# Simulation Results Related to Stochastic Electrodynamics

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**Abstract.** Stochastic electrodynamics (SED) is a classical theory of nature advanced significantly in the 1960s by Trevor Marshall and Timothy Boyer. Since then, SED has continued to be investigated by a very small group of physicists. Early investigations seemed promising, as SED was shown to agree with quantum mechanics (QM) and quantum electrodynamics (QED) for a few linear systems. In particular, agreement was found for the simple harmonic electric dipole oscillator, physical systems composed of such oscillators and interacting electromagnetically, and free electromagnetic fields with boundary conditions imposed such as would enter into Casimir-type force calculations. These results were found to hold for both zero-point and non-zero temperature conditions. However, by the late 1970s and then into the early 1980s, researchers found that when investigating nonlinear systems, SED did not appear to provide agreement with the predictions of QM and QED. A proposed reason for this disagreement was advocated by Boyer and Cole that such nonlinear systems are not sufficiently realistic for describing atomic and molecular physical systems, which should be fundamentally based on the Coulombic binding potential. Analytic attempts on these systems have proven to be most difficult. Consequently, in recent years more attention has been placed on numerically simulating the interaction of a classical electron in a Coulombic binding potential, with classical electromagnetic radiation acting on the classical electron. Good agreement was found for this numerical simulation work as compared with predictions from QM. Here this worked is reviewed and possible directions are discussed. Recent simulation work involving subharmonic resonances for the classical hydrogen atom is also discussed; some of the properties of these subharmonic resonances seem quite interesting and unusual.

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## INTRODUCTION

This article will present a brief history of the classical atomic theory of nature that has come to be known as stochastic electrodynamics. The early successes of this theory and its apparent later failings will be discussed, ending with some recent successes and suggestions for why SED may still be applicable.

The theory of SED has now been around for sometime, initially obtaining startling and impressive results in the 1960s and early 70s. At first there appeared indications that SED might provide a more fundamental basis for quantum mechanics (QM) and quantum electrodynamics (QED). For researchers first becoming familiar with SED, this idea is likely to be most surprising, particularly when it is learned that SED involves only classical physics consisting of Maxwell's equations and the relativistic version of Newton's equation of motion.

However, physicists investigating the theoretical predictions of SED observed noticeable difficulties by 1976, with even more serious consequences by the early to mid-1980s. The small group of physicists that were once investigating SED became even smaller.

In this introduction, we first summarize some of this early history. The next section then turns to describing conceptually how the classical theory of SED might account for phenomena as apparently "quantum mechanically" distant from classical ideas as atomic stability, the de Broglie wave length, and diffraction patterns that particles exhibit. A more quantitative examination is provided in the subsequent sections, including a review of the main equations in SED and a brief discussion of the difficulties of the nonlinear analysis. The last sections then turn to recent results and methods that look promising, plus possible directions for future work.

As for the key origin of the theory of SED, it's main advance began, independently, by Trevor Marshall [1], [2], [3] and Timothy Boyer [4], [5] in the 1960s . Earlier insights and initial beginnings of this work are more complicated to follow, as there are some ties to far earlier work by Planck, Nernst, and Einstein;

Ref. [6] does a nice job describing these connections. Indeed, Ref. [6] represents a very complete review [7] of SED up until about 1995, while other useful shorter reviews exist such as in Refs. [8], [9], and [10]. A semipopular scientific discussion is available in a Scientific American article in 1985 [11].

We now turn to summarizing some of the SED developments since 1963. The work begun by Marshall and Boyer in the 1960s, and continued by them and others, found an array of agreements with predictions between SED versus QM and QED for linear systems [8], such as for (1) calculations of ensemble averages of free electromagnetic fields [12], (2) systems of electric dipole simple harmonic oscillators (SHO) [12], [10], including the complicated situation of van der Waals at any distance [13], [14], [15], [16], and for (3) Casimir/van der Waals type situations [17], [18], [19], [20], [6], [21], and [22]. Yet other classically unexpected and traditionally credited quantum mechanical phenomena have been found from SED, such as diamagnetism [1], [23], and the thermal effects of acceleration through the vacuum [24], [25], [26], [27], [15], [28].

Interestingly enough, some of these results from SED actually preceded calculations carried out in QED, such as for retarded der Waals forces between electric dipole oscillators at temperatures $T \geq 0$ [14], the repulsive Casimir-type force prediction between a perfectly conducting plate and an infinitely permeable plate [20], which has recently gained some attention in the micro-electro-mechanical (MEMS) community due to its potential application of effecting stiction in MEMS devices [29], and a number of predicted phenomena involving the thermal effects of acceleration through the vacuum [25], [26], [27], [15], including spatially extended systems [28]. Nevertheless, most physicists today are quite unaware of these classical physical results; probably for the most part, this fact is quite justified.

The reason? There are probably two main ones. First, despite some of the impressive calculational results just mentioned, a direct connection was not found to either Schrödinger's wave equation or Dirac's relativistic wave equation; instead, deep calculations could be carried out for systems with a linear equation of motion, like for the electric dipole SHO, or for free electromagnetic fields in linear dielectric media. Second, when Boyer began pushing more deeply on nonlinear systems in 1976 [30], discrepancies were found between SED and QM/QED predictions. A relatively concentrated effort then began on nonlinear oscillators, and the hydrogen atom in particular [31], [32], [33], [34], [35], [36], [10], [6], revealing the core difficulties of SED as applied to more realistic nonlinear atomic systems in nature. These analyses of SED predicted clear disagreements with physical observation, such as that (1) a single hydrogen atom will ionize at $T = 0$, (2) line spectra for hydrogen was not apparently arising out of the theory, and (3) instead of an equilibrium between a Planckian plus ZP spectrum and nonlinear charged oscillators, equilibrium was found only when the obviously physically incorrect Rayleigh-Jeans spectra was present [30], [35], [36].

Since the early to mid 1980s, most research in SED was largely abandoned. Probably it is safe to say that most researchers familiar with SED concluded that this theory provided a much better description of physical processes than does conventional classical electrodynamics without the consideration of ZP and Planckian electromagnetic radiation [37], [38], but that SED was erroneous in key important areas and could not come close to predicting the wide range of phenomena predicted by QM/QED.

