Simple New Keynesian Model: Foundations

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Outline

- Preferences and Technology.
  - First best, efficient allocations (benchmark for later analysis).
- Flexible Price Equilibrium
  - Efficient, as long as a monopoly distortion is eliminated.
    - Complete monetary neutrality.
- Equilibrium in sticky price version of the model (New Keynesian model).
  - Private sector equilibrium conditions.
  - Ramsey Equilibrium.
  - Linearization (first order perturbation)
  - Taylor rule
  - Properties of Taylor rule equilibrium

Preferences and Technology

- Household utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}$$

- Goods production
  - Final goods:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

  - Intermediate goods:

$$Y_{i,t} = \exp(a_t) N_{i,t}, \ a_t = \rho a_{t-1} + \varepsilon_t^a$$

- Labor:

$$N_t = \int_0^1 N_{it} di.$$

First Best Allocations

- Must solve two problems:
  - For any given N_t, how should employment be allocated across sectors?
  - What should the level of total employment, N_t, be?

Efficient Sectoral Allocation of Employment

- Consider a given level of N_t.
- Can show: efficiency dictates allocating employment *equally* across sectors:

$$N_{it} = N_t, \ i \in [0, 1] \, .$$

- Consider, for example, the class of deviations from equal:

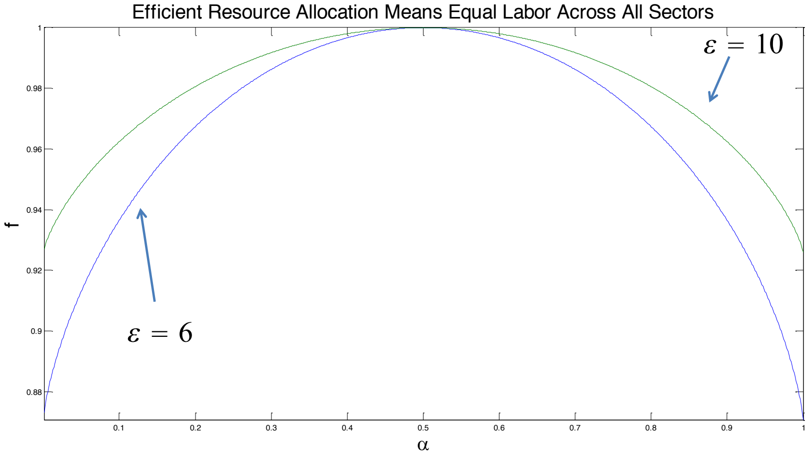
$$N_{it} = \begin{cases} 2\alpha N_t & i \in \left[0, \frac{1}{2}\right] \\ 2(1-\alpha) N_t & i \in \left[\frac{1}{2}, 1\right] \end{cases} , \ 0 \leq \alpha \leq 1.$$

This allocation is consistent with N_t:

$$\int_0^1 N_{it} di = \frac{1}{2} 2\alpha N_t + \frac{1}{2} 2(1-\alpha) N_t = N_t$$

$$
\begin{aligned}
Y_t &= \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= \left[\int_0^{\frac{1}{2}} Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t}\left[\int_0^{\frac{1}{2}} N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 N_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t}\left[\int_0^{\frac{1}{2}} \left(2\alpha N_t\right)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 \left(2\left(1-\alpha\right)N_t\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t}N_t\left[\int_0^{\frac{1}{2}} \left(2\alpha\right)^{\frac{\varepsilon-1}{\varepsilon}} di + \int_{\frac{1}{2}}^1 \left(2\left(1-\alpha\right)\right)^{\frac{\varepsilon-1}{\varepsilon}} di\right]^{\frac{\varepsilon}{\varepsilon-1}} \\
&= e^{a_t}N_t\left[\frac{1}{2}\left(2\alpha\right)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2}\left(2\left(1-\alpha\right)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} = e^{a_t}N_t f\left(\alpha\right)
\end{aligned}
$$

$$f(\alpha) = \left[\frac{1}{2}(2\alpha)^{\frac{\varepsilon-1}{\varepsilon}} + \frac{1}{2}(2(1-\alpha))^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}}$$



Economy with Efficient *N* Allocation

- Efficiency dictates

$$N_{it} = N_t \text{ all } i$$

- So, with efficient production:

$$Y_t = e^{a_t} N_t$$

- Resource constraint:

$$C_t \leq Y_t$$

- Preferences:

$$E_0 \sum_{t=0}^{\infty} \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}, \ \varepsilon_t^{\tau} \sim iid,$$

Efficient Determination of Labor

- Lagrangian:

$$\max_{C_t,N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \overbrace{u(C_t, N_t, \tau_t)}^{=\log C_t - \exp(\tau_t)\frac{N_t^{1+\varphi}}{1+\varphi}} + \lambda_t [e^{a_t} N_t - C_t] \right\}$$

- First order conditions:

$$u_c(C_t, N_t, \tau_t) = \lambda_t,\ u_n(C_t, N_t, \tau_t) + \lambda_t e^{a_t} = 0$$

- or:

$$u_{n,t} + u_{c,t} e^{a_t} = 0$$

$$\overbrace{\frac{-u_{n,t}}{u_{c,t}}}^{\text{marginal cost of labor in consumption units} = \frac{-\frac{du}{dN_t}}{\frac{du}{dC_t}} = \frac{dC_t}{dN_t}} = \overbrace{e^{a_t}}^{\text{marginal product of labor}}$$

Efficient Determination of Labor, cont'd

- Solving the fonc's:

$$\frac{-u_{n,t}}{u_{c,t}} = e^{a_t}$$

$$C_t \exp(\tau_t) N_t^{\varphi} = e^{a_t}$$

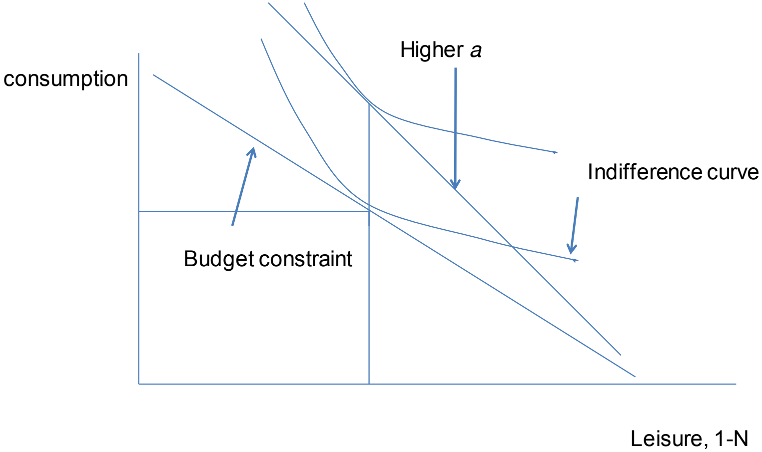
$$e^{a_t} N_t \exp(\tau_t) N_t^{\varphi} = e^{a_t}$$

$$\rightarrow N_t = \exp\left(\frac{-\tau_t}{1+\varphi}\right)$$

$$\rightarrow C_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right)$$

- Note:
  - Labor responds to preference shock, *not* to tech shock

Response to a Jump in *a*



Key Features of First Best

- Employment does not respond to a_t.
  - This result is only *somewhat* sensitive to the absence of capital.
  - In RBC models response of N_t to a_t positive, but small.
    - One reason for the small response is that income effects mitigate substitution effects.
- First best consumption not a function of intertemporal considerations.
  - Consumption is a function only of current realization of shocks.
    - Time series representations of shocks irrelevant.
    - For example, the impact of ε_t^a and ε_t^τ on $E_t a_{t+j}$ and $E_t \tau_{t+j}$, $j > 0$, respectively have no effect on first best consumption.
  - The parameter, β, does not enter first-best N_t and C_t.

