PH 201-4A spring 2007

Simple Harmonic Motion

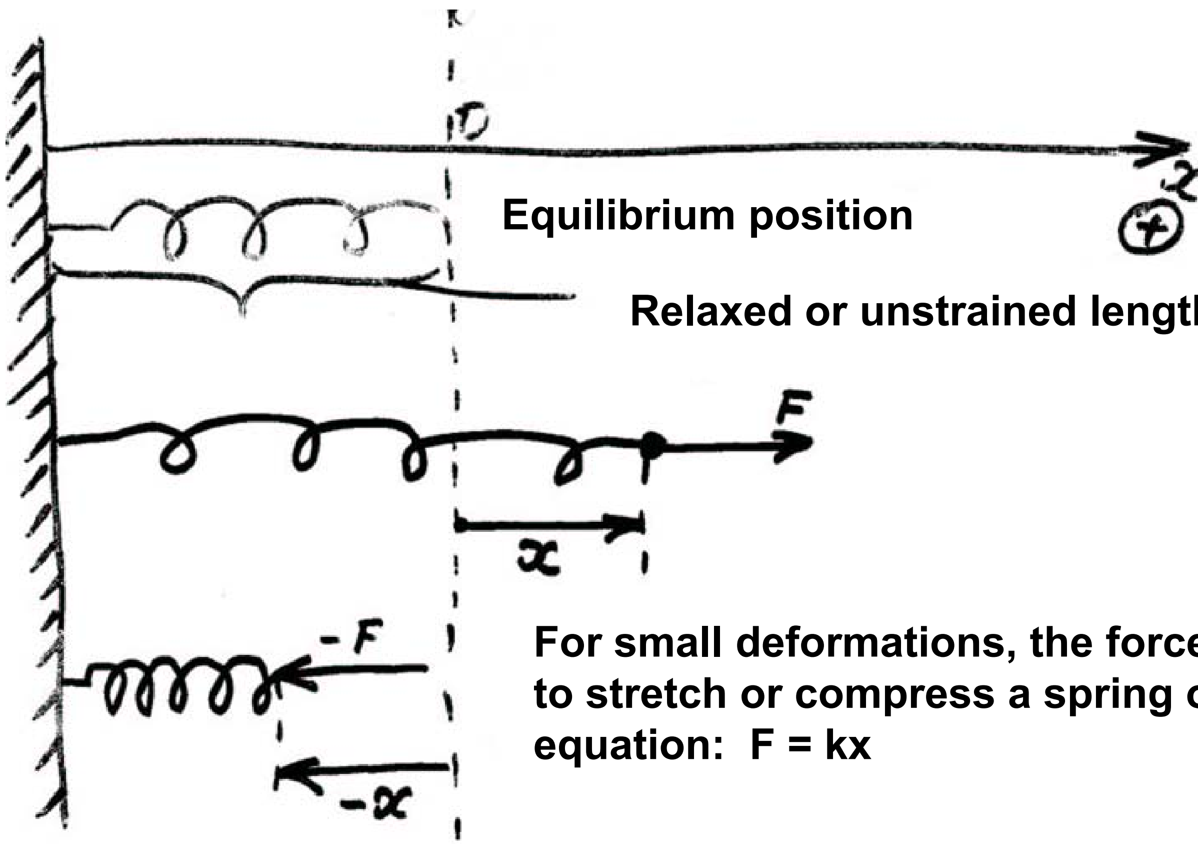
Lectures 24-25

Chapter 10

(Cutnell & Johnson, Physics 7th edition)

The Ideal Spring

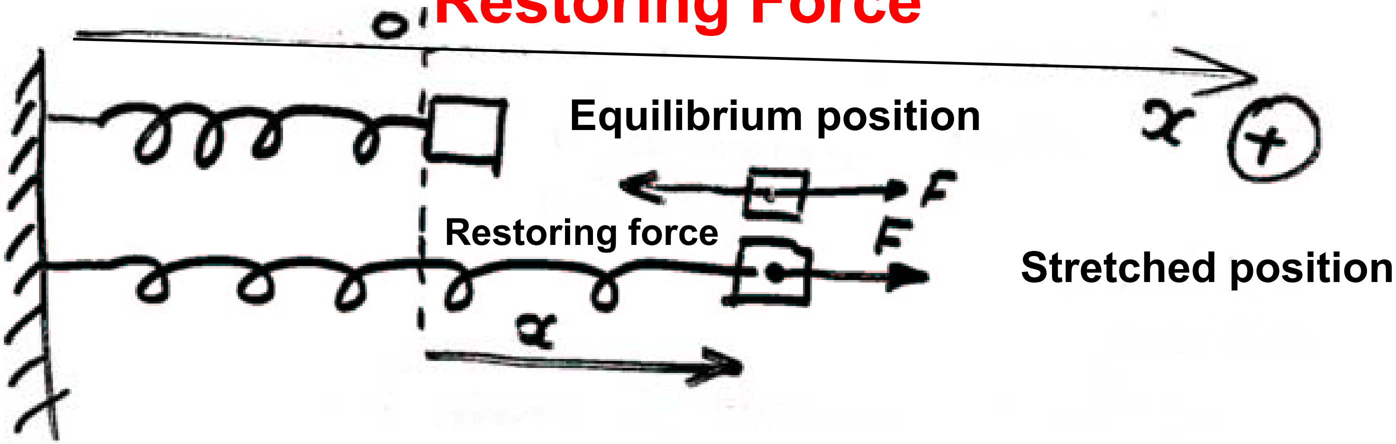
Springs are objects that exhibit elastic behavior. It will return back to its original length after being stretched or compressed.



For small deformations, the force "F" required to stretch or compress a spring obeys the equation: F = kx

- **<u>x - displacement</u> of the spring from its unstrained length**
- **<u>k – spring constant</u> [N/m] unit**
- **A spring that behaves according to the relationship <u>F = kx</u> it is said to be an <u>ideal spring</u>**

Restoring Force



- **To stretch or compress a spring a force F must be applied**
- **Newton's 3rd Law: Every action has an equal in magnitude and opposite reaction.**

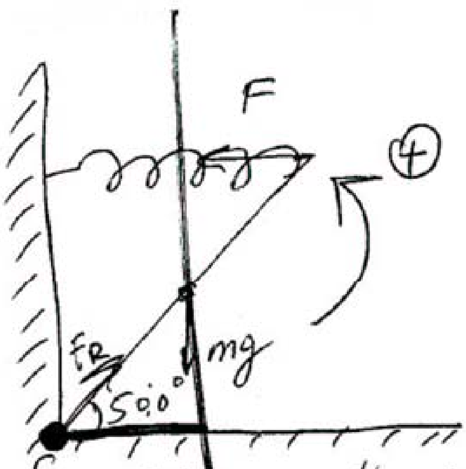
The reaction force that is applied by the spring to the agent that does the pulling or pushing is called restoring force

Hooke's Law: $F = -kx$

F = restoring force k = spring constant
x = displacement from unstrained length

The restoring force is always opposite to the displacement of the spring

Problem 9: A 10.1 kg uniform board is wedged into a corner and held by a spring at 50.0° angle, as the drawing shows. The spring has a spring constant of 176 N/m and is parallel to the floor. Find the amount by which the spring is stretched from its unstrained length.



