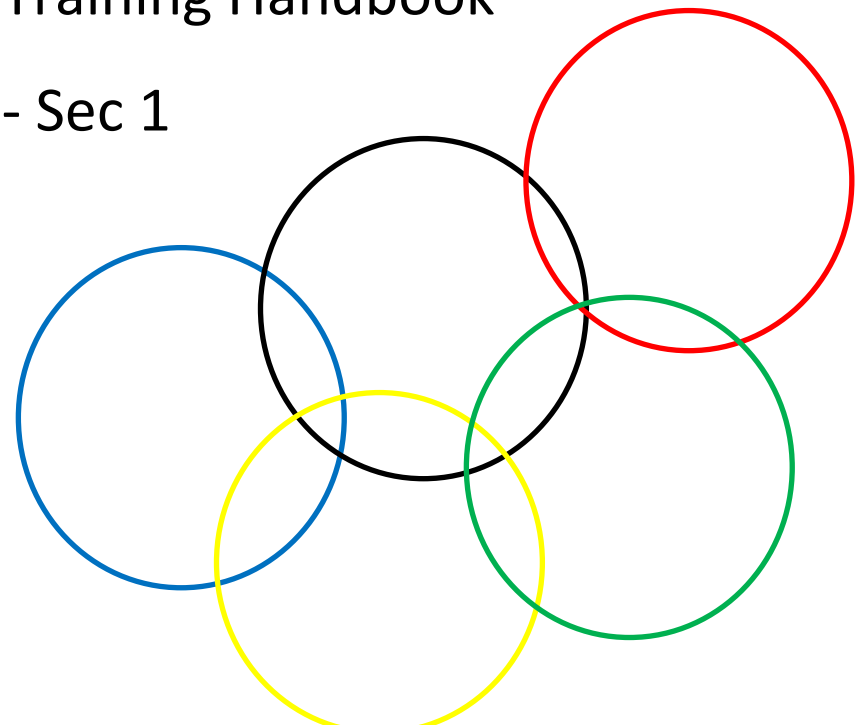
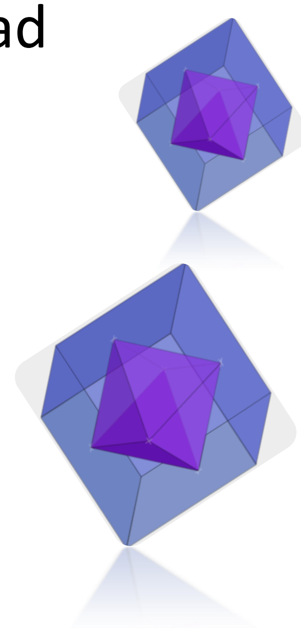
Develop The Maths Genius in You

Singapore Mathematical Olympiad Training Handbook - Sec 1





Includes Questions from other Olympiads

Dear young students of mathematics

Mathematics is a wonderful subject. It is one of the most useful ways to develop your mind.

The material in front of you has been developed over the years in training talented pupils in this subject in a top secondary girl school.

If you are a Primary 6 or even Primary 5 pupil who is seeking challenges or a Secondary 1 pupil who is looking for ways to develop your mathematics talent, look no further. Pick up a pencil and have a go at it.

By the end, I hope you enjoy what I had put together here for you.

Cheers

Mr Ang K L , 2012

Contents

Worksheet

Tropical Revision

Problem Solving

From Arithmetic to Algebra

What is Algebra?

Algebra is a study of number properties in the form of alphabetical representation.

Ex 1. The average of three numbers A, B, and C is a. If the average of A and B is b, what is the value of C?

Ex 2. The cost of a shirt, a hat and a pair of shoe is \$$a$. A shirt is \$$b$ more expensive than a pair of shoe. A shirt is \$$c$ more expensive than a hat. Find the cost of a pair of shoe.

The operations in algebra are the same as those in arithmetic.

Ex 3. There are two piles of printing papers on a table. The first pile is a kg more than the second pile. If b kg are used in each pile, the first pile will be β times of second pile. Find the weights in each pile.

Ex 4. To complete a job, A alone took a days, B alone took b days. Find the number of days it would take both of them to complete the job together.

Ex 5. To complete a job, *A* alone took *a* days, *B* alone took *b* days. Now that *A* alone had completed *c* days $(c < a)$, then *B* alone complete the rest of the job. How many days would *B* take to complete the rest of the job?

Ex 6. There are *a* number of chicken and rabbits, with *b* number of legs. How many rabbits are there?

Practices

1. There are two baskets of apples. If *a* apples were moved from first basket to the second basket, then the two baskets would have the same number of apples. If *b* apples were moved from second basket to the first basket, then the first basket would have twice as many as the second basket. Find the number of apples in each basket.

$$\begin{bmatrix} 1st \to 4a+3b \\ 2nd \to 2a+3b \end{bmatrix}$$

2. After a Math test, the top 10 pupils had an average of *a*. The top 8 pupils had an average of *b*. The ninth pupil had *c* more than the tenth pupil. Find the score of the tenth pupil.

$$\left[\frac{10a-8b-c}{2}\right]$$

3. There are some 4 ¢ and 8 ¢ stamps, with a total amount of $*a*. If there are *b* pieces more 8 ¢ stamps than 4 ¢ stamps, find the number of 8 ¢ stamps and the number of 4 ¢ stamps are there.

$$\left[\frac{25a+b}{3}\right]$$

4. A fast car and a slow truck departed from town X and town Y respectively, towards each other. It took *a* hours for the fast car to reach town Y. The slow truck took *b* hours to reach town X. If the fast car travelled *m* km more than the slow truck in an hour, how long would it take for the two to meet on their journey?

$$\left[\frac{ab}{a+b}\right]$$

5. To complete a job, Mr A alone and Mr B alone took *a* and *b* $(<a)$ days respectively. To complete this job, Mr A did a few days, then Mr B took over to complete the rest. It took c days $(b<c<a)$ in total to get it done. Find the number of days each took to complete this job.

$$\left[\frac{a(c-b)}{a-b}\right]$$

Formulating Equations

To solve problems with algebra will generally require the forming of equation(s). An equation is an expression of two equal quantities that are divided by the sign "=".

In order for this to be possible, we will learn how to translate from words into algebra.

Ex. 1 A horse and a donkey met on their way. The donkey said to the horse: "If you transfer one bag to me, my load would have been twice of your load." The horse replied: "If you transfer one bag to me, our load would have been even." Find the number of bags on the donkey.

Here are the steps to take to form an equation:

Let the number of bags on the donkey be	x
The donkey said to the horse: "If you transfer one bag to me, my load would have been twice of your load."	$x+1$, donkey $\left(\frac{x+1}{2}\right)$, horse
(we can then tell that the horse actually had)	$\left(\frac{x+1}{2}\right)+1$
The horse replied: "If you transfer one bag to me, donkey our load would have been even."	$x-1$, $\left(\frac{x+1}{2}\right)+1+1$

(From the last statement, we know that the number of bags on the horse is the same as the number of bag that is on the donkey)

$$x - 1 = \left(\frac{x+1}{2}\right) + 1 + 1$$ (This is your first equation)

$$x - 1 = \left(\frac{x+1}{2}\right) + 2$$

$$\underline{\quad +1 \qquad\qquad +1 \quad}$$ (This is the concept of balancing)

$$x = \left(\frac{x+1}{2}\right) + 3$$

$$x = \frac{x}{2} + \frac{1}{2} + 3$$

$$x \quad = \quad \frac{x}{2} \quad + \frac{7}{2}$$

$$\underline{-\frac{x}{2} \qquad -\frac{x}{2}}$$ (Again, apply the balance concept)

$$\frac{x}{2} \quad = \quad \frac{7}{2}$$

$$\frac{x}{2} \times 2 \quad = \quad \frac{7}{2} \times 2$$ (Once again, balancing)

$$x = 7$$

Can you formulate this problem differently? Let’s try!

Ex 2. A, B, C, and D together, have 45 books. If A has 2 less, B has 2 more, C has double, and D is halved, then each would have the same number of books. How many books has A?

Let "each would have the same number of books" Be	x
"A has 2 less", then A actually has	$x+2$
"B has 2 more", then B actually has	$x-2$
"C has double", then C actually has	$\frac{x}{2}$
"D is halved", then D actually has	$2x$

"A, B, C, and D together, have 45 books" Use this to form an equation:

$$(x+2)+(x-2)+\frac{x}{2}+2x=45$$

We can then simplify this equation into:

$$4\frac{1}{2}x=45 \Leftrightarrow \frac{9}{2}x=45$$

$$\frac{9}{2}x\times 2=45\times 2$$ (Remember the concept of balancing)

$$x=10$$

Ex. 3 A group of students was to clean up to two areas in their school. Area A was $1\frac{1}{2}$ times of Area B. In the morning (half of a day), the number of students cleaning Area A was 3 times that of the number of students in Area B. In the afternoon (another half of a day), $\frac{7}{12}$ of the students worked in Area A while the rest of them in Area B. At the end of the day, Area A was done, but Area B still needed 4 students to work one more day before it was done. How many were there in this group of students?

Ex. 4 Jug A contained 11 litres of pure honey, and Jug B contained 15 litres of pure water. Some honey from Jug A was poured into Jug B, the mixture was well stirred. Next, some mixture from Jug B was poured into Jug A. At the end, Jug A still contained 62.5% of honey by volume and Jug B contained 25% of honey by volume. If the total volume remained the same, how much had the mixture been poured into Jug A?

Practices

1. There are two warehouses; with the first one has three times the number of TV sets than the second one. If 30 sets were transferred from the first to the second, then the second one would have $\frac{4}{9}$ that of the first one. Find the number of TV sets in the second one. $[130]$

2. There were 140 black chocolate bars and white chocolate bars on shelve. After one quarter of the black chocolate bars was sold, the storekeeper added another 50 white chocolate bars on the shelve. Then, the number of white chocolate bars would be twice the number of black ones. Find the number of black chocolate bars at first. $[76]$

3. Mr A and Mr B were to depart from the same place to town X. Mr A walked at a speed of 5 km/h. After he had departed for one and a half hour, Mr B cycled to town X. It took Mr B 50 minutes to arrive at town X together with Mr A. Find the speed of Mr B. $[14]$

4. A and B departed together to town Y, B on foot, and A by bicycle. A's speed is 1 km/h more than thrice of B. Upon arriving at town Y, A rested for an hour before returning. On the return trip, A met B when B had already walked for two and a half hours. If town Y was $14\frac{3}{4}$ km away from their departure point, find the speeds of the two and how far had they each travelled before they met again? $[13, 4]$

5. A motorist departed at 9 am from town A to town B. He planned to arrive at 12 noon. An hour later, he realized that he would be late by 20 minutes with his current speed. As such, he increased his speed by 35 km/h and in so doing, arrived at exactly 12 noon. Find the original speed of the motorist and the distance between the two towns. $[210,700]$

6. The numbers of pupils in two groups are in the ratio of 4 : 1. If 15 pupils are transferred from the first group to the second group, then, there will have same number of pupils in each group. How many pupils are to be transferred from the first group to the second group so that the ratio becomes 3 : 7?

$[25]$

7. There are two candles, one thick and the other one thin, but are of equal length. The thick one can last 5 hours. The thin one can last 4 hours. If the two candles are lighted together, how long will it take for the thick one to be 4 times that of the thin one? $\left[3\frac{3}{4}\right]$

Algebra with Arithmetic

In solving many mathematical problems, the arithmetic approach seek to develop a better understand of the problem over the algebraic counterpart. In combining the use of the two approaches, one can usually find solution to a problem much easier.

Ex 1. A car is traveling from town X to town Y. If the speed of the car is increased by 20%, it arrives at town Y one hour earlier than as planned. If it has, at first, travelled for 120 km with the original speed, then increases its speed by 25% for the rest of the journey, it will arrive 40 minutes earlier instead. Find the distance between town X and town Y.

Method 1, Arithmetic approach

Method 2, Algebraic approach,

Ex. 2 A job can be done by Mr A alone in 9 days, Mr B alone in 6 days. Now that Mr A has done 3 days of the job, how many days will it take Mr B to complete the job, without Mr A?

Method 1, Arithmetic approach

Method 2, Algebraic approach,

Ex. 3 For a project, team A can complete the project in 10 days. Team B can complete the project in 30 days. Now that both team are working on the project. But team A has two rest days, and team B has 8 days of rest. Find the number of days it will take them to complete the project.

Ex 4. A project will take 63 days by team A, and then another 28 days by team B to complete. If both teams are to work on this project together, it will take 48 days to complete. If team A is to work 42 days, how many days will it take team B alone to complete the rest of this project?

Ex. 5 There are two water filling pipes, A and B and one drain pipe, C connected to a pool. It takes 3 hours to empty a full pool with all 3 pipes open. It takes just one hour to empty this pool with pipe A and pipe C only. It takes 45 minutes to empty this pool with pipe B and pipe C only. If the filling rate of pipe A is 1 m^3/min more than pipe B, find the fill rate and drain rate of each pipe.

Ex. 6 A jug contained some liquid(water and alcohol mixture). After a cup of water was added, the concentration of alcohol in the jug became 25%. After another cup of pure alcohol was added into the jug, the concentration of alcohol was 40%. How many cups of liquid were there in the jug at first?

(Concentration of alcohol by volume $= \dfrac{amount\ of\ \ alcohol}{amount\ of\ \ liquid}$)

Ex. 7 Two teams were working on writing a book. Team A wrote $\frac{1}{3}$ of the book in 4 days. Then team B joined the project. With team A, they finally completed the book in 3 days. If team B wrote 75 pages of the book, find the number of pages in this book.

Practices

1. A project will take Mr A and Mr B 12 days to complete. Now that both of them work for 4 days, with the rest to be completed by Mr A in 10 days, find the number of days each take to complete this project by himself. $[15, 60]$

2. In another project, if Mr A works on it for 2 days and Mr B works on it for 5 days, $\frac{4}{15}$ of the project will be completed. But if Mr A works on it for 5 days and Mr B works on it for 2 days, then $\frac{19}{60}$ of the project will have been completed. Find the number of days each take to complete this project by him alone. $[20, 30]$

3. A tank is filled from empty to full by pipe A in 12 minutes. It only takes pipe B 5 minutes to drain it completely. Pipe C takes 6 minutes to fill this tank. If pipe A is open to fill an empty tank for the first few minutes before pipe B and C are open, it will take 18 minutes to fill this tank. How long has pipe A been open before the other two pipes are open? $[3]$

4. The amount of work done by Mr B in a day took Mr A one-third of a day to do. The amount of work done by Mr C in a day took Mr B $\frac{3}{4}$ of a day to do. Now, each day, 2 of them were to work on a project. It took Mr A 4 days, Mr B 3 days and Mr C 3 days to complete this project. Find the number of days Mr A alone took to complete this project. $\left[5\frac{3}{4}\right]$

System of Equations

When there are 2 or more unknowns, it often requires 2 or more equations to be set up. The ways of solving these equations are the lesson for today.

Ex.1 A fraction, after being simplified, is $\frac{2}{3}$. If a integer is added to both the numerator and its denominator of this fraction, it becomes $\frac{8}{11}$. If one is added to this integer, and the new integer is subtracted from both the numerator and the denominator of this fraction, it becomes $\frac{5}{9}$. Find this fraction.

Method 1, Algebraic approach,

Method 2, Arithmetic approach,

Ex. 2 Mr *A*, Mr *B*, and Mr *C* took part in a bicycle race. Mr *A* finished 12 minutes earlier than Mr *B*. Mr *B* finished 3 minutes before Mr *C*. If Mr *A* was 5 km/h faster than Mr *B*, and Mr *B* was 1 km/h faster than Mr *C*, find the distance of their race.

Method 1, Algebraic approach,

Method 2, Arithmetic approach,

Ex. 3 A red ballpoint pen costs 19 cents, and a blue ballpoint pen costs 11 cents. Now that I pay a total of $2.80 for 16 pens, how many are blue pens?

Ex 4. Clerk *A* takes 6 hours to type a report and clerk *B* takes 10 hours to type the same report. If clerk *A* starts to type for a few hours then hand over the rest of the typing to clerk *B*, it will take 7 hours in total to complete. Find the number of hours each takes to type this report?

Ex. 5 Two iron ores, the first ores contain 68% of iron and the second ores contains 63% of iron. Now that 100 tones of 65% of iron ores are required, how much of each ores is to be used?

Practices

1. A delivery order took a fleet of trucks over a number of days to fill. If 6 fewer trucks were used, then it would extend 3 days to complete the order. If 4 additional trucks were used, then it would have shortened one day to complete. Find the number of days and the number of the truck in this fleet.

[5,16]

2. There were 360 door gifts for Founder's day to be assembled over the weekend. Team A produced 112% of its quota; team B produced 110% of its quota. As a result, there were 400 door gifts assembled in total. Find the number of gift in excess of the quota from each team. [24,16]

3. 30 English books and 24 Math books cost $83.40 in total. An English book costs 44 cents more than a Math book. Find the cost of an English book. [1.74]

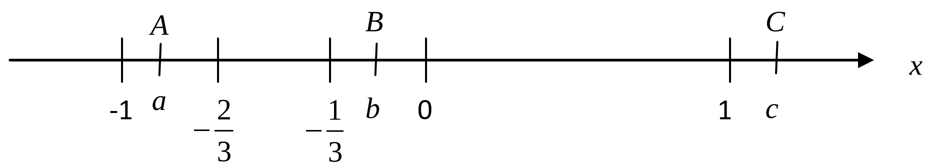
4. A chemical of 80% concentration is to be mixed with the same chemical of 90% concentration to produce a 84% concentration chemical. For a 500 litres of this mixed chemical, find the amount used by each chemical. [300,200]

5. Alcohol of 72% concentration is mixed with alcohol of 58% concentration to make a 62% mixture. If 15 litres more of each alcohol are added to this mixture, the concentration becomes 63.25%. Find the amount of each alcohol used in the first mixture. [12,30]

Rational Number and some of its operations

Numbers and number line

Ex.1 Observe the diagram,



a, *b*, and *c* are values correspond to point *A*, *B*, and *C* respectively. Arrange $\frac{1}{ab}$, $\frac{1}{b-a}$ and $\frac{1}{c}$ in ascending order.

Ex. 2 By adding operators '+' or '-' between the numbers 1 2 3 4 $\cdots$ 1990, What is the least non-negative value.

Ex. 3 Find the sum of $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots\frac{1}{9800}$.