One outstanding objection to this conclusion has remained, however. As first suggested by Boyer [39] and as analyzed further by Cole [40], all of the key physical effects should arise from electromagnetic interactions for atomic systems. Examining other nonlinear binding potentials, other than ones arising from Coulombic binding potentials, have no relation to real physical atomic systems. Even though one can place any potential function in Schrödinger's equation and attempt to solve it, SED does not need to match these solutions, as they have little relationship, in detail, to the real physical world of atomic systems. Instead, realistic binding potentials must be examined. Moreover, for perturbation analyses, if one assumes that the small effect of the electric charge is a key part of the perturbation analysis, then this effect must be consistently carried out for the radiation reaction as well as for the binding potential and the effect of the ZP field acting on the orbiting charge [40]. Earlier work on the classical hydrogen atom had such questionable perturbation expansions in their analyses, possibly invalidating their conclusions. Properly accounting for these objections into an improved analytic, or even semi-analytic, reanalysis of SED, has been quite difficult.

This synopsis of the history of SED brings us to the most recent results [41], as well as additional new results that will be described in a later section here. Before turning to these results, however, we will next turn to the core physical ideas and motivation for pursuing SED, followed by a discussion of the nonlinear stochastic differential equations in SED.

## PHYSICAL IDEAS BEHIND SED

Marshall and Boyer proposed that atomic physical processes could be accurately described within classical physics provided one takes into account the appropriate classical electromagnetic stochastic radiation fields acting on classical charged particles. This stochastic radiation, as well as the stochastic motion of the charged particles, arises in a purely natural way at all temperatures, including even at absolute zero temperature, due to the interaction between charges and radiation. The key idea here is that if thermodynamic equilibrium of classical charged particles is at all possible, then a thermodynamic radiation spectrum must also exist and must be an essential part of the thermodynamic system of charged particles and radiation. As first clearly revealed by a relatively simple analysis in 1975 by Boyer [9], the problem of atomic collapse for a classical hydrogen atom that helped turn physicists, such as Planck, Einstein, and Bohr, from classical physics in the early 1900s, might not be the unaddressable "roadblock" for classical physics, but instead might actually be the central element for understanding how quantum mechanical phenomena emerges from fully classical physics.

To quickly see qualitatively how this might be possible, we can consider a classical electron spiraling about an oppositely charged classical nucleus. More and more energy will be radiated by the spiralling classical electron, with its radius $r$ to the nucleus therefore becoming smaller and smaller. For an initial circular orbit of $r = 0.5\,\mathring{\mathrm{A}}$, one can show $r \to 0$ in about $1.3 \times 10^{-11}\,\mathrm{s}$ [42]. However, that's not the end of the story. If one wants to seriously consider a possible scenario where statistical equilibrium might exist between classical charged particles and classical electromagnetic radiation, then the large amount of radiation emitted must be a key part to analyzing this possible statistical equilibrium situation. Certainly such radiation would effect the behavior of another nearby similar classical atom, with the result being that this second spiraling electron would undergo a much more complicated stochastic like-motion in the course of its spiral, and, of course the energy it radiates off would effect the spiralling motion of the first classical atom in a similar way. If we now consider a large ensemble of such systems, with all systems effecting each other, it clearly becomes increasingly more complicated to predict what will be the net effect.

Probably one's first natural reaction to this posed situation is that, yes, in detail the problem becomes far more complicated, as does any problem when more and more degrees of freedom are introduced; nevertheless, the end result should be the same: namely, lots of radiation and lots of collapsed orbits. However, according to Earnshaw's theorem [43], a system of classical charged particles cannot exist in a static, stable equilibrium configuration. Hence, if an equilibrium situation is at all possible for classical charged particles, then the particles must follow fluctuating trajectories in space. Fluctuating trajectories of charged particles necessarily imply that fluctuating radiation must also be present, and we again come back to the realization that if equilibrium is possible, then the presence of both particles and radiation, all fluctuating, is a natural, necessary, and essential component of this situation. Moreover, this property must be true at whatever temperature characterizes the equilibrium situation, including the temperature of absolute zero, $T = 0$. These properties correspond to what we observe in nature, since at $T = 0$, atomic and molecular motion does not cease, but rather there is a "zero-point" motion.

This observation constitutes the only difference between SED and traditional classical electrodynamics, namely, that the assumption is not imposed on the analysis that at $T = 0$, radiation must vanish [9]. If one examines the thermodynamic issues surrounding this question, particularly as in Refs. [16], [44], [21], and [45], but also as supported and discussed from other perspectives in Refs. [4], [5], and [22], there is nothing contained within the three laws of classical thermodynamics that suggest that fluctuating motion for classical systems should *not* exist for systems of classical charged particles at $T = 0$. Rather, as first brought out by Boyer in 1969 [4], the inclusion of classical electromagnetic ZP radiation in any thermodynamic argument is a critical component of classical physical analysis. The insight into this reasoning was deduced in Ref. [4] by reinvestigating a detailed thermodynamic analysis originally introduced by Einstein and Hopf in 1910 [46].

References [16], [44], [21], [47], [48], and [22] proved that for the electrodynamic systems investigated, only one spectral functional form of classical electromagnetic radiation conforms with the following thermodynamic definition of absolute zero temperature: namely, that no heat should flow during reversible thermodynamic operations. The spectral form deduced was precisely that of classical electromagnetic ZP radiation, with the energy per unit volume per unit angular frequency interval being

$$\rho(\omega) = \frac{\hbar\omega^3}{2\pi^2 c^3} = \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{2} \quad . \tag{1}$$

The constant $\hbar$ effectively sets the scale of the classical electromagnetic ZP radiation.

Returning to the idea of the classical hydrogen atom, if a single classical hydrogen atom existed, then the spiralling classical electron about a classical charged nucleus must be in equilibrium with the random radiation field having the spectrum of Eq. (1). Boyer's argument in 1975 [9] explained the basic idea. As the electron's orbit decays, the electron spirals faster. Each average orbit interacts with a wide range of frequencies of the ZP radiation, but will interact most strongly at the frequency corresponding to the period of the orbit due to a resonance-like effect. Hence, as the electron spirals inward toward the nucleus, it will interact more and more strongly with increasingly higher frequencies of the ZP fields. From Eq. (1), the higher the frequency, the stronger the fields of the ZP radiation fields are likely to be. Hence the likelihood increases that work will be done on the electron, which will result in an orbit that starts expanding on average. However, once the orbit becomes larger, then lower frequency parts of the ZP fields become more effective in modifying the spiralling classical electron's motion. These fields are weaker. Hence, the electron then becomes more likely to radiate more energy than it is to pick up energy from the ZP fields, thereby increasing the probability that the orbit will decrease in size. This pattern of increasing and decreasing orbital size can continue indefinitely, with the electron following a fluctuating orbital motion, i.e., a stochastic motion, characterized by some average radius. In this way, atomic stability might arise. Indeed, by making the scale factor of the ZP radiation spectrum correctly correspond with Planck's constant, Ref. [9] estimated this average radius to be the Bohr atomic radius.