Reasoning

- In order to find the amount of the spring stretches we need to calculate the force $|F| = |kx|$ acting on the spring
- Since the board is in equilibrium, the net torque acting on it is zero.
- taking the axis of rotation to be at the corner and assuming the board has a length L

$$\Sigma \tau_c = 0$$

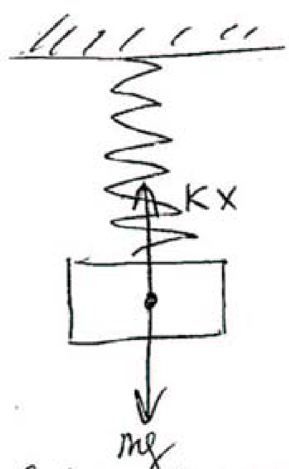
Solution

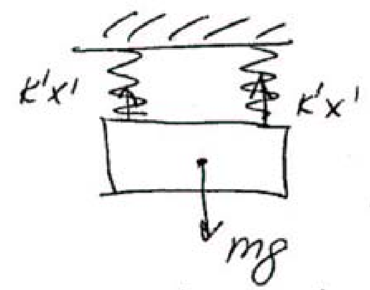
$$F \cdot L \sin 50.0^\circ - mg\left(\frac{L}{2}\right)\cos 50.0^\circ = 0$$

$$F = \frac{mg \cos 50.0^\circ}{2 \sin 50.0^\circ} = \frac{mg}{2 \tan 50.0^\circ}$$

$$\Rightarrow x = \frac{F}{k} = \frac{mg}{2k \tan 50.0^\circ} = \frac{(10.1\,kg)(9.80\,m/s^2)}{2(176\,N/m)\tan 50.0^\circ} = 0.236\,m$$

An object is attached to the lower end of a 100-coil spring that is hanging from the ceiling. The spring stretches by 0.160 m. The spring is then cut into two identical springs of 50 coils each. As the drawing shows, each spring is attached between the ceiling and the object. By how much does each spring stretch?





$mg = kx$

before the spring is cut

$mg = k'x' + k'x'$

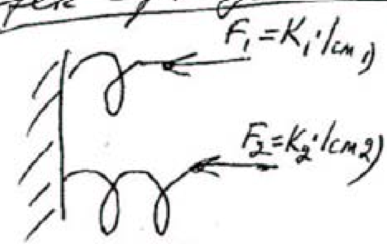
after it is cut

$\Rightarrow kx = 2k'x'$

Shorter springs are stiffer springs

$k' = 2k$

$\Rightarrow x' = \frac{x}{4} = \frac{0.160m}{4} = 0.040m$



1) a coil is compressed by 1 cm

2) one coil is compressed by 0.5 cm

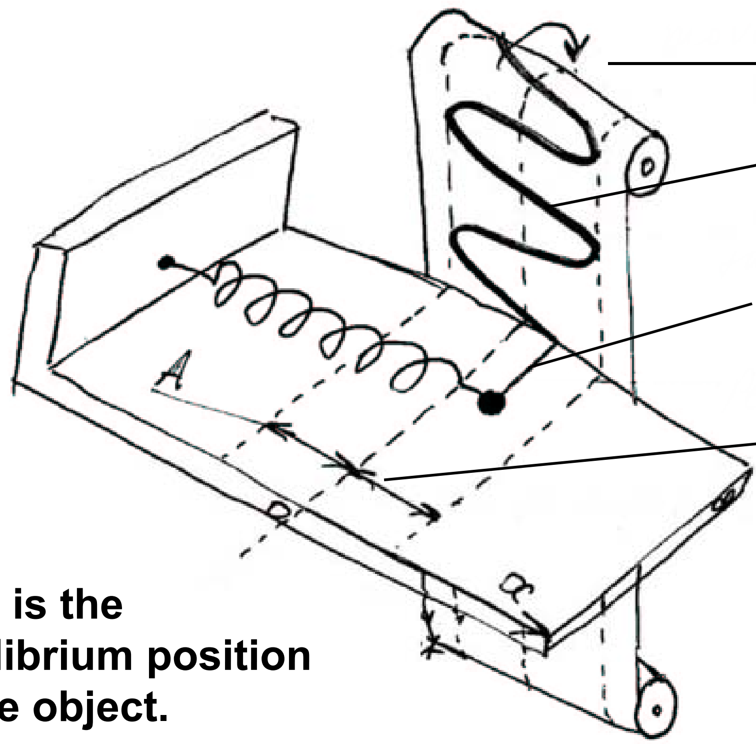
$\Rightarrow F_1 = 2F_2$

$2F_2 = k_1 \times 1$

$F_2 = k_2 \times 1$

$\Rightarrow K_1 = 2K_2$ — shorter springs are stiffer

When an object attached to a horizontal spring is moved from its equilibrium position and released, the restoring force F = -kx leads to simple harmonic motion



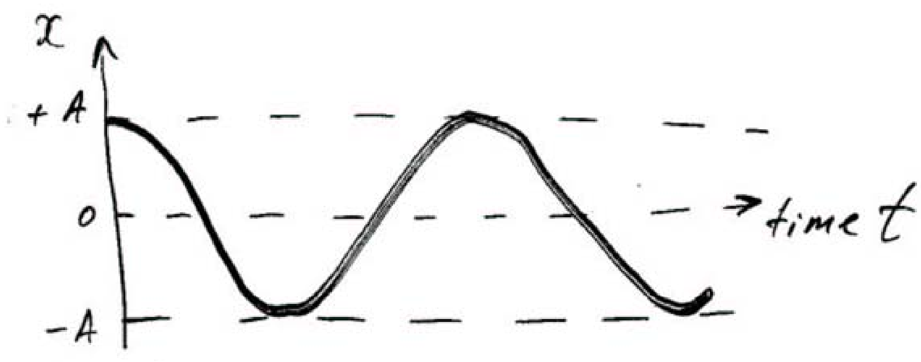
Moving strip of paper at a steady rate

Record position of the vibrating object vs. time

pen

A = amplitude – the maximum displacement from equilibrium

x = 0 is the equilibrium position of the object.



Position as a function of time has the shape of trigonometric sine or cosine function

Oscillations

Periodic motion – the motion of a particle or a system of particles is periodic, or cyclic, if it repeats again and again at regular intervals of time.

