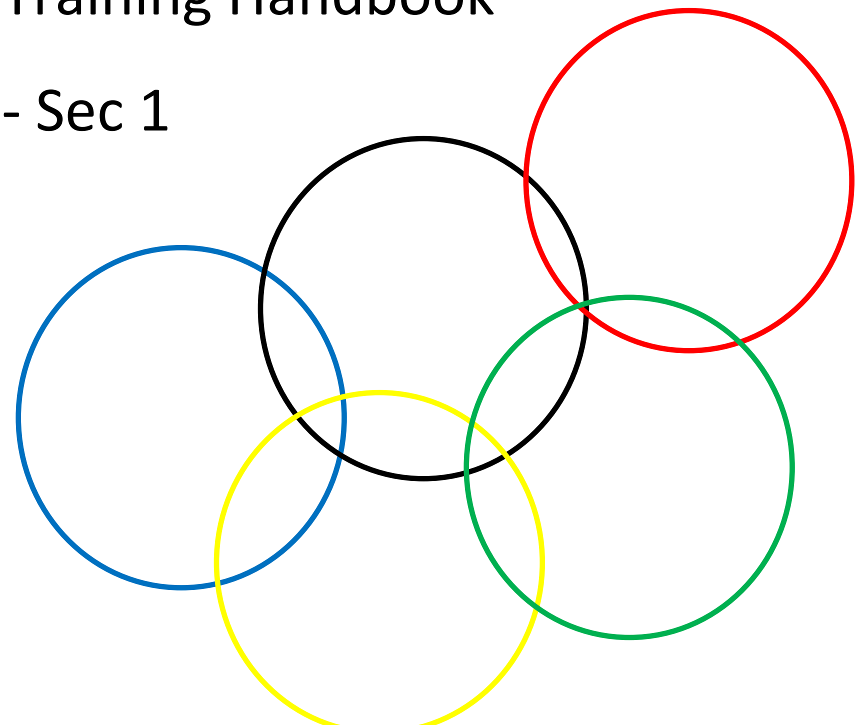
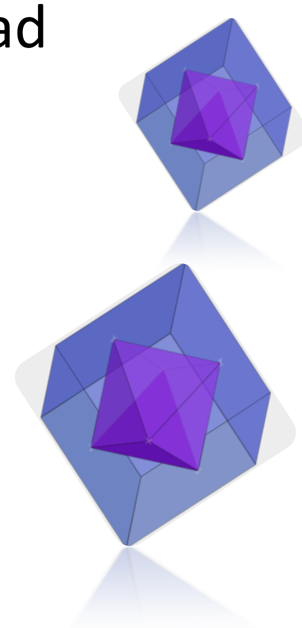
Develop The Maths Genius in You

# Singapore Mathematical Olympiad Training Handbook - Sec 1






Includes Questions from other Olympiads

Dear young students of mathematics

Mathematics is a wonderful subject.  It is one of the most useful ways to develop your mind.

The material in front of you has been developed over the years in training talented pupils in this subject in a top secondary girl school.

If you are a Primary 6 or even Primary 5 pupil who is seeking challenges or a Secondary 1 pupil who is looking for ways to develop your mathematics talent, look no further.  Pick up a pencil and have a go at it.

This handbook contains copyrighted material so it should strictly be for your personal use.

By the end, I hope you enjoy what I had put together here for you.

Cheers

Mr Ang K L , 2012

# Contents

## Worksheet



## Tropical Revision



## Problem Solving

# From Arithmetic to Algebra

## What is Algebra?

Algebra is a study of number properties in the form of alphabetical representation.

Ex 1. The average of three numbers $A$, $B$, and $C$ is $a$. If the average of $A$ and $B$ is $b$, what is the value of $C$?

Ex 2. The cost of a shirt, a hat and a pair of shoe is \$$a$. A shirt is \$$b$ more expensive than a pair of shoe. A shirt is \$$c$ more expensive than a hat. Find the cost of a pair of shoe.