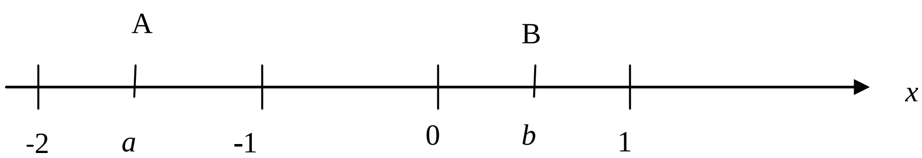
Ex. 4 Evaluate $1+2+2^2+2^3+\cdots+2^{2000}$

Ex. 5 Compare the value of $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{2000}{2^{2000}}$ and 2.

.

Practices

1. Refer to the diagram below, two points A and B are on a number line correspond to value a and b. If $x=\frac{a-5b}{a+5b}$, which value, x or -1 is larger?



2. Calculate $1+2-3+4+5-6+7+8-9+\cdots+97+98-99$.

3. Derive the formula for $1\times1+2\times2\times1+3\times3\times2\times1+\cdots+n\times n\times(n-1)\times\cdots\times2\times1$.

4. In this number pattern:

$$1;\frac{1}{2},\frac{2}{2},\frac{1}{2};\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{2}{3},\frac{1}{3};\frac{1}{4},\cdots$$

(a) which position is $\frac{7}{10}$?

(b) which number is on the 400th position?

5. Calculate $1+\frac{3}{2}+\frac{5}{4}+\cdots\frac{2\times1991+1}{2^{1991}}$.

6. Given that O is the origin on a number line. Point A and B are positions on 1 and 2. Let P_1 be the mid-point on AB, P_2 be the mid-point on AP_1, $\cdots$, P_{100} be the mid-point on AP_{99}, find the value of $P_1+P_2+P_3+\cdots+P_{100}$.

7. Evaluate $-1-(-1)^1-(-1)^2-(-1)^3-\ldots-(-1)^{99}-(-1)^{100}$.

8. Find the sum of $\frac{1}{10}+\frac{1}{40}+\frac{1}{88}+\frac{1}{154}+\frac{1}{238}$.

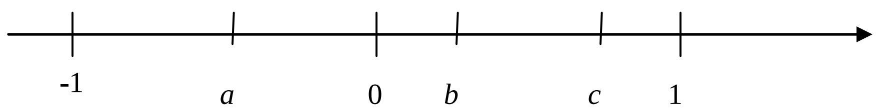
9. Evaluate $1^2-2^2+3^2-4^2+\ldots-2008^2+2009^2$.

10. The sum of $\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\frac{1}{4\times5\times6}+...+\frac{1}{13\times14\times15}+\frac{1}{14\times15\times16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

Comparing Rational numbers

By number line

Number a, b, c are on a number line as shown in the diagram below



Of these 4 numbers, $-\frac{1}{a}$, $-a$, $c-b$, $c+a$, which is the largest?

On the number line, a number on the left is always less than a number on its right.

All positive number is always larger than 0, “zero”.

All negative number is less than 0.

Therefore all positive number is larger than all negative number.

In the diagram above, $-1<a<0,\ 0<b<c<1$.

$\therefore -1<c+a<1,\ c-b<1-0=1$

$\because -1<a<0,\ \therefore 0<-a<1,\ -\frac{1}{a}>1$.

$\therefore$ Of these numbers $-\frac{1}{a}$, $-a$, $c-b$, $c+a$, $-\frac{1}{a}$ is the largest.

Alternative Method,

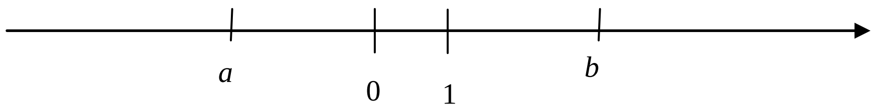
Let $a=-\frac{1}{2},\ b=\frac{1}{4},\ c=\frac{3}{4}$,

$-\frac{1}{a}=2,\ -a=\frac{1}{2},\ c-b=\frac{3}{4}-\frac{1}{4}=\frac{1}{2},\ c+a=\frac{3}{4}+\left(-\frac{1}{2}\right)=\frac{1}{4}$

Therefore $-\frac{1}{a}$ is the largest.

Practice

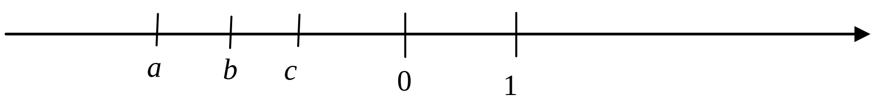
1. $a, b \in Q$, as shown on a number line below



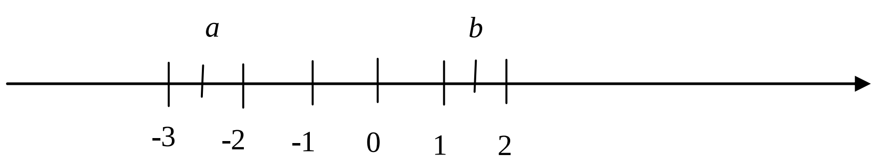
A. $\frac{1}{a} < 1 < \frac{1}{b}$ B. $\frac{1}{a} < \frac{1}{b} < 1$ C. $\frac{1}{b} < \frac{1}{a} < 1$ D. $1 < \frac{1}{b} < \frac{1}{a}$

2. a, b, c are value on the number line below, find the largest of these 3 numbers

$$\frac{1}{a-b},\ \frac{1}{c-b},\ \frac{1}{a-c}.$$



3. a, b are shown on a number line below, which one of the inequality is incorrect?



A. $|a| > |b|$

B. $a^2 > b^2$

C. $a > -b$

D. $-a > b$

By the difference of the numbers

Ex. If a, b are rational, and $b<0$, then what is the relative sizes of a, $a-b$, $a+b$?

Given two numbers x and y, if $x-y>0$, then $x>y$; if $x-y=0$, then $x=y$; if $x-y<0$, then $x<y$.

$$a-(a-b)=a-a+b=b<0\text{, therefore } a<a-b.$$

$$a-(a+b)=a-a-b=-b>0\text{, therefore } a>a+b.$$

$$\therefore a+b<a<a-b$$

Practice

1. Comparing a and $\frac{a}{3}$.

2. Comparing $A=\frac{7890123456}{8901234567}$ and $B=\frac{7890123455}{8901234566}$.

3. Comparing $-\frac{2^{2000}+1}{2^{2001}+1}$ and $-\frac{2^{2001}+1}{2^{2002}+1}$.

By division

Comparing $P=\frac{99^9}{9^{99}}$ and $Q=\frac{11^9}{9^{90}}$.

$$\frac{a}{b}>1 \text{ and } b>0 \Rightarrow a>b$$

$$\frac{a}{b}<1 \text{ and } b>0 \Rightarrow a<b$$

$$\frac{a}{b}=1 \Rightarrow a=b$$

$$\frac{P}{Q}=\frac{99^9}{9^{99}}\times\frac{9^{90}}{11^9}=\frac{(11\times 9)^9}{9^{99}}\times\frac{9^{90}}{11^9}=\frac{11^9\times 9^9\times 9^{90}}{9^{99}\times 11^9}=1$$

$$\therefore P=Q$$

Practice

1. Comparing 3^{555}, 4^{444}, and 5^{333}.

2. Given that $m<0$, $-1<n<0$, comparing m, mn, mn^2.

3. If $n>1$, comparing $\frac{n}{n-1}$, $\frac{n-1}{n}$, $\frac{n}{n+1}$.

4. If $ab < 0$, then then the relation in sizes of $(a-b)^2$ and $(a+b)^2$ is
   (a) $(a-b)^2 < (a+b)^2$
   (b) $(a-b)^2 = (a+b)^2$
   (c) $(a-b)^2 > (a+b)^2$
   (d) Not determined

Application of basic Algebra

Find the sum of $\left(1^2+3^2+5^2+7^2+\cdots+99^2\right)-\left(2^2+4^2+6^2+8^2+\cdots+100^2\right)$.

We can re-arrange the above expression into $1^2-2^2+3^2-4^2+5^2-6^2+\cdots+99^2-100^2$.

There is a pattern of $(2n-1)^2-(2n)^2=4n^2-4n+1-4n^2=-4n+1$. Therefore,

$$\left(1^2+3^2+5^2+7^2+\cdots+99^2\right)-\left(2^2+4^2+6^2+8^2+\cdots+100^2\right)$$

$$=1^2-2^2+3^2-4^2+5^2-6^2+\cdots+99^2-100^2$$

$$=-4(1)+1-4(2)+1-4(3)+1\cdots-4(50)+1$$

$$=\left\{-4(1+2+3+4+\cdots+50)\right\}+50$$

$$=-4\left\{\frac{50(50+1)}{2}\right\}+50$$

$$=-5050$$

Practices

1. Find the sum of $1^2-2^2+3^2-4^2+5^2-6^2+\cdots-1998^2+1999^2$.

2. Find the product of $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{6^2}\right)$.

3. Find the sum of $(-1)+(-1)^2+(-1)^3+\cdots+(-1)^{99}+(-1)^{100}$.

Evaluate simple algebraic expression(I)

Calculate $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$.

Let $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=a$, and $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1+a$; $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=1+a+\frac{1}{5}$; then

$$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$$

$$=(1+a)\left(a+\frac{1}{5}\right)-\left(1+a+\frac{1}{5}\right)a$$

$$=a+\frac{1}{5}+a^2+\frac{1}{5}a-a-a^2-\frac{1}{5}a$$

$$=\frac{1}{5}$$

Try it out!

1. From 2009, subtract half of it at first, then subtract $\frac{1}{3}$ of the remaining number, next subtract $\frac{1}{4}$ of the remaining number, and so on, until $\frac{1}{2009}$ of the remaining number is subtracted. What is the remaining number?

2. Evaluate

$$\left(\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2009}\right)\left(1+\frac{1}{2}+...+\frac{1}{2008}\right)-\left(1+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2009}\right)\left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2008}\right).$$

Practices

1. Given that $y=ax^4+bx^2+c$, when $x=-5$, $y=3$. Find the value of y when $x=5$.

2. Given that when $x = 7$, the value of the expression $ax^5 + bx - 8$ is 4. Find the value of $\frac{a}{2}x^5 + \frac{b}{2}x + 3$, when $x = 7$.

3. Given the expression $ax^5 + bx + c$ has a value of 8 and 1 when $x = -3$ or 0 respectively, find the value of the expression when $x = 3$.

Evaluate simple algebraic expression(II)

Given that $\frac{a+b}{a-b}=7$, find the value of $\frac{2(a+b)}{a-b}-\frac{a-b}{3(a+b)}$.

Since $\frac{a+b}{a-b}=7$, then we have $\frac{a-b}{a+b}=\frac{1}{7}$.

$$\therefore \quad \frac{2(a+b)}{a-b}-\frac{a-b}{3(a+b)}=2\left(\frac{a+b}{a-b}\right)-\frac{1}{3}\left(\frac{a-b}{a+b}\right)$$

$$=2(7)-\frac{1}{3}\left(\frac{1}{7}\right)$$

$$=13\frac{20}{21}$$

Practice

1. If the expression $2y^2+3y+7=2$, then find the value of $4y^2+6y-9$.

2. Given that $a=2000x+1999$, $b=2000x+2000$, and $c=2000x+2001$, then find the value of $(a-b)^2+(b-c)^2+(c-a)^2$.

3. Let $ab^2=6$, find the value of $ab\left(ab^3+a^2b^5-b\right)$.

Evaluate simple algebraic expression(III)

If $ab=1$, find the value of $\frac{a}{a+1}+\frac{b}{b+1}$.

$\because \quad ab=1, \therefore \quad a=\frac{1}{b}$

$$\frac{a}{a+1}+\frac{b}{b+1}=\frac{\frac{1}{b}}{\frac{1}{b}+1}+\frac{b}{b+1}$$

$$=\frac{1}{1+b}+\frac{b}{b+1}$$

$$=\frac{b+1}{b+1}$$

$$=1$$

Practice

1. If $x+y=2z$, and $x\neq y$, then find the value of $\frac{x}{x-z}+\frac{y}{y-z}$.

2. If $x-y=2$, and $2y^2+y-4=0$, find the value of $\frac{x}{y}-y$.

3. Given that $x-2y=2$, find the value of $\frac{3x-y-6}{4x-y-8}$.

4. If $\frac{1}{x}-\frac{1}{z}=4$, find the value of $\frac{2x+4xz-2z}{x-z-2xz}$.

Evaluate simple algebraic expression(IV)

If $x+y=2z$, $x\neq y$, then find the value of $\frac{x}{x-z}+\frac{y}{y-z}$.

Given that $x+y=2z$, $x\neq y$, we can choose $x=1$, $y=3$, and $z=2$ to find the value of the expression.

$$\frac{x}{x-z}+\frac{y}{y-z}=\frac{1}{1-2}+\frac{3}{3-2}=-1+3=2$$

Practice

1. Given that $x-2y=2$, find the value of $\frac{3x-y-6}{4x-y-8}$.

2. Given that $f(x,y)=3x+2y+m$, and $f(2,1)=18$, find the value of $f(3,-1)$.

3. If $a+b+c=0$, and $\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}=0$, find the value of $\frac{bc+b-c}{b^2c^2}+\frac{ca+c-a}{c^2a^2}+\frac{ab+a-b}{a^2b^2}$.

Evaluate simple algebraic expression(V)

If $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, and $3x-2y+z=18$, find the value of $x+5y-3z$.

From $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, let $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=k$.

Then we have $x=3k$, $y=4k$, $z=5k$.

$$3x-2y+z=18$$

$$3(3k)-2(4k)+(5k)=18$$

$$k=3,\ x=9,\ y=12,\ z=15$$

$$\therefore\ x+5y-3z=9+5(12)-3(15)=24$$

Practice

1. If $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, and $4x-5y+2x=10$, find the value of $2x-5y+z$.

2. If $x=\frac{y}{2}=\frac{z}{3}$, and $x+y+z=12$, find the value of $2x+3y+4z$.

3. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, find the value of $x+y+z$.

4. SMO 2009 Junior Paper

   Given that $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$ and $a-b+2\neq 0$, find the value of $ab-a+b$.

Odd and Even Integers

Integers can be divided into two sets of number. The set with numbers that can be divided by 2 is called Even; and the rest that cannot be divided by 2 exactly, is called Odd.

An Even integer can be denoted as $2n$, where n is an integer.

An Odd integer can be denoted as $2n+1$, where n is an integer.

An integer must be either Odd or Even, but not both.

Some of the other properties:

(1) Odd $\neq$ Even
(2) Odd + Even = Even + Odd = Odd; Odd – Even = Even – Odd = Odd.
(3) Even + Even = Even; Even – Even = Even.
(4) If $a \times b =$ even, then at least one of the factors is even.
(5) The product of two consecutive integers must be even, $n \times (n+1) =$ even.
(6) If the sum or difference of integers is odd, then there must be at least an Odd integer in the sum or difference.
(7) If the sum or difference of integers is even, then the number of odd integers must be even.
(8) If the sum or difference of integers is odd, then the number of odd integers must be odd.
(9) If the product of integers is odd, then all the numbers in the product must be odd.
(10) If the product of integers id even, then there must be at least one even integer.

Ex. 1 If a, b, c are random integers, then among the three numbers $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$, there are

(A) all non-integers. (B) at least an integer.

(C) at least two integers. (D) all integers.

Ex. 2 Given that the sum of 100 positive integers is 10000, the number of odd integers is more than the number of even integers, what is the most number of even integers?

(A) 49 (B) 48 (C) 47 (D) 46

Ex. 3 Given that a and b are consecutive integers and that $c = ab$, $N^2 = a^2 + b^2 + c^2$, what is the parity of N^2?

(A) Odd (B) Even (C) maybe Odd or Even (D) none of the above

Ex. 4 There are *n* number: x_1, x_2, x_3, $\cdots$; x_n. Each of these numbers is either a 1 or -1.

If $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$, what must be truth about n?

(A) Even (B) Odd (C) multiple of 4 (D) cannot tell at all.

Ex. 5 Given that positive integers $p, q, p-q$ are primes, and also that $p+q$ is even,

evaluate the value of $\left(1+\frac{1}{2}\right)^p \left(1-\frac{1}{3}\right)^q$.

Ex. 6 In this number pattern: $1, 2, 5, 13, 34, 89, \cdots$, starting from the second number, the sum of any two adjacent number is equal to three times the middle number. What should be the parity of the 2003$^{\text{rd}}$ number?

Ex. 7 Given three integers x, y, z with two odds and one even, prove that $(x+1)(y+2)(z+3)$ must be even.

Practices

1. Given 2003 consecutive positive integers: $1, 2, 3, 4, \cdots, 2003$, if either a '+' or '$-$' operator is added in between any two numbers, will the result be odd or even? [even]

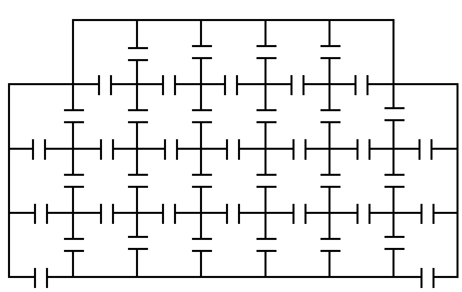
2. In a Math competition, there are 40 questions. A correct answer scores 5 points; a nil return scores 1 point; and a wrong answer deducts 1 point. Should the total score of all the competitors be even or odd. [even]

3. 30 books are to be packed into 5 boxes. Each box must have odd number of books. How can this be done? [impossible]

4. Is it possible to arrange these 10 numbers: 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, on a line such that there is a number in between the two '1's; there are two numbers in between the two '2's; $\cdots$; there are five numbers in between the two '5's? [impossible]

5. Is it possible to visit all the 26 rooms just once without re-entering any one of them?

[impossible]



Prime and Composite Numbers

Among all the positive integers, 1 is the only one that has only one positive factor, and that is itself.

All positive integers greater than 1 have at least two positive factors.

If an positive integer has only two factors, that is, 1 and the number itself, then this integer is called a Prime number, or Prime in short.

All non-prime integers are collectively called Composite Number, or Composite in short.

By definition, 1 is NOT a prime. There is only one even prime, 2, which is also the smallest prime. The smallest composite is 4.

1 is also NOT a composite.

All positive integer, other than 1, can be prime factorise

$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$ where $p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, \cdots, p_k^{a_k}$ are distinct primes.

The number of positive factor of N can be found with this expression:

Number of positive factor of N = $(a_1+1)(a_2+1)(a_3+1)\cdots(a_k+1)$

Ex. 1 If p and p^3+5 are prime, what is p^5+7?