Example:

- **The orbital motion of a planet**
- **The uniform rotational motion of a phonograph turntable**
- **Back and forth motion of a piston in an automobile engine**
- **Vibrations of a guitar string**

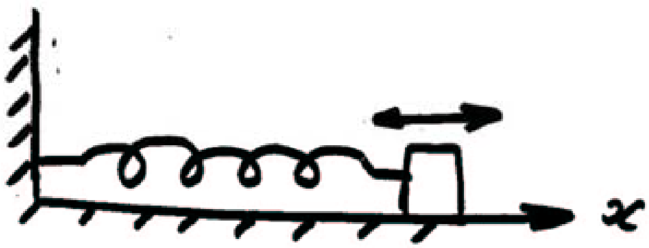
Oscillation – back and forth or swinging periodic motion is called an oscillation

Simple Harmonic Motion

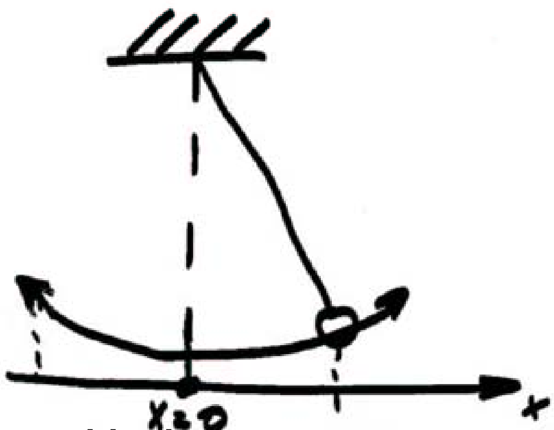
- **Simple harmonic motion is a special kind of one dimensional periodic motion**
- **The particle moves back and forth along a straight line, repeating the same motion again and again**

Simple harmonic motion – the particles position can be expressed as a cosine or a sine function of time.

Cosines and sines are called harmonic functions => we call motion of the particle harmonic.



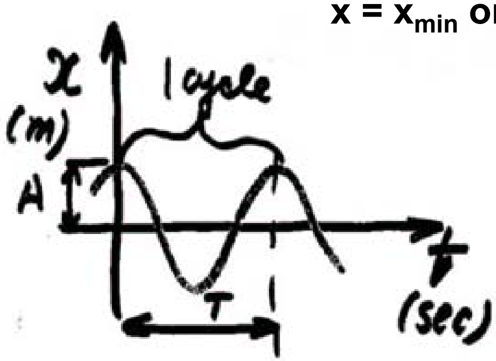
The motion of a mass oscillating back and forth in response to the push and pull of a spring – simple harmonic



midpoint **turning point of the motion**
$x = x_{min}$ or x_{max}

The motion of a pendulum is approximately simple harmonic

A – amplitude of the motion the distance between the midpoint (x = 0) and either of the turning points (x = +A; x = -A)



ω – angular frequency [rad/sec]
ωt – "angle" [rad]

$$x = A\cos(\omega t) = A\cos[\omega t + 2\pi] = A\cos[\omega(t + \underbrace{\tfrac{2\pi}{\omega}}_{T})]$$

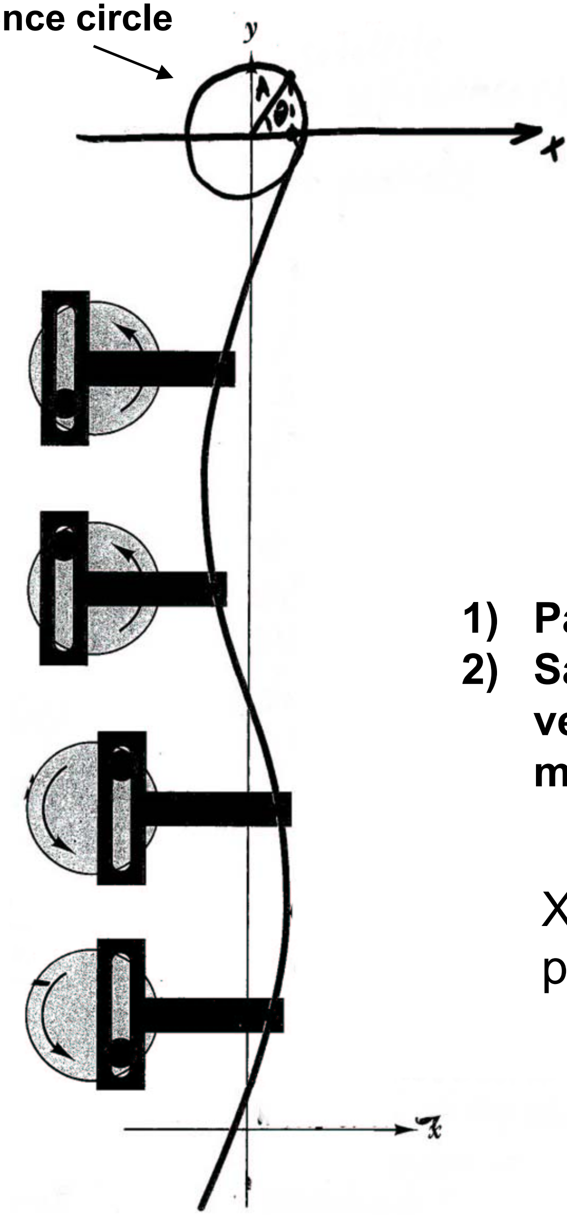
$T = \frac{2\pi}{\omega}$ – period of the motion

$f = \frac{1}{T} = \frac{\omega}{2\pi}$ – frequency of the motion

[cycle/sec] = 1 herts = 1 Hz

Special Geometrical Relationship Between Simple Harmonic Motion and Uniform Circular Motion

Reference circle



Simple mechanism for generating simple harmonic motion from uniform circular motion.

A slotted arm placed over a peg which is attached to a wheel in uniform circular motion.

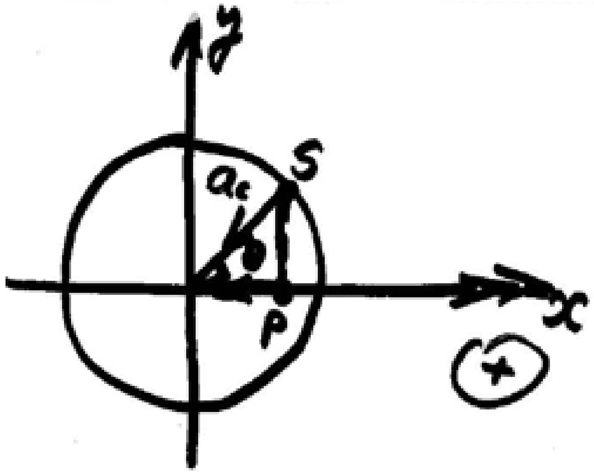
The slot is vertical and the arm is constrained to move the horizontal peg – "satellite" midpoint of the slot in the arm – "particle"

1) **Particle – SHM $x = \cos(\omega t)$**
2) **Satellite particle in uniform circular motion with angular velocity "ω" along a circle radius "A" centered at midpoint $\theta = \omega t$ $x_{sat} = A\cos\theta = A\cos(\omega t)$**

X_{sat} always coincides with the x coordinate of the particle. They have exactly the same x motion.