Thus, the problem of atomic collapse may not be a barrier for classical physics, but, just the opposite: full consideration of this problem, when taking into account the natural stochastic equilibrium between classical charged particles and electromagnetic radiation, may prove to be the basis for how quantum mechanical phenomena arises in the first place. Indeed, as fairly recently shown, and as will be discussed in more detail in a later section, the idea just described qualitatively does appear to hold quantitatively. This result was obtained by numerically tracking a classical charged particle interacting with classical electromagnetic ZP radiation, under the action of a Coulombic binding potential. The computationally intensive numerical results have produced a close agreement with the ground state of hydrogen as predicted by Schrödinger's equation [41].

Undoubtedly, though, even once this qualitative idea is understood, a huge number of questions arise, such as: "What about the quantized energy states?" After all, the argument just presented only addressed the ground state of hydrogen. Also: "What about the spectral lines?" Clearly the previous description of the ground state from the point of view of SED does not involve any sort of quantized energy level. The electron's energy varies continuously according to SED, as electromagnetic energy is constantly being radiated due to the continual fluctuating motion of the electron and as positive and negative work continually act on the classical electron's motion by the ZP fields.

Perhaps another question that immediately comes to mind is, "If the electromagnetic ZP fields are constantly acting on all particles, then why aren't these fields more obviously observed? Also, "What about the wavelike properties of particles that are known to exist, such as from the interference patterns observed by electrons during scattering from crystals?"

These and many more questions have been addressed qualitatively, and to a reasonable extent in SED [6], [9]. For example, regarding the wave-like property of particles, it is well known from the study of the Casimir force that the presence of matter effects the fluctuation patterns of ZP radiation. In the presence of a "slit", the ZP pattern will likewise be modified. As particles pass through a set of slits, the rapidly fluctuating fields will have an average impact on the particle motion that should result in the diffraction and interference patterns seen in nature. Moreover, the answer to the question of, "If the real (i.e., not virtual) ZP fields act constantly on all particles, then why isn't this effect more apparent?", is simply that the effect is apparent, since this would be the important origin of the stability of atoms and molecules. Indeed, most of quantum mechanical phenomena would spring from this origin, so the observation of quantum mechanical phenomena would then constitute the effect of the ZP field's interaction with matter.

Similar qualitative answers exist to most of the other posed questions. Quantitative answers exist for these questions for most linear systems of interest, such as for the electric dipole SHO, where fully satisfactory agreement has been found between QED and SED [12], including the consideration of excited states [49]. Although there are certainly yet further qualitative and conceptual ideas to be better understood, what is sorely needed are further quantitative answers. SED is a very unforgiving theory. There are no free parameters to adjust. Either it works by agreeing with nature, or it does not. If it "works", then SED may indeed be a more fundamental physical basis for QED, in the sense that QED can be deduced from SED,

but not the other way around. SED, if correct, will show in detail how the Heisenberg uncertainty relation appears to arise in terms of measurements, how quantized energy levels appear as a dominant characteristic of atomic and molecular states, and how spectral lines arise in the midst of the otherwise homogenous and isotropic background of ZP plus Planckian radiation for systems in thermodynamic equilibrium.

Before leaving this more qualitative discussion, perhaps three points should be made. First, one might wonder whether, even if many of the points of SED are valid for atomic physics, then how would this possibly apply to the other huge realms of physics that capture the attention of most modern theoretical physicists, including the weak and strong interactions in particle physics, as well as the expected connection to gravitation? Since SED has a long ways to go before possibly quantitatively addressing even atomic physics, then the following comment is really quite speculative, but, it perhaps should still be stated. The key underlying idea of SED that, if successful, could be applied to areas of physics in addition to the interaction of electrodynamics, particularly for the weak and strong interactions, is that a consistent examination of thermodynamic equilibrium of all interacting fields and particles involved, including at $T = 0$, is absolutely essential, and would give rise to the quantum mechanical phenomena observed. Nonequilibrium situations would provide the appearance of "excited states".

Secondly, if SED is fundamentally correct, then many if not all of the odd quantum mechanical paradoxes, such as those posed by Schrödinger's cat and the "measurement problem" in QM, no longer appear as paradoxes within SED. In a very real sense, SED constitutes a "hidden variables" theory, but a very natural and even essential one, with the fluctuations of charged particles being a consequence of the interaction with the also necessarily fluctuating electromagnetic radiation in equilibrium with the charged particles.

Lastly, yes, there will be differences between SED predictions and QM/QED. SED is not simply a version of QM/QED that recasts quantum theory in a different light. There will be testable differences. Two sets of experiments have been suggested by Boyer and Marshall, although neither have been carried out to date. Boyer's suggestions involves the Bohm-Aharonov effect [50], [51], [52], [53], [54], [55], [56], [57], [58], [59]. Marshall's experimental suggestions involves work he has advanced with colleagues, particularly E. Santos, in "stochastic optics," aiming in particular at a local model explanation of the Bell inequalities [60], [61]. At the end of the 1990s, Marshall developed with colleagues a local physical explanation of entanglement involving a nonlinear optical process often called spontaneous parametric down conversion [62], [63], [64], [65], [66], [67]. This work led them to the prediction of a new phenomena, still to be tested for, involving a natural explanation in terms of unquantized light and local quantities that they have termed spontaneous parametric up conversion [68], [69], [70], [71].

Because of the significantly different physical basis of SED versus QM and QED, there are likely to be many more differences between the two theories that can be experimentally tested. This last statement will likely become clearer by the end of this article.