(A) Prime (B) Prime or Composite (C) Composite

(D) Not prime or composite

Ex. 2 Given three primes p, q, r that satisfy $p+q=r$ and $p<q$, find the value of p.

(A) 2 (B) 3 (C) 7 (D) 13

Ex. 3 Given that n is a positive integer such that $n+3$ and $n+7$ are both primes. Find the remainder when n is divided by 3.

Ex. 4 If positive integers $n_1 > n_2$, and $n_1^2 - n_2^2 - 2n_1 - 2n_2 = 19$, find the values of n_1, and of n_2.

.

Ex. 5 If p, q, r are primes, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1661}{1986}$, find the value of $p+q+r$.

Ex. 6 If p is a prime not less than 5, and $2p+1$ is also a prime, prove that $4p+1$ is a composite.

Practices

1. Given x, y are primes, find the number of ordered pairs of equation $x + y = 1999$.

   (A) 1 pair (B) 2 pairs (C) 3 pairs (D) 4 pairs

2. Given that m, n are distinct primes, and $p = m + n + mn$ with p being the minimum value, evaluate the expression $\dfrac{m^2 + n^2}{p^2}$.

3. Given that p, q are primes and $p = m + n$, $q = mn$, where m, n are positive integers, evaluate the expression $\dfrac{p^p + q^q}{m^n + n^m}$.

4. If m, n are primes that satisfy $5m + 7n = 129$, find the values of $m + n$.

5. Given that $p, p + 2, p + 6, p + 8, p + 14$ are primes, find the number of p.

6. **SMO question 2009**

   Let p and q represent two consecutive prime numbers. For some fixed integer n, the set {n – 1, 3n – 19, 38 – 5n, 7n – 45} represents {p , 2p, q, 2q}, but not necessarily in that order. Find the value of n.

Divisibility

A **divisibility rule** is a method that can be used to determine whether a number is evenly divisible by other numbers. Divisibility rules are a shortcut for testing a number's factors without resorting to division calculations. Although divisibility rules can be created for any base, only rules for decimal are given here.

The rules given below transform a given number into a generally smaller number while preserving divisibility by the divisor of interest. Therefore, unless otherwise noted, the resulting number should be evaluated for divisibility by the same divisor.

For divisors with multiple rules, the rules are generally ordered first for those appropriate for numbers with many digits, then those useful for numbers with fewer digits.

If the result is not obvious after applying it once, the rule should be applied again to the result.

Divisor	**Divisibility Condition**	**Examples**
1	Automatic.	Any integer is divisible by 1.
2	The last digit is even (0, 2, 4, 6, or 8).	1,294: 4 is even.
3	The sum of the digits is divisible by 3. For large numbers, digits may be summed iteratively.	405: 4 + 0 + 5 = 9, which clearly is divisible by 3. 16,499,205,854,376 sums to 69, 6 + 9 = 15, 1 + 5 = 6, which is clearly divisible by 3.
4	The number obtained from these examples must be divisible by 4, as follows:	
	If the tens digit is even, the last digit is divisible by 4 (0, 4, 8).	168: 6 is even, and 8 is divisible by 4.
	If the tens digit is odd, the last digit plus 2 is divisible by 4 (2, 6).	5,496: 9 is odd, and 6+2 is divisible by 4.
	If the number formed by the last two digits is divisible by 4.	2,092: 92 is divisible by 4.
5	The last digit is 0 or 5.	490: the last digit is 0.
6	It is divisible by 2 and by 3.	24: it is divisible by 2 and by 3.
	Add the last digit to four times the sum of all other digits.	198: $(1 + 9) \times 4 + 8 = 48$
7	The number obtained from these examples must be divisible by 7, as follows:	
	Form the alternating sum of blocks of three from right to left.	1,369,851: $851 - 369 + 1 = 483 = 7 \times 69$

	Double the number with the last two digits removed and add the last two digits.	364: (3 × 2) + 64 = 70.
	Add 5 times the last digit to the rest.	364: 36 + (5 × 4) = 56.
	Subtract twice the last digit from the rest.	364: 36 − (2 × 4) = 28.
	The number obtained from these examples must be divisible by 8, as follows:	
8	If the hundreds digit is even, examine the number formed by the last two digits.	624: 24.
	If the hundreds digit is odd, examine the number obtained by the last two digits plus 4.	352: 52 + 4 = 56.
	Add the last digit to twice the rest.	56: (5 × 2) + 6 = 16.
9	The sum of the digits is divisible by 9. For larger numbers, digits may be summed iteratively. Result at the final iteration will be 9.	2,880: 2 + 8 + 8 + 0 = 18: 1 + 8 = 9.
10	The last digit is 0.	130: the last digit is 0.
	The number obtained from these examples must be divisible by 11, as follows:	
11	Form the alternating sum of the digits.	918,082: 9 - 1 + 8 - 0 + 8 - 2 = 22.
	Add the digits in blocks of two from right to left.	627: 6 + 27 = 33.
	Subtract the last digit from the rest.	627: 62 - 7 = 55.
12	It is divisible by 3 and by 4.	324: it is divisible by 3 and by 4.
	Subtract the last digit from twice the rest.	324: (32 × 2) − 4 = 60.
	The number obtained from these examples must be divisible by 13, as follows:	
13	Add the digits in alternate blocks of three from right to left, then subtract the two sums.	2,911,272: − (2 + 272) + 911 = 637
	Add 4 times the last digit to the rest.	637: 63 + (7 × 4) = 91, 9 + (1 × 4) = 13.
14	It is divisible by 2 and by 7.	224: it is divisible by 2 and by 7.
	Add the last two digits to twice the rest. The answer must be divisible by 7.	364: (3 × 2) + 64 = 70.

15	It is divisible by 3 and by 5.	390: it is divisible by 3 and by 5.
	The number obtained from these examples must be divisible by 16, as follows:	
16	If the thousands digit is even, examine the number formed by the last three digits.	254,176: 176.
	If the thousands digit is odd, examine the number formed by the last three digits plus 8.	3,408: 408 + 8 = 416.
	Sum the number with the last two digits removed, times 4, plus the last two digits.	176: $(1 \times 4) + 76 = 80$.
17	Subtract the last two digits from twice the rest.	3213: $(2 \times 32) - 13 = 51$.
	Alternately add and subtract blocks of two digits from the end, doubling the last block and halving the result of the operation, rounding any decimal end result as necessary.	20,98,65: $(65 - (98 \times 2)) : 2 + 40 = -25.5 = 255 = 15 \times 17$
	Subtract 5 times the last digit from the rest.	221: $22 - (1 \times 5) = 17$.
18	It is divisible by 2 and by 9.	342: it is divisible by 2 and by 9.
19	Add twice the last digit to the rest.	437: $43 + (7 \times 2) = 57$.
20	It is divisible by 10, and the tens digit is even.	360: is divisible by 10, and 6 is even.
	If the number formed by the last two digits is divisible by 20.	480: 80 is divisible by 20.

Ex.1 The greatest integer that can divide the sum of any three consecutive integers is:

(A) 1 (B) 2 (C) 3 (D) 6 .

Ex. 2 A six-digit integer $\overline{568abc}$ is divisible by 3, 4, 5. Find the smallest number $\overline{abc}$.

Ex. 3 Prove that $1999^{2001}+2001^{1999}$ is a multiple of 10.

Ex. 4 Find the remainder when $\overline{1234567891011\cdots200120022003}$ is divided by 9.

Ex. 5 Given that $A=1\times3\times5\times7\times9\times11\times13\times\cdots\times1999\times2001\times2003$, find the last three digits of A.

Ex. 6 Given natural numbers from 1 to 1995 inclusively, find the number of a from these 1995 numbers that $(a+1995)$ can divide exactly $1995\times a$.

Practices

1. Find the number of 7-digit number $\overline{1287xy6}$ that is multiple of 72.

2. If n is a positive integer, $n+3, n+7$ are primes, find the remainder when n is divided by 3.

3. Find the last digit of $1988^{1989}+1989^{1988}$.

4. Let a_n be the last two digits of 7^n, find the value of $a_1+a_2+a_3+\cdots+a_{2002}+a_{2003}$.

5. Prove that $53^{53}-33^{33}+3^{1998}+4^{1998}$ can be divided exactly by 5.

6. **SMO Question 1995**

   A natural number greater than 11 gives the same remainder (not zero) when divided b 3, 5, 7 and 11. Find the smallest possible value of this natural number.

7. **SMO Question 1998**

   Find the smallest positive integer *n* sch that $100 \le n \le 1100$ and

   $1111^n+1222^n+1333^n+1444^n$ are divisible by 10.

8. **SMO Question 1999**

   When the three numbers 1238, 1596 and 2491 are divided by a positive integer *m*, the remainders are all equal to a positive integer *n*. Find *m* + *n*.

9. **SMO Question 2001**

   Find the smallest positive integer *k* sch that $2^{69}+k$ is divisible by 127.

10. **SMO Question 2003**

How many integers n between 1 and 2003 (inclusive) are there such that $1^n + 2^n + 3^n + 4^n + 5^n$ is divisible by 5?

11. **SMO Question 2009**

m and n are two positive integers satisfying $1 \le m \le n \le 40$. Find the number of pairs of (m, n) such that their product is divisible by 33.

Ratio and its properties

If John has 75 cents and Mary has 50 cents, the ratio of the amount of money of John to Mary is

75 : 50 or 3 : 2.

You know this very well. If we let the amount of money that John has be J and the amount of money that Mary has be M, then here is how we can express this ratio

$$J : M = 3 : 2.$$

Alternatively, we can also write the ratio as

$$\frac{J}{M} = \frac{3}{2}.$$

Now consider this: let say that the ratio of the amount of money of John and Mary be 3 : 2, what is the amount of money of each of them?

The ratio is $\frac{J}{M} = \frac{3}{2}$. The amount of money of John is $J = 3m$; the amount of Mary is $M = 2m$, where m is the common unit.

We can see that $m = \frac{J}{3} = \frac{M}{2}$. Therefore this ratio of John and Mary can be expressed as

$$\frac{J}{3} = \frac{M}{2}.$$

These are all equivalent statement of the same thing.

$$J : M = 3 : 2 \quad \Leftrightarrow \quad \frac{J}{M} = \frac{3}{2} \quad \Leftrightarrow \quad \frac{J}{3} = \frac{M}{2}$$

If the ratio of $A : B : C = a : b : c$, it will be easier to express it as $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$, rather than

$$\frac{A}{B} = \frac{a}{b} \quad \text{and} \quad \frac{B}{C} = \frac{b}{c}.$$

Can you prove this relationship, $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$?

In general, given that $a:b=c:d \Leftrightarrow ad=bc$

(i) $\frac{a}{c}=\frac{b}{d}$

(ii) $\frac{b}{a}=\frac{d}{c}$

(iii) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a+b}{b}=\frac{c+d}{d}$. Can you prove this?

(iv) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a-b}{b}=\frac{c-d}{d}$. Can you prove this?

(v) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. Can you prove this?

(vi) Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots=\frac{n}{m}=\frac{a+c+e\cdots+n}{b+d+f+\cdots+m}$; if $b+d+f+\cdots+m\neq 0$

The above extension of the basic ratio can be helpful in solving ratio related problems.

Ex. 1 Given that $\frac{a}{b}=\frac{2}{3}$, which of the following is/are not a true statement?

(A) $a=2,b=3$ (B) $a=2k,b=3k\left(k\neq 0\right)$ (C) $3a=2b$

(D) $a=\frac{2}{3}b$

Ex. 2 Fill in the blank.

(A) Given that $\frac{a}{b}=\frac{3}{5}$, $\frac{a+2b}{2a-b}=$ __________ .

(B) Given that $\frac{3a-4b}{2a-3b}=\frac{7}{4}$, $\frac{a}{b}=$ __________ .

(C) Given that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{3}{4}$, $\frac{3a+2c-e}{3b+2d-f}=$ __________ .

(D) Given that $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$, $\frac{xy+yz+zx}{x^2+y^2+z^2}=$ __________ .

Ex. 3 Which of the following is/are true statements,

(A) if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{b}=\frac{c+m}{d+m}$;

(B) if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2}{b^2}=\frac{c^2}{d^2}$;

(C) if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+c}{b}=\frac{b+c}{d}$;

(D) if $\frac{a^2}{b^2}=\frac{c^2}{d^2}$, then $\frac{a}{b}=\frac{c}{d}$.

Ex. 4 (A) Given $\frac{x}{y}=2$, find the value of $\frac{2x^2-3xy+y^2}{x^2+2y^2}$.

. (B) if $6x^2-5xy+y^2=0$, where $(x, y\neq 0)$, find the value of $\frac{2x-3y}{2x+3y}$.

Ex. 5 Given that $x:y=3:5$ and $y:z=2:3$, find the value of $\dfrac{x+y-z}{2x-y+z}$.

Ex. 6 Given that $2x-3y+z=0$, $3x-2y-6z=0$ and $xyz\neq 0$, find the value of,

(A) $x:y:z$, (B) $\dfrac{x^2+y^2+z^2}{2x^2+y^2-z^2}$.

<u>Practices</u>

1. Given that $x = \frac{c}{a+b} = \frac{a}{b+c} = \frac{b}{a+c}$, find the value of x.

2. Given that $\frac{1}{x} - \frac{1}{y} = 3$, find the value of $\frac{2x+3xy-2y}{x-2xy-y}$.

3. Given that $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$, find the value of $\frac{a}{b} + \frac{b}{a}$.

4. Given that $\frac{ab}{a+b} = \frac{3}{2}$, $\frac{bc}{b+c} = 3$, $\frac{ca}{c+a} = \frac{1003}{1004}$, find the value of $\frac{abc}{ab+bc+ac}$.

5. Given that $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a^2+b^2}{ab} = \frac{c^2+d^2}{cd}$.

6. Given that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that $\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} = \frac{(x+y+z)^3}{(a+b+c)^2}$.

7. **SMO Question 1998**

Three boys, Tom, John and Ken, agreed to share some marbles in the ratio of 9: 8 : 7 respectively. John then suggested that the should share the marbles in the ratio 8: 7: 6 instead. Who should then get more marbles than before and who world get less than before if the ratio is changed?

8. **SMO Question 2009**

There are two models of LCD television on sale. One is a 20 inch standard model while the other is a 20 inch widescreen model. The ratio of the length to the height of the standard model is 4: 3, while that of the widescreen model is 16: 9. Television screens are measured b the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is A: 300, find the value of A.

Perfect Squares

Perfect squares are natural numbers: 1^2, $2^2, 3^2, 4^2$, …., n^2. In general, we call these numbers – Square numbers.

The last digit of a square cannot be 2, 3, 7, or 8. It can only be 0, 1, 4, 5, 6, 9.

Here is the list of the last two digits of a square numbers:

00	01	21	41	61	81
25	09	29	49	69	89
	04	24	44	64	84
16	36	56	76	96	

These are the 22 possibilities of a square numbers.

Notice that the tens-digit of an odd square number must be even.

The tens-digit of a square number ends in 6, must be odd.

The tens-digit of a square number ends in 4, must be even.

We can make use of these rules to deduce whether a number is square or not square.

In addition, here are the other properties of a square:

1. Each even square can be expressed as a multiple of 4 (eg. $4k$).
2. Each odd square can be expressed as a multiple of 4 plus 1 (eg $4k+1$).
3. Each odd square can be expressed as a multiple of 8 plus 1 (eg $8k+1$).
4. Each $3n+2$ integer cannot be a square.
5. Each $5n+2$ or $5n+3$ integer cannot be a square.
6. Each $8n+2$, $8n+3$, $8n+5$, $8n+6$, $8n+7$ integer cannot be a square.
7. Each $9n+2$, $9n+3$, $9n+5$, $9n+6$, $9n+8$ integer cannot be a square.
8. If the tens digit and ones digit of an integer are odd, it is not a square.
9. If the tens digit is even and ones digit is 6 of an integer, it is not a square.

Lastly, if a natural number N, such that $n^2 < N < (n+1)^2$, then N is not a square.

Ex.1 A number is formed using 300 digits of 2 and some digit 0, can it be arrange into a square?

Ex. 2 Prove that there is no square number in this number pattern: 11, 111, 1111 , 11111, … .

Ex. 3 Find a 4-digit square number in the form of $\overline{aabb}$.

Ex. 4 How many square numbers are there from 1 to 1000 inclusively?

Ex. 5 Given a 2-digit number N, by adding another 2 digits on its left, becomes the square of N. Find all such positive integers, N.

Practices

1. Prove that for any given natural number n, $n(n+1)$ is not a square.

2. Find the number of 4-digit number, after added 400, it becomes a square.

3. Prove that $\underbrace{11\cdots1}_{m\ digits}\times\underbrace{10\cdots5}_{m+1\ digits}+1$ is a perfect square.

4. Find all square numbers with common last 4 digits.

5. Given that x and y are positive integer, find all the order pair of (x, y) that satisfy the equation $x^2+y^2=7$

6. Given that n is a positive integer, find the smallest n so that $2008n$ is a square.

7. **China 1992**

   If x and z are positive integers, prove that the vales of x^2+z+1 and z^2+4x+3 cannot be both perfect squares at the same time.

8. **SMO Question 2000**

   Evalate $\sqrt{\underbrace{11\cdots1}_{2000 digits}-\underbrace{222...22}_{1000 digits}}$.

2. **SMO Question 2000**

   Let n be a positive integer such that n + 88 and n – 28 are both perfect squares. Find all possible values of n.

Modular Arithmetic

CONGRUENCE

What is Congruence?

Two integers are congruent modulo *m* if and only if they have the same remainders after division by *m*.

Let *m* be a fixed positive integer. If $a, b, \in Z$, we say that "*a* is **congruent** to *b* **modulo** *m*" and write

$a \equiv b \pmod{m}$ whenever $m \mid (a-b)$.

If $m \nmid (a-b)$, we write $a \not\equiv b \pmod{m}$.

Another way to look at Congruency

The condition for a to be congruent to b modulo *m* is equivalent to the condition that

$a = b + km$ for some $k \in Z$.

For example, $7 \equiv 3 \pmod{4}$, $-6 \equiv 14 \pmod{10}$, $121 \equiv 273 \pmod{2}$, but $5 \not\equiv 4 \pmod{3}$ and $21 \not\equiv 10 \pmod{2}$.

True or False?

1. $88 \equiv 54 \pmod{17}$ ( )
2. $185 \equiv 392 \pmod{23}$ ( )
3. $12 \equiv 23 \pmod{7}$ ( )
4. $101 \equiv 72 \pmod{3}$ ( )

Properties of congruency

(i) $a \equiv a \pmod{m}$ (reflexive property)

(ii) $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$ (symmetric property)

(ii) $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$ (transitive property)

Can you prove these properties?