The Instantaneous Acceleration in Simple Harmonic Motion

The instantaneous acceleration of a particle in simple harmonic motion is proportional to the instantaneous distance x, but is in the opposite direction.



$$x_s = x_p \Rightarrow$$

$$a_{sx} = a_{px}$$

$$a_{sx} = a_{cent} \cdot \cos\theta =$$

$$= -\omega^2 A \cos\theta$$

$$\left(a_{cent} = \omega^2 R = \frac{v^2}{R} \right)$$

$$v = \omega R$$

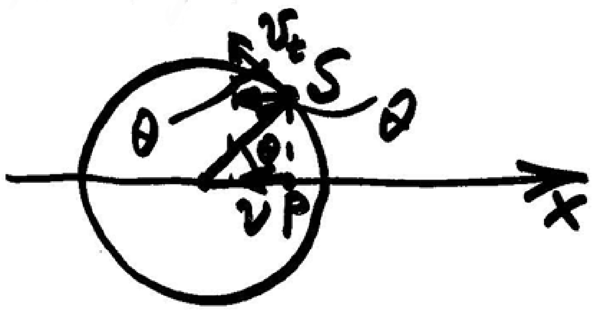
$$a_{px} = -\omega^2 A \cos\theta$$

$$\theta = \omega t$$

$$\Rightarrow \quad a = -\omega^2 A \cos(\omega t) = -\omega^2 x$$

The Speed of Simple Harmonic Motion

The velocity of the mass attached to the end of the spring can be found with the help of the reference circle.



Satellite particle

ωs = ωp = ω R = A

The point S moves with the tangential velocity Vt. The x component of this velocity is the velocity of the point P and => the velocity of the mass m

$$v = -v_t \sin\theta \;;$$

for satellite $v_t = \omega R = \omega A$

$$\Rightarrow \boxed{v = -\omega A \sin \omega t}$$

$$\sin^2\theta + \cos^2\theta = 1$$ trigonometric identity

$$\sin^2\theta = 1 - \cos^2\theta \qquad \sin\theta = \pm\sqrt{1-\cos^2\theta}$$

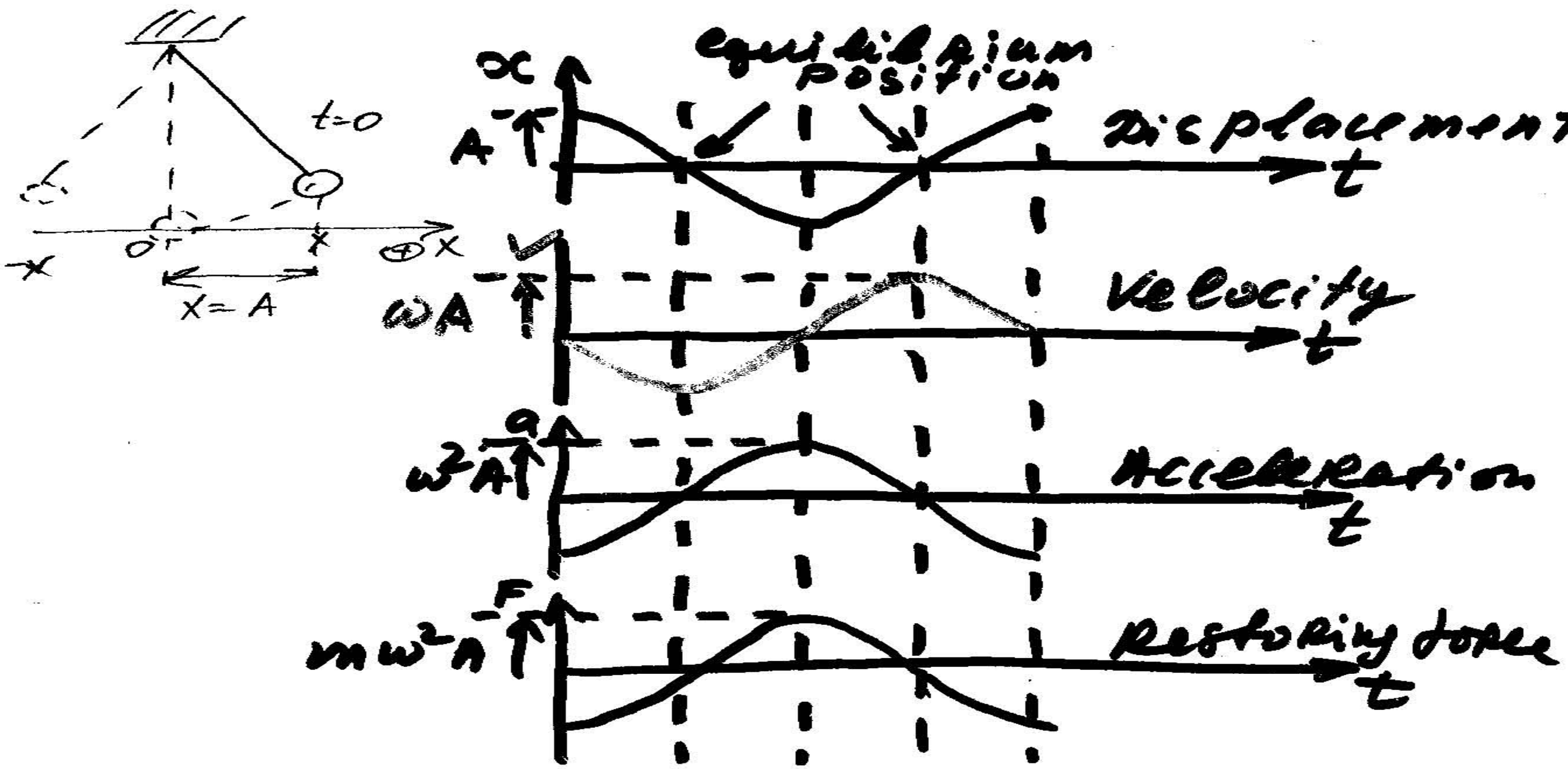
$$\cos\theta = \frac{x}{A} \Rightarrow \sin\theta = \pm\sqrt{1-\frac{x^2}{A^2}}$$

$$\boxed{v = \pm\,\omega A\sqrt{1-\frac{x^2}{A^2}} = \pm\,\omega\sqrt{A^2-x^2}}$$

velocity at any displacement.