The operations in algebra are the same as those in arithmetic.

Ex 3. There are two piles of printing papers on a table. The first pile is $a$ kg more than the second pile. If $b$ kg are used in each pile, the first pile will be $\beta$ times of second pile. Find the weights in each pile.

Ex 4. To complete a job, A alone took $a$ days, B alone took $b$ days. Find the number of days it would take both of them to complete the job together.

Ex 5. To complete a job, $A$ alone took $a$ days, $B$ alone took $b$ days. Now that $A$ alone had completed $c$ days $(c < a)$, then $B$ alone complete the rest of the job. How many days would $B$ take to complete the rest of the job?

Ex 6. There are $a$ number of chicken and rabbits, with $b$ number of legs. How many rabbits are there?

**Practices**

1. There are two baskets of apples. If *a* apples were moved from first basket to the second basket, then the two baskets would have the same number of apples. If *b* apples were moved from second basket to the first basket, then the first basket would have twice as many as the second basket. Find the number of apples in each basket.

$$\begin{bmatrix} 1st \to 4a+3b \\ 2nd \to 2a+3b \end{bmatrix}$$

2. After a Math test, the top 10 pupils had an average of *a*. The top 8 pupils had an average of *b*. The ninth pupil had *c* more than the tenth pupil. Find the score of the tenth pupil.

$$\left[\frac{10a-8b-c}{2}\right]$$

3. There are some 4 ¢ and 8 ¢ stamps, with a total amount of $*a*. If there are *b* pieces more 8 ¢ stamps than 4 ¢ stamps, find the number of 8 ¢ stamps and the number of 4 ¢ stamps are there.

$$\left[\frac{25a+b}{3}\right]$$

4. A fast car and a slow truck departed from town X and town Y respectively, towards each other. It took *a* hours for the fast car to reach town Y. The slow truck took *b* hours to reach town X. If the fast car travelled *m* km more than the slow truck in an hour, how long would it take for the two to meet on their journey?

$$\left[\frac{ab}{a+b}\right]$$

5. To complete a job, Mr A alone and Mr B alone took *a* and *b* $(< a)$ days respectively. To complete this job, Mr A did a few days, then Mr B took over to complete the rest. It took c days $(b < c < a)$ in total to get it done. Find the number of days each took to complete this job.

$$\left[\frac{a(c-b)}{a-b}\right]$$

# Formulating Equations

To solve problems with algebra will generally require the forming of equation(s). An equation is an expression of two equal quantities that are divided by the sign "=".

In order for this to be possible, we will learn how to translate from words into algebra.

Ex. 1 A horse and a donkey met on their way. The donkey said to the horse: "If you transfer one bag to me, my load would have been twice of your load." The horse replied: "If you transfer one bag to me, our load would have been even." Find the number of bags on the donkey.

Here are the steps to take to form an equation:

| | |
|---|---|
| Let the number of bags on the donkey be | $x$ |
| The donkey said to the horse: "If you transfer one bag to me, my load would have been twice of your load." | $x+1$, donkey<br>$\left(\frac{x+1}{2}\right)$, horse |
| (we can then tell that the horse actually had) | $\left(\frac{x+1}{2}\right)+1$ |
| The horse replied: "If you transfer one bag to me, donkey our load would have been even." | $x-1$,<br>$\left(\frac{x+1}{2}\right)+1+1$ |

(From the last statement, we know that the number of bags on the horse is the same as the number of bag that is on the donkey)

$$x-1=\left(\frac{x+1}{2}\right)+1+1$$ (This is your first equation)

$$x-1=\left(\frac{x+1}{2}\right)+2$$

$$\underline{\quad +1 \qquad\qquad +1 \quad}$$ (This is the concept of balancing)

$$x=\left(\frac{x+1}{2}\right)+3$$

$$x=\frac{x}{2}+\frac{1}{2}+3$$

$$x \quad = \quad \frac{x}{2} \quad +\frac{7}{2}$$

$$\underline{-\frac{x}{2} \qquad -\frac{x}{2}}$$ (Again, apply the balance concept)

$$\frac{x}{2} \quad = \quad \frac{7}{2}$$

$$\frac{x}{2}\times 2 \quad = \quad \frac{7}{2}\times 2$$ (Once again, balancing)

$$x=7$$

Can you formulate this problem differently?  Let's try!