Modular Arithmetic

If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then

(1) $a + b \equiv a' + b' \pmod{m}$

(2) $a - b \equiv a' - b' \pmod{m}$

(3) $ab \equiv a'b' \pmod{m}$

(4)If $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$, then it follows that $a \equiv b \pmod{m}$

Can you provide proof for these operations?

Congruences occur in everyday life. The short hand of a clock indicates the hour modulo 12, while the long hand indicates the minute modulo 60. For example, 20 hours after midnight, the clock indicates 8 o'clock because $20 \equiv 8 \pmod{12}$. In determining which day of the week a particular date falls, we apply congruence modulo 7. Two integers are congruent modulo 2 if and only if they have the same parity; that is if and only if they are both odd or both even.

The idea of congruence is not radically different from divisibility, but its usefulness lies in its notation, and the fact that congruence, with respect to a fixed modulus, has many of the properties of ordinary equality.

Example 1

What is the remainder when 2^{37} is divided by 7?

Example 2

What is the remainder when $4^{10} \times 7^{7}$ is divided by 5?

Example 3

Find the remainder when 56976 is divided by 4.

Tests for Divisibility

Congruences can be used to prove some of the familiar tests for divisibility by certain integers.

The test of divisibility by 4 works because $100 \equiv 0 \pmod{4}$. Given an integer $n \times 100 + \overline{ab}$, where n is an integer and a, b are the last two digits of this number, $100n + \overline{ab} \equiv \overline{ab} \pmod{4}$. Therefore the remainder when $n \times 100 + \overline{ab}$ is divided by 4 is the same as that of $\overline{ab}$ when divided by 4.

Example 4

Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

Example 5

Prove that a number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

Fermat's Little Theorem

If p is a prime number that does not divide the integer a, then

$$a^{p-1} \equiv 1 \pmod{p}$$

In addition, for any integer a and prime p,

$$a^{p} \equiv a \pmod{p}$$

Exercise

1. If a and b are divided by 5, it gives a remainder 1 and 4 respectively. Find the remainder when $3a-b$ is divided by 5.

2. If today is a Saturday, what is the day of the week after 10^{2000} days?

3. Find the remainder when 1993^{1994} is divided by 7.

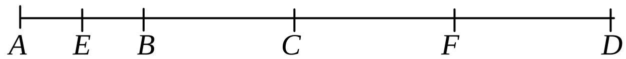
4. Prove that $3^{2000}+4^{1993} \equiv 0 \pmod{5}$

5. **Moscow 1982**

   Find all the positive integers n such that $n.2^{n}+1$ is divisible by 3.

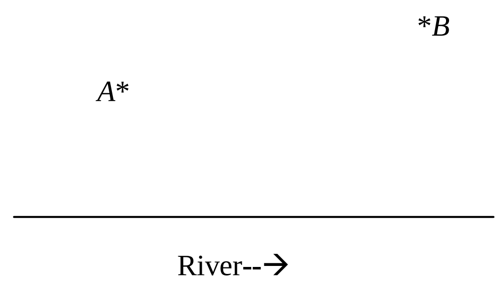
Lines and Angles

Ex.1 Given that $AB : BC : CD = 2 : 3 : 4$ with E and F being the mid points on AB and CD respectively, if $EF = 12$ cm, find the length of AD.



Ex. 2 Given that $AB = 10$ cm on a straight line, C is a point on the same line that $AC = 2$ cm. If M and N are the midpoints on AB and AC respectively, find the length of MN.

Ex. 3 If Jenny is to travel from point A to point B but need to reach the river bank to wash her hands, what is the shortest path for her?



Ex. 4 A wire of length 10 cm is to be bent to form a pentagon, what is the longest side of this shape?

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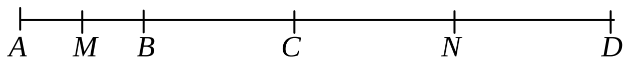
Ex. 5 For an analog clock, if the time is from 0230 to 0250, how much does each hour hand and minute hand have to move, in angles?

Ex. 6 For an angle *A*, if the ratio of its complementary angle to its supplementary angle is 2 : 7, find the measure of angle *A*.

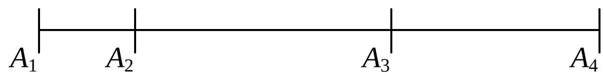
Ex 7. For an analog clock starting at 5 O'clock, how much will the hour hand move before the minute hand overlap it?

Practices

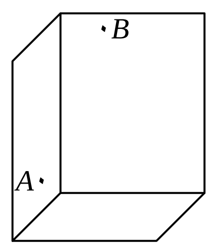
1. For the given diagram, *B* and *C* are two points on *AD*. *M* and *N* are the midpoints on *AB* and *CD* respectively. If $MN = a$ and $BC = b$, find the length *AD*.



2. For the given diagram, A_2 and A_3 are two points on the line A_1A_4. If $A_1A_2 = a$, $A_1A_3 = a^2$, $A_1A_4 = a^3$, find the sum of length of all the line segment between A_1 and A_4.



3. Two points *A* and *B* are on two sides of a box as show in the diagram, find the shortest line to draw from *A* to *B*.



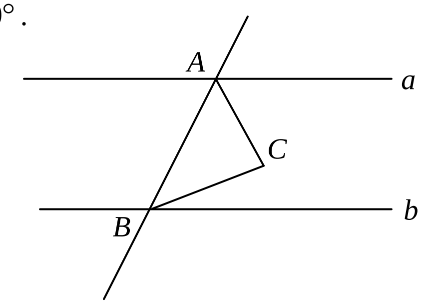
5. If the difference in two supplementary angles is 28°, find the complementary angle of the smaller angle.

6. From 6 O'clock to 7 O'clock, when will the hour and minute hands are at 90° apart?

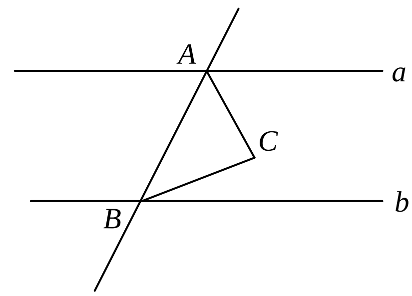
7. From 4 O'clock to 6 O'clock, when will the hour and minute hands are at 120° apart?

Parallel Lines

Ex.1 Given the diagram below, $a // b$, line AB intersects a and b at A and B .IF CA bisects $\angle BAa$ and BC bisects $\angle ABb$, prove that $\angle C = 90°$.



What about straight lines a, b have been intersected by another straight line AB. CA bisects $\angle BAa$ and BC bisects $\angle ABb$. If $\angle C = 90°$, are the lines a, b parallel to one another?



Ex. 2 Given the diagram below, $AA_1 // BA_2$ find the measure of $\angle A_1 - \angle B_1 + \angle A_2$.

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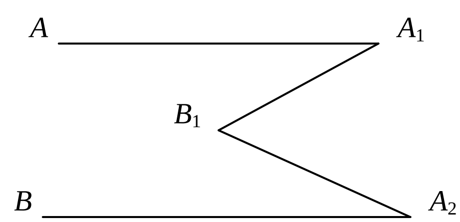
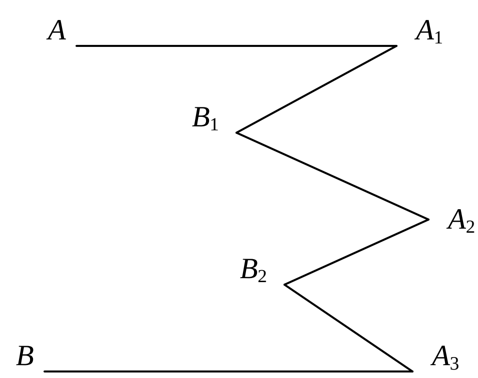
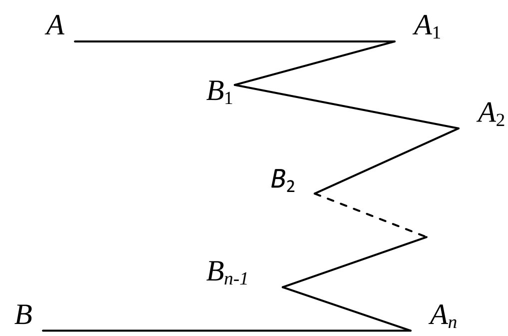


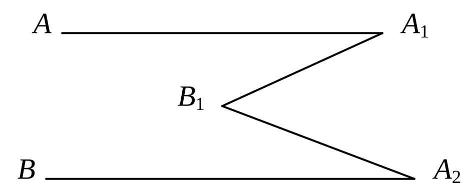
What about this if $AA_1 // BA_3$?



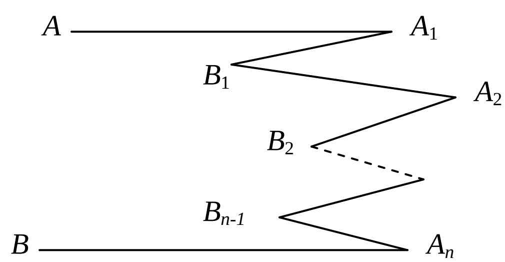
And how about this if $AA_1 // BA_n$?



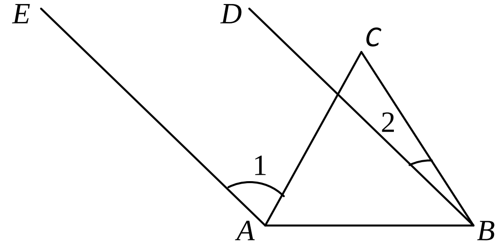
Question : If $\angle A_1 - \angle B_1 + \angle A_2 = 0$, is $AA_1 // BA_2$?



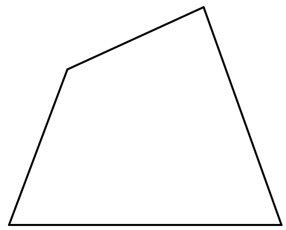
Question : If $\angle A_1 + \angle A_2 + \cdots + \angle A_n = \angle B_1 + \cdots + \angle B_{n-1}$, is $AA_1 // BA_n$?



Ex. 3 For the given diagram below, $AE // BD$, $\angle 1 = 3 \times \angle 2$, and $\angle 2 = 25°$. Find $\angle C$.

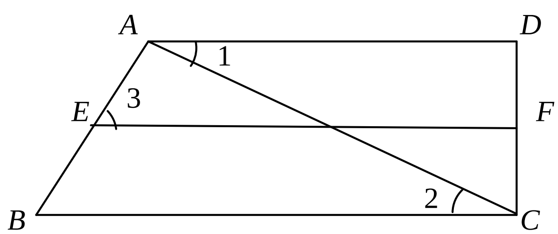


Ex. 4 Prove that the sum of interior angles of the quadrilateral is $360°$.



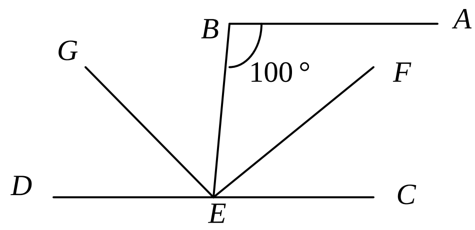
Question : What about the sum of all interior angles of a polygon of n sides?

Ex. 5 For the diagram below, $\angle 1 = \angle 2$, $\angle D = 90°$, $EF \perp CD$, show that $\angle 3 = \angle B$.

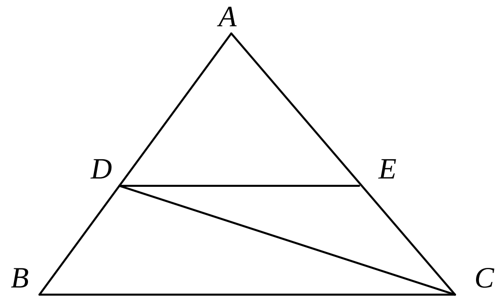


Practices

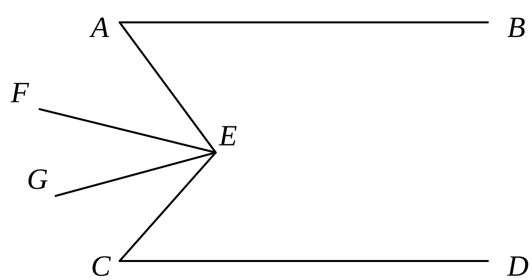
1. For the given diagram, $AB//CD$, $\angle GEF = 90°$, $\angle B = 100°$, EF bisects $\angle BEC$, and $EG \perp EF$. Find $\angle BEG$ and $\angle DEG$.



2. For the given diagram, CD bisects $\angle ACB$. If $\angle ACB = 40°$, $\angle B = 70°$ and $DE//BC$, find $\angle EDC$ and $\angle BDC$.

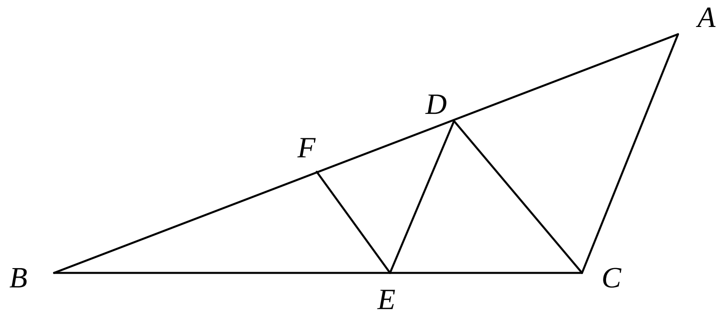


3. For the given diagram, $AB//CD$, $\angle BAE = 30°$, $\angle DCE = 60°$, and EF, EG trisects $\angle AEC$ equally. Are EF or EG parallel to AB? And why?



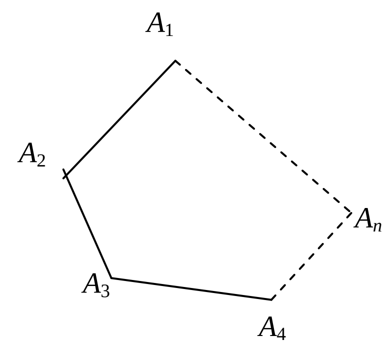
7. Prove that the sum of internal angle of a pentagon is 540 °.

8. For the given diagram, CD bisects $\angle ACB$, $DE // AC$ and $CD // EF$. Prove that EF bisects $\angle DEB$.

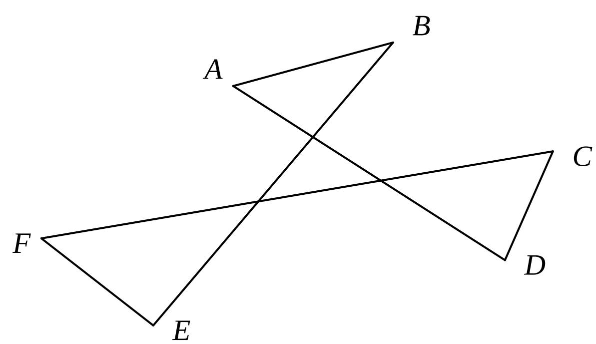


Triangles

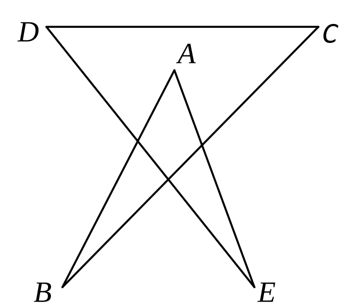
Ex.1 Show that the sum of interior angles of a *n*-gon is given as $180(n-2)°$.



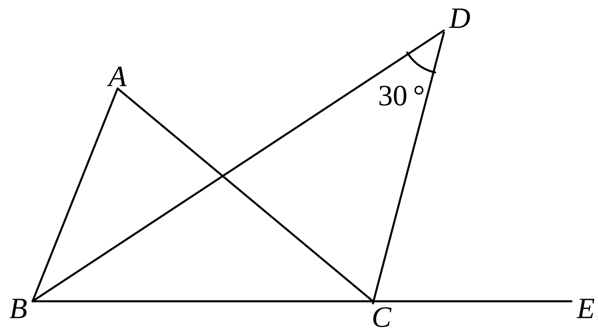
Ex. 2 Find the sum of angles, $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F$.



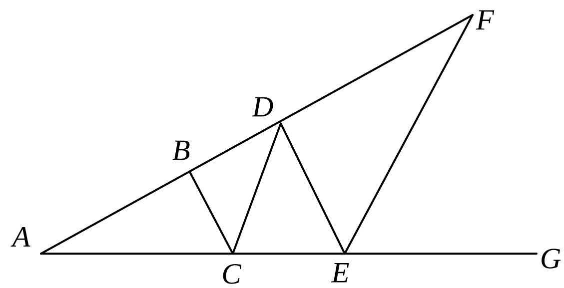
Ex. 3 Find the sum of angles, $\angle A+\angle B+\angle C+\angle D+\angle E$.



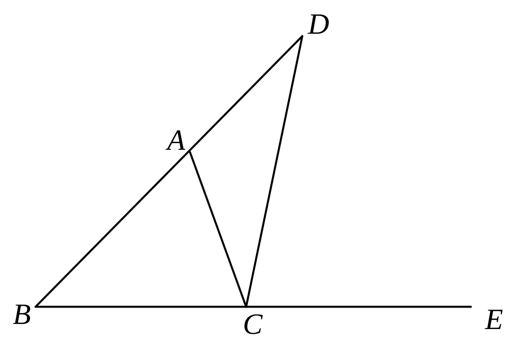
Ex. 4 In this ΔABC, the angle bisector of $\angle B$ and angle bisector of $\angle ACE$ intersects at D. If $\angle D = 30°$, find $\angle A$.



Ex. 5 For the diagram below, $\angle A = 10°$, $\angle ABC = 90°$, $\angle ACB = \angle DCE$, $\angle ADC = \angle EDF$, $\angle CED = \angle FEG$, find $\angle F$.

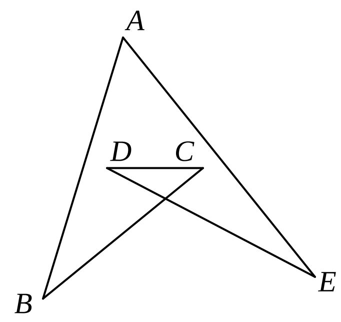


Ex. 6 For the diagram ΔABC, BA extend to intersect the angle bisector of $\angle ACE$ at D. Show that $\angle BAC > \angle B$.