Kinematic Equations for Simple Harmonic Motion

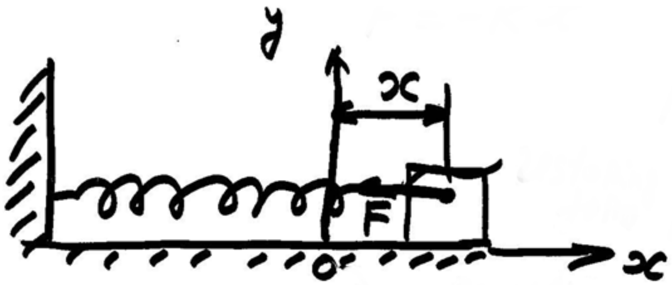
$$x = A\cos\omega t$$
$$v = -\omega A\sin\omega t$$
$$v = \pm\,\omega\sqrt{A^2 - x^2}$$
$$a = -\omega^2 A\cos\omega t$$
$$F = -m\omega^2 A\cos\omega t$$



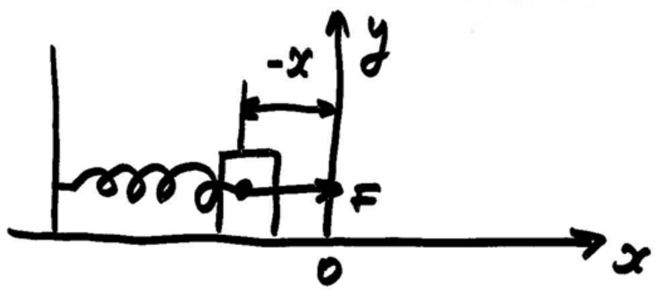
The Simple Harmonic Oscillator

Consists of a mass coupled to an ideal, massless spring which obeys Hook's Law.

$$F = -kx$$



After positive displacement of the mass the spring pulls the mass back toward the equilibrium position – the relaxed length of the spring.



The mass overshoots the equilibrium position.

Mass attached to a spring sliding back and forth on a frictionless surface

$ma = -kx$; $a = -\omega^2 x$

The equations become identical if

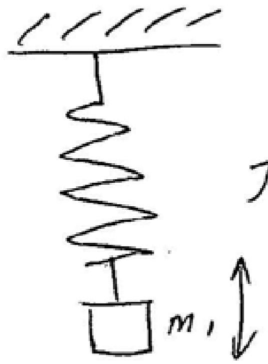
$\omega^2 = k/m \quad \omega = \sqrt{k/m}$

frequency $= \omega/2\pi = 1/2\pi\sqrt{k/m}$; $T = 1/\text{freq} = 2\pi\sqrt{m/k}$

Isochronism – frequency of the oscillator is the same, regardless of the amplitude

$-m\omega^2 x = -kx \qquad \omega^2 = k/m$

<u>Problem 90</u>: When an object of mass m_1 is hung on a vertical spring and set into vertical simple harmonic motion, its frequency is 12.0 Hz. When another object of mass m_2 is hung on the spring along with m_1, the frequency of the motion is 4.00 Hz. Find the ratio m_2/m_1 of the masses.



$$f_1 = \frac{1}{2\pi}\sqrt{\frac{K}{m_1}}$$

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{K}{m_1+m_2}}$$

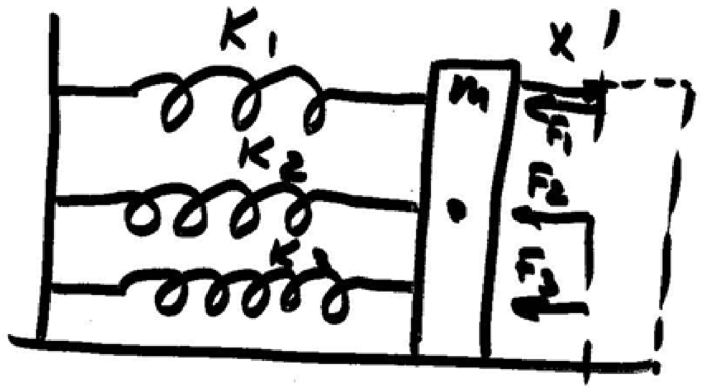
$$\Rightarrow \frac{f_1}{f_2} = \sqrt{\frac{m_1+m_2}{m_1}} = 3.00$$

$$\left(\sqrt{\frac{m_1+m_2}{m_1}}\right)^2 = (3.00)^2 \; ; \Rightarrow \frac{m_1+m_2}{m_1} = 9.00 \; ; \; m_1+m_2 = 9.00\,m_1$$

$$m_2 = 8.00\,m_1$$

$$\frac{m_2}{m_1} = 8.00$$

Three springs with force constants k_1 = 10.0 N/m, k_2 = 12.5 N/m, and k_3 = 15.0 N/m are connected in parallel to a mass of 0.500 kg. The mass is then pulled to the right and released. Find the period of the motion.



$$F_{tot} = F_1 + F_2 + F_3 = K_1 X + K_2 X + K_3 X =$$

$$= (K_1 + K_2 + K_3)\, X = K_{Equivalent}\, X$$

$$T = 2\pi\sqrt{\frac{m}{K_E}} = 2\pi\sqrt{\frac{m}{K_1 + K_2 + K_3}} =$$

$$= 2\pi\sqrt{\frac{0.500\,kg}{10.0 + 12.5 + 15}} = 0.726\,s$$

A mass of 0.300 kg is placed on a vertical spring and the spring stretches by 10.0 cm. It is then pulled down an additional 5.00 cm and then released. Find:

a) K; b) ω; c) frequency; d) T; e) max velocity; f) a_{max};
g) F_{max} (max restoring force); h) V for x = 2.00 cm;
i) The equations for displacement, velocity and acceleration at any time

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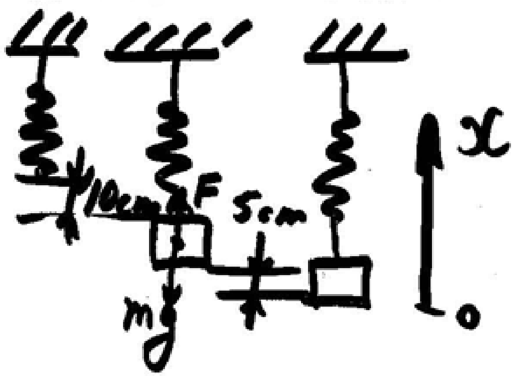


a) $F - mg = 0$; $F = mg$; $\Rightarrow K = \frac{F}{x} = \frac{mg}{x} = \frac{(0.300\,kg)(9.80\,m/s^2)}{0.100\,m} = 29.4\,N/m$

b) $\omega = \sqrt{\frac{K}{m}} = \sqrt{\frac{29.4}{0.300}} = 9.90\,rad/s$

c) $f = \frac{\omega}{2\pi} = \frac{9.90\,rad/s}{2\pi\,rad} = 1.58\,\frac{cycles}{s} = 1.58\,Hz$

d) $T = \frac{1}{f} = \frac{1}{1.58\,Hz} = 0.633\,s$

e) $v_{max} = \omega A = (9.90\,rad/s)(5.00 \times 10^{-2}\,m) = 0.495\,m/s$

f) $a_{max} = \omega^2 A = (9.90\,rad/s)^2(5.00 \times 10^{-2}\,m) = 4.90\,m/s^2$

g) $F_{max} = K X_{max} = KA = (29.4\,N/m)(5.00 \times 10^{-2}\,m) = 1.47\,N$

h) $v = \pm \omega\sqrt{A^2 - x^2} = \pm\left(9.90\ \text{rad/s}\right)\sqrt{\left(5.00\times10^{-2}\right)^2 - \left(2.00\times10^{-2}\right)^2}$