Ex 2. A, B, C, and D together, have 45 books. If A has 2 less, B has 2 more, C has double, and D is halved, then each would have the same number of books. How many books has A?

| | |
|---|---|
| Let "each would have the same number of books" Be | $x$ |
| "A has 2 less", then A actually has | $x+2$ |
| "B has 2 more", then B actually has | $x-2$ |
| "C has double", then C actually has | $\frac{x}{2}$ |
| "D is halved", then D actually has | $2x$ |

"A, B, C, and D together, have 45 books" Use this to form an equation:

$$(x+2)+(x-2)+\frac{x}{2}+2x=45$$

We can then simplify this equation into:

$$4\frac{1}{2}x=45 \Leftrightarrow \frac{9}{2}x=45$$

$$\frac{9}{2}x\times 2=45\times 2$$ (Remember the concept of balancing)

$$x=10$$

Ex. 3 A group of students was to clean up to two areas in their school. Area A was $1\frac{1}{2}$ times of Area B. In the morning (half of a day), the number of students cleaning Area A was 3 times that of the number of students in Area B. In the afternoon (another half of a day), $\frac{7}{12}$ of the students worked in Area A while the rest of them in Area B. At the end of the day, Area A was done, but Area B still needed 4 students to work one more day before it was done. How many were there in this group of students?

Ex. 4 Jug A contained 11 litres of pure honey, and Jug B contained 15 litres of pure water. Some honey from Jug A was poured into Jug B, the mixture was well stirred. Next, some mixture from Jug B was poured into Jug A. At the end, Jug A still contained 62.5% of honey by volume and Jug B contained 25% of honey by volume. If the total volume remained the same, how much had the mixture been poured into Jug A?

**<u>Practices</u>**

1. There are two warehouses; with the first one has three times the number of TV sets than the second one. If 30 sets were transferred from the first to the second, then the second one would have $\frac{4}{9}$ that of the first one. Find the number of TV sets in the second one. $[130]$

2. There were 140 black chocolate bars and white chocolate bars on shelve. After one quarter of the black chocolate bars was sold, the storekeeper added another 50 white chocolate bars on the shelve. Then, the number of white chocolate bars would be twice the number of black ones. Find the number of black chocolate bars at first. $[76]$

3. Mr A and Mr B were to depart from the same place to town X. Mr A walked at a speed of 5 km/h. After he had departed for one and a half hour, Mr B cycled to town X. It took Mr B 50 minutes to arrive at town X together with Mr A. Find the speed of Mr B. $[14]$

4. A and B departed together to town Y, B on foot, and A by bicycle. A‘s speed is 1 km/h more than thrice of B. Upon arriving at town Y, A rested for an hour before returning. On the return trip, A met B when B had already walked for two and a half hours. If town Y was $14\frac{3}{4}$ km away from their departure point, find the speeds of the two and how far had they each travelled before they met again? $[13, 4]$

5. A motorist departed at 9 am from town A to town B. He planned to arrive at 12 noon. An hour later, he realized that he would be late by 20 minutes with his current speed. As such, he increased his speed by 35 km/h and in so doing, arrived at exactly 12 noon. Find the original speed of the motorist and the distance between the two towns. $[210{,}700]$

6. The numbers of pupils in two groups are in the ratio of 4 : 1. If 15 pupils are transferred from the first group to the second group, then, there will have same number of pupils in each group. How many pupils are to be transferred from the first group to the second group so that the ratio becomes 3 : 7?