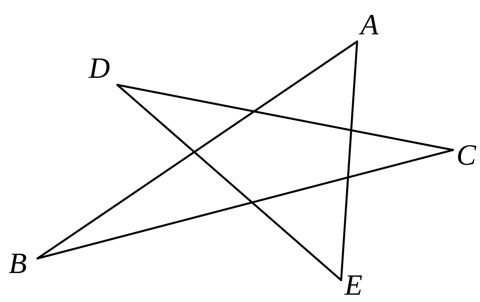


Practices

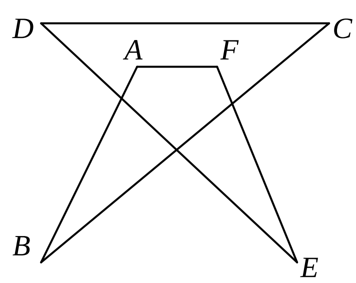
1. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E$.



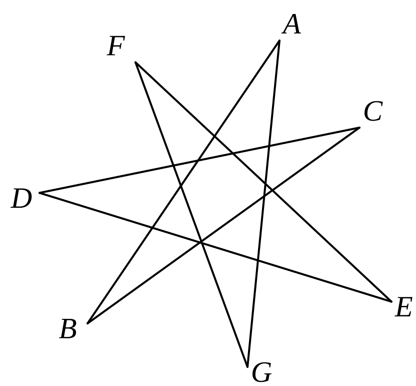
2. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E$.



3. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.



4. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$.



5. If the sum of all interior angles of a polygon is as following respectively, find the number of sides of each polygon.
   (a) $1260°$ (b) $2160°$

6. Show that the sum of all exterior angles of a polygon is $360°$.

7. **China 1998**

   In a right angle triangle *ABC*, $\angle ACB = 90^{\circ}$, *E*, *F* are on *AB* such that $AE = AC$, $BF = BC$, find $\angle ECF$ in degrees.

8. **AHSME 1996**

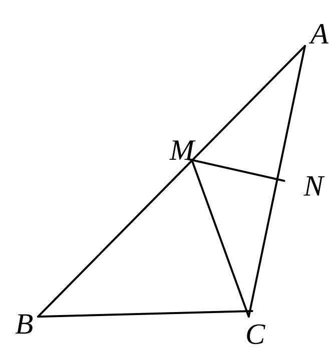
   Triangles *ABC* and *ABD* are isosceles with $AB = AC = BD$, and *AC* intersects *BD* at *E*. If *AC* is perpendicular to *BD*, find the value of $\angle C + \angle D$.

9. **SMO Question 1995**

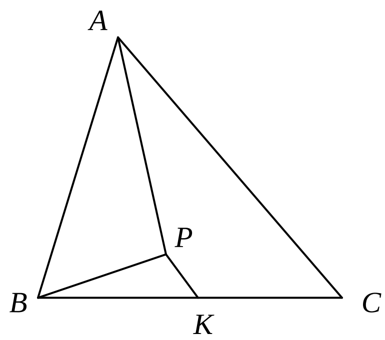
   In a right angled triangle, the lengths of the adjacent sides are 550 and 1320. What is the length of the hypotenuse?

10. **SMO Question 1996**

    In the figure below, *ABC*, *M* and *N* are points on *AB* and *AC* respectively such that $AM : MB = 1: 3$ and $AN : NC = 3: 5$. What is the ratio of $[MNC] : [ABC]$?



11. **SMO Question 1998** In the figure below, AP is the bisector of $\angle BAC$ and BP is perpendicular to AP. Also, K is the midpoint of BC. Suppose that $AB = 8$ cm and $AC = 26$ cm. Find the length of PK in cm.



12. **SMO Question 2000**

Determine the number of acute angled triangles having consecutive integer sides and perimeter not more than 2000.

13. **SMO Question 2001**

Two of the three altitudes of a right-angled triangle are of lengths 12 and 15. Find the largest possible length of the third altitude of the triangle.

Areas

Area of triangle, $S_{\Delta} = \frac{1}{2}HB$ where *H* = height of a triangle, *B* = the base that is $\perp$ to its height.
Area of a parallelogram, $S_{\lozenge} = HB$
Area of a trapezium, $S = \frac{1}{2}(A+B)H$ where *A* and *B* are the lengths of the parallel sides

Some properties of area of triangles:

(1) common base and common height of two triangles have the same area.

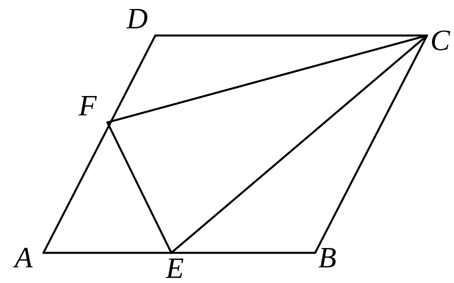
(2) the ratio of the areas of two triangle is the ratio of the products of the base and height of each triangle.

(3) the ratio of the areas of two common base triangles is the same as the ratio of its heights.

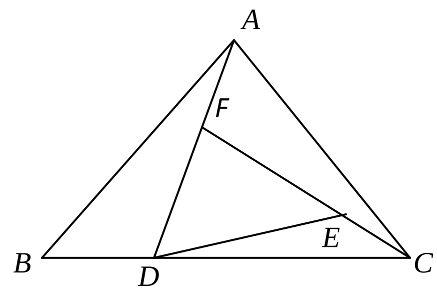
(4) the ratio of the areas of two common height triangles is the same as the ratio of its bases.

Ex.1 Given a ΔABC, the three sides of a, b, c corresponds to heights 4, 5, and 3. Find the ratio of sides $a : b : c$.

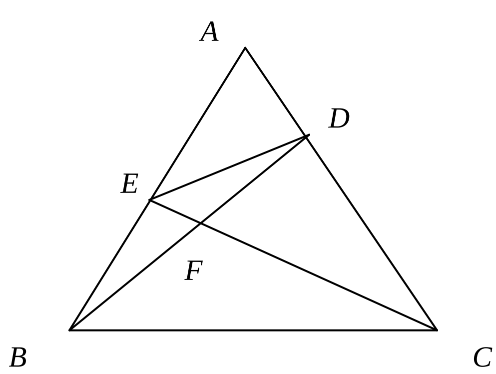
Ex. 2 Given the parallelogram $ABCD$ of area 64 square cm, *E* and *F* are midpoints of *AB* and *AD* respectively. Find the area of ΔCEF.



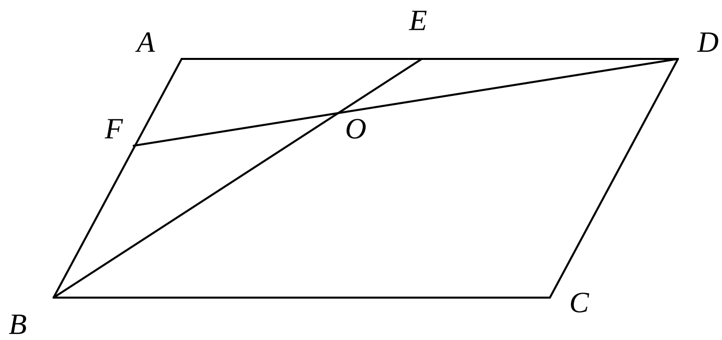
Ex. 3 In this ΔABC of area 1 unit, $BD = \frac{1}{2} DC$, $AF = \frac{1}{2} FD$, $CE = \frac{1}{2} EF$, find the area of ΔDEF.



Ex. 4 In this ΔABC, *E* is a mid point on *AB*, *D* is a point on *AC* such that *AC*: *DC* = 2 : 3, and *BD* and *CE* intersect at *F*. If the area of triangle *ABC* is 40 units, find the area of *AEFD*.

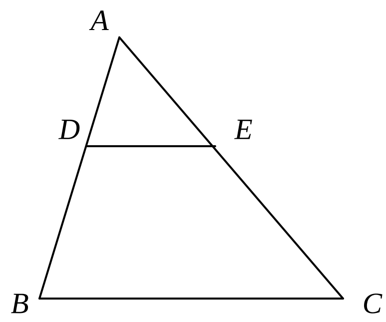


Ex. 5 For the diagram below, *ABCD* is parallelogram. *E* and *F* are a point on *AD* and *AB* respectively. If $BE = DF$ and *BE* and *DF* intersects at *O*, prove that *C* is equip-distant from *BE* and *DF*.

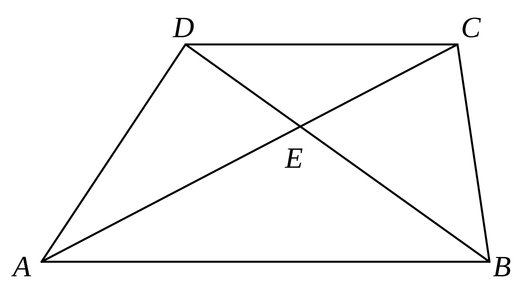


Practices

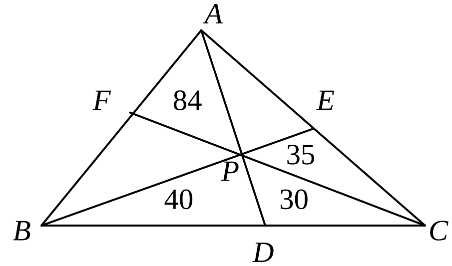
1. For the given diagram, in this ΔABC, $EF//BC$, $AE:EB=m:1$, show that $AF:FC=m:1$.



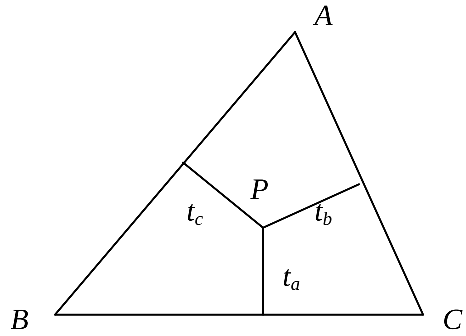
2. For the given diagram, ABCD is a trapezium. $AB//CD$. If the area of ΔDCE is a quarter of the area of ΔDCB, find the fraction of the area of ΔDCE to that of the area of ΔABD.



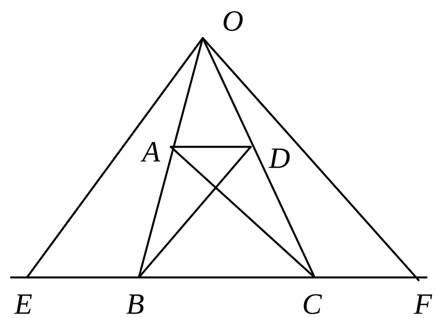
3. For the given diagram, in this ΔABC, *P* is a point within this triangle. *AP*, *BP* and *CP* intersect on the opposite sides at *D*, *E* and *F* respectively. Given the area of 4 of the six smaller triangles, find the area of ΔABC.



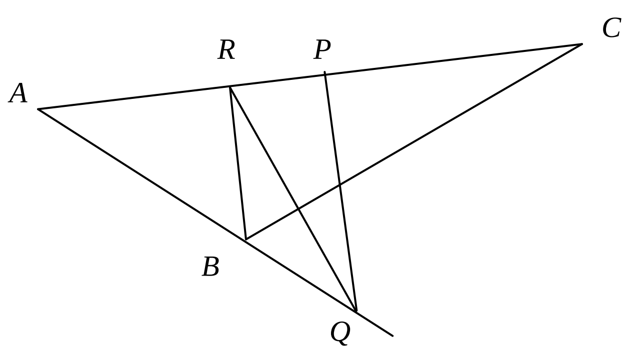
4. For the given diagram, P is a point within this triangle. Given that the altitudes from the vertices A, B and C are h_a, h_b and h_c respectively, and the perpendicular distances from P to each sides as t_a, t_b and t_c, show that $\frac{t_a}{h_a}+\frac{t_b}{h_b}+\frac{t_c}{h_c}=1$.



5. For the given diagram, $ABCD$ is a trapezium. BA and CD produce a intersection at O. $OE//DB$, $OF//AC$. E, B, C and F are on a straight line. Show that $BE=CF$.

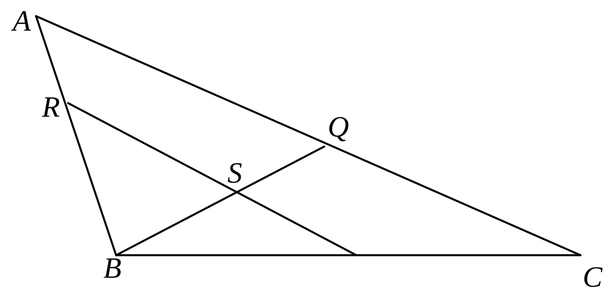


6. For the given diagram, P is a mid-point on AC in the ΔABC. $PQ\perp AC$ and intersects at the AB produce at Q. $BR\perp AC$. Show that $S_{\Delta ARQ}=\frac{1}{2}S_{\Delta ABC}$.



7. **SMO Question 1998**

In the figure below, triangle ABC is a right-angles triangle with $\angle B = 90^o$. Suppose that $\dfrac{BP}{CP} = \dfrac{AQ}{CQ} = 2$ and AC is parallel to RP. If the area of triangle BSP is 4 square nits, find the area of triangle ABC in square units.



8. **SMO Question 1999**

In triangle ABC, D, E, F are points on the sides BC, AC and AB respectively such that $BC = 4CD$, $AC = 5AE$ and $AB = 6BF$. Suppose the area of triangle ABC is 120 cm^2. What is the area of triangle DEF?

9. **SMO Question 2004**

In a triangle ABC, $AB = 2BC$ and P lies within the triangle ABC such that $\angle APB = \angle BPC = \angle CPA$. Given that $\angle ABC = 60^o$ and that $[ABC] = 70$ cm^2, find $[APC]$.

Solving Diophantine Equations

What is a Diophantine equation?

A Diophantine equation is concerned with only integer solutions. In some cases, it concerns only with positive integer solutions. What make solving Diophantine equation interesting and challenging is that there are more unknowns than available equations. For example, you may have 2 equations but three unknowns. The usual method of solving system of equations will then need further development. You will also need to have a strong understanding in Number Properties. So, enjoy the lesson.

Ex.1 Jane spent five dollars on a number of erasers and pencils. An eraser cost 30 cents, a pencil cost one dollar. How many erasers and pencils did Jane buy?

Ex. 2 Find the integer solutions of $11x + 15y = 7$.

Ex. 3 Find the non-negative integer solutions of $6x+22y=90$.

Ex. 4 Find the positive integer solutions of $7x+19y=213$.

.

Ex. 5 Find the integer solutions of $37x+107y=25$.

Ex. 6 Find the number of ways of using numerous 5 or 7 only to sum up to exactly 142.

Ex. 7 Find the integer solutions to $9x + 24y - 5z = 1000$.

Practices

1. Find the integer solutions to:

   (a) $72x+157y=1$, (b) $9x+21y=144$, (c) $103x-91y=5$

2. Find the positive integer solutions to:

   (a) $3x-5y=19$, (b) $12x+5y=125$,

3. Find the integer solutions to:

   (a) $5x+8y+19z=50$, (b) $39x-24y+9z=78$.

4. Find the integer solutions to $2x+5y+7z+3t=10$.

5. Find the positive integer solutions to $5x+7y+2z=24$ and $3x-y-4z=4$.

6. **China**

   m and *n* are integers satisfying $3m + 2 = 5n + 3$ and $30 < 3m + 2 < 40$. find the vale of *mn*.

7. **AHSME 1992**

   If *k* is a positive integer such that the equation in *x* $kx-12=3k$ has an integer root, then what is *k*?

8. **RSMO 1983**

   Given that a pile of 100 small weights have a total weight of 500g, and the weight of a small weight is 1g, 10g or 50g. Find the number of each kind of weights in the pile.

9. **China 2001**

   How man pairs of (*x*, *z*) of two integers that satisfy the eqation $x^2-z^2=12$?

Topical revision (1) on Properties of Rational Number

1. Given that $|a| = 5$, $|b| = 3$, and $|a-b| = b-a$, find the value of $a+b$.

2. What is the unit digit of $2^{1999}+3^{2000}$?

3. The numbers 49, 29, 9, 40, 22, 15, 53, 33, 13, 47 are grouped in pairs so that the sum of each pair is the same. Which number is paired with 15?

4. $n! = n\times(n-1)\times(n-2)\times\cdots\times3\times2\times1$, is called n factorial. Find the smallest positive integer n such that $n!$ is divisible by 990.

5. Let $A=\overline{6a3}$ and $B=\overline{2b5}$ be two 3 digit numbers. If $A+B$ is divisible by 9, find the value of $a+b$.

6. Find the sum of digits of $22222+33333^2$.

7. Given that $x=-\dfrac{\pi}{3}$, find the value of $|x+1|-|x+2|+|x+3|-|x+4|$.

8. Find the value of x such that $|x+2|+|x-3|$ is minimum.

SMO

2004

9. What is the last two digits of 9^{2004}?

10. The number *A* is formed by putting consecutive integers from 2004 to 2040 together. What is the remainder when *A* is divided by 9?

11. It is known that $n = 100^{100} - 1$ has 100 digit 9. How many digits 9 are there in n^3?

12. Find the number of digits in *N* where *N* is the product of all the positive divisors of 100, 000, 000.

2009

13. *m* and *n* are two positive integers of reverse order (for example 123 and 321) such that mn = 1446921630. Find the vale of $m + n$.

Answers:

1. -2 or -8
2. 9
3. 47
4. 11
5. 2 or 11
6. 10
7. $\frac{2}{3}\pi - 4$
8. $-2 \le x \le 3$
9. 61
10. 6
11. 199
12. 325
13. 79497

Topical revision (2) on Integral Algebraic Expressions

1. Evaluate $2002^2 - 2001^2 + 2000^2 - 1999^2 + \cdots + 2^2 - 1$.

2. If $2|3a-2b| + (4b-12)^2 = 0$, find the value of $\frac{1}{4}a^{2b-1} - \left(a^3 + \frac{1}{2}a^b + 4\right)$.

3. If $a < b < c$, $ac < 0$, and $|c| < |b| < |a|$, find the least value of $|x-a| + |x-b| + |x+c|$.

4. Find the sum of digits of the value of $777777777777777^2 - 222222222222223^2$.

5. If a, b, c are rational numbers, $a+b+c=0$, and $abc > 0$, find the value of $\frac{b+c}{|a|} + \frac{c+a}{|b|} + \frac{a+b}{|c|}$.