Markets

- First, consider flexible prices.
  - Equilibrium efficient as long as government subsidy neutralizes monopoly power distortion.
  - Monetary policy relevant to real allocations.
- Second, introduce price-setting frictions in goods market.
  - Labor market has flexible wages.
  - We'll see that equilibrium could be first best, or could be really bad. It all depends on the quality of monetary policy.

Households

- Problem:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right), \ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^{\tau}$$

$$\text{s.t. } P_t C_t + B_{t+1} \leq W_t N_t + R_{t-1} B_t + \text{Profits net of taxes}$$

- First order conditions ($\bar{\pi}_t \equiv P_t / P_{t-1}$):

$$\begin{aligned} \frac{1}{C_t} &= \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \\ \exp(\tau_t) C_t N_t^{\varphi} &= \frac{W_t}{P_t}. \end{aligned}$$

Final Goods

- Final good firms:
  - maximize profits:

$$P_t Y_t - \int_0^1 P_{i,t} Y_{i,t} dj,$$

subject to:

$$Y_t = \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj \right]^{\frac{\varepsilon}{\varepsilon-1}} .$$

  - Foncs:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon} \rightarrow \overbrace{P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}}}^{\text{"cross price restrictions"}}$$

Intermediate Good Producers

- Demand curve for monopoly producer of $Y_{i,t}$:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon}.$$

- Production function:

$$Y_{i,t} = \exp\left(a_t\right) N_{i,t},\ \Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a$$

- Competitive in labor market:
  - Wage rate, W_t, taken as given.
  - (Real) marginal cost of production:

$$s_t = \frac{\left(1-\nu\right) W_t / P_t}{e^{a_t}},$$

  where ν is a government subsidy to firms.

- Optimizing monopolist sets price as a markup, $\varepsilon / \left(\varepsilon - 1\right)$, over marginal cost:

$$P_{i,t} = \frac{\varepsilon}{\varepsilon - 1} P_t s_t.$$

Features of Flexible Price Equilibrium

- Prices are all equal:

$$P_{i,t} = P_t, \text{ for all } t,$$

so sectoral allocation of employment efficient and:

$$1 = \frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1} \frac{(1-\nu) W_t / P_t}{e^{a_t}}.$$

- Aggregate employment, N_t, might not be efficient:

$$\overbrace{\frac{-u_{N,t}}{u_{c,t}}}^{\text{cost of labor effort}} \overbrace{=}^{\text{by household optimization}} \frac{W_t}{P_t} = e^{a_t} \frac{1}{\frac{\varepsilon}{\varepsilon-1}(1-\nu)} \; (*)$$

- If $\nu = 0$:
  - Actual marginal cost $<$ actual marginal product of labor.
  - Households get 'wrong' signal from the market about their actual productivity, e^{a_t}.

Profits, Government Budget Constraint and Aggregate Resources (Walras' Law)

- Government budget constraint and profits:

$$\begin{aligned} \text{taxes} &= \nu W_t \int_i N_{i,t} di \\ \text{profits} &= \int_i \left[P_{i,t} Y_{i,t} - (1-\nu) W_t N_{i,t} \right] di \end{aligned}$$

- Household budget constraint at equality:

$$P_t C_t = W_t N_t + \int_i \left[P_{i,t} Y_{i,t} - (1-\nu) W_t N_{i,t} \right] di + \nu W_t \int_i N_{i,t} di$$

$$\overbrace{=}^{\text{because } N_{i,t}=N_t,\ Y_{i,t}=Y_t} \quad W_t N_t + P_t Y_t - (1-\nu) W_t N_t + \nu W_t N_t$$

$$= P_t Y_t$$

$$\rightarrow C_t = Y_t.$$

- Using efficiency of sectoral resource allocation: $C_t = e^{a_t} N_t$. (**)

Equilibrium

- Sequences, $\{P_t, P_{i,t}, W_t/P_t, R_t, C_t, N_t\}_{t=0}^{\infty}$, such that
  - Households' problems are solved
  - Firms' problems are solved
  - Markets clear.
- By equations $(^*)$, $(^{**})$

$$
\begin{aligned}
C_t &= e^{a_t} N_t, \\
\exp(\tau_t) C_t N_t^{\varphi} &= e^{a_t} \frac{1}{\frac{\varepsilon}{\varepsilon-1}(1-\nu)}.
\end{aligned}
$$

- Then,

$$
\begin{aligned}
C_t &= e^{a_t} N_t \\
N_t &= \left[\frac{\exp(-\tau_t)}{\frac{\varepsilon}{\varepsilon-1}(1-\nu)} \right]^{\frac{1}{1+\varphi}}.
\end{aligned}
$$

Properties of Equilibrium

- Efficient if, and only if, monopoly power extinguished so that real wage correctly signals marginal product of work:

$$\frac{\varepsilon}{\varepsilon - 1}(1 - \nu) = 1,$$

  so that equilibrium employment and consumption are first best:

$$\begin{aligned} N_t &= \exp\left(-\frac{\tau_t}{1+\varphi}\right),\ C_t = \exp\left(a_t - \frac{\tau_t}{1+\varphi}\right) \\ \exp(a_t) &= \frac{W_t}{P_t} = \exp(\tau_t) C_t N_t^{\varphi}. \end{aligned}$$

- Evidently, $C_t, N_t, W_t/P_t$ are uniquely determined.
  - The objects, R_t, $\bar{\pi}_{t+1}$, P_t are not uniquely determined.
  - Presumably, monetary policy can determine these variables. But, note that monetary policy is ineffective and uninteresting in the flexible price case, since consumption and employment are completely determined without regard to monetary policy.