$[25]$

7. There are two candles, one thick and the other one thin, but are of equal length. The thick one can last 5 hours. The thin one can last 4 hours. If the two candles are lighted together, how long will it take for the thick one to be 4 times that of the thin one? $\left[3\frac{3}{4}\right]$

# Algebra with Arithmetic

In solving many mathematical problems, the arithmetic approach seek to develop a better understand of the problem over the algebraic counterpart. In combining the use of the two approaches, one can usually find solution to a problem much easier.

Ex 1. A car is traveling from town X to town Y. If the speed of the car is increased by 20%, it arrives at town Y one hour earlier than as planned. If it has, at first, travelled for 120 km with the original speed, then increases its speed by 25% for the rest of the journey, it will arrive 40 minutes earlier instead. Find the distance between town X and town Y.

Method 1, Arithmetic approach

Method 2, Algebraic approach,

Ex. 2 A job can be done by Mr A alone in 9 days, Mr B alone in 6 days. Now that Mr A has done 3 days of the job, how many days will it take Mr B to complete the job, without Mr A?

Method 1, Arithmetic approach

Method 2, Algebraic approach,

Ex. 3 For a project, team A can complete the project in 10 days. Team B can complete the project in 30 days. Now that both team are working on the project. But team A has two rest days, and team B has 8 days of rest. Find the number of days it will take them to complete the project.

Ex 4. A project will take 63 days by team A, and then another 28 days by team B to complete. If both teams are to work on this project together, it will take 48 days to complete. If team A is to work 42 days, how many days will it take team B alone to complete the rest of this project?

Ex. 5 There are two water filling pipes, A and B and one drain pipe, C connected to a pool. It takes 3 hours to empty a full pool with all 3 pipes open. It takes just one hour to empty this pool with pipe A and pipe C only. It takes 45 minutes to empty this pool with pipe B and pipe C only. If the filling rate of pipe A is 1 $m^3/\text{min}$ more than pipe B, find the fill rate and drain rate of each pipe.

Ex. 6 A jug contained some liquid(water and alcohol mixture). After a cup of water was added, the concentration of alcohol in the jug became 25%. After another cup of pure alcohol was added into the jug, the concentration of alcohol was 40%. How many cups of liquid were there in the jug at first?

(Concentration of alcohol by volume $= \dfrac{amount\ of\ \ alcohol}{amount\ of\ \ liquid}$ )

Ex. 7 Two teams were working on writing a book. Team A wrote $\frac{1}{3}$ of the book in 4 days. Then team B joined the project. With team A, they finally completed the book in 3 days. If team B wrote 75 pages of the book, find the number of pages in this book.

**Practices**

1. A project will take Mr $A$ and Mr $B$ 12 days to complete. Now that both of them work for 4 days, with the rest to be completed by Mr $A$ in 10 days, find the number of days each take to complete this project by himself. $[15, 60]$

2. In another project, if Mr $A$ works on it for 2 days and Mr $B$ works on it for 5 days, $\frac{4}{15}$ of the project will be completed. But if Mr $A$ works on it for 5 days and Mr $B$ works on it for 2 days, then $\frac{19}{60}$ of the project will have been completed. Find the number of days each take to complete this project by him alone. $[20, 30]$

3. A tank is filled from empty to full by pipe $A$ in 12 minutes. It only takes pipe $B$ 5 minutes to drain it completely. Pipe $C$ takes 6 minutes to fill this tank. If pipe $A$ is open to fill an empty tank for the first few minutes before pipe $B$ and $C$ are open, it will take 18 minutes to fill this tank. How long has pipe $A$ been open before the other two pipes are open? $[3]$

4. The amount of work done by Mr $B$ in a day took Mr $A$ one-third of a day to do. The amount of work done by Mr $C$ in a day took Mr $B$ $\frac{3}{4}$ of a day to do. Now, each day, 2 of them were to work on a project. It took Mr $A$ 4 days, Mr $B$ 3 days and Mr $C$ 3 days to complete this project. Find the number of days Mr $A$ alone took to complete this project. $\left[5\frac{3}{4}\right]$

# System of Equations

When there are 2 or more unknowns, it often requires 2 or more equations to be set up. The ways of solving these equations are the lesson for today.