6. If $abc < 0$, $a+b+c > 0$, then when $x = \frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|}$, find the value of $x^{2002} - x + 2002$.

7. Given that $x + y = 10$, $x^3 + y^3 = 100$, find the value of $x^2 + y^2$.

8. If $(2000-a)(1998-a) = 1999$, then find the value of $(2000-a)^2 + (1998-a)^2$.

SMO Questions

2004

9. If $1000^{2004} - 2004$ is expressed as an integer, what is the sum of its digit?

10. Find the exact value of

$$\frac{1}{10^{-2004}+1} + \frac{1}{10^{-2003}+1} + \frac{1}{10^{-2002}+1} + \ldots + \frac{1}{10^{2002}+1} + \frac{1}{10^{2003}+1} + \frac{1}{10^{2004}+1}.$$

11. If $m^2 + m - 3 = 0$, what is the smallest possible value of $m^3 + 4m^2 + 2004 = 0$?

12. Let x and z be two integers such that $x > z$. Suppose that $x + z = 20$ and $x^2 + z^2 = 328$. Find $x^2 - z^2$.

13. Find the sum of $\frac{2004}{1\times 2} + \frac{2004}{2\times 3} + \ldots + \frac{2004}{2003\times 2004}$.

14. Given that a, b, c and d are real numbers such that $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{a}$. Find the largest possible vale of $\frac{ab-3bc+ca}{a^2-b^2+c^2}$.

Answers:

1. 2005003
2. -8
3. $-c-a$
4. 74
5. 1
6. 10
7. 40
8. 4002
9. 18031
10. 2004.5
11. 2013
12. 320
13. 2003
14. 3

Topical revision (3) on Solving Linear Equations

1. Find the number of different solutions of the equation $\left|x-|2x+1|\right|=3$.

2. Solve $\frac{1}{9}\left\{\frac{1}{7}\left[\frac{1}{5}\left(\frac{x+2}{3}+4\right)+6\right]+8\right\}=1$.

3. If $496=2^m-2^n$, where m, n are integers, find the value of $m+n$.

4. If there are infinite roots of x for the equation $m(x-1)=2001-n(x-2)$, then find the value of $m^{2001}+n^{2001}$.

5. Solve the equation $|2x-1|-|x-2|=9$.

6. For all value of k, $x=-1$ is always the solution of the equation $\frac{kx+a}{2}-\frac{2x-bk}{3}=1$, find the value of a and of b.

7. Given that the two equations $3\left[x-2\left(x-\frac{a}{3}\right)\right]=4x$ and $\frac{3x+a}{12}-\frac{1-5x}{8}=1$ has the same solution, find the value of x.

8. If $\frac{1}{x+\frac{1}{5}}=\frac{5}{3}$, then find the value of x.

SMO 2009

9. Find the vale of the smallest positive integer m such that the equation $x^2+2(m+5)x+(100m+9)=0$ has only integer solutions.

Answers:

1. 2
2. 1
3. 13
4. 0
5. -10 or 8
6. $a = \frac{2}{3}$, $b = \frac{3}{2}$
7. $\frac{27}{28}$
8. $\frac{2}{5}$
9. 90

Topical revision (4) on Solving dual variables Linear Equations

1. If a three digit number $\overline{kmn}$ satisfy $64k+8m+n=403$, find this number.

2. If $2^x+3^y=41$, where x and y are positive integers, find the value of $x+y$.

3. If p and q are positive integers, how many pairs of (p,q) satisfy $2p+3q=25$?

4. The product of two positive integers p and q is 100. What is the largest value of $p+q$?

5. How many two-digit numbers are exactly seven times the sum of their digits?

6. Find the number of all ordered pairs of (a,b) where a and b are integers that satisfy $|ab|+|a+b|=1$.

7. If $x+y=6$, $2x+3y=2k+1$ and $3x-2y=4k+3$, find the value of k.

8. How many integer solution are there for the equation $|x-2y-3|+|x+y+1|=1$.

Answers:

1. 623
2. 7
3. 4
4. 101
5. 4
6. 6
7. 5
8. 2

Topical revision (5) on Solving Linear Inequalities

1. How many x are there that satisfy $\frac{x-1}{3} < \frac{5}{7} < \frac{x+4}{5}$, where x is an integer?

2. If $|x| \le 3$, $y \le 1$, $|z| \le 4$ and $|x-2y+z| = 9$, find the value of $x^2y^4z^6$.

3. Find the sum of all the integers n for which $2000 < 5^{n-1} < 20000$.

4. Solve the inequality $\frac{8x+1}{5x+3} > 1$.

5. If all integer solutions of system of inequalities $\begin{cases} 9x-a \ge 0 \\ 8x-b < 0 \end{cases}$ are 1, 2 and 3, where a and b are integers. How many ordered pairs of (a,b) are there?

6. Solve the inequality $1 \le |3x-5| \le 2$.

7. Given that $\begin{cases} 3x+2y = 4a+3 \\ 2x+3y = a+7 \\ x+y > 0 \end{cases}$, find the possible value range of a.

8. Solve the inequality $|x-5|-|x+3| \le 2$.

SMO
2004

9. Let a and b be positive integers such that $\frac{2}{3} < \frac{a}{b} < \frac{5}{6}$. Find the vale of $a + b$ when b has the minimum value.

2009

10. Let k be a real number. Find the maximum vale of k such that the following inequality holds: $\sqrt{x-2} + \sqrt{7-x} \ge k$.

11. Given that x and z are both negative integers satisfying the equation $z = \frac{10x}{10-x}$, find the maximum of z.

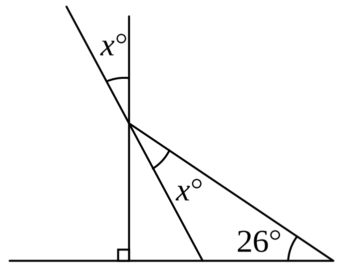
12. Given that $z = (x - 16)(x - 14)(x + 14)(x + 16)$, find the minimum value of z.

Answers:

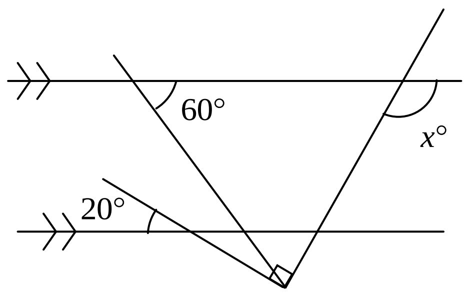
1. 4
2. 36864
3. 13
4. $x < -\frac{3}{5}$ or $x > \frac{3}{5}$
5. 72
6. $1 \le x \le \frac{4}{3}$ or $2 \le x \le \frac{7}{3}$
7. $a > -2$
8. $x \ge 0$
9. 16
10. $\sqrt{10}$
11. -5
12. -900

Topical revision (6) on parallel lines

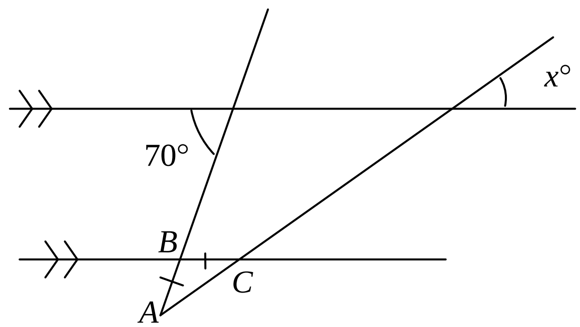
1. Find the value of x.



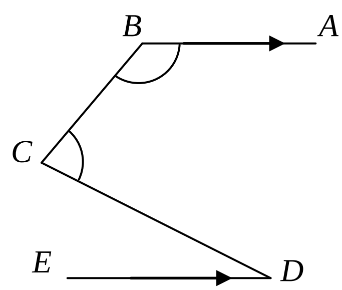
2. Find the value of x.



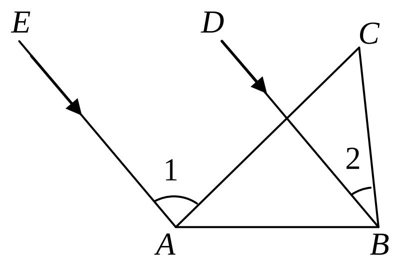
3. IF $AB = CB$, find the value of x.



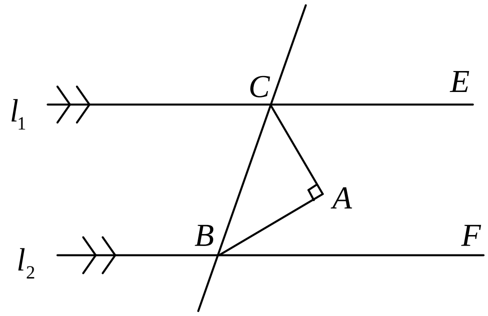
4. As illustrated in the diagram below, $\angle ABC = 120°$, $\angle BCD = 85°$ and $AB // DE$. Find $\angle CDE$.



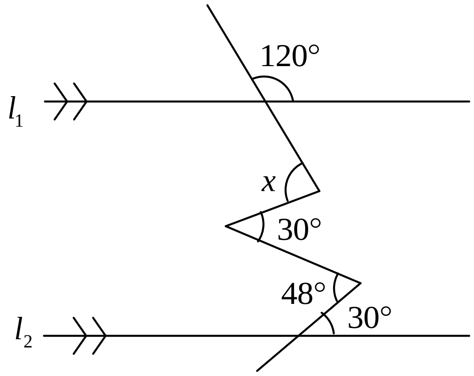
5. As illustrated in the diagram below, $\angle 1 = 3 \times \angle 2$, $\angle 2 = 28°$ and $AE \parallel BD$. Find $\angle C$.



6. As shown in the diagram below, $l_1 \parallel l_2$, ΔABC is a right triangle, $\angle A = 90°$ and $\angle ABF = 25°$. Find $\angle ACE$.



7. As shown in the diagram below, $l_1 \parallel l_2$, Find $\angle x$.

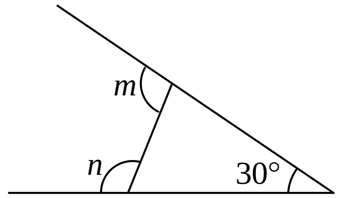


<u>Answers:</u>

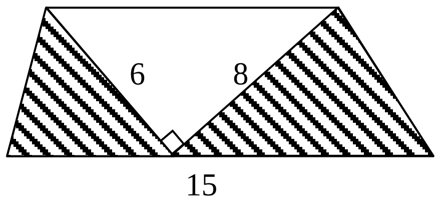
1. 32°
2. 110°
3. 35°
4. 25°
5. 28°
6. 65°
7. 72°

Topical revision (7) on triangles

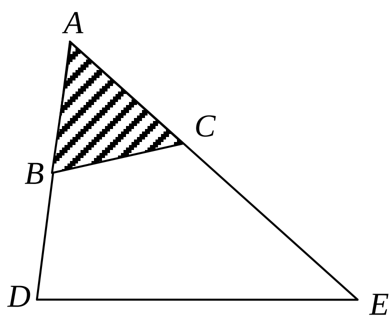
1. In the diagram below, $\angle m = \frac{2}{3}\angle n$ Find the value of m.



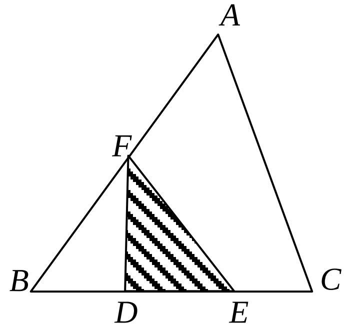
2. A trapezium has been divided into three triangles, find the total areas of the two shaded triangles.



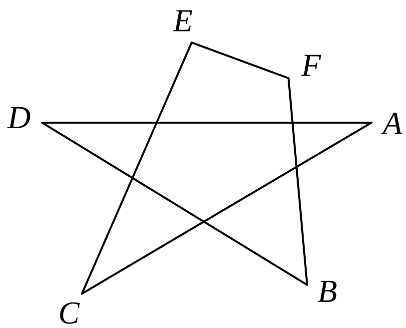
3. IF $AD = 2AB$, $AE = 3AC$ find the ratio of the area of ΔABC to the area of ΔADE.



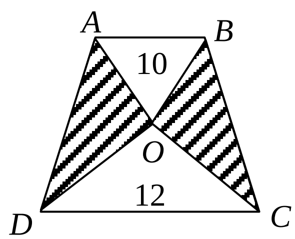
4. In the ΔABC, if $BD = DE = EC$, $BF = FA$ and the area of $\Delta FDE = 1$ square unit, find the area of ΔABC.



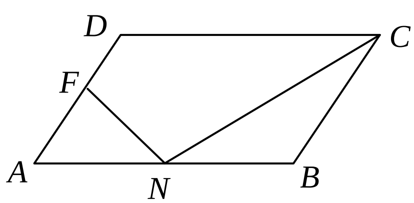
5. Find $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.



6. As shown in the diagram below, $ABCD$ is a trapezium. If the areas of ΔABO and ΔDOC are 10 and 12 square units respectively, find are of the shaded triangles.

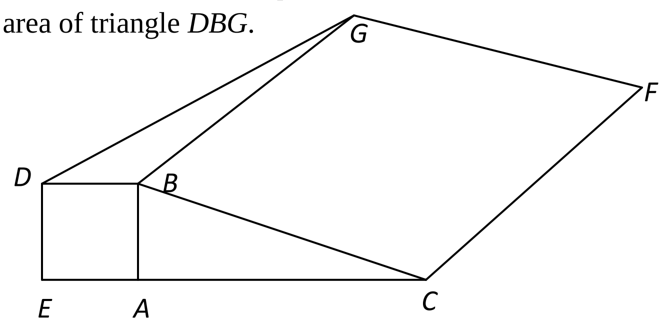


7. As shown in the diagram below, $ABCD$ is a parallelogram. F is the midpoint of AD and N is the midpoint of AB. Find the ratio of the area of ΔAFN to the area of quadrilateral $FNCD$.

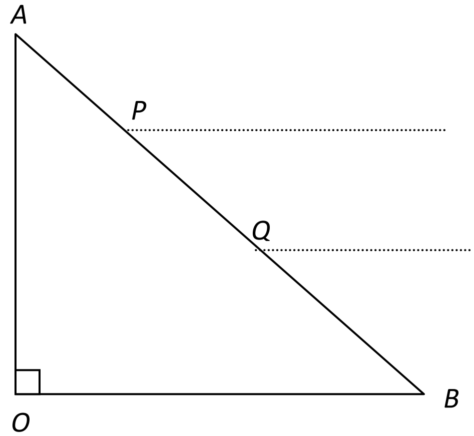


SMO
2009

8. *ABC* is a right angled triangle with $\angle BAC = 90^{o}$. A square is constructed on the side of *AB* and *BC* as shown. The area of the square *ABDE* is 8 cm^2 and the area of the square *BCFG* is 26 cm^2. Find the area of triangle *DBG*.



9. In the diagram, *OAB* is a triangle with $\angle AOB = 90^o$ and $OB = 13$ cm. *P* and *Q* are two points on *AB* such that $26AP = 22PQ = 11QB$. If the vertical height of $PQ = 4$ cm, find the area of the triangle *OPQ*.



10. Three sides *OAB*, *OAC* and *OBC* of a tetrahedron *OABC* are right angled triangles. Given that *OA* is seven, *OB* is two and *OC* is six, find the vale of

$$(\text{Area of } OAB)^2 + (\text{Area of } OAC)^2 + (\text{Area of } OBC)^2 + (\text{Area of } ABC)^2 .$$

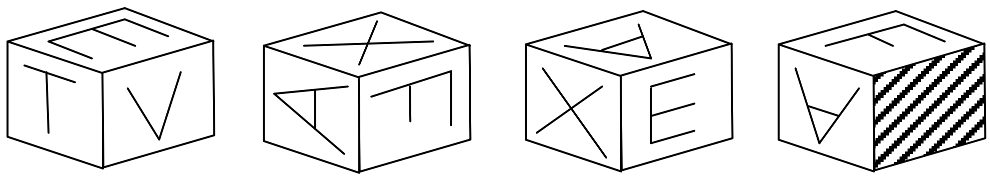
<u>Answers:</u>

1. $86°$
2. 36
3. $1:6$
4. 6
5. $360°$
6. 23
7. $\frac{1}{5}$
8. 6 cm^2
9. 26 cm^2
10. 1052

Topical revision (8) on Problem solving

1. The entire contents of a jug can exactly fill 9 small glasses and 4 large glasses of juice. The entire contents of the jug could instead fill 6 small glasses and 6 large glasses. If the entire contents of the jug is used to fill only large glasses, find the maximum number of large glasses that can be filled fully.

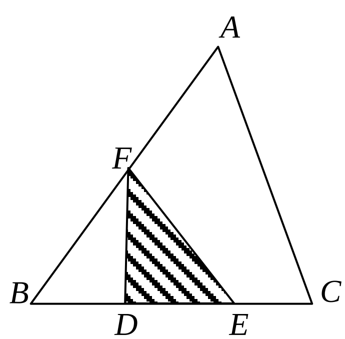
2. A different letter is painted on each face of a cube. This cube is shown below in three different positions. What letter belongs to the shaded face on the last cube?



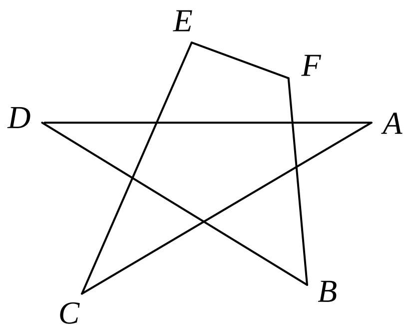
3. In the 4×4 square shown below, each row, column, and diagonal should contain each of the digits 1, 2, 3, and 4. Find the value of $K+N$

1	F	G	H
T	2	J	K
L	M	3	N
P	Q	1	R

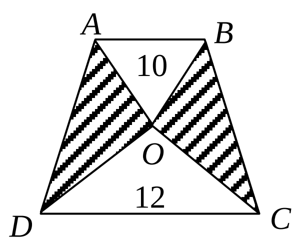
4. In the ΔABC, if $BD = DE = EC$, $BF = FA$ and the area of $\Delta FDE = 1$ square unit, find the area of ΔABC.



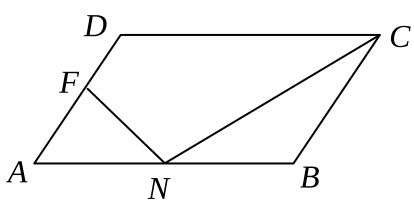
5. Find $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.