Real Interest Rate in Equilibrium

- The real interest rate, $R_t/\bar{\pi}_{t+1}$.
  - Absent uncertainty, $R_t/\bar{\pi}_{t+1}$ determined uniquely:
$$\frac{1}{C_t} = \beta \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}.$$
  - With uncertainty, household intertemporal condition simply places a single linear restriction across all the period $t+1$ values for $R_t/\bar{\pi}_{t+1}$ that are possible given period t.
- The real interest rate, $\tilde{r}_t$, on a risk free one-period bond that pays in $t+1$ *is* uniquely determined:
$$\frac{1}{C_t} = \tilde{r}_t \beta E_t \frac{1}{C_{t+1}}.$$
- By no-arbitrage, the following weighted average of $R_t/\bar{\pi}_{t+1}$ across period $t+1$ states of nature is uniquely determined:
$$\tilde{r}_t = \frac{E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}}}{E_t \frac{1}{C_{t+1}}}.$$

Real Interest Rate, Continued

- The object, $\tilde{r}_t$, plays no role in the computation of the equilibrium allocations.
- But, $\tilde{r}_t$ is important from an economic perspective since it plays a key role in the consumption decisions of households in the model.

Real Interest Rate, Continued

- To understand the economics of the model, it is useful to examine response of $\tilde{r}_t$ to shocks:

$$\tilde{r}_t = \frac{1}{\beta}\frac{1}{E_t \frac{C_t}{C_{t+1}}} \to \tilde{r}_t = \frac{1}{\beta}\frac{1}{E_t \exp\left[a_t - a_{t+1} + \frac{\tau_{t+1}-\tau_t}{1+\varphi}\right]}$$

$$= \frac{1}{\beta}\frac{1}{\exp\left[E_t\left(a_t - a_{t+1}\right) + E_t\left(\frac{\tau_{t+1}-\tau_t}{1+\varphi}\right) + V\right]},$$

where $c_t \equiv \log C_t$ and assuming ε^a_{t+1} and ε^τ_{t+1} are Normally distributed (V is very small).

- Then,

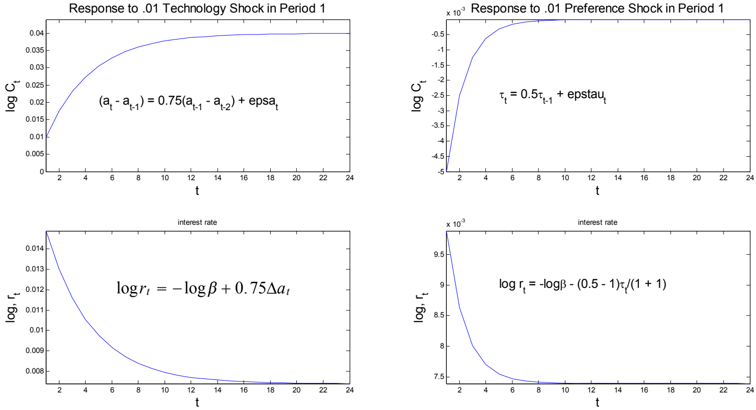
$$r^*_t \equiv \log \tilde{r}_t = -\log\left(\beta\right) + E_t\left[a_{t+1} - a_t - \left(\frac{\tau_{t+1}-\tau_t}{1+\varphi}\right) - V\right]$$

Real Interest Rate, Continued

$$r_t^* = -\log(\beta) + E_t\left[a_{t+1} - a_t - \left(\frac{\tau_{t+1} - \tau_t}{1+\varphi}\right) - V\right]$$

- The response of r_t^* to ε_t^a or ε_t^τ depends how the forecast of future a_t and τ_t responds.
- Get opposite response of r_t^* to ε_t^a in trend stationary case: $a_t = \rho a_{t-1} + \varepsilon_t^a$.
- Suppose technology jumps 1 percent (i.e., ε_t^a, a_t rise by 0.01).
  - In difference stationary case (with $\rho > 0$), aggregate demand (i.e., C_t), absent a change in r_t^*, responds by more than 1 percent, for consumption smoothing reasons. A jump in r_t^* ensures that aggregate demand only rises by the efficient amount, 1 percent.
  - In the trend stationary case, aggregate demand rises by less than 1 percent, absent a change in r_t^*. A fall in r_t^* helps to boost aggregate demand so that it rises by the efficient amount, 1 percent.

Dynamic Properties of the Model



Real Interest Rate, Continued

- Because of its attractive properties as a guide to aggregate demand for goods (here, just C_t), will later refer to r_t^* as the 'natural rate of interest'.
- When we study equilibrium with price setting frictions, welfare-reducing discrepancies between equilibrium and first-best allocations will be traced to deviations between the actual and natural rate of interest.

Outline

- Preferences and Technology. (done)
  - First best, efficient allocations (benchmark for later analysis).
- Flexible Price Equilibrium (done)
  - Efficient, as long as a monopoly distortion is eliminated.
    - Complete monetary neutrality.
- Equilibrium in sticky price version of the model (next).
  - Private sector equilibrium conditions.
  - Ramsey Equilibrium.
  - Linearization (first order perturbation)
  - Taylor rule
  - Properties of Taylor rule equilibrium

The Model with Price Setting Frictions on Intermediate Good Producers

- Household and final good producers unchanged from before:

$$\begin{aligned} \frac{1}{C_t} &= \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \\ \exp(\tau_t) C_t N_t^{\varphi} &= \frac{W_t}{P_t}. \end{aligned}$$

$$\begin{aligned} Y_t &= \left[\int_0^1 Y_{i,t}^{\frac{\varepsilon-1}{\varepsilon}} dj\right]^{\frac{\varepsilon}{\varepsilon-1}}, Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}}\right)^{\varepsilon} \\ &\rightarrow \ P_t = \left(\int_0^1 P_{i,t}^{(1-\varepsilon)} di\right)^{\frac{1}{1-\varepsilon}} \end{aligned}$$

- Intermediate good firms' problem changes.

Intermediate Good Producers

- Demand curve for i^{th} monopolist:

$$Y_{i,t} = Y_t \left(\frac{P_t}{P_{i,t}} \right)^{\varepsilon}.$$

- Production function:

$$Y_{i,t} = \exp(a_t) N_{i,t}, \; a_t = \rho a_{t-1} + \varepsilon_t^a$$

- Calvo Price-Setting Friction:

$$P_{i,t} = \begin{cases} \tilde{P}_t & \text{with probability } 1-\theta \\ P_{i,t-1} & \text{with probability } \theta \end{cases}.$$

- Real marginal cost:

$$s_t = \frac{\frac{d\text{Cost}}{d\text{worker}}}{\frac{d\text{output}}{d\text{worker}}} = \frac{\overbrace{(1-\nu)}^{\text{minimize monopoly distortion by setting } = \frac{\varepsilon-1}{\varepsilon}} \frac{W_t}{P_t}}{\exp(a_t)}$$

Intermediate Good Firm

- Present discounted value of firm profits:

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\upsilon_{t+j}}^{\text{marginal value of dividends to household}=u_{c,t+j}/P_{t+j}} \overbrace{\left[\overbrace{P_{i,t+j} Y_{i,t+j}}^{\text{revenues}} - \overbrace{P_{t+j} s_{t+j} Y_{i,t+j}}^{\text{total cost}} \right]}^{\text{period } t+j \text{ profits sent to household}}$$

