simple harmonic oscillator $$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta \Rightarrow SHO$$

Angular frequency for SHO is independent of amplitude

$$\omega_0 \cong \sqrt{g / l}$$

Demonstration:
U-tube Oscillations

Worked Example: Fluid Oscillations in a U-tube

A U-tube open at both ends to atmospheric pressure is filled with an incompressible fluid of density r. The cross-sectional area A of the tube is uniform and the total length of the column of fluid is L. A piston is used to depress the height of the liquid column on one side by a distance x, and then is quickly removed. What is the frequency of the ensuing simple harmonic motion? Assume streamline flow and no drag at the walls of the U-tube. The gravitational constant is g.

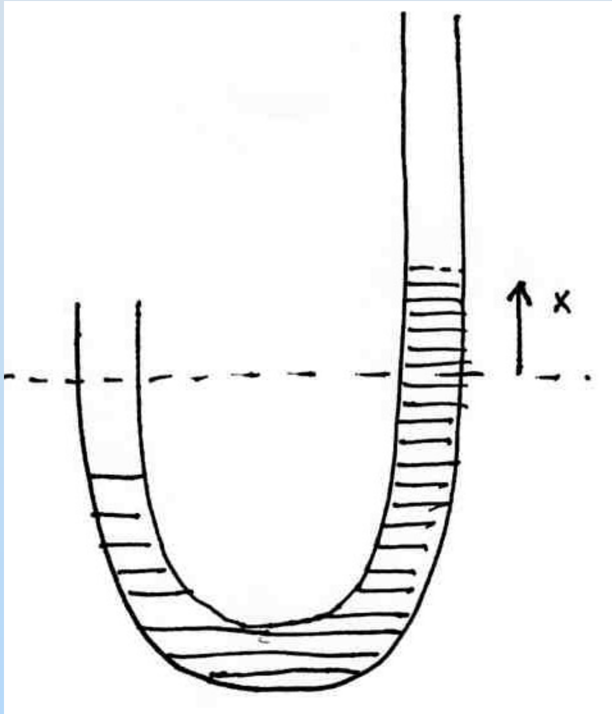
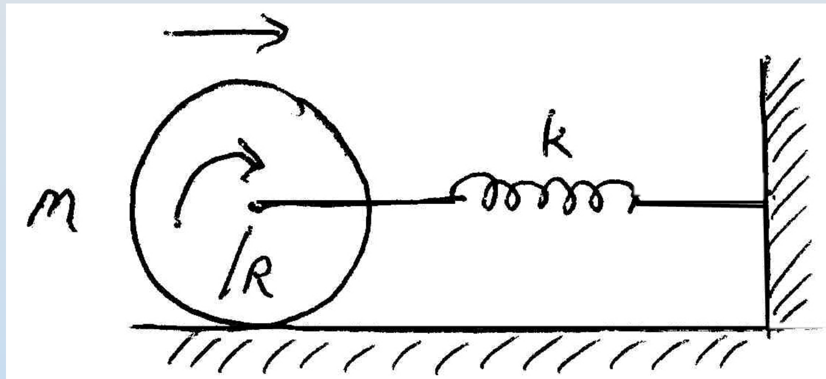


Table Problem: Rolling and Oscillating Cylinder



Attach a solid cylinder of mass M and radius R to a horizontal massless spring with spring constant k so that it can roll without slipping along a horizontal surface. At time t, the center of mass of the cylinder is moving with speed V_{cm} and the spring is compressed a distance x from its equilibrium length. What is the period of simple harmonic motion for the center of mass of the cylinder?