6. As shown in the diagram below, $ABCD$ is a trapezium. If the areas of ΔABO and ΔDOC are 10 and 12 square units respectively, find are of the shaded triangles.



7. As shown in the diagram below, $ABCD$ is a parallelogram. F is the midpoint of AD and N is the midpoint of AB. Find the ratio of the area of ΔAFN to the area of quadrilateral $FNCD$.



SMO
2004

8. If I were to start writing positive integers in the order of 1, 2, 3, … with the condition that each digit (0 to 9) can only be used 100 times, which is the first integer that cannot be written?

9. Some unit cubes are assembled to form a larger cube and then some of the faces of the larger cube are painted. After the paint dries, the larger cube is disassembled into the unit cubes and it is fond that 96 of those have no paint on an of their faces. How man faces of the larger cube were painted?

2009

10. 2009 students are taking a test which comprises ten true or false questions. Find the minimum number of answer scripts required to guarantee two scripts with a least nine identical answers.

<u>Answers:</u>

1. $86°$
2. 36
3. $1:6$
4. 6
5. $360°$
6. 23
7. $\frac{1}{5}$
8. 163
9. 4
10. 513

Problem solving Practice #1

1. In figure 1, A and B are two points on the number line which corresponds to the value a and b respectively. Which of the following statement is incorrect?

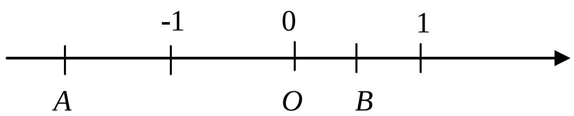


Figure 1

(A) $\frac{a}{b} < 0$, (B) $a - b < 0$, (C) $|a| < b$, (D) $|b| > a$.

2. For the four statements below, how many are truth?

(i) Rational number set consists of negative and positive rational numbers only.

(ii) Rational number set consists of fractions and integers only.

(iii) Positive rational numbers consist of positive fractions and positive integers only.

(iv) Negative rational numbers consist of negative fractions and negative integers only.

(A) 1, (B) 2, (C) 3, (D) 4.

3. For the three statements below, how many are truth?

(i) On a plane, two non-overlapping and non-parallel line must intersect each other.

(ii) On a plane, two non-overlapping and non-parallel rays must intersect each other.

(iii) On a plane, two non-overlapping and non-parallel line segments must intersect each other.

(A) 1, (B) 2, (C) 3, (D) 0.

4. Find the solution of x, given that $|2006x| - a = 0$.

(A) $\frac{1}{2006}a$, (B) $-\frac{1}{2006}a$, (C) $\pm\frac{1}{2006}a$, (D) none of them.

5. In figure 2, $AB // CD$, the line segments in between these parallel lines are distinct in length, find the number of pairs of triangle with the same area.

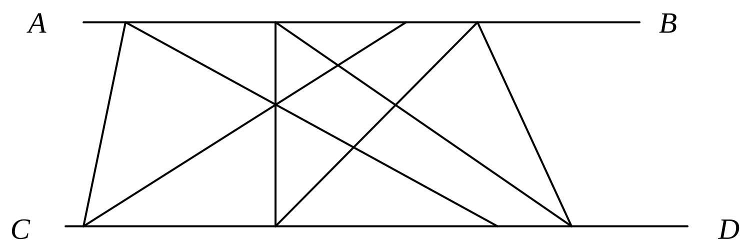


Figure 2

(A) 1, (B) 2, (C) 3, (D) 4.

6. Using six unit length sticks, what is the most number of equilateral triangles of unit length can you arrange?

(A) 1, (B) 2, (C) 3, (D) 4.

7. In figure 3, let $\angle ABC = \alpha$, $\angle DEF = \beta$, $\angle CGH = \gamma$, then the relationship of α, β, γ is

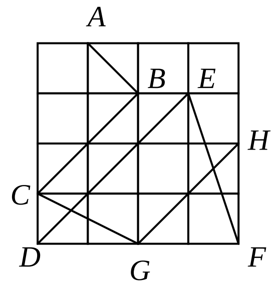


Figure 3

(A) $\beta < \alpha < \gamma$, (B) $\beta < \gamma < \alpha$, (C) $\alpha < \gamma < \beta$, (D) $\alpha < \beta < \gamma$.

8. Given that $a, b \in Q$, with $ab > a$ and $a - b > b$, how many of the three statements is/are true?

(i) $a < 1$ and $b < 1$.

(ii) $ab < 0$.

(iii) $a \neq 0$ and $b \neq 0$.

(A) 1, (B) 2, (C) 3, (D) 0.

9. Let $a \neq 0$, m is a positive odd integer, how many of the following statements is/are correct?

(i) $(-1)^m a$ is the additive inverse of a.

(ii) $(-1)^{m+1} a$ is the additive inverse of a.

(iii) $(-a)^m$ is the additive inverse of a^m.

(iv) $(-a)^{m+1}$ is the additive inverse of a^{m+1}.

(A) 1, (B) 2, (C) 3, (D) 4.

10. Given that $p=|x+1|+|x-3|$, where p is a constant for some range of value of x, find the value of p.

(A) 1, (B) 2, (C) 3, (D) 4.

Problem solving Practice #2

1. In a ΔABC, $\angle A$ is an acute angle. How many of the following four statements is/are correct?

   (i) $\angle A + \angle B$ is a supplementary angle of $\angle C$.

   (ii) $\angle A + \angle B$ is a complementary angle of $\angle C$.

   (iii) $\angle C + \angle B - 90°$ is a supplementary angle of $\angle A$.

   (iv) $\angle C + \angle B - 90°$ is a complementary angle of $\angle A$.

   (B) 1, (B) 2, (C) 3, (D) 0.

2. Given the solution of the equation $mx + 2 = 2(m - x)$ satisfy $\left|x - \frac{1}{2}\right| - 1 = 0$, find the value of m.

   (B) 10 or $\frac{2}{5}$, (B) 10 or $-\frac{2}{5}$, (C) -10 or $\frac{2}{5}$, (D) -10 or $-\frac{2}{5}$.

3. For all $x \in Q$, $|x-1| + |x-2| + |x-9| + |x-10| + |x-11| \geq m$, find the largest value of m.

   (A) 18, (B) 22, (C) 33, (D) none of them.

4. Find the value of $\dfrac{2006}{12342006^2 - 12342005 \times 12342007}$.

5. Find the sum of $1 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \cdots + 99^2 - 100^2$.

6. Given that $|n| < 401$, where $n \in Z$, find the number of n that is a perfect square.

7. Find the number of integers in between $-2\frac{3}{5}$ and $4\frac{3}{7}$.

8. If $a+b=3$, $a^2b+ab^2=-30$, then find the value of a^2-ab+b^2.

9. In figure 1, $\angle AOB=\angle COD=90°$, $\angle BOC=6\angle BOD$, find the measure of $\angle BOD$.

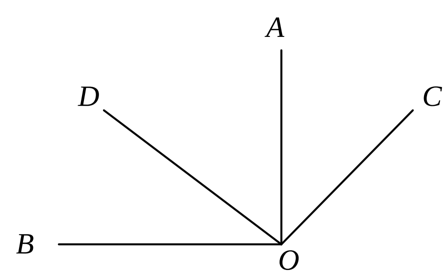


Figure 1

Problem solving Practice #3

1. In a triangle, the ratio of the exterior angles is $3:4:5$. Find the measure of the largest exterior angle.

2. A four-digit number with the ones-digit being three times that of the thousand-digit, the sum of digit is 8. Find the number of such numbers.

3. Let $a,b \in Q$, $a+b=20$, if the least value of a^2+b^2 is m and the largest value of ab is n, then find the value of $m+n$.

4. In figure 1, the areas of square $ABDE$, $CAFG$ and $BCHK$ are 25 cm^2, 16 cm^2 and 9 cm^2 respectively. Find the area of the shaded region.

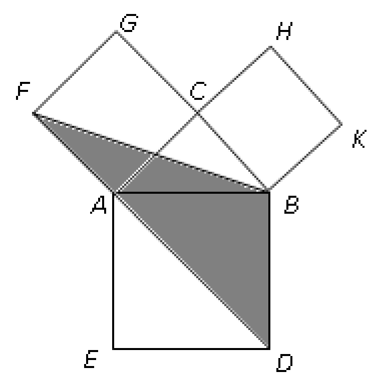


Figure 1

5. Given that $mn \neq 0$, $m-2$ is the multiplicative inverse of $-\frac{1}{4}(n+2)$. Find the value of $\frac{1}{m}-\frac{1}{n}$.

6. Define $F(m)=-1-2-3-\cdots-2m-(2m+1)+2+4+\cdots+2m$, find the value of $F(100)$.

7. Given that,

$$\begin{cases} 3x+5y=1 \\ 4x-5y=-2 \end{cases},$$

find the value of $2x^3y^4+x^2y^5$.

8. What is the 2006[th] digit after the decimal point when $\frac{1}{13}$ is written as decimal number?

9. If $(x-1)^2+(y+1)^2+|z|=0$, find the value of $(3xy^2-2xyz)-(2xy^2+x^2z)+(xyz+x^2z)$.

10. Given that $P=a^2b^2+5$, $Q=2ab-a^2-4a$, if $P>Q$, what is the condition that real number a, b must satisfy?

Problem solving Practice #4

1. If $a \in \Re$, satisfies $(a-2005)^2 + (2006-a)^2 = 2007$, find the value of $(a-2005)(a-2006)$.

2. From 1 to 2006 inclusively, how many positive integers are co-prime with 26?

3. For these two equations to have solution(s), find the value that a cannot take.

$$\begin{cases} ax + 2y = 1 \\ x + 2y = 2 \end{cases}$$

4. A student paid \$14.85 for two kind of item A and B. Each item A costs \$2.16, and each item B costs \$4.23. Together, how many items of A and B had the student brought

5. Given that $n, k, t \in N$, $n = 37k + 5 = 41t + 11$, find the least positive integer of n.

6. In figure 1, $AB // CD$, $\angle ADC = 90°$, $AB = AD$, the length of the square $DEFG$ is 6 cm, find the area of triangle BGE.

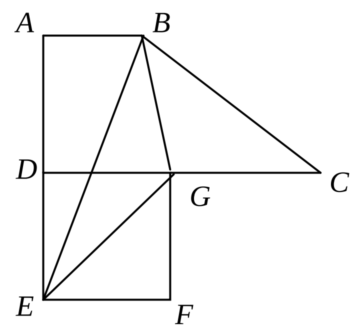


Figure 1

7. Given that $7 \times \overline{ABCXYZ} = 6 \times \overline{XYZABC}$ where $A, X \neq 0$,

if different letters represent different digits, find the number $\overline{ABCXYZ}$.

8. Given that x, y are both two-digit natural numbers, $\overline{xy}$ is a perfect square. If $x - y = 1$, find the number $\overline{xy}$.

9. In figure 2, $ABCD$ is a trapezium. $BC \perp AB$, $CD = 3$, $AB = 9$. E and F are a point on AB and BC respectively. The three equal areas of ΔADE, quadrilateral $EDCF$ and ΔEFB are S_1, S_2 and S_3 respectively. If $CF = 2$, find the area of the trapezium $ABCD$.

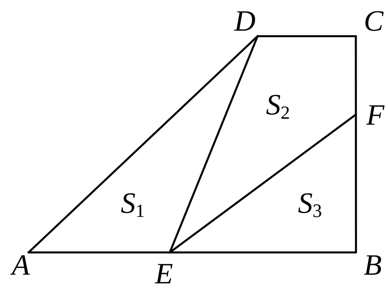


Figure 2

10. Given that $2006 < \overline{xyzw} < 5000$, where x, y, z, w are not necessary distinct digit. If $y \le 7$, $z \le 8$, $w \le 9$, find the number of $\overline{xyzw}$ that have the sum of digits equal to 20.

Problem solving Practice #5

1. Given two 2-digit numbers, HCF=8, LCM=96, find the sum of these two numbers.

   (A) 56, (B) 78, (C) 84, (D) 96.

2. Given that a, b, $c \in N^{+}$, are the three sides of a triangle. The LCM of these numbers is 60. The HCF of a, b is 4, of b, c is 3. Find the value of $a+b+c$.

   (A) 30, (B) 31, (C) 32, (D) 33.

3. In this number sequence 1, 2, 3, ⋯⋯, 100, find the number of numbers in this sequence that is a multiple of 2, but not a multiple of 3.

   (A) 33, (B) 34, (C) 35, (D) 37.

4. Given a 7-digit number 7175624, if the last 4 digits of this number are to be rearranged at random, how many of these distinct numbers are divisible by 3.

   (A) 24, (B) 12, (C) 6, (D) 0.

5. If a positive integer a and 1995 are congruence to mod 6, the possible value of a can be:

   (A) 25, (B) 26, (C) 27, (D) 28.

6. Let n be a positive integer such that $19n+14 \equiv 10n+3 \pmod{83}$, the smallest value of n is:

   (A) 4, (B) 8, (C) 16, (D) 32.

7. A positive integer n, when divided by 3 or 4 or 5 produces remainder 2, 3, 4 respectively. Find the least value of n.

8. Let $[a,b]$ denotes LCM of a and b. If $[x,y]=6$ and $[y,z]=15$ find the number of possible groups of (x,y,z).

9. A 4-digit number is a multiple of 9, the first 3-digit only is a multiple of 4. Find the ones digit of the largest of this number.

10. Given a 11-digit number, from left to right, the first k digits can be divided by k ($k = 1, 2, 3, \cdots, 10, 11$). Find the least of this number.

11. Let n be a positive integer, find the remainder when $3^{2n} + 8$ is divided by 8.

12. Find the ones digit of this sum: $1^4 + 2^4 + 3^4 + 4^4 + \cdots + 1994^4 + 1995^4$.

13. Given two positive integers that the sum of these two numbers is 667, the quotient of the LCM divided by GCD is 120. Find these two numbers.

14. Given that the sum of two positive integers is 40, the sum of the LCM and GCD is 56. Find these two numbers

15. A 5-digit number $\overline{4H97H}$ can be divided exactly by 12. The last two digits, $\overline{7H}$ is a multiple of 6. Find the digit H .

16. If a , b , c , d are distinct integers such that $(x-a)(x-b)(x-c)(x-d) = 9$, prove that $4 | (a+b+c+d)$.

17. A number consists of five factors of 2; three factors of 3; two factors of 5; and one factor of 7. Find the largest two-digit factor of this number.

18. Find the remainder when 2^{400} is divided by 11.

19. Prove that $5 | 3^{1980} + 4^{1981}$.

Problem solving Practice #6

1. For this set of integers, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, let the number of prime number be x, the number of even numbers be y, the number of perfect squares be z and the number of composite number be u, the value of $x+y+z+u$ is

   (A) 17, (B) 15, (C) 13, (D) 11.

2. Let n be a integer greater than 1, which of the following expressions cannot be a perfect square.

   (A) $3n^2-3n+3$, (B) $5n^2-5n-5$, (C) $9n^2-9n+9$, (D) $11n^2-11n-11$.

3. Given three numbers, one is the least odd prime, another is the largest prime less than 50, the last one is the smallest prime larger than 60. The sum of these three number is

   (A) 101, (B) 110, (C) 111, (D) 113.

4. Given that the sum of two primes is 49, the sum of the multiplicative inverses is

   (A) $\frac{94}{49}$, (B) $\frac{49}{94}$, (C) $\frac{86}{45}$, (D) $\frac{45}{86}$.

5. If a, b are positive integers, and $56a+392b$ is a perfect square, then the least value of $a+b$ is

   (A) 6, (B) 7, (C) 8, (D) 9.

6. Let a, b, c be distinct primes and $a+b=c$, if $a<b<c$, then a is

   (A) 2, (B) 3, (C) 5, (D) 7.

7. If m^2+m+7 is a perfect square where $m \in N$, find the product of all possible m,

8. If $s-45$ and $s+35$ are perfect squares, find the possible value of s.

9. If $p^2+d=125$ where p is a prime and d is a positive odd integer, find the value of pd.

10. Let p be a prime and p^2+2 is also a prime, then find the value of p^4+1997.

11. Let n be a positive integer, if $n+3$ and $n+7$ are both prime, then find the remainder when n is divided by 3.

12. Given two positive integers n_1 and n_2 with $n_1^2-n_2^2=79$, find the values of n_1 and of n_2.

13. If a, b, c, d are all primes, with $10<c<d<20$, $c-a>p$ that p is a prime larger than 2, $d^2-c^2=a^3b(a+b)$, find these four prime numbers.

14. Given that $\overline{xyz}=n^2$ and $xyz=n-1$, find $\overline{xyz}$.

15. Let n_1 and n_2 be two primes that are greater than 3, with $n_1^2-1=M$ and $n_2^2-1=N$, find the least GCD of M and N.

16. Show that there are infinitely number of positive integer n, such that n^2+n+41 is a composite number.

17. Given that p and $8p^2+1$ are primes, show that $8p^2-p+2$ is also a prime.

Problem solving Practice #7

1. Find the number of odd digits in the product of two ten-digit numbers: 1111111111 and 9999999999,

   (A) 7, (B) 8, (C) 9, (D) 10.

2. If the sum of positive integers: n, $n+1$, and $n+2$ does not result in any carrying over, the number n is called a jointed number. Eg. 12+13+14 does not produce any carrying over, therefore 12 is a jointed number. But 13+14+15 produces carrying over, so 13 is not a jointed number. Find the number of jointed number that is less than 100.

   (A) 9, (B) 11, (C) 12, (D) 15.

3. In this number pattern: 2, 22, 222, 2222, $\cdots$, find the tens digit of the sum of first 27 numbers.

   (A) 9, (B) 7, (C) 5, (D) 3.

4. Find the ones digit of $1993^{2002}+1995^{2002}$.

   (A) 6, (B) 4, (C) 5, (D) 3.

5. If $3\times\overline{BIDFOR}=4\times\overline{FORBID}$, where each letter is a distinct digit, find the 6 digit number $\overline{BDFIOR}$.

6. Given an eight digits number $\overline{141a28b3}$, if it is divisible by 99, find the value of $a+b$.

7. If $ab\times\overline{ab}=\overline{bbb}$, find the value of a, and of b.

8. Given a three digit number $\overline{abc}$, with $a<b<c$, find the number of such numbers.

9. Given a six digit number $\overline{25xy52}$ where $x, y>7$ that both y and x are digit from 1 to 9, if this number can be divided exactly by 11, find the number $\overline{1xy5}$.

