Name: ______________________ Date: ______________________

Level: X — Simplifying Rational Expressions

Definition: rational expression

A ***rational expression*** is an algebraic expression in fraction form, with *polynomials* in the numerator and denominator such that at least one variable appears in the denominator.

$$\text{e.g.}\quad \frac{1}{x} \quad\text{or}\quad \frac{2x^2+1}{-x^3+4x^2-3x-5}$$

"Dietary" Restrictions

Definition: domain of an algebraic expression

The ***domain*** of an algebraic expression is the set of real numbers that the variable is permitted to have. It may be helpful to think of the domain as the "diet" of the expression.

Exercise D1: Determine the domain of each of the following expressions.

$$\text{a)}\ \frac{x-4}{2} \qquad \text{b)}\ \sqrt[4]{x} \qquad \text{c)}\ \frac{x-1}{x+3}$$

a) Solution: Notice that $\frac{x-4}{2} = \frac{x}{2} - \frac{4}{2} = \frac{1}{2}x - 2$. This is a *linear* polynomial! There are no domain/dietary restrictions for polynomials. Therefore, the domain is given by the complete set of real numbers, $\mathbb{R}$.

b) Solution: Recall that whenever we have an nth root where the n is *even*, the expression under the radical must be nonnegative. For the expression $\sqrt[4]{x}$, $n = 4$. Thus, the domain restrictions are: $x \nless 0$. In other words, the domain is: $x \in [0, \infty)$

c) Solution: Here, we have a *rational expression*. Because the numerator is a polynomial, there are no restrictions here. For the denominator, we require that $x + 3 \neq 0$. Therefore, the domain is the set of real numbers such that $x + 3 \neq 0$. In other words, $x \in \mathbb{R}$ such that $x \neq -3$ or equivalently, $x \in (-\infty, -3) \cup (-3, \infty)$.

Exercise D2: Determine if the following statement is TRUE or FALSE. $\frac{x+2}{x+2} = 1$

Solution: FALSE. We know that if a is some real number, where $a \neq 0$, then $\frac{a}{a} = 1$. Recall that the rational expression $\frac{x+2}{x+2}$ changes as x changes. Furthermore, in the case where $x = -2$, we have $\frac{-2+2}{-2+2} = \frac{0}{0}$. This is an *indeterminate form*, which does not always equal 1. The correct statement is:

$$\frac{x+2}{x+2} = 1, \quad \text{provided } x \neq -2$$

Addition and Subtraction

Suppose you were asked to simplify the following rational expression:

$$\frac{1}{x+1} - \frac{6}{x^2+2x+1} + \frac{1}{x^2-1}$$

Goal # 1 Understanding the importance of an LCD (Least Common Denominator)

When attempting to simplify an algebraic expression, it may be helpful to recall our approach with *numerical* examples. We know that in order to add or subtract *fractions*, we must first have a **common denominator**. Below is a common property to help us achieve this goal:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} \qquad \textbf{(Property 1)}$$

Although this property guarantees that we get a common denominator, we may or may not have found the Least Common Denominator (LCD). So, in a moment, we will explore how to find it since LCDs have the potential to drastically simplify our computations later on.

Consider the following example: $\frac{5}{12} - \frac{5}{18}$

Using Property 1, we have:

$$\begin{aligned} \frac{5}{12} - \frac{5}{18} &= \frac{5(18) - 5(12)}{12(18)} \\ &= \frac{90-60}{216} \\ &= \frac{30}{216} \\ &= \frac{5}{36} \end{aligned}$$

Notice that 216 is *six* times larger than the reduced denominator of 36! If you found it obvious that the LCD was 36, you're off to a great start!

Goal # 2 Finding LCDs Systematically

Note that LCDs are not always easy to identify. So, we will explore a more systematic approach to finding them! If b and d denote denominators, we have the following cases:

Case I: One denominator is a *factor* of the other. e.g. $\frac{2}{3} - \frac{5}{18}$

$\Rightarrow$ LCD = Larger denominator
In the example, 3 is a factor of 18, so the LCD=18!

Case II: The greatest common factor of the denominators is 1. i.e. GCD $(b, d) = 1$
This means that the denominators have *no common prime factors*. e.g. $\frac{1}{4} + \frac{5}{21}$

$\Rightarrow$ LCD $= b \cdot d$ (the product of the two denominators)
In the example, $4 = 2^2$ and $21 = 3 \cdot 7$. Therefore, the LCD $= 4 \cdot 21 = 84$!