- Each of the $1-\theta$ firms that can optimize price choose $\tilde{P}_t$ to optimize

$$E_t \sum_{j=0}^{\infty} \beta^j \overbrace{\theta^j}^{\substack{\text{in selecting price, firm only cares about} \\ \text{future states in which it can't reoptimize}}} \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}].$$

Intermediate Good Firm Problem

- Substitute out the demand curve:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} [\tilde{P}_t Y_{i,t+j} - P_{t+j} s_{t+j} Y_{i,t+j}]$$

$$= E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [\tilde{P}_t^{1-\varepsilon} - P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon}].$$

- Differentiate with respect to $\tilde{P}_t$:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon} [(1-\varepsilon)(\tilde{P}_t)^{-\varepsilon} + \varepsilon P_{t+j} s_{t+j} \tilde{P}_t^{-\varepsilon-1}] = 0,$$

- or

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \upsilon_{t+j} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

Intermediate Good Firm Problem

- Objective:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j \frac{u'(C_{t+j})}{P_{t+j}} Y_{t+j} P_{t+j}^{\varepsilon+1} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$\rightarrow E_t \sum_{j=0}^{\infty} (\beta\theta)^j P_{t+j}^{\varepsilon} \left[\frac{\tilde{P}_t}{P_{t+j}} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0.$$

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \right] = 0,$$

$$\tilde{p}_t = \frac{\tilde{P}_t}{P_t},\ X_{t,j} = \begin{cases} \frac{1}{\bar{\pi}_{t+j}\bar{\pi}_{t+j-1}\cdots\bar{\pi}_{t+1}}, j \geq 1 \\ 1, j = 0. \end{cases} ,\ X_{t,j} = X_{t+1,j-1} \frac{1}{\bar{\pi}_{t+1}}, j > 0$$

Intermediate Good Firm Problem

- Want $\tilde{p}_t$ in:

$$E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \left[\tilde{p}_t X_{t,j} - \frac{\varepsilon}{\varepsilon-1} s_{t+j} \right] = 0$$

- Solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t}$$

- But, still need expressions for K_t, F_t.

$$
\begin{aligned}
K_t &= E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \sum_{j=1}^{\infty} (\beta\theta)^{j-1} \left(\frac{1}{\bar{\pi}_{t+1}} X_{t+1,j-1} \right)^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta \overbrace{E_t E_{t+1}}^{=E_t \text{ by LIME}} \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} \overbrace{E_{t+1} \sum_{j=0}^{\infty} (\beta\theta)^j X_{t+1,j}^{-\varepsilon} \frac{\varepsilon}{\varepsilon - 1} s_{t+1+j}}^{\text{exactly } K_{t+1}!} \\
&= \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}
\end{aligned}
$$

- From previous slide:

$$K_t = \frac{\varepsilon}{\varepsilon - 1} s_t + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}} \right)^{-\varepsilon} K_{t+1}.$$

- Substituting out for marginal cost:

$$\frac{\varepsilon}{\varepsilon - 1} s_t = \frac{\varepsilon}{\varepsilon - 1}(1 - \nu) \frac{\overbrace{W_t/P_t}^{d\text{Cost}/d\text{labor}}}{\underbrace{}_{}\overbrace{\exp(a_t)}^{d\text{Output}/d\text{labor}}}$$

$$= \frac{\varepsilon}{\varepsilon - 1}(1 - \nu) \frac{\overbrace{\exp(\tau_t) N_t^{\varphi} C_t}^{=\frac{W_t}{P_t} \text{ by household optimization}}}{\exp(a_t)}.$$

In Sum

- solution:

$$\tilde{p}_t = \frac{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{-\varepsilon} \frac{\varepsilon}{\varepsilon-1} s_{t+j}}{E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon}} = \frac{K_t}{F_t},$$

- Where:

$$K_t = (1-\nu_t)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t) N_t^{\varphi} C_t}{\exp(a_t)} + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{-\varepsilon} K_{t+1}.$$

$$F_t \equiv E_t \sum_{j=0}^{\infty} (\beta\theta)^j (X_{t,j})^{1-\varepsilon} = 1 + \beta\theta E_t \left(\frac{1}{\bar{\pi}_{t+1}}\right)^{1-\varepsilon} F_{t+1}$$

To Characterize Equilibrium

- Have equations characterizing optimization by firms and households.
- Still need:
  - Expression for all the prices. Prices, $P_{i,t},\ 0 \leq i \leq 1$, will all be different because of the price setting frictions.
  - Relationship between aggregate employment and aggregate output not simple because of price distortions:

$$Y_t \neq e^{a_t} N_t, \text{ in general}$$

- This part of the analysis is the reason why it made Calvo famous – it's not easy.

Going for Prices

- Aggregate price relationship

$$P_t = \left[\int_0^1 P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

$$= \left[\int_{\text{firms that reoptimize price}} P_{i,t}^{(1-\varepsilon)} di + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

$$\overbrace{=}^{\text{all reoptimizers choose same price}} \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di \right]^{\frac{1}{1-\varepsilon}}$$

- In principle, to solve the model need all the prices, $P_t,\ P_{i,t},\ 0 \leq i \leq 1$
  - Fortunately, that won't be necessary.

Key insight

$$\int_{\text{firms that don't reoptimize price in } t} P_{i,t}^{(1-\varepsilon)} di$$

$$\overbrace{=}^{\text{add over prices, weighted by \# of firms posting that price}}$$

$$\int \left[\overbrace{f_{t-1,t}(\omega)}^{\text{'number' of firms that had price, } P(\omega)\text{, in } t-1 \text{ and were not able to reoptimize in } t} P(\omega)^{(1-\varepsilon)} \right] d\omega$$

Applying the Insight

- By Calvo randomization assumption

$$f_{t-1,t}(\omega) = \theta \times \overbrace{f_{t-1}(\omega)}^{\text{total 'number' of firms with price } P(\omega) \text{ in } t-1}, \text{ for all } \omega$$

- Substituting:

$$\int_{\text{firms that don't reoptimize price}} P_{i,t}^{(1-\varepsilon)} di = \int f_{t-1,t}(\omega) P(\omega)^{(1-\varepsilon)} d\omega$$

$$= \theta \int f_{t-1}(\omega) P(\omega)^{(1-\varepsilon)} d\omega$$

$$= \theta P_{t-1}^{(1-\varepsilon)}$$

Expression for $\tilde{p}_t$ in terms of aggregate inflation

- Conclude that this relationship holds between prices:

$$P_t = \left[(1-\theta)\tilde{P}_t^{(1-\varepsilon)} + \theta P_{t-1}^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}.$$

  – Only two variables here!