## BRIEF DESCRIPTION OF THE MAIN EQUATIONS IN SED

The electromagnetic fields in SED are described by Maxwell's equations. Most quantitative analyses in SED have treated charged particles as classical charged point particles following the classical relativistic Lorentz-Dirac equation of motion [72], [73], [74], including radiation reaction:

$$m\frac{d^2z^\mu}{d\tau^2} = \frac{2q^2}{3c^3}\left[\frac{d^3z^\mu}{d\tau^3} - \frac{1}{c^2}\left(\frac{d^2z^\lambda}{d\tau^3}\frac{d^2z_\lambda}{d\tau^3}\right)\frac{dz^\mu}{d\tau}\right] + F^\mu \quad , \tag{2}$$

where $z^\mu$ is the four-vector space-time position of the particle, $m$ is the particle's normalized mass, $q$ is its charge, $\tau$ is the particle's proper time, $c$ is the speed of light, and $F^\mu$ represents the sum of all four-vector forces acting on the particle. For atomic physics problems usually addressed in SED, $F^\mu$ contains the force of the binding potential, the Lorentz force due to the stochastic radiation fields, and any other externally applied forces. Most analyses in SED have employed the nonrelativistic approximation to Eq. (2), as the approximate situation has often first been tackled where the speed of the particle is taken to be small compared to the speed of light. However, certainly there has been a focus in SED on tackling the high velocity situations [75], [76], [35], [36], [24], [25], [26], [27], [15], [28]. The lengthy calculations involved in such work (e.g., see Refs. [35] and [36]), indicate the difficulty of this type of analysis involving relativistic particle dynamics under the influence of a broad spectral radiation.

For problems where one assumes a thermal spectrum is present (i.e., Planckian plus ZP) that effects the particle motion, then the radiation fields in SED are typically represented in the following way. (More general methods are available for particles within cavities of arbitrary shapes as discussed in Ref. [22].) For a rectilinear-piped region in empty space, with dimensions $L_x$, $L_y$, and $L_z$, along the three Cartesian axes, then a representation of the electric and magnetic fields in this region of space can be written as a sum of plane waves:

$$\mathbf{E}(\mathbf{x},t) = \sum_{n_x,n_y,n_z=-\infty}^{\infty} \sum_{\lambda=1,2} \frac{\hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda}}{(L_x L_y L_z)^{1/2}} \left[ A_{\mathbf{k_n},\lambda} \cos(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) + B_{\mathbf{k_n},\lambda} \sin(\mathbf{k_n}\cdot\mathbf{x} - \omega_{\mathbf{n}} t) \right] \quad , \tag{3}$$

with the same expression for $\mathbf{B}(\mathbf{x},t)$, but with $\hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda}$ replaced by $\left(\hat{\mathbf{k}} \times \hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda}\right)$. Here, periodic boundary conditions are imposed, so

$$\mathbf{k_n} = \frac{2\pi n_x}{L_x}\hat{\mathbf{x}} + \frac{2\pi n_y}{L_y}\hat{\mathbf{y}} + \frac{2\pi n_z}{L_z}\hat{\mathbf{z}} \quad , \tag{4}$$

where $n_x$, $n_y$, and $n_z$ are integers, $\omega_{\mathbf{n}} = c|\mathbf{k_n}|$, $\mathbf{k_n}\cdot\hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda} = 0$, and $\hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda}\cdot\hat{\boldsymbol{\varepsilon}}_{\mathbf{k_n},\lambda'} = 0$ for $\lambda \neq \lambda'$. . Typical calculations in SED are carried out this way, where at the end of the calculations, one takes the limit of $L_x$, $L_y$, and $L_z$ going to infinity.

Moreover, $A_{\mathbf{k_n},\lambda}$ and $B_{\mathbf{k_n},\lambda}$ are real variables that are held fixed for any given situation. However, as in statistical mechanics, if we consider $N$ such regions of space, then from one box in space to another, the realized values of $A_{\mathbf{k_n},\lambda}$ and $B_{\mathbf{k_n},\lambda}$ would follow a statistical distribution. In SED, this distribution is taken to be a Gaussian distribution in compliance with the usual notions of the origin of the normal distribution in relation to random processes. However, some discussion still exists on these points, as initially first analyzed by Planck [77], later by Einstein and Hopf [78], and as covered in more modern times in Refs. [4], [6], and [79]. Thus, the usual assumption is that over an ensemble of such similar systems, with angle brackets representing an ensemble average, then for thermal radiation, $\langle A_{\mathbf{k_n},\lambda}\rangle = \langle B_{\mathbf{k_n},\lambda}\rangle = 0$, $\langle A_{\mathbf{k_n},\lambda} B_{\mathbf{k_{n'}},\lambda'}\rangle = 0$, and

$$\left\langle A_{\mathbf{k_n},\lambda} A_{\mathbf{k_{n'}},\lambda'}\right\rangle = \left\langle B_{\mathbf{k_n},\lambda} B_{\mathbf{k_{n'}},\lambda'}\right\rangle = 0 \quad , \quad \text{if} \quad \mathbf{n} \neq \mathbf{n'} \text{ or } \lambda \neq \lambda' \quad , \tag{5}$$

but

$$\langle A_{\mathbf{k_n},\lambda} A_{\mathbf{k_n},\lambda}\rangle = \langle B_{\mathbf{k_n},\lambda} B_{\mathbf{k_n},\lambda}\rangle = f(\omega_{\mathbf{n}},T) \quad , \tag{6}$$

so that these last correlation functions are a function of frequency and temperature, for a thermodynamic equilibrium situation. As can be shown, the relation between $f(\omega_{\mathbf{n}},T)$ and the spectral energy density $\rho(\omega,T)$ is given by the following [21], [22]:

$$\rho(\omega,T) = \frac{\omega^2}{\pi^2 c^3}\frac{f(\omega,T)}{4\pi} \quad , \tag{7}$$

with $f(\omega,T) \to 2\pi\hbar\omega$ as $T \to 0$.

Much of the early classic thermodynamic analysis of radiation in a cavity, such as the deduction of the Stéfan-Boltzmann law and the Wien displacement law, can be carried out in a similar way as done by these researchers and as elegantly described in Planck's treatise of Ref. [77]. However, as analyzed by Cole in Ref. [21] and discussed more generally in Ref. [80], these early thermodynamic analyses subtly imposed, without possibly intending to, that all radiation vanishes as $T \to 0$. These derivations can be reconstructed in a very natural way without this assumption [21], thereby enabling one to deduce the functional form of the spectral energy density and its limit as $T \to 0$ [44], [21], [47], [48], [22].