$= \pm 0.454\ \text{m/s}$

where v is positive when moving up
negative down

i) $x = A\cos\omega t = \left(5.00\times10^{-2}\ \text{m}\right)\cos\left[\left(9.90\ \text{rad/s}\right)t\right]$

$v = -\omega A\sin\omega t = -\left(9.90\ \text{rad/s}\right)\left(5.00\times10^{-2}\ \text{m}\right)\sin\left[\left(9.90\ \frac{\text{rad}}{\text{s}}\right)t\right]$

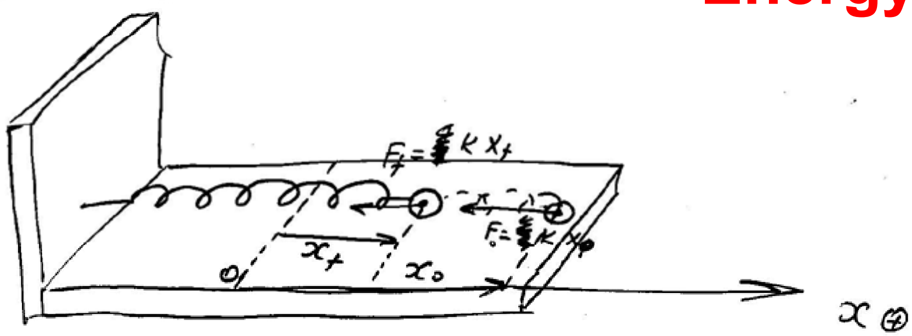
$= -0.495\ \sin 9.90t$

$a = -\omega^2 A\cos\omega t =$

$= -\left(9.90\ \text{rad/s}\right)^2\left(5.00\times10^{-2}\ \text{m}\right)\cos\left[\left(9.90\ \text{rad/s}\right)t\right] =$

$= -\left(4.90\ \text{m/s}^2\right)\cos\left[\left(9.90\ \text{rad/s}\right)t\right]$

Energy and Simple Harmonic Motion. Elastic Potential Energy.



- Energy is the capacity of the object to do work
- A spring has potential energy when it is stretched or compressed and can do work on an object that is attached to the spring. (elastic potential energy)
- When the object attached to one end of a stretched spring is released, the spring pulls the object from its initial position x_0 to its final position x_f.
- The work done by a constant force $W = (F\cos\theta)s$
- $s = x_0 - x_f$ magnitude of the displacement
- Force has changing magnitude because the dependence of the spring force on x is linear $F_{av} = ½(kx_0 + kx_f)$
- W_{elast} done by the average spring force

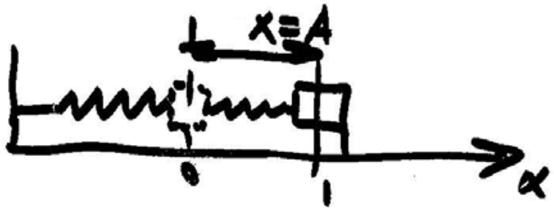
$W_{el} = (F\cos\theta)s = ½(kx_0 + kx_f)\cos 0°(x_0 - x_f) = 1/2kx_0^2 - 1/2kx_f^2$

The elastic potential energy $PE_{elastic} = U$ is the energy that the spring has by virtue of being stretched or compressed.

$PE_{elastic} = 1/2kx^2$

Conservation of Energy for the Simple Harmonic Oscillator

The force exerted by a spring is a conservative force. External nonconservative forces (friction) do no net work



$$E_{tot} = K + U = const$$

$$E_{tot} = \frac{1}{2}mv^2 + \cancel{\frac{1}{2}I\omega^2} + \cancel{mgh} + \frac{1}{2}kx^2$$

$$E_{tot_1} = U_1 = \frac{1}{2}kA^2;\ K_1 = 0$$

$$E_{tot_0} = K_1 = \frac{1}{2}mv^2;\ U_0 = 0$$

$$\frac{1}{2}kA^2 = E_{tot} = \frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k(A\cos\omega t)^2 +$$

$$+ \frac{1}{2}m(-\omega A\sin\omega t)^2$$

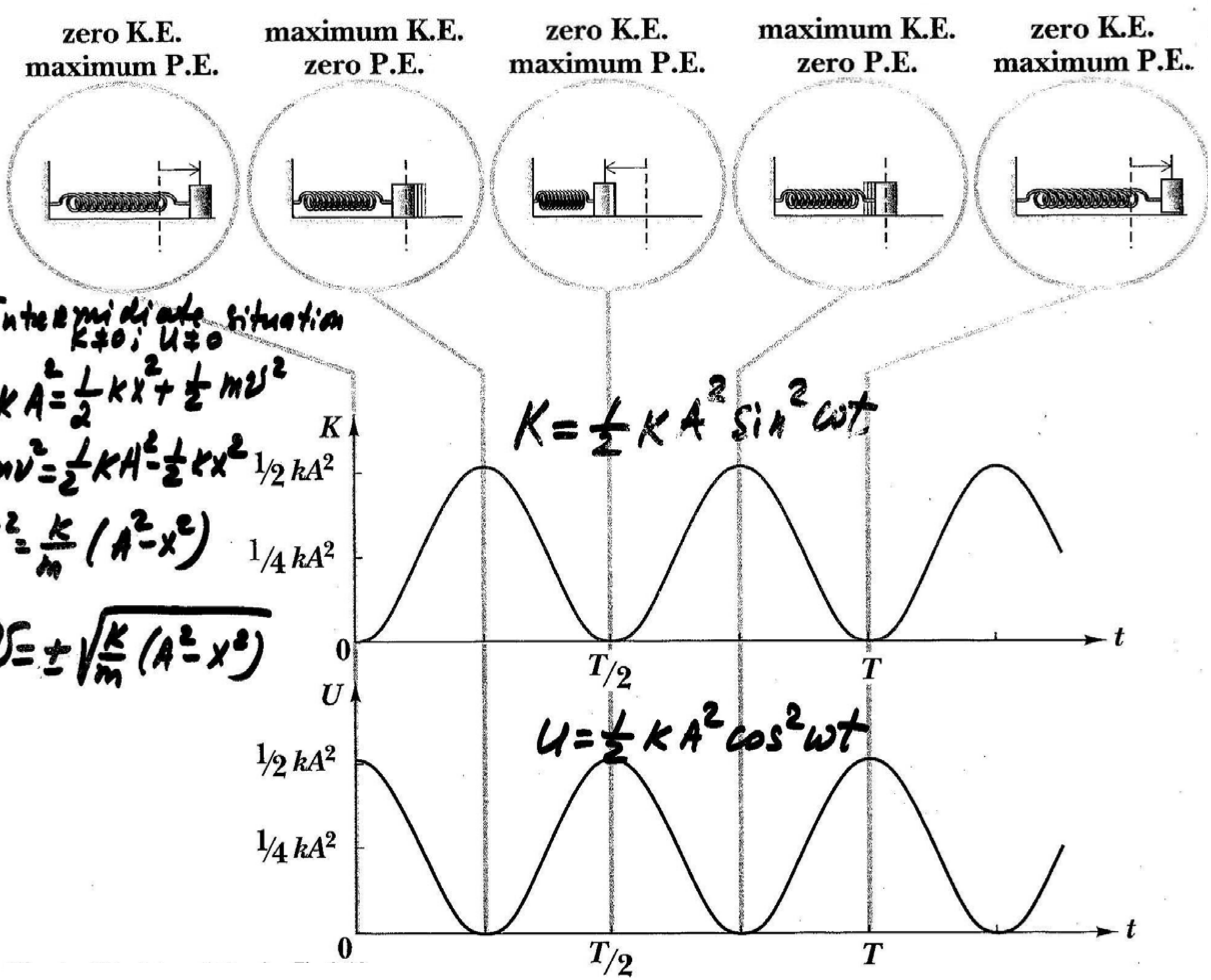
OR. $E_{tot} = \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}m\omega^2A^2\sin^2\omega t$