- Divide by P_t:

$$1 = \left[(1-\theta)\tilde{p}_t^{(1-\varepsilon)} + \theta\left(\frac{1}{\bar{\pi}_t} \right)^{(1-\varepsilon)} \right]^{\frac{1}{1-\varepsilon}}$$

- Rearrange:

$$\tilde{p}_t = \left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right]^{\frac{1}{1-\varepsilon}}$$

Relation Between Aggregate Output and Aggregate Inputs

- Technically, there is no 'aggregate production function' in this model
  - If you know how many people are working, *N*, and the state of technology, *a*, you don't have enough information to know what *Y* is.
  - Price frictions imply that resources will not be efficiently allocated among different inputs.
    - Implies *Y* low for given *a* and *N*. How low?
    - Tak Yun (JME) gave a simple answer.

Tak Yun Algebra

$$Y_t^* = \int_0^1 Y_{i,t} di \left(= \int_0^1 A_t N_{i,t} di \overset{\text{labor market clearing}}{\overbrace{=}} A_t N_t \right)$$

$$\overset{\text{demand curve}}{\overbrace{=}} Y_t \int_0^1 \left(\frac{P_{i,t}}{P_t} \right)^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} \int_0^1 (P_{i,t})^{-\varepsilon} di$$

$$= Y_t P_t^{\varepsilon} (P_t^*)^{-\varepsilon}$$

Calvo insight

- Where: $P_t^* \equiv \left[\int_0^1 P_{i,t}^{-\varepsilon} di \right]^{\frac{-1}{\varepsilon}} = \left[(1-\theta) \tilde{P}_t^{-\varepsilon} + \theta (P_{t-1}^*)^{-\varepsilon} \right]^{\frac{-1}{\varepsilon}}$

Relationship Between Agg Inputs and Agg Output

- Rewriting previous equation:

$$Y_t = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} Y_t^*$$

$$= p_t^* e^{a_t} N_t,$$

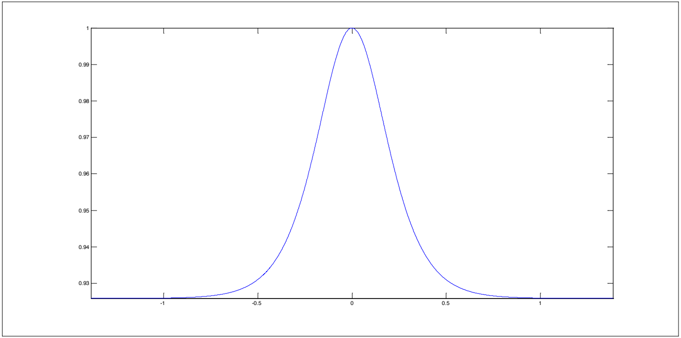
- 'efficiency distortion':

$$p_t^* : \begin{cases} \leq 1 & \\ = 1 & P_{i,t} = P_{j,t}, \text{ all } i,j \end{cases}$$

Example of Efficiency Distortion

$$P_{j,t} = \begin{cases} P^1 & 0 \le j \le \alpha \\ P^2 & \alpha \le j \le 1 \end{cases} . \; p_t^* = \left(\frac{P_t^*}{P_t}\right)^{\varepsilon} = \left(\frac{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{-\varepsilon}\right]^{\frac{-1}{\varepsilon}}}{\left[\alpha + (1-\alpha)\left(\frac{P^2}{P^1}\right)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}} \right)^{\varepsilon}$$

$\alpha = 0.5, \varepsilon = 10$



$\log P^1/P^2$

Collecting Equilibrium Conditions

- Price setting:

$$K_t = (1-\nu)\frac{\varepsilon}{\varepsilon-1}\frac{\exp(\tau_t)N_t^{\varphi}C_t}{A_t} + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon}K_{t+1} \quad (1)$$

$$F_t = 1 + \beta\theta E_t\bar{\pi}_{t+1}^{\varepsilon-1}F_{t+1} \quad (2)$$

- Intermediate good firm optimality and restriction across prices:

$$\overbrace{\frac{K_t}{F_t}}^{=\tilde{p}_t \text{ by firm optimality}} = \overbrace{\left[\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta}\right]^{\frac{1}{1-\varepsilon}}}^{=\tilde{p}_t \text{ by restriction across prices}} \quad (3)$$

Equilibrium Conditions

- Law of motion of (Tak Yun) distortion:

$$p_t^* = \left[(1-\theta)\left(\frac{1-\theta\bar{\pi}_t^{(\varepsilon-1)}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta\bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right]^{-1} \quad (4)$$

- Household Intertemporal Condition:

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \quad (5)$$

- Aggregate inputs and output:

$$C_t = p_t^* e^{a_t} N_t \quad (6)$$

- 6 equations, 8 unknowns:

$$\nu,\ C_t,\ p_t^*, N_t, \bar{\pi}_t, K_t, F_t, R_t$$

- System under determined!

Underdetermined System

- Not surprising: we added a variable, the nominal rate of interest.
- Also, we're counting subsidy as among the unknowns.
- Have two extra policy variables.
- One way to pin them down: compute optimal policy.

Ramsey-Optimal Policy

- 6 equations in 8 unknowns…..
  - Many configurations of the 8 unknowns that satisfy the 6 equations.
  - Look for the best configurations (Ramsey optimal)
    - Value of tax subsidy and of *R* represent optimal policy

- Finding the Ramsey optimal setting of the 6 variables involves solving a simple Lagrangian optimization problem.

Ramsey Problem

$$\max_{\nu, p_t^*, C_t, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log C_t - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{1t} \left[\frac{1}{C_t} - E_t \frac{\beta}{C_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta \theta F_{t+1} - F_t]$$

$$+ \lambda_{4t} \left[(1-\nu) \frac{\varepsilon}{\varepsilon - 1} \frac{C_t \exp(\tau_t) N_t^{\varphi}}{e^{a_t}} + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1-\theta \bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right]$$

$$+ \lambda_{6t} [C_t - p_t^* e^{a_t} N_t] \}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{\nu, p_t^*, N_t, R_t, \bar{\pi}_t, F_t, K_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{1t} \left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t} \beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1 - \theta (\bar{\pi}_t)^{\varepsilon - 1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon - 1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

$$+ \lambda_{3t} [1 + E_t \bar{\pi}_{t+1}^{\varepsilon - 1} \beta \theta F_{t+1} - F_t]$$

$$+ \lambda_{4t} \left[(1-\nu) \frac{\varepsilon}{\varepsilon - 1} \exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta \theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

$$+ \lambda_{5t} \left[F_t \left(\frac{1 - \theta \bar{\pi}_t^{\varepsilon - 1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \}$$

Solving the Ramsey Problem (surprisingly easy in this case)