For atomic systems, the binding potential of electrons to the nucleus must be of Coulombic form. To date, the attempts on the simplest of atomic systems, namely, hydrogen, have been carried out for the nonrelativistic approximation of Eq. (2), namely,

$$m\frac{d^2\mathbf{z}}{dt^2} = -\frac{e^2\mathbf{z}}{|\mathbf{z}|^3} + \frac{2}{3}\frac{e^2}{c^3}\frac{d^3\mathbf{z}}{dt^3} + (-e)\left\{\mathbf{E}[\mathbf{z}(t),t] + \frac{\dot{\mathbf{z}}}{c}\times\mathbf{B}[\mathbf{z}(t),t]\right\} \quad , \tag{8}$$

where $\mathbf{z}(t)$ is the three-vector for the position of the classical electron at time $t$, and $-e$ is the charge of the particle. To tackle the three-dimensional isotropic SHO, one would just replace the nonlinear Coulombic

binding force of $-e^2\mathbf{z}/|\mathbf{z}|^3$ with the linear SHO binding force of $-m\omega_0^2\mathbf{z}$. Much of early SED dealt with the latter system, leading, as mentioned, to agreement with QED for van der Waals forces at all distances between SHO electric dipole oscillators [13], [81], [15].

In some ways it seems odd that just a simple change of the linear binding potential $-m\omega_0^2\mathbf{z}$ to the physically far more important nonlinear form of $-e^2\mathbf{z}/|\mathbf{z}|^3$ should make such a significant difference in the difficulty of the analysis. However, as is known in the study of nonlinear stochastic processes [82], [83], such problems are notoriously difficult ones. Indeed, the area of "chaotic theory" has been built around the analysis of nonlinear problems, but generally only for very simple forcing functions, rather than for the infinitely wide spectral Lorentz force due to the classical electromagnetic random radiation fields acting on the charged particle. Moreover, regarding perturbation methods, which are often a common means for tackling such problems [82], [6], [35], [36], the problem is not nearly so simple as taking some limit of a small charge $e$, as might be done in QED for scattering analysis, since SED needs to deduce both ground states and perturbations to them from applied radiation. As can be seen in Eq. (8), the assumption of small $e$ effects both the binding potential, the radiation reaction, and the Lorentz force; all of this must be consistently treated, as analyzed in some detail in Ref. [40].

# RECENT RESULTS INVOLVING THE DEDUCTION OF THE GROUND STATE OF HYDROGEN FROM SED

## Introductory comments

During the past few years, a fair amount of effort has been placed on trying to sidestep many of the analytical difficulties encountered in the past. Since perturbation methods to date involving the charge $e$ have been suspect [39], [40], and since it is highly desirable to examine a range of problems, not necessarily just thermodynamic equilibrium situations, and since computational methods have indeed been so very helpful in exploring nonlinear problems in a wide range of areas, this tactic was undertaken, as reported in Refs. [41], [42], [84], [85], [86], and [87]. Back in 1988 to 1989, much of Ref. [42] was carried out, plus some attempts on Ref. [41], but with computers so very much slower then, questions existing on classical chaotic behavior, and uncertainty at the likelihood of making good progress on numerical efforts involving a very broadband radiation source, resulted in this work being put aside until the recent set of events to be discussed here. Despite significantly faster computational systems today than in the late 1980s, still approximation methods had to be implemented, as described here, to result in reasonable length CPU times for simulation runs.

## Description of recently published results for the ground state of hydrogen

Here we briefly describe the results published in Ref. [41] involving the simulation investigation of whether SED predicts the quantum mechanical ground state for hydrogen. Eleven cases of a classical electron following Eq. (8) were numerically solved, assuming classical electromagnetic ZP radiation exerting a classical Lorentz force on the classical electron [i.e., Eq. (1) with Eqs. (6) and (3)] in addition to the Coulombic binding force of $-e^2\mathbf{z}/|\mathbf{z}|^3$. To significantly reduce computational time, each particle was confined to trajectories lying within a single plane containing the classical nucleus. Also to reduce computational time, only a subset of the infinitely wide spectral range of radiation in Eq. (3) was included in the simulation; this constraint was relaxed considerably for the numerical results that will be described in the next subsection. A total of 55 CPU days were invested in the final runs reported here, using 11 Pentium 4 PCs, each with 1.8 GHz processing speed and 512 MB of RAM. All electrons were begun in circular orbits of radius $r = 0.53\,\text{Å}$; their trajectories then evolved over time to considerably different paths. Several algorithms were experimented with, but the one that was finally used was a Runge-Kutta 5th order algorithm, with an adaptive stepsize. Each of the amplitudes of the plane waves in Eq. (3) were selected by a random number generator using a Gaussian distribution obeying Eqs. (5)-(7) and (1) for $T = 0$. These amplitudes for $A_{\mathbf{k_n},\lambda}$ and $B_{\mathbf{k_n},\lambda}$ were then held fixed throughout the remainder of the simulation.

Figure 1 shows a plot of radius versus time for a typical trajectory. As can be seen, the trajectory appears to have followed somewhat of a "random walk", although that is not really an accurate description, since the "memory" of classical electromagnetic zero-point radiation is very specific [15] and quite different from the white noise spectra assumed in traditional Brownian motion type analysis, which one normally thinks of in relation to "random walk" analyses [88]. (Indeed, the difference between these spectral functions is what enables the classical electromagnetic ZP spectrum to yield the complex thermal effects of acceleration associated with the Unruh-Davies relationship [24].) The classical electron's orbit spirals in and out in radius, gaining some elliptical characteristics, and sampling the region of space near the classical nucleus.

Most notably, the electron does not spiral inward to the nucleus. The basic idea of Boyer in Ref. [9] is indeed confirmed: as the electron decays inward toward the radius, the higher frequencies it more effectively interacts with from the ZP radiation cause, on average, more positive work to be done than negative work, thereby increasing the likelihood that the electron will spiral outward. Upon spiralling outward, however, the lower frequency components of the ZP radiation are then not so effective at maintaining this net positive work being done on the electron, and the classical electron is then likely to decay back down in its orbit.

Surely this result alone must surprise most physicists, since here we find that classical physics actually provides the solution to atomic collapse. As earlier believed, classical physics provided only the problem of atomic collapse, with no means to move beyond this difficulty, unless one turned to quantum theory. Moreover, this classical mechanism that prevents atomic collapse, arises in a very natural way, as a consequence of the thermodynamic equilibrium between fluctuating electromagnetic radiation and fluctuating charged particle motion. To emphasize this point, a second curve is shown in Fig. 1, illustrating that atomic collapse would occur in about $1.3 \times 10^{-11}$ s if no ZP radiation was acting.