since $\omega^2 = \frac{k}{m}$

$$E_{tot} = \frac{1}{2}kA^2\cos^2\omega t + \frac{1}{2}m\frac{k}{m}A^2\sin^2\omega t =$$

$$= \underbrace{\frac{1}{2}kA^2\cos^2\omega t}_{PE} + \underbrace{\frac{1}{2}kA^2\sin^2\omega t}_{KE} =$$

$$= \frac{1}{2}kA^2(\underbrace{\cos^2\omega t + \sin^2\omega t}_{=1}) = \frac{1}{2}kA^2$$



Intermidiate situation
$K \neq 0;\ U \neq 0$

$$\frac{1}{2}kA^2=\frac{1}{2}kx^2+\frac{1}{2}mv^2$$

$$\frac{1}{2}mv^2=\frac{1}{2}kA^2-\frac{1}{2}kx^2$$

$$v^2=\frac{k}{m}(A^2-x^2)$$

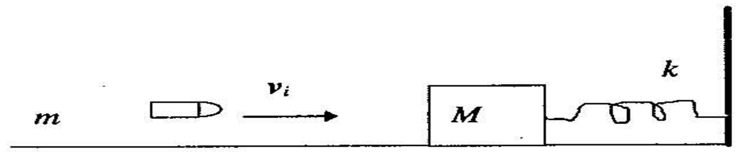
$$v=\pm\sqrt{\frac{k}{m}(A^2-x^2)}$$

$$K=\frac{1}{2}kA^2\sin^2\omega t$$

$$U=\frac{1}{2}kA^2\cos^2\omega t$$

6. A bullet of mass m =15 g is fired into a block of mass M = 985 g, which is attached to an uncompressed spring of force constant k = 1000 N/m. The spring is anchored to a wall, and the block rests on a horizontal frictionless surface as shown in Fig. After the bullet embeds itself in the block, the block compresses the spring a maximum distance of x = 12 cm.

Find the initial velocity of the bullet?



Given: inelastic

$m = 15\,g = 15 \times 10^{-3}\,kg$

$M = 985\,g = 985 \times 10^{-3}\,kg$

$K = 1000\,N/m$

$x = 12\,cm = 0.12\,m$

Find $v_i = ?$

1) After the collision

$K_E = \frac{1}{2}(m+M)v_f^2$ was transformed into potential energy of the spring $U = \frac{Kx^2}{2}$

2) mechanical energy is conserved

$$\Rightarrow \frac{1}{2}(m+M)v_f^2 = \frac{1}{2}Kx^2$$

$$\frac{1}{2}(15\times10^{-3} + 985\times10^{-3})v_f^2 = \frac{1}{2}1000\times(0.12\,m)^2$$

$$v_f = 3.79\ m/s$$

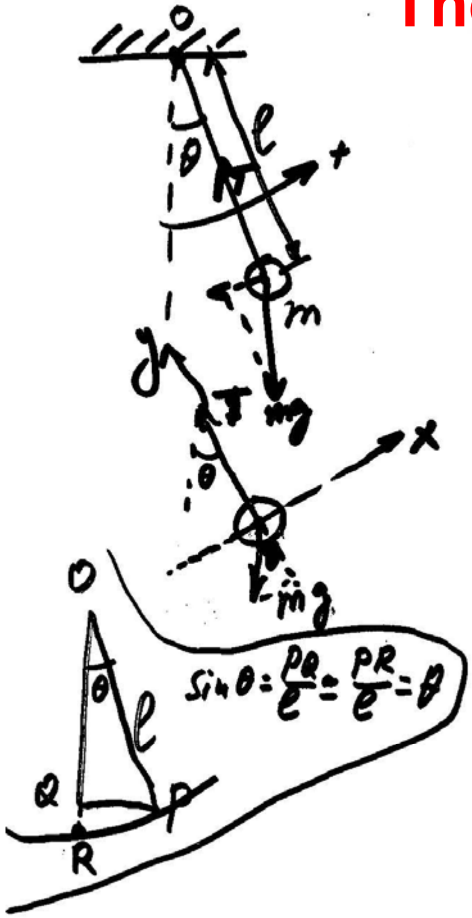
3) Conservation of momentum during inelastic collision

$$m v_i = (m+M)v_f$$

$$v_i = \frac{(m+M)v_f}{m} = \frac{(15\times10^{-3} + 985\times10^{-3})(3.79)}{15\times10^{-3}} =$$

$$= 253\ m/s$$

The Simple Pendulum



A simple pendulum consists of a bob suspended by a string or a rod.