- First, substitute out consumption everywhere

$$\max_{v,p_t^*,N_t,R_t,\bar{\pi}_t,F_t,K_t} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \left(\log N_t + \log p_t^* - \exp(\tau_t)\frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

defines R →
$$+\lambda_{1t}\left[\frac{1}{p_t^* N_t} - E_t \frac{e^{a_t}\beta}{p_{t+1}^* e^{a_{t+1}} N_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}} \right]$$

$$+\lambda_{2t}\left[\frac{1}{p_t^*} - \left((1-\theta)\left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right]$$

defines F →
$$+\lambda_{3t}[1 + E_t \bar{\pi}_{t+1}^{\varepsilon-1} \beta\theta F_{t+1} - F_t]$$

defines tax →
$$+\lambda_{4t}\left[(1-v)\frac{\varepsilon}{\varepsilon-1}\exp(\tau_t) N_t^{1+\varphi} p_t^* + E_t \beta\theta \bar{\pi}_{t+1}^{\varepsilon} K_{t+1} - K_t \right]$$

defines K →
$$+\lambda_{5t}\left[F_t \left(\frac{1-\theta\bar{\pi}_t^{\varepsilon-1}}{1-\theta} \right)^{\frac{1}{1-\varepsilon}} - K_t \right] \Bigg\}$$

Solving the Ramsey Problem, cnt'd

- Simplified problem:

$$\max_{\bar{\pi}_t, p_t^*, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \{ \left(\log N_t + \log p_t^* - \exp(\tau_t) \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

$$+ \lambda_{2t} \left[\frac{1}{p_t^*} - \left((1-\theta) \left(\frac{1-\theta(\bar{\pi}_t)^{\varepsilon-1}}{1-\theta} \right)^{\frac{\varepsilon}{\varepsilon-1}} + \frac{\theta \bar{\pi}_t^{\varepsilon}}{p_{t-1}^*} \right) \right] \}$$

- First order conditions with respect to $p_t^*, \bar{\pi}_t, N_t$

$$p_t^* + \beta \lambda_{2,t+1} \theta \bar{\pi}_{t+1}^{\varepsilon} = \lambda_{2t},\ \bar{\pi}_t = \left[\frac{(p_{t-1}^*)^{\varepsilon-1}}{1-\theta+\theta(p_{t-1}^*)^{\varepsilon-1}} \right]^{\frac{1}{\varepsilon-1}},\ N_t = \exp\left(-\frac{\tau_t}{\varphi+1} \right)$$

- Substituting the solution for inflation into law of motion for price distortion:

$$p_t^* = \left[(1-\theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}.$$

Solution to Ramsey Problem

Eventually, price distortions eliminated, regardless of shocks

$$p_t^* = \left[(1-\theta) + \theta (p_{t-1}^*)^{(\varepsilon-1)} \right]^{\frac{1}{(\varepsilon-1)}}$$

When price distortions gone, so is inflation.

$$\bar{\pi}_t = \frac{p_{t-1}^*}{p_t^*}$$

Efficient ('first best') allocations in real economy

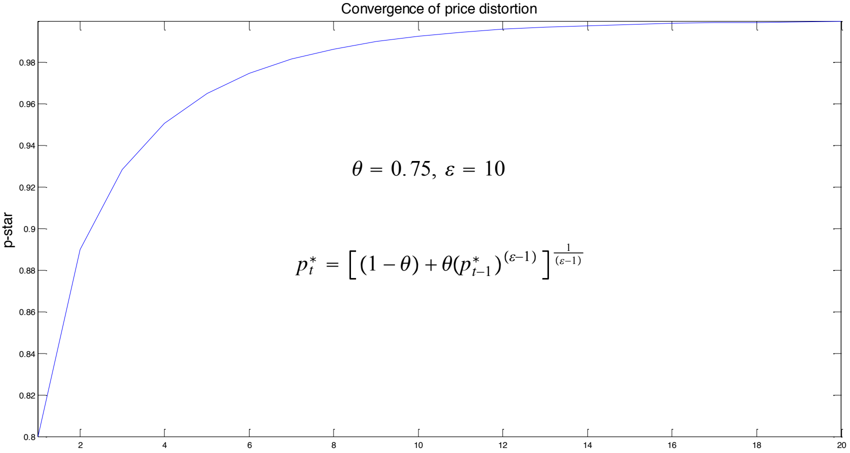
$$N_t = \exp\left(-\frac{\tau_t}{1+\varphi}\right)$$

$$1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$$

$$C_t = p_t^* e^{a_t} N_t.$$

Consumption corresponds to efficient allocations in real economy, eventually when price distortions gone

Eventually, Optimal (Ramsey) Equilibrium and Efficient Allocations in Real Economy Coincide



- The Ramsey allocations are eventually the best allocations in the economy without price frictions (i.e., ‘first best allocations’)

- Refer to the Ramsey allocations as the ‘natural allocations’….
  - Natural consumption, natural rate of interest, etc.

Linearizing around Efficient Steady State

- In steady state (assuming $\bar{\pi} = 1$, $1 - \nu = \frac{\varepsilon - 1}{\varepsilon}$)

$$p^* = 1,\ K = F = \frac{1}{1 - \beta\theta},\ s = \frac{\varepsilon - 1}{\varepsilon},\ \Delta a = \tau = 0,\ N = 1$$

- Linearizing the Tack Yun distortion, (4), about steady state: $\hat{p}_t^* = \theta \hat{p}_{t-1}^* \rightarrow$

$$\boxed{\hat{p}_t^* = 0,\ t \text{ large}}$$

- Denote the output gap in ratio form by X_t:

$$X_t \equiv \frac{C_t}{A_t \exp\left(-\frac{\tau_t}{1+\varphi}\right)} = p_t^* N_t \exp\left(\frac{\tau_t}{1+\varphi}\right),$$

so that (using $x_t \equiv \hat{X}_t$):

$$\boxed{x_t = \hat{N}_t + \frac{d\tau_t}{1+\varphi}}$$

NK IS Curve

- The intertemporal Euler equation, (5), after substituting for C_t in terms of X_t:

$$\frac{1}{X_t A_t \exp\left(-\frac{\tau_t}{1+\varphi}\right)} = \beta E_t \frac{1}{X_{t+1} A_{t+1} \exp\left(-\frac{\tau_{t+1}}{1+\varphi}\right)} \frac{R_t}{\bar{\pi}_{t+1}}$$

$$\frac{1}{X_t} = E_t \frac{1}{X_{t+1} R^*_{t+1}} \frac{R_t}{\bar{\pi}_{t+1}},$$

where

$$R^*_{t+1} \equiv \frac{1}{\beta} \exp\left(a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right)$$

then, use

$$\widehat{z_t u_t} = \hat{z}_t + \hat{u}_t, \; \widehat{\left(\frac{u_t}{z_t}\right)} = \hat{u}_t - \hat{z}_t$$

to obtain:

$$\hat{X}_t = E_t \left[\hat{X}_{t+1} - \left(\hat{R}_t - \widehat{\bar{\pi}}_{t+1} - \hat{R}^*_{t+1} \right) \right]$$

NK IS Curve

- Let:

$$R_t \equiv \exp(r_t)$$
$$\to \hat{R}_t = \frac{d\exp(r_t)}{\exp(r)} = \frac{\exp(r)\,dr_t}{\exp(r)} = dr_t \equiv r_t - r.$$