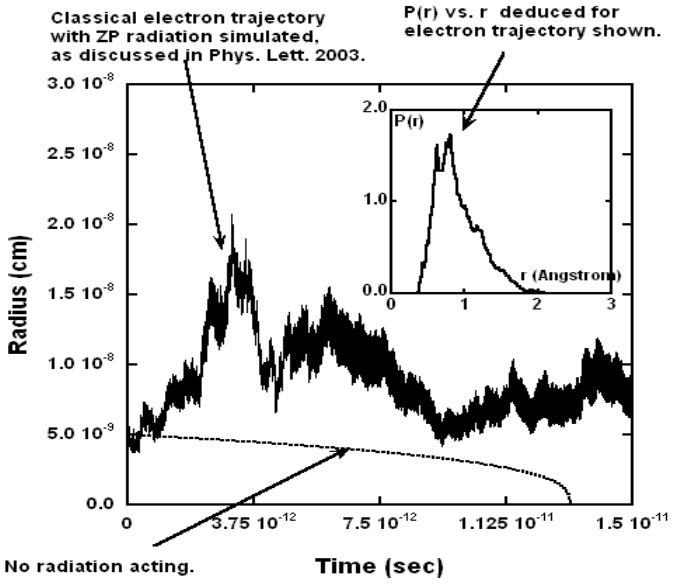


**Figure 1.** Typical plot of $r$ vs. $t$ for one trajectory realization of an electron in a classical hydrogen atom, via the simulation methods described here. The inset shows the probability density $P(r)$ vs. $r$ computed for this particular trajectory. A second curve of $r$ vs. $t$ illustrates that if no ZP radiation was acting, then atomic collapse would occur in about $1.3 \times 10^{-11}$ s. The starting trajectories for each of the two plots began as circular orbits with radius 0.5 Å.

However, the results in Ref. [41] go beyond this point, as seen in Figs. 2(a) through 2(d). In each figure, the histogram of the ensemble of 11 "electrons" is displayed up to the average time indicated for the 11 particles. In each plot, the probability density for the ground state of hydrogen, $P(r) = 4\pi r^2 |\Psi(\mathbf{x})|^2 = \frac{4r^2}{a_B^3} \exp\left(-\frac{2r}{a_B}\right)$,

as predicted by Schrödinger's wave function, is also displayed ($a_B = \hbar^2/me^2$ is the Bohr radius). As can be seen, in Fig. 2(a), the probability density distribution for the simulations still has a very close signature of the original starting position of $r = 0.53\,\text{Å}$. However, in each succeeding time sequence shown, the simulation results show a very clear convergence to the predictions of Schrödinger's equation. By the time of $t = 7.252 \times 10^{-12}\,\text{s}$ in Fig. 2(d), a very close comparison is indeed obtained. Presumably, with more electrons simulated, and with the simulation going out yet farther in time, the results would be even closer.

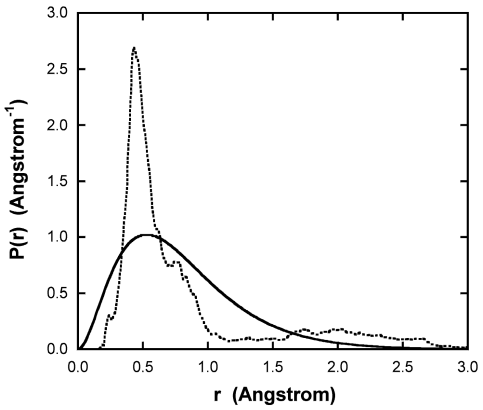


**Figure 2(a).** $t = 1.417 \times 10^{-12}\,\text{s}$.

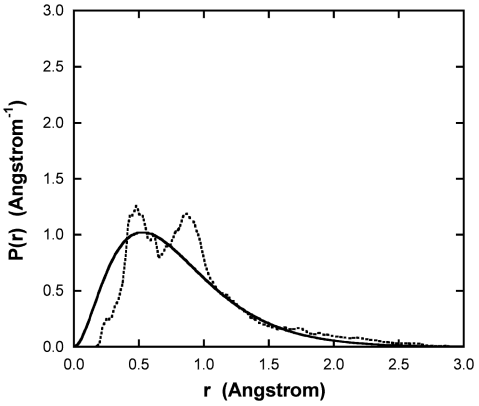


**Figure 2(b).** $t = 4.500 \times 10^{-12}\,\text{s}$.

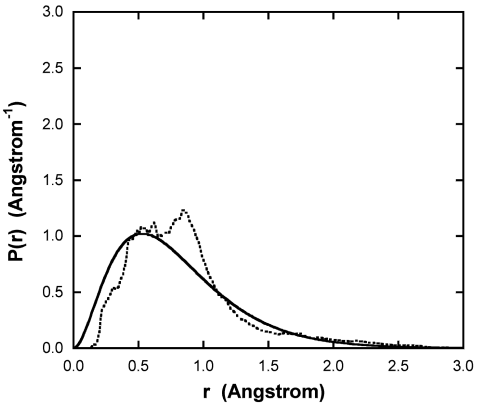


**Figure 2(c).** $t = 5.705 \times 10^{-12}\,\text{s}$.

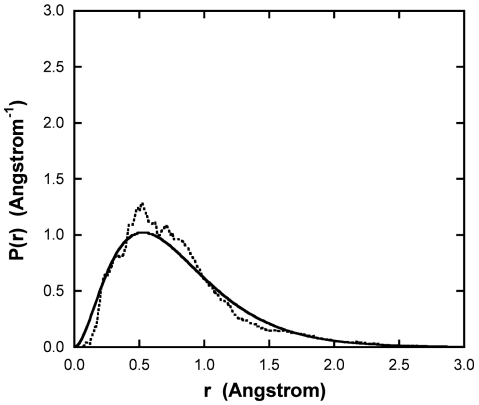


**Figure 2(d).** $t = 7.252 \times 10^{-12}\,\text{s}$.

**Figure 2.** Plots of the radial probability density vs. radius. The solid line was calculated from the ground state of hydrogen via Schrödinger's equation. The dotted curves are the simulation results, calculated as a time average for all eleven simulation runs, up to the times indicated below each figure.

## Approximations used in simulation method

The main difficulty in the simulations is properly representing the electromagnetic radiation fields. Their correlation properties, as a function of both space and time [15], are strongly suspected to be intimately connected with the ability to achieve equilibrium with the classical hydrogen atom. However, a straightforward summation over all plane waves, or even a broad but finite subset of them, introduces costly computational expenses, since numerically, this roughly needs to be repeated at each iterative step.