String is massless bob – m

Gravity provides restoring force

$$I = m\ell^2$$

$$I\alpha = \tau$$

$$\tau = -mg\ell\sin\theta$$

$$m\ell^2\alpha = -mg\ell\sin\theta$$

$$\ell\alpha = -g\sin\theta$$

$\sin\theta \approx \theta$ for θ small $\leq 10^\circ$

$$\Rightarrow \ell\alpha = -g\theta$$

Compare $ma = -kx$

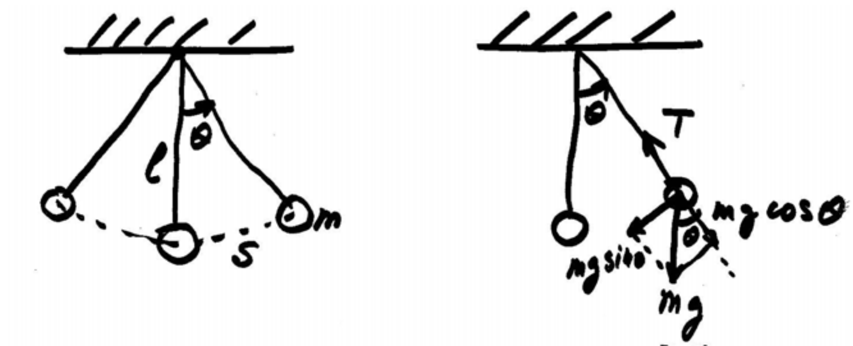
$$\left[x = A\cos(\omega t) = A\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)\right]$$

$$\Rightarrow \theta = A\cos(\omega t) = A\cos\left(\sqrt{\tfrac{g}{\ell}}\,t\right)$$

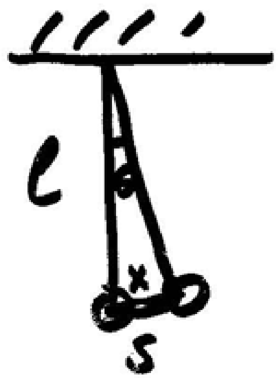
$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\ell}{g}}}$$

The Simple Pendulum

A simple pendulum is a bob that is attached to a string and allowed to oscillate. The bob is assumed to behave like a particle of mass “m”, and the string is massless.



Restoring force = -mgsinθ   Apply 2nd Newton’s Law: F = ma =>
-mgsinθ = ma =>
The tangential acceleration of the bob is: a = -gsinθ



$\sin\theta = \frac{x}{l}$; $\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{l}$

$x \approx s$ for $\theta < 10^\circ$

$\therefore \ \theta \approx \sin\theta$

$\Rightarrow a = -g\theta = -g\frac{s}{l} = \underline{-g\frac{x}{l}}$

$$a = -\frac{g}{\ell}\, x$$

For simple harmonic motion of a spring the acceleration was found to be $a = -\frac{K}{m}\, x$

Use the equations developed for the vibrating spring to describe the motion of the pendulum.

$$\frac{k}{m} = \frac{g}{\ell} \quad \text{or} \quad K_{pendulum} = \frac{mg}{\ell}$$

$$\Rightarrow T_p = 2\pi\sqrt{\frac{m}{k_p}} = 2\pi\sqrt{\frac{m\ell}{mg}} \Rightarrow$$

$$\boxed{T = 2\pi\sqrt{\frac{\ell}{g}}}$$

<u>Example 1</u>: The pendulum can be used as a very simple device to measure the acceleration of gravity at a particular location.

- **measure the length "l" of the pendulum and then set the pendulum into motion**
- **measure "T" by a clock**
- $$g = \frac{4\pi^2}{T^2}\,\ell$$

Example 2: The Length of a Pendulum. A student is in an empty room. He has a piece of rope, a small bob, and a clock. Find the volume of the room.

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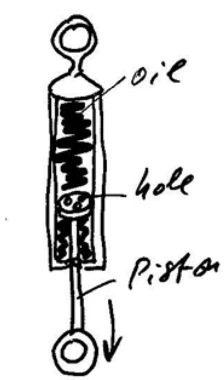
1. From the piece of rope and a bob we make a simple pendulum
2. We set pendulum into motion
3. We measure period “T” by a clock
4. We calculate the length of the pendulum (rope)

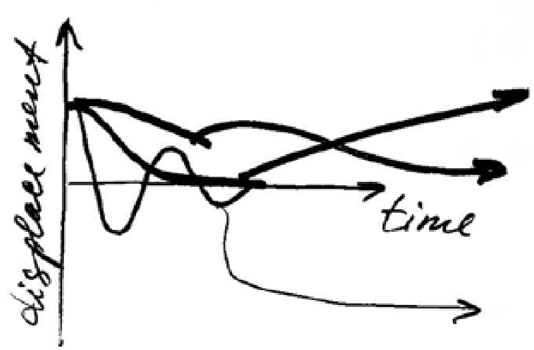
$$\ell = \frac{T^2}{4\pi^2} g \Leftarrow T = 2\pi\sqrt{\frac{\ell}{g}}$$

5. With a help of the rope of the known length we measure the dimensions of the room a x b x h and its volume V = a x b x h

Damped Harmonic Motion

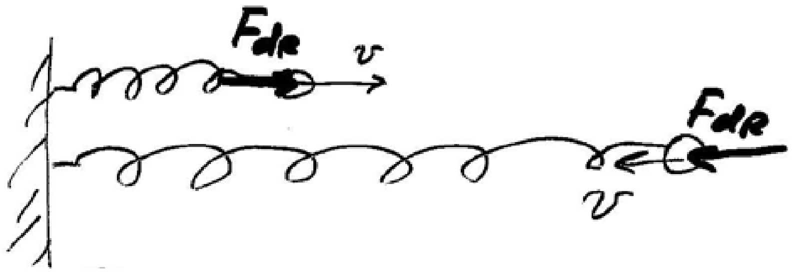
• In simple harmonic motion an object oscillates with a constant amplitude because there is no mechanism for dissipating energy.
• In practice, oscillating mechanical systems lose energy in a variety of ways via friction and the amplitude of oscillation decreases as time passes until motion gradually dies away.
• The decrease of amplitude is called damping and the motion is called damped harmonic motion.
• Example: Suspension system of an automobile uses shock absorbers. When the piston moves in response to a bump in the road, holes in the piston head permit the piston to pass through the oil. Viscous forces that arise during this movement cause the damping.
• The smallest degree of damping that completely eliminates the oscillations critical damping.
• When damping exceeds critical value – motion over damped
• When damping is less than critical value, the motion is under damped.





Driven Harmonic Motion and Resonance

• To set an object on an ideal spring into simple harmonic motion we must apply a force that stretches or compresses the spring initially.



• Suppose that this force is applied at all times, not just for a brief initial moment. We push and pull the ball back and forth.
• The resultant motion is driven harmonic motion because the additional force controls or drives the behavior of the object.
• The frequency at which the spring system naturally oscillates is called natural frequency.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$$

• When $f_{dr} = f_n$ resonance will occur.
• Resonance is the condition in which a time dependent force can transmit large amounts of energy to an oscillating object (driving force has the same direction as velocity) leading to a large amplitude motion.