Case III: The denominators have prime factors in common. e.g. $\frac{5}{12} - \frac{5}{18}$

In this case, we have a set of strategies for finding the LCD!

$\boxed{\text{S1}}$ **Factor each denominator (D_1 and D_2). In other words, express D_1 and D_2 as a product of their prime factors (i.e. the "parts" that make them).**

> **Why?** So that we can easily see the *minimum* requirements needed in order to "include" every denominator in the LCD.

For example:

$D_1 : 12 = 2^2 \cdot 3^1$ $\qquad$ $D_2 : 18 = 2^1 \cdot 3^2$

$\boxed{\text{S2}}$ **Construct the LCD by multiplying together the *highest* power of each prime that appears among the denominators.**

> **Why?** We know that every denominator must be a factor of the LCD, so by taking the highest exponent on each prime, we guarantee that no denominator will end up missing some of its "parts." For example, if we include 2^2, we will cover the requirement for "2" for both D_1 and D_2. However, if we had only take 2^1, only D_2 will be satisfied by the LCD.

$$\text{LCD } = 2^2 \cdot 3^2 = 36$$

$\boxed{\text{S3}}$ **Multiply each fraction's numerator and denominator by the missing factors needed to create the LCD, then simplify the expression.**

> **Why?** Remember that we cannot change the original problem. So, "what we do to the bottom, we must always do to the top," which simply means we're multiplying by 1. Our goal was just to manipulate the expression so that we have an LCD.

$$\begin{aligned}\left(\frac{5}{2^2 \cdot 3^1}\right)\left(\frac{3}{3}\right) - \left(\frac{5}{2^1 \cdot 3^2}\right)\left(\frac{2}{2}\right) &= \frac{5(3) - 5(2)}{2^2 \cdot 3^2} \\ &= \frac{15 - 10}{36} \\ &= \frac{5}{36}\end{aligned}$$

Remember: Every strategy has a purpose!

Goal #3 Because variables are just *numbers in disguise*, we can use the same strategic approach (above) for the rational expression first proposed:

$$\frac{1}{x+1} - \frac{6}{x^2+2x+1} + \frac{1}{x^2-1}$$

S1 Factor the denominators.

$D_1: \quad (x+1)$ (A "prime" polynomial factor)

$D_2: \quad x^2+2x+1 = (x+1)^2$ (Perfect Square; requires factoring)

$D_3: \quad x^2-1 = (x+1)(x-1)$ (Difference of Squares; requires factoring)

S2 Construct the LCD by multiplying together the *highest* power of each factor that appears among the denominators

$$\text{LCD } = (x+1)^2\,(x-1)^1$$

Q: What would our denominator have been if we had used Propery 1?
A: $(x+1)(x^2+2x+1)(x^2-1)\ldots$ a polynomial of degree 5 instead of 3, which also means a lot more work when simplifying the numerator!

S3 Multiply each fraction's numerator and denominator by the missing factors needed to create the LCD, then *simplify* the expression. Be sure to *state restrictions* before cancelling. This is important now that our expressions involve variables!

$$\frac{1}{x+1} - \frac{6}{(x+1)^2} + \frac{1}{(x+1)(x-1)} = \frac{(1)\,(x+1)(x-1)}{(x+1)(x+1)(x-1)} - \frac{6\,(x-1)}{(x+1)^2\,(x-1)} + \frac{(1)\,(x+1)}{(x+1)(x-1)(x+1)}$$

$$= \frac{(x+1)(x-1) - 6\,(x-1) + 1\,(x+1)}{(x+1)^2(x-1)}$$

$$= \frac{x^2-1-6x+6+x+1}{(x+1)^2(x-1)}$$

$$= \frac{x^2-5x+6}{(x+1)^2(x-1)}$$

$$= \frac{(x-2)(x-3)}{(x+1)^2(x-1)}$$

Multiplication and Division

Recall that to multiply two fractions, we simply multiply the numerators together and write the product on top. Then multiply the denominators together and write the product on the bottom. Algebraically:

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

Division, on the other hand, is performed by multiplying the fraction in the numerator by the *reciprocal* of the fraction in the denominator. **Note that we require $b\neq 0$, $c\neq 0$ *and* $d\neq 0$**. Why? Algebraically, we have:

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

Consider the following exercise: $\dfrac{1}{x^2+7x+10}\cdot\dfrac{x+3}{x}\div\dfrac{x+3}{x^2-25}$

General Strategy Begin by factoring the expressions in the numerators and denominators because it will make it easier to **simplify/reduce** the fraction at the end! **Be sure to state the restrictions** on the *fraction that you are dividing by, when multiplying by the reciprocal* and whenever you *cancel expressions!*

$$\begin{aligned}
\frac{1}{x^2+7x+10}\cdot\frac{x+3}{x}\div\frac{x+3}{x^2-25} &= \frac{1}{(x+2)(x+5)}\cdot\frac{x+3}{x}\div\frac{x+3}{(x+5)(x-5)}\\
&= \frac{x+3}{(x+2)(x+5)(x)}\cdot\frac{(x+5)(x-5)}{x+3} && \text{provided} \quad x\neq -3,\pm 5\\
&= \frac{(x+3)(x-5)(x+5)}{x(x+3)(x+2)(x+5)}\\
&= \frac{\cancel{(x+3)}(x-5)\cancel{(x+5)}}{x\cancel{(x+3)}(x+2)\cancel{(x+5)}} && \text{provided} \quad x\neq -3,-5\\
&= \boxed{\frac{x-5}{x(x+2)},\ \text{provided } x\neq -3,\pm 5}
\end{aligned}$$

Note: Restrictions may also include $x\neq 0,-2$. However, this is not required since the expressions x and $x+2$ are still visible in the denominator, making these restrictions evident when looking at the final, simplified expression.

Here, it is also important to note that the only reason we were able to "cancel" the above expressions is because we first **acknowledged the restrictions**: $x+3\neq 0$ and $x-5\neq 0$. Consequently, $x\neq -3,5$ and thus, $\dfrac{x+3}{x+3}=\dfrac{x+5}{x+5}=1$. Furthermore, we know that multiplying by 1 does not change the value of our final expression!

Bringing Things Together

Exercise: Simplify the following expression: $\dfrac{x + \frac{x-4}{x-1}}{x - \frac{x^2-6}{1-x}} = \dfrac{N}{D}$

Breaking the problem into smaller parts:

$$N = x + \frac{x-4}{x-1}$$

$$\overset{\text{LCD}}{=} \frac{x(x-1)}{(x-1)} + \frac{x-4}{x-1}$$

$$= \frac{x(x-1) + x-4}{x-1}$$

$$\overset{\text{Expand}}{=} \frac{x^2 - \cancel{x} + \cancel{x} - 4}{x-1}$$

$$\overset{\text{collect}}{=} \frac{x^2-4}{x-1}$$

$$\overset{\text{factor}}{=} \frac{(x-2)(x+2)}{x-1}$$

$$D = x - \frac{x^2-6}{1-x}$$

$$\overset{\text{LCD}}{=} \frac{x(1-x)}{(1-x)} - \frac{x^2-6}{1-x}$$

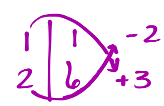
$$= \frac{x(1-x) - (x^2-6)}{1-x}$$

$$\overset{\text{expand}}{=} \frac{x - x^2 - x^2 + 6}{1-x}$$

$$\overset{\text{collect}}{=} \frac{-2x^2 + x + 6}{1-x}$$

$$\overset{\text{factor}}{=} \frac{-(2x^2 - x - 6)}{1-x}$$

$$= \frac{2x^2 - x - 6}{x-1}$$



$$\overset{\text{factor}}{=} \frac{(2x+3)(x-2)}{x-1}$$

Combining results:

$$\frac{N}{D} = \frac{(x-2)(x+2)}{x-1} \div \frac{(2x+3)(x-2)}{x-1}$$

flip!

$$= \frac{(x-2)(x+2)}{x-1} \cdot \frac{x-1}{(2x+3)(x-2)}$$

provided $x \neq 1, 2, -3/2$

$$= \frac{\cancel{(x-2)}(x+2)\cancel{(x-1)}}{\cancel{(x-2)}(2x+3)\cancel{(x-1)}}$$

$$= \frac{x+2}{2x+3}$$ provided $x \neq 1, 2, -3/2$