The following scheme was therefore implemented to ease the computational burden, at the risk of less accurately describing the detailed physical behavior. First, the classical electron was confined to orbits in a single plane containing the classical nucleus. Second, the approximation was made that radiation exerting the most influence on the orbit, would largely arise from the frequencies nearest the instantaneous and approximate circular orbit of the electron. The incentive for this approximation came from a study of the resonance properties of electromagnetic waves on the electron's classical orbit as described in Refs. [42], [84], and particularly as illustrated in Fig. 9 in Ref. [86]. In this way, only a band of plane waves needed to be considered at each instant of time. For this reason, the term "window" approximation was applied to this algorithm [41]. Moreover, since plane waves normally incident to the orbit exert the most influence on the orbit, then only these waves were treated, by selecting the parallelepiped domain encompassing the classical atom to be long and narrow, and restricting the net range of angular frequencies to be considered in the sum in Eq. (3) to be in the range of $4.6 \times 10^{11}\,\mathrm{s}^{-1} \lesssim \omega \lesssim 5.0 \times 10^{17}\,\mathrm{s}^{-1}$. This range was expected to adequately pertain to the main orbital motion for approximately circular orbits of radii ranging between $0.1\,\text{Å}$ to $1.06 \times 10^{-5}\,\mathrm{cm}$, since for unperturbed circular orbits, $\omega = \left(\frac{e^2}{mr^3}\right)^{1/2}$.

## More recent results: "no-window" approximation

Although certainly very satisfying that the "window" approximation yielded the results shown in Fig. 2, still the window algorithm was implemented only due to computational reasons, and there remained a concern that this approximation resulted in missed physical effects. Hence, a new set of runs were undertaken, using a bank of 30 PCs with Pentium 4 processors, 1.8 GHz processing speed, and 512 MB of RAM. The same procedure was followed as before, but now without the narrow window approximation just discussed, so that each electron constantly experienced the full effects of the sum of plane waves in Eq. (3) within the regime of $4.6 \times 10^{11}\,\mathrm{s}^{-1} \lesssim \omega \lesssim 5.0 \times 10^{17}\,\mathrm{s}^{-1}$. Consequently, the CPU times rose drastically, by about two orders of magnitude. Moreover, ensuring a sufficiently high numerical precision was absolutely essential, otherwise "ionization" effects occurred due to numerical instability. Improving the numerical precision reduced this effect, but of course also greatly increased the CPU time. The final run of 30 days on 30 PCs (i.e., 900 CPU days) entailed a sufficiently high numerical precision in the adaptive time step 5th order Runge Kutta algorithm, that no ionization events occurred. Figures 3 and 4 summarize the results. Fortunately, the main physical effects predicted by the "window" algorithm results in Fig. 2 were largely confirmed by the "no-window" results in Fig. 3.

One key change in the runs that was made is that subensembles of electrons were started at different initial radii, as indicated in Fig. 4. It should be emphasized that this simulation experiment has yet to be fully completed, as full access to the 30 PCs could not be continued after one month. Still, the results seem to provide reasonable agreement with the predictions of QM.

These plots show the time average, up to the times indicated, for each of the four subensembles of classical electrons starting at the initial radii indicated. The average of all of these subensembles is displayed in Fig. 3. In Fig. 4, two simulation curves are shown in each plot, one halfway through the time evolution of the subensemble, and the other displaying the final result reached at the end of the simulation experiment. For comparison purposes, the result from Schrödinger's equation is also displayed for the ground state in each of these plots.

Despite nearly 900 CPU days of processing time, the electrons evolved only about 1/10 as far in time as compared with the "no-window" simulation experiments reported in the last subsection that required only 55 CPU days. Nevertheless, a number of interesting observations can be made. First, as can be seen by comparing Fig. 2(a) and 4(b) (both had particles starting at $r = 0.53\,\text{Å}$), which correspond the closest match in time for the electron time evolution between the two runs, the "no-window" result of Fig. 4(b) evolved

faster toward the Schrödinger equation result than the "window" algorithm result of Fig. 2(a). In some ways this seems reasonable, since the additional spectral range of frequencies acting on the electron is sure to cause more dispersion in the evolution of the trajectories. Fortunately, however, this extra dispersive effect appears to push the distribution in the right direction, namely, toward the Schrödinger ground state, as opposed to something quite different.

Second, the combination of all the subensemble results of Fig. 4 with initial radii of 0.2 Å (7 electrons), 0.53 Å (14 electrons), 1.0 Å (8 electrons), and 2.0 Å (1 electron), as summarized in Fig. 3, evolves even faster toward the Schrödinger ground state than the subensemble of electrons beginning at $r = 0.53$ Å in Fig. 4(b). This result is as it should be if the entire ensemble is expected to evolve to the Schrödinger ground state, as the number of members in the initial subensembles were chosen on purpose to correspond to what one expects roughly the ground state distribution to be, thereby providing an additional check on the consistency of these ideas.

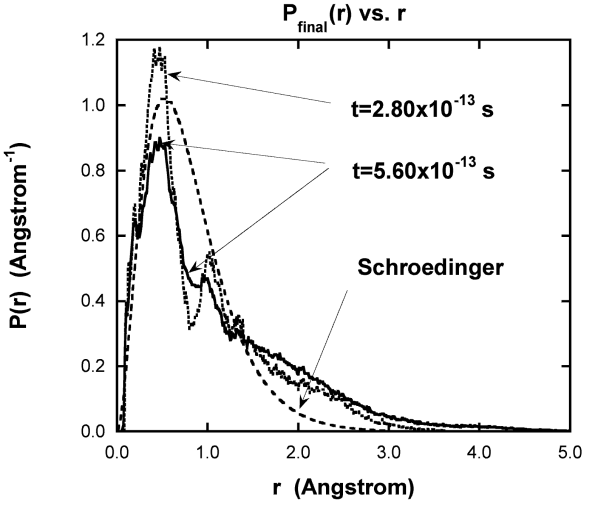


**Figure 3.** $P(r)$ vs. $r$ for the entire ensemble of 30 particles, with 7 starting at 0.2 Å, 14 starting at 0.53 Å, 8 starting at 1.0 Å, and 1 starting at 2.0 Å. $P(r)$ for the 30 particles at the average time of $5.60 \times 10^{-13}$ s, as well as at half that time, or, $2.80 \times 10^{-13}$ s, are displayed.