10. Find the right most digit of 43^{43}.

11. Find the ones digit of $2^{m+2000} - 2^{m}$ where $m \in Z^{+}$.

12. Given two positive integers n_1 and n_2 with $\frac{1}{n_1} + \frac{1}{n_2} = \frac{1}{8}$, find the values of n_1 and of n_2.

13. Let x be a 5-digit odd positive integer, each digit 2 is replaced with digit 5 and each digit 5 is replaced with digit 2, the new number is y. If $y = 2(x+1)$, find the number x.

14. The sum of a number and its reflected number is 10879. A reflected number of 13 is 31. If the first two digits of the original number is identical, find this number.

15. Find all the two digit numbers such that when it is multiplied by 2, 3, 4, 5, 6, 7, 8, 9, the sum of digit of the product remains the same.

16. Find the ones digit of this sum: $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 123456789^2$.

17. Find the number $\overline{abcd}$ such that $\overline{abcd} \times 9 = \overline{dcba}$.

Problem solving Practice #8

1. If $|a|^2 = 1$, then the value of $\frac{a}{|a|}$ is

(A) 1, (B) -1, (C) 1 or -1, (D) none of these.

2. The number of integer solutions to the equation $|x-2|+|x-3|=1$ is

(A) More than 3, (B) 1, (C) 2, (D) 3.

3. In these four statements, which are the correct ones?

a. There exists one and only one positive integer that is the same as its additive inverse.

b. There exists one and only one rational number that is the same as its additive inverse.

c. There exists one and only one positive integer that is the same as its multiplicative inverse.

d. There exists one and only one rational number that is the same as its multiplicative inverse.

(A) a & b, (B) b & c, (C) c & d, (D) d & a.

4. Let $y = ax^{15} + bx^{13} + cx^{11} - 5$, where a, b, c are constants. When $x = 7$, $y = 7$. The value of y when $x = -7$ is

(A) -7, (B) -17, (C) 17, (D) uncertain.

5. If a, b, c, d are integers, b is positive, that satisfy $a+b=c$, $b+c=d$, $c+d=a$, then the largest value of $a+b+c+d$ is

(A) -1, (B) 0, (C) 1, (D) -5.

6. Let $a < 0$, $x \le \frac{a}{|a|}$, find the value of $|x+1|-|x-2|$.

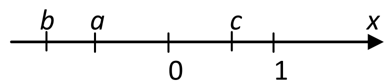
7. Let a and b be two points on a number line with $a \le b$, if the distance of x from a is twice that of the distance of x from b, find the value x.

8. If $|a+6|$ and $(m-3)^2$ are additive inverse, find the value of a^m.

9. Find the sum of : $\frac{1}{1+2}+\frac{1}{1+2+3}+\frac{1}{1+2+3+4}+\cdots+\frac{1}{1+2+3+\cdots+100}$.

10. If $a \in Q$, find the least value of $(-a)+|a|+|-a|+(-a)$.

11. If a, b, c are points on the number line as shown below, simplify $|a+b|-|b-1|-|a-c|-|1-c|$.



12. Simplify $|x+5|+|2x-3|$.

13. Given that $(2a-1)^2+|b+1|=0$, find the value of $\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^{2002}$.

14. If $abc \neq 1$, find all the possible value of $\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}$.

15. Let $x \in Q$, find the least value of $\left|x-\frac{100}{221}\right|+\left|x+\frac{95}{221}\right|$.

16. Given that a, b are additive inverses and c, d are multiplicative inverses, if the absolute value of x is 1, find the value of $a+b+x^2-cdx$.

17. Given that $|ab|+|a+b|=1$, find all the ordered pair of (a,b) where $a,b \in Z$.

18. Find the range of value of x such that $2x+|4-5x|+|1-3x|+6$ is a constant, and also the value of this constant.

19. Given that the equation $|x|=ax+1$ has only one negative root but no positive root, find the range of value of a.

Problem solving Practice #9

1. If $2y^2+3y+7=2$, then the value of $4y^2+6y-9$ is

(A) 1, (B) -19, (C) -9, (D) 9.

2. In this expression ab^2, if each value of a, b is reduced by 25%, then the value of the expression is

(A) reduced by 50%, (B) reduced by 75%,
(C) reduced by $\frac{37}{64}$, (D) reduced by $\frac{27}{64}$.

3. Given a two digit number, two less than three times the sum of digits produces the same number. This two digit number is

(A) 26, (B) 28, (C) 36, (D) 38.

4. Given this expression $|x+1|+|x+2|+|x+3|+|x+4|$, the least value for all values of x is

(A) 1, (B) 2, (C) 3, (D) 4.

5. Let a, b, c be real numbers that satisfy $a+b+c=0$, $abc=1$, then the value of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ is

(A) an integer, (B) 0, (C) negative, (D) non-zero.

6. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x+y+z}=1$, Which of the following statements is correct?

(A) At least one of the three is 1, (B) Each of the three is 1,
(C) none of the three is 1, (D) none of the above.

7. Debra went up a hill at an average speed of v and came down by the same route at speed u. Find the average speed of the whole round trip.

8. Given that $|y-3|+(x-2)^2=0$, find the value of $x^x+y^y-x^y-y^x$.

9. Let a, b, c, d be integers, and $m = a^2 + b^2$, $n = c^2 + d^2$, then express mn as a sum of two squares.

10. Let an binary operation $x \nabla y$ be defined as $x \nabla y = \dfrac{x+y}{x-y}$. Find the value of $(11 \nabla 12) \nabla (19 \nabla 31)$.

11. If $2x^2 - 3x - 1 \equiv a(x-1)^2 + b(x-1) + c$, then find the value of $\dfrac{a+b}{c}$.

12. If $(x-a)(x-4) - 1 \equiv (x+b)(x+c)$ and a, b, c are integers, then find a.

13. If a, b, c are distinct real numbers, simplify $\dfrac{2a-b-c}{(a-b)(a-c)} + \dfrac{2b-c-a}{(b-c)(b-a)} + \dfrac{2c-a-b}{(c-a)(c-b)}$,

14. Given that $x - 2y = 2$, find the value of $\dfrac{3x+y-6}{4x-y-8}$.

15. If $abc = 1$, find the value of $\dfrac{a}{ab+a+1} + \dfrac{b}{bc+b+1} + \dfrac{c}{ca+c+1}$.

16. Given that $a+b+c=0$, find the value of $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right) + b\left(\dfrac{1}{c}+\dfrac{1}{a}\right) + c\left(\dfrac{1}{a}+\dfrac{1}{b}\right) + 3$.

17. Given that $ax+by=7$, $ax^2+by^2=49$, $ax^3+by^3=133$, $ax^4+by^4=406$, find the value of $1999(x+y) + 6xy - \dfrac{17}{2}(a+b)$.

18. Given that $(2x-1)^5 = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$, find the value of $a_1 + a_2 + a_3 + a_4 + a_5$.

Problem solving Practice #10

1. If $m = 10x^3 - 6x^2 + 5x - 4$, $n = 2 + 9x^3 + 4x - 2x^2$ then the value of $19x^3 - 8x^2 + 9x - 2$ is

(A) $m + 2n$, (B) $m - n$, (C) $3m - 2n$, (D) $m + n$.

2. If the expansion of this expression $(a + b - x)^2$ does not contain any x term, , then the value of the a, b should be

(A) $a = b$, (B) $a = 0$ or $b = 0$, (C) $a = -b$, (D) none of these.

3. If $m^2 = m + 1$, $n^2 = n + 1$, $m \neq n$, then the value of $m^5 + n^5$ is

(A) 5, (B) 7, (C) 9, (D) 11.

4. Given that $x^2 - 6x + 1 = 0$, the value of $x^2 + \frac{1}{x^2}$ is

(A) 32, (B) 33, (C) 34, (D) 35.

5. Let $\frac{a^3 + b^3 + c^3 - 3abc}{a + b + c} = 3$, then the value of $(a - b)^2 + (b - c)^2 + (a - b)(b - c)$ is

(A) 1, (B) 2, (C) 3, (D) 4.

6. Let $f(x) = x^2 + mx + n$, where $m, n \in Z$, if $f(x)$ is a factor of $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, the value of m, n are

(A) $m = 2$, $n = 5$ (B) $m = -2$, $n = 5$, (C) $m = 2$, $n = -5$, (D) $m = -2$, $n = -5$.

7. Let a, b, c be non-zero real numbers, find the value of $\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{ab}{|ab|} + \frac{bc}{|bc|} + \frac{ca}{|ca|} + \frac{abc}{|abc|}$.

8. Let $(ax^3 - x + 6)(3x^2 + 5x + b) \equiv 6x^5 + 10x^4 - 7x^3 + 13x^2 + 32x - 12$, find the value of a, b.

9. Find the remainder when $x^4 - x^3 + 3x^2 - 10$ is divided by $x + 2$.

10. If $x+y-2$ is a factor of $x^2+axy+by^2-5x+y+6$, find the value of $a+b$.

11. Evaluate $(2^1+1)(2^2+1)(2^4+1)(2^8+1)(2^{16}+1)(2^{32}+1)(2^{64}+1)+1$.

12. Given that a, b, c satisfy $\frac{c-a}{2(a-b)}=\frac{2(b-c)}{c-a}$, find the value of $a+b-2c$.

13. Let a, b, c be integers, 11 divides $7a+2b-5c$, prove that $11|3a-7b+12c$.

14. Simplify $(4x^4-6x^2+2)(5x^3-2x^2+x-1)$.

15. Simplify $(8x^2-2x+x^4-14)\div(x+1)$.

16. Given that $\frac{a}{a^2+a+1}=6$, find the value of $\frac{a^2}{a^4+a^2+1}$.

17. Given that $x+y+z=3$, $x^2+y^2+z^2=29$, $x^3+y^3+z^3=45$, find the value of $x^4+y^4+z^4$.

18. Find the value of a, b such that $b+ax-3x^2+6x^3+2x^4$ is divisible by $2x^2-4x+1$.

19. Let $P(x)=x^4+ax^3+bx^2+cx+d$ where a, b, c, d are constants, $P(1)=1993$, $P(2)=3986$, $P(3)=5979$, find the value of $\frac{1}{4}[P(11)+P(-7)]$.

20. Let $f(x)$ be a 2nd degree polynomial, it produces a remainder 2, 28 when it is divided by $x-1$ and $x-3$ respectively. If $x+1$ divides $f(x)$, find $f(x)$.

Problem solving Practice #11

1. If a, b are rational numbers, $a^{2001}+b^{2001}=0$, then

(A) $a=b=0$, (B) $a-b=0$, (C) $a+b=0$, (D) $ab=0$.

2. If $a^2+b^2+c^2=9$, then the largest value of $(a-b)^2+(b-c)^2+(c-a)^2$ is

(A) 27, (B) 18, (C) 15, (D) 12.

3. Find the value of $a^2+b^2+c^2-ab-bc-ca$ given that

$$\begin{cases} a = 2001x+2002 \\ b = 2001x+2003 \text{ is} \\ c = 2001x+2004 \end{cases}$$

(A) 0, (B) 1, (C) 2, (D) 3.

4. Given that $\frac{b+c-a}{a+b-c}=\frac{c+a-b}{b+c-a}=\frac{a+b-c}{c+a-b}=q$, the value of q^3-q^2+q is

(A) 1, (B) $1-q$, (C) $1-q^3$, (D) $1-2q^2$.

5. Given that $x+y+z=10$, $x^2+y^2+z^2=38$, $x^3+y^3+z^3=160$, then the value of abc is

(A) 24, (B) 30, (C) 36, (D) 42.

6. Given that $a-b=2$, $b-c=-3$, $c-d=5$, find the value of $(a-c)(b-d)\div(a-d)$.

7. Evaluate $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{9^2}\right)\left(1-\frac{1}{10^2}\right)$.

8. Given that a, b, c, d are non-zero real numbers, $a \neq b$, $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{a}$, find the value of $\frac{a+b+c+d}{b+c+d-a}$.

9. Given that $a = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{1024}$, find the value of $\frac{1}{a-1}$.

10. Prove that $2(a-b)(a-c) + 2(b-c)(b-a) + 2(c-a)(c-b) = (b-c)^2 + (c-a)^2 + (a-b)^2$.

11. Lagrange's identity – Prove that

$$\left(a^2+b^2+c^2\right)\left(m^2+n^2+k^2\right)-(am+bn+ck)^2=(an-bm)^2+(bk-cn)^2+(cm-ak)^2$$

12. Given that $14\left(a^2+b^2+c^2\right)=(a+2b+3c)^2$, prove that $a:b:c=1:2:3$.

13. If $\frac{a}{x^2-yz}=\frac{b}{y^2-zx}=\frac{c}{z^2-xy}$, prove that $ax+by+cz=(x+y+z)(a+b+c)$.

14. Given that a, b, c, d satisfy $a+b=c+d$, $a^3+b^3=c^3+d^3$, prove that $a^{2001}+b^{2001}=c^{2001}+d^{2001}$.

15. Given that $a+b+c=abc$,

prove that $a\left(1-b^2\right)\left(1-c^2\right)+b\left(1-a^2\right)\left(1-c^2\right)+c\left(1-a^2\right)\left(1-b^2\right)=4abc$.

16. Given that $a^3+b^3+c^3=(a+b+c)^3$, prove that $a^{2n+1}+b^{2n+1}+c^{2n+1}=(a+b+c)^{2n+1}$ where $n\in N^+$.

17. Given that a, b, c are positive real numbers, $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=3$, prove that $a=b=c$.

Problem solving Practice #12

1. Solve the equation $\frac{x}{2}+\frac{x}{6}+\frac{x}{12}+\cdots\frac{x}{4006002}=2001$, then x is

(A) 2000, (B) 2001, (C) 2002, (D) 2003.

2. If the solution of x in this equation $\frac{2}{3}x-3k=5(x-k)+1$ is negative, then range of value of x is

(A) $k>\frac{1}{2}$, (B) $k<\frac{1}{2}$, (C) $k=\frac{1}{2}$, (D) none of these.

3. Given that $xyz\neq 0$ and

$\begin{cases} x+3y+5z=0 \\ 2x+3y+z=0 \end{cases}$, then the value of $\frac{x^2+y^2-2z^2}{3x^2+2y^2+z^2}$ is

(A) $\frac{67}{23}$, (B) $\frac{23}{67}$, (C) $-\frac{23}{67}$, (D) none of these.

4. Find the number of ordered pairs that satisfies $\frac{1}{x}+\frac{1}{y}=\frac{1}{1987}$.

(A) 0, (B) 3, (C) 3, (D) none of these.

5. If $\frac{2x-a}{3}>\frac{a}{2}-1$ and $\frac{x}{a}<5$ have the common solution of x, then the range of value of a is

(A) any real, (B) -3, (C) $=-\frac{2}{5}$, (D) $>-\frac{2}{5}$.

6. Let x, y, z be positive real numbers that satisfy this system of inequality

$$\begin{cases} \frac{11}{6}z < x+y < 2z \\ \frac{3}{2}x < y+z < \frac{5}{3}x \\ \frac{5}{2}y < x+z < \frac{11}{4}y \end{cases}$$, Arrange x, y, z in ascending order.

(A) $x < y < z$, (B) $y < z < x$, (C) $z < x < y$, (D) none of these.

7. Solve the equation of x, $\frac{x-b-c}{a}+\frac{x-c-a}{b}+\frac{x-a-b}{c}=3$ where $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\neq 0$.

8. Given that $2a(x+5)=3x+1$ has no solution, find the range of value of a.

9. Given that these two system of equations share the same solution of x, y,

$\begin{cases} ax-2by=2 \\ 2x-y=7 \end{cases}$ $\begin{cases} 3ax-5by=9 \\ 3x-y=11 \end{cases}$, find the value of a and of b.

10. Find the integer solution of $4x+7y=20$.

11. Find the range of solution of x, given that $4x-2+\frac{1}{x-5}>\frac{1}{x-5}+3x+2$.

12. If $\frac{3x-1}{2}-\frac{7}{3}\geq x-\frac{5+2x}{3}$, let the minimum value and maximum value of $|3-x|-|x+2|$ be a, b respectively, find the value of ab.

13. Solve $1-\frac{x-\frac{1+x}{3}}{3}=\frac{x}{2}-\frac{2x-\frac{10-7x}{3}}{2}$.

14. Solve the equation of x, given that $\frac{x-n}{m}-\frac{x-m}{n}=\frac{m}{n}$, $(mn \neq 0)$.

15. Solve $\frac{x+1}{20}=\frac{y+1}{21}=\frac{x+y}{17}$.

16. Solve the system of equations,

$$\begin{cases} 5x-y+3z=a \\ 5y-z+3x=b \\ 5z-x+3y=c \end{cases}.$$

17. Solve $4x+7y=20$ for non-negative integer solutions.

18. Solve for positive integer solutions of

$$\begin{cases} 5x+7y+9z=52 \\ 3x+5y+7z=36 \end{cases}.$$

19. Solve the inequalities,

a. $$\begin{cases} 4x+5 \geq 8x+a \\ 4x-7 \leq 6x-b \end{cases}$$

b. $$|x+5|-|3x-2| \leq 2$$

20. Find the range of value of k for the system of equation

$\begin{cases} kx+4y=8 \\ 3x+2y=6 \end{cases}$ to have positive solution.

Problem solving Practice #13

1. Begin with 2003, then subtracts half this amount from this number, then subtracts $\frac{1}{3}$ of the remainder from this remainder, then subtracts $\frac{1}{4}$ of the new remainder from this new remainder, … , then repeating until subtracting $\frac{1}{2003}$ of the previous remainder from the previous remainder. What is this last remainder?

2. Solve the equations

$$\begin{cases} x:y:z = 1:2:7 \\ 2x - y + 3z = 21 \end{cases}.$$

3. Let n be a positive integer, the difference of the product of the odd number which is one more than n and n and the product of the odd number that is one less than n and n is 140. Find the number n.

4. Given that a, b, c are distinct prime numbers, $ab^bc + a = 2000$, find the sum of $a+b+c$.

5. If $|x| = 5$, $|y| = 3$, and $|x-y| = y-x$, find the value of $(x+y)^{|x+y|}$, giving your answer in the form of a^n where a is a prime and n is a positive integer.

6. Given that $\frac{1}{x} + \frac{1}{y} = 2$, find the value of $\frac{3x-5xy+3y}{-x+3xy-y}$.

7. If $2a = 6b = 3c$, and $ab+bc+ca = 99$, then find the value of $2a^2 + 12b^2 + 9c^2$.

8. Given that $\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{1333} - \frac{1}{1334} + \frac{1}{1335}$ where p and q are co-prime of each other, show that p is a multiple of 2003.