- Also,

$$R^*_{t+1} = \exp\left(\log R^*_{t+1}\right)$$
$$\to \hat{R}^*_{t+1} = \frac{d\exp\left(\log R^*_{t+1}\right)}{\exp\left(\log R^*\right)} = d\log R^*_{t+1}$$
$$= \log R^*_{t+1} - \overbrace{\log R^*}^{=r \text{ in efficient steady state, with } \bar{\pi}=1}$$

- So, (letting $r^*_t \equiv E_t \log R^*_{t+1}$)

$$E_t\left(\hat{R}_t - \hat{R}^*_{t+1}\right) = dr_t - E_t d\log R^*_{t+1} = r_t - r^*_t.$$

NK IS Curve

- Substituting

$$E_t\left(\hat{R}_t - \hat{R}^*_{t+1}\right) = r_t - r^*_t$$

into

$$\hat{X}_t = E_t\left[\hat{X}_{t+1} - \left(\hat{R}_t - \widehat{\bar{\pi}}_{t+1} - \hat{R}^*_{t+1}\right)\right],\ x_t \equiv \hat{X}_t,$$

and using

$$\widehat{\bar{\pi}}_{t+1} = \pi_{t+1},\ \text{when } \bar{\pi} = 1,$$

we obtain NK IS curve:

$$\boxed{x_t = E_t x_{t+1} - E_t\left[r_t - \pi_{t+1} - r^*_t\right]}$$

- Also,

$$r^*_t = -\log(\beta) + E_t\left[a_{t+1} - a_t - \frac{\tau_{t+1} - \tau_t}{1+\varphi}\right].$$

Linearized Marginal Cost

- Marginal cost (using $da_t = a_t$, $d\tau_t = \tau_t$ because $a = \tau = 0$):

$$
\begin{aligned}
s_t &= (1-\nu)\frac{\bar{w}_t}{A_t},\ \bar{w}_t \equiv \frac{W_t}{P_t} = \exp(\tau_t) N_t^{\varphi} C_t \\
&\rightarrow \widehat{\bar{w}}_t = \tau_t + a_t + (1+\varphi)\hat{N}_t
\end{aligned}
$$

- Then,

$$
\boxed{\hat{s}_t = \widehat{\bar{w}}_t - a_t = (\varphi+1)\left[\tfrac{\tau_t}{\varphi+1} + \hat{N}_t\right] = (\varphi+1)\, x_t}
$$

Linearized Phillips Curve

- Log-linearize equilibrium conditions, (1)-(3), around steady state:

$$\hat{K}_t = (1 - \beta\theta)\,\hat{s}_t + \beta\theta\left(\varepsilon\widehat{\bar{\pi}}_{t+1} + \hat{K}_{t+1}\right) \quad (1)$$

$$\hat{F}_t = \beta\theta\,(\varepsilon - 1)\,\widehat{\bar{\pi}}_{t+1} + \beta\theta\hat{F}_{t+1} \quad (2)$$

$$\hat{K}_t = \hat{F}_t + \frac{\theta}{1-\theta}\widehat{\bar{\pi}}_t \quad (3)$$

- Substitute (3) into (1)

$$\hat{F}_t + \frac{\theta}{1-\theta}\hat{\pi}_t = (1 - \beta\theta)\,\hat{s}_t + \beta\theta\left(\varepsilon\widehat{\bar{\pi}}_{t+1} + \hat{F}_{t+1} + \frac{\theta}{1-\theta}\hat{\pi}_{t+1}\right)$$

- Simplify the latter using (2), to obtain the NK Phillips curve:

$$\boxed{\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{s}_t + \beta\pi_{t+1}}$$

The Equilibrium Conditions

$x_t = x_{t+1} - \left[r_t - \pi_{t+1} - r_t^*\right]$
$\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\hat{s}_t + \beta\pi_{t+1}$
$\hat{s}_t = (\varphi + 1)\, x_t$
$r_t^* = -\log(\beta) + E_t\left[a_{t+1} - a_t - \frac{\tau_{t+1}-\tau_t}{1+\varphi}\right]$

Taylor rule (in deviation from steady state):

$$r_t = \alpha r_{t-1} + (1-\alpha)\left[\phi_\pi \pi_t + \phi_x x_t\right]$$

Shocks:

$$\Delta a_t = \rho \Delta a_{t-1} + \varepsilon_t^a,\ \tau_t = \lambda \tau_{t-1} + \varepsilon_t^\tau.$$

Solving the Model

$$s_t = \begin{pmatrix} \Delta a_t \\ \tau_t \end{pmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \lambda \end{bmatrix} \begin{pmatrix} \Delta a_{t-1} \\ \tau_{t-1} \end{pmatrix} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_t^\tau \end{pmatrix}$$

$$s_t = P s_{t-1} + \epsilon_t$$

$$\begin{bmatrix} \beta & 0 & 0 & 0 \\ \frac{1}{\sigma} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t+1} \\ x_{t+1} \\ r_{t+1} \\ r^*_{t+1} \end{pmatrix} + \begin{bmatrix} -1 & \kappa & 0 & 0 \\ 0 & -1 & -\frac{1}{\sigma} & \frac{1}{\sigma} \\ (1-\alpha)\phi_\pi & (1-\alpha)\phi_x & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \pi_t \\ x_t \\ r_t \\ rr^*_t \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \pi_{t-1} \\ x_{t-1} \\ r_{t-1} \\ r^*_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} s_{t+1} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\sigma\psi\rho & -\frac{1}{\sigma+\varphi}(1-\lambda) \end{pmatrix} s_t$$

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

Solving the Model

$$E_t[\alpha_0 z_{t+1} + \alpha_1 z_t + \alpha_2 z_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0$$

$$s_t - P s_{t-1} - \epsilon_t = 0.$$

- Solution:

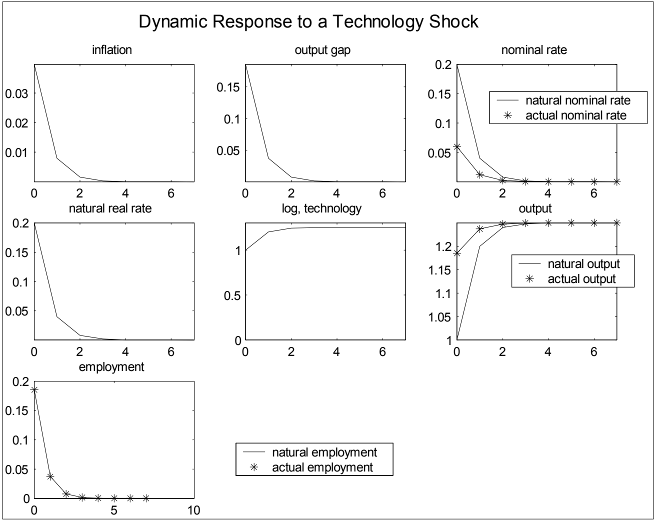
$$z_t = A z_{t-1} + B s_t$$

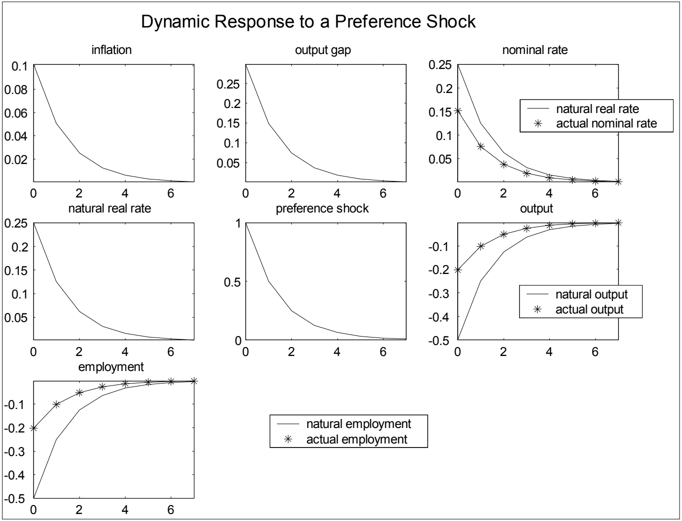
- As before:

$$\alpha_0 A^2 + \alpha_1 A + \alpha_2 I = 0,$$

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1)B] = 0$$

$$\phi_x = 0,\ \phi_\pi = 1.5,\ \beta = 0.99,\ \varphi = 1,\ \rho = 0.2,\ \theta = 0.75,\ \alpha = 0,\ \delta = 0.2,\ \lambda = 0.5.$$





Dynamic Response to a Preference Shock
inflation
output gap
nominal rate
natural real rate
actual nominal rate
natural real rate
preference shock
output
natural output
actual output
employment
natural employment
actual employment

Why Is Output Inefficiently High or Low Sometimes?

- Brief answer: the Taylor rule sets the wrong interest rate (should be the natural rate).
- Households equate costs with the *private* benefit of working, W_t:

$$\frac{-u_{N,t}}{u_{c,t}} = \exp(\tau_t) C_t N_t^{\varphi} = \frac{W_t}{P_t}.$$

  - So, the only possible reason efficiency may not occur is if W_t/P_t does not correspond to the actual marginal product of labor.
- The relationship between W_t/P_t and labor productivity may be understood by studying the markup of price over marginal cost.

Price over Marginal Cost (Markup)

- i^{th} firm sets $P_{i,t}$ as a markup, $\mu_{i,t}$, over marginal cost

$$P_{i,t} = \overbrace{\mu_{i,t}}^{\text{markup}} \times \frac{\overbrace{(1-\nu)\,W_t}^{\text{marginal cost}}}{e^{a_t}},$$

  where we have been setting $1-\nu = \frac{\varepsilon-1}{\varepsilon}$.

- In the flexible price version of the model the markup is trivial, $\mu_{i,t} = \frac{\varepsilon-1}{\varepsilon}$ so

$$P_{i,t} = P_t = \frac{W_t}{e^{a_t}},$$

  and W_t/P_t corresponds to the marginal product of labor.

Price over Marginal Cost (Markup)

- In the sticky price version of the model, the markup is more complicated.
  - firms currently setting prices, do so to get the markup to be $\varepsilon / (\varepsilon - 1)$ on average in the current and future periods (see earlier discussion)
  - for firms not able to set prices in the current period, the markup is

$$\mu_{i,t} = \frac{P_{i,t-1}}{\frac{(1-\nu)W_t}{e^{a_t}}}.$$

    - for these firms the markup moves inversely with a shock to marginal cost.
- Need to look at some aggregate of all markups.

Price over Marginal Cost (Markup)

- Weighted average markup, μ_t:

$$\mu_t \equiv \left(\int_0^1 \mu_{i,t}^{(1-\varepsilon)} di \right)^{\frac{1}{1-\varepsilon}} = \frac{P_t}{\frac{(1-\nu)W_t}{e^{a_t}}} = \frac{1}{\frac{1-\nu}{e^{a_t}} e^{\tau_t} C_t N_t^{\varphi}}$$

$$= \frac{1}{(1-\nu)\, e^{\tau_t} p_t^* N_t^{1+\varphi}} = \frac{1}{(1-\nu)\, p_t^*} \left(\frac{e^{-\frac{\tau_t}{1+\varphi}}}{N_t} \right)^{1+\varphi}.$$

- Consider the output gap:

$$X_t = \frac{C_t}{e^{a_t - \tau_t/(1+\varphi)}} = \frac{p_t^* e^{a_t} N_t}{e^{a_t - \tau_t/(1+\varphi)}} = p_t^* \frac{N_t}{e^{-\tau_t/(1+\varphi)}}$$

- Use this to substitute into the markup

$$\mu_t = \frac{(p_t^*)^{\varphi}}{(1-\nu)} X_t^{-(1+\varphi)}.$$

μ_t moves inversely with the output gap.

Price over Marginal Cost (Markup)

- The preceding implies:

$$\frac{-u_{N,t}}{u_{c,t}} \overbrace{=}^{\text{source of efficiency}} \frac{W_t}{P_t} = \frac{e^{a_t}}{(1-\nu)\,\mu_t} = e^{a_t}\,(p_t^*)^{-\varphi}\,X_t^{1+\varphi}.$$

- Suppose $p_t^* = 1$.
- When the output gap is high, $X > 0$, then the markup is low and the real wage *exceeds* e^a.
  - There is too much employment in this case because the private benefit to the workers of working exceeds the actual benefits (the difference comes out of lump sum profits).
- Why are firms willing to produce with a low markup? Because on average it is high $(\mu = \varepsilon/\,(\varepsilon - 1))$ and so cutting it (not too much!) allows them to still get some profits out of the workers.
- Interpretation: model implies markups are countercyclical.
  - Nekarda and Ramey ('The Cyclical Behavior of the Price-Cost Markup', UCSD 2013) argue that the evidence does not support this.

Wrap Up

- Suppose monetary policy puts the interest rate below the natural rate, driving up the output gap.
- The low interest rate gives people an incentive to spend more.
- This raises costs. Pices go up too, but less because of the stickiness. So, markups,

$$\mu_t = \frac{P_t}{MC_t},$$

  go down ($MC_t = (1-\nu) W_t / e^{a_t}$).
- Low markup means high wage when there is a monetary shock, encouraging more work.
- Profits are reduced, but presumably remain positive:

$$\begin{aligned} \text{profits} &= P_{i,t}Y_{i,t} - (1-\nu) W_t N_{i,t} \\ &= P_{i,t}Y_{i,t} - e^{a_t} MC_t \times N_{i,t} \\ &= [P_{i,t} - MC_t] Y_{i,t} \\ &= \left[\mu_{i,t} - 1\right] MC_t \times Y_{i,t} \end{aligned}$$