Finally, and perhaps most interesting, the peaks of each subensemble distribution in Figs. 4(a)-4(d) do not move very fast, although the spread of each peak occurs at a rate commensurate with the distribution beginning at the Bohr radius of $r \approx 0.53$ Å. In hindsight, this is roughly as it should be, as the key factors governing the peaks is the rate of radiation that is constantly being given off by the radiation reaction, versus the average work done by the ZP radiation on the electron. Further testing of subensembles going considerably farther out in time, such as to $t \approx 10^{-9}$ s, might enable a deeper understanding of "excited states" to be studied within SED, and might reveal the expected phenomena of line spectra for hydrogen. The time of approximately $10^{-9}$ s is mentioned here since it is the approximate order of magnitude from QM for the decay of several of the excited states (i.e., $2p \rightarrow 1s$) to the ground state for hydrogen [89].

## CONCLUDING REMARKS

Although the classical hydrogen probability distribution analysis, as discussed in earlier sections, is undoubtedly the most "eye opening" of the past recent simulation work to be discussed here, it seems important to

mention another broad class of related nonlinear phenomena that has been researched in a series of articles in Refs. [42], [84], [85], and [86]. As shown there, a surprising set of phenomena involving quasistable situations exist for near circular and near elliptical orbits, under various states of simple applied electromagnetic fields. These effects have been analyzed both computationally, analytically where possible, and with perturbation methods.

In addition, more recent results [87] have been found for very pronounced "subharmonic" resonances of the classical hydrogen atom. More specifically, If the classical electron is in a near circular orbit with radius $r$, and has an angular frequency of $\omega = \left(\frac{e^2}{mr^3}\right)^{1/2}$, then by applying circularly polarized (CP) plane waves with frequencies of either $2\omega$, $3\omega$, $4\omega$, etc., of a sufficient amplitude as discussed in Ref. [87], can result in powerful resonant behavior. These resonances are highly nonlinear effects of the applied radiation amplitudes; they also depend strongly on the phase of the radiation in relation to the state of the orbit.

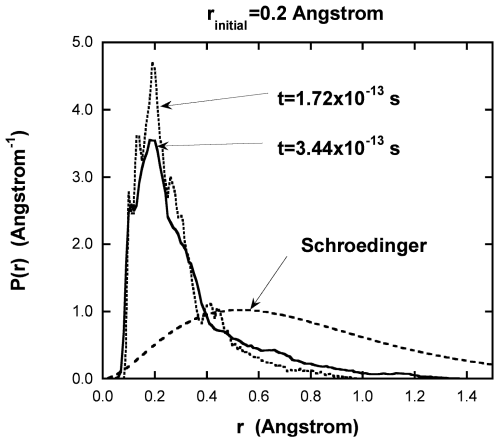


**Figure 4(a).** 7 particles.

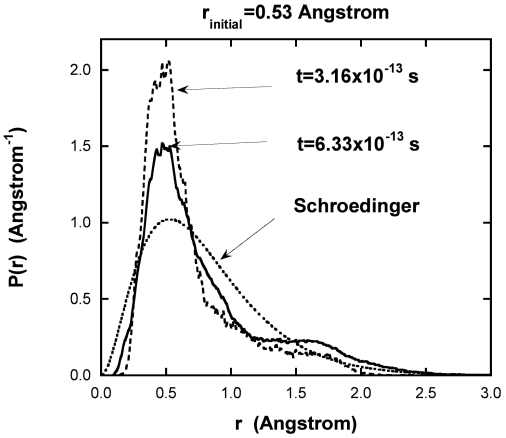


**Figure 4(b).** 14 particles.

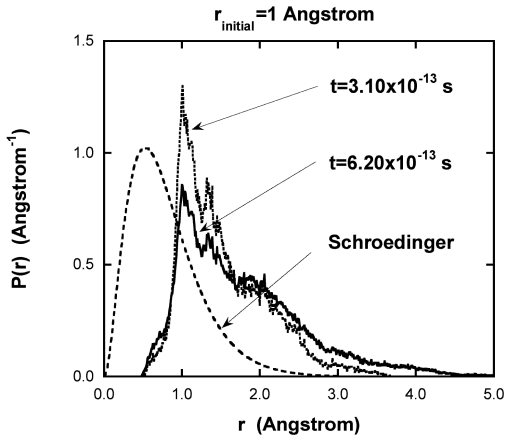


**Figure 4(c).** 8 particles.

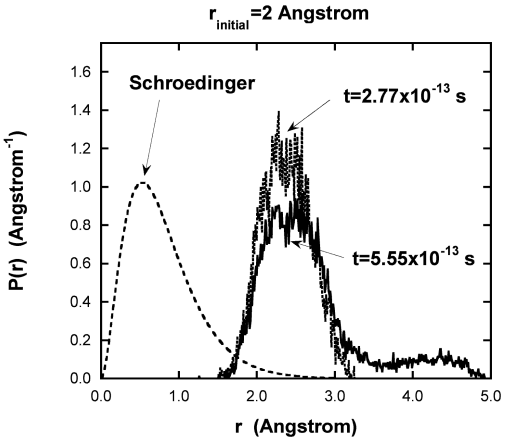


**Figure 4(d).** 1 particle.

**Figure 4.** These plots show the time average, up to the times indicated, for each of the four subensembles of classical electrons starting at the initial radii indicated. The average of all of these subensembles is displayed

in Fig. 3. In each figure above, two simulation curves are shown, one halfway through the time evolution of the subensemble, and the other displaying the final result reached at the end of the simulation experiment. In each plot, the result from Schrödinger's equation is also displayed for the groundstate.

It seems likely that all of these effects will play a role in the full behavior of the classical hydrogen atom. These results can lead in a wide variety of directions, from (1) aiding in forming more approximate, but faster algorithms, as in the case of the "window" algorithm, (2) establishing deeper connections into underlying statistical mechanical notions that will need to be investigated to serve as a fuller theoretical basis for these ideas, and even (3) help to form technological ideas for manipulating and making use of atomic systems for storing information, changing chemical reactions via unusual applied signals of electromagnetic radiation, better controlling ionization, plasma etching, ion implantation, and a range of other possibilities. These effects of radiation on atoms are gaining attention in the literature, both experimentally and theoretically, such as the nonlinear effects of applied fields on ionization rates for Rydberg atoms [90] and the action of fast applied electromagnetic fields on Rydberg atomic systems [91]. Indeed, very recent experimental results reported in Ref. [92] appear to closely agree with simulation predictions [42], [85], [86].

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