Ex.1 A fraction, after being simplified, is $\frac{2}{3}$. If a integer is added to both the numerator and its denominator of this fraction, it becomes $\frac{8}{11}$. If one is added to this integer, and the new integer is subtracted from both the numerator and the denominator of this fraction, it becomes $\frac{5}{9}$. Find this fraction.

Method 1, Algebraic approach,

Method 2, Arithmetic approach,

Ex. 2 Mr *A*, Mr *B*, and Mr *C* took part in a bicycle race. Mr *A* finished 12 minutes earlier than Mr *B*. Mr *B* finished 3 minutes before Mr *C*. If Mr *A* was 5 km/h faster than Mr *B*, and Mr *B* was 1 km/h faster than Mr *C*, find the distance of their race.

Method 1, Algebraic approach,

Method 2, Arithmetic approach,

Ex. 3 A red ballpoint pen costs 19 cents, and a blue ballpoint pen costs 11 cents. Now that I pay a total of \$2.80 for 16 pens, how many are blue pens?

Ex 4. Clerk *A* takes 6 hours to type a report and clerk *B* takes 10 hours to type the same report. If clerk *A* starts to type for a few hours then hand over the rest of the typing to clerk *B*, it will take 7 hours in total to complete. Find the number of hours each takes to type this report?

Ex. 5 Two iron ores, the first ores contain 68% of iron and the second ores contains 63% of iron. Now that 100 tones of 65% of iron ores are required, how much of each ores is to be used?

**Practices**

1. A delivery order took a fleet of trucks over a number of days to fill. If 6 fewer trucks were used, then it would extend 3 days to complete the order. If 4 additional trucks were used, then it would have shortened one day to complete. Find the number of days and the number of the truck in this fleet.
$[5,16]$

2. There were 360 door gifts for Founder's day to be assembled over the weekend. Team A produced 112% of its quota; team B produced 110% of its quota. As a result, there were 400 door gifts assembled in total. Find the number of gift in excess of the quota from each team. $[24,16]$

3. 30 English books and 24 Math books cost \$83.40 in total. An English book costs 44 cents more than a Math book. Find the cost of an English book. $[1.74]$

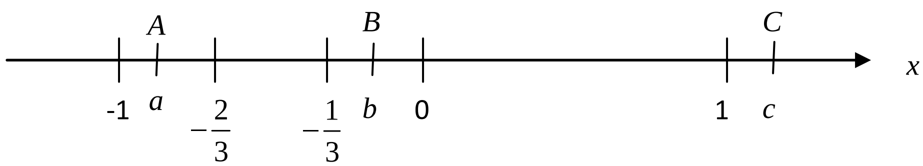
4. A chemical of 80% concentration is to be mixed with the same chemical of 90% concentration to produce a 84% concentration chemical. For a 500 litres of this mixed chemical, find the amount used by each chemical. $[300,200]$

5. Alcohol of 72% concentration is mixed with alcohol of 58% concentration to make a 62% mixture. If 15 litres more of each alcohol are added to this mixture, the concentration becomes 63.25%. Find the amount of each alcohol used in the first mixture. $[12,30]$

# Rational Number and some of its operations

## Numbers and number line

Ex.1 Observe the diagram,




$a$, $b$, and $c$ are values correspond to point $A$, $B$, and $C$ respectively. Arrange $\frac{1}{ab}$, $\frac{1}{b-a}$ and $\frac{1}{c}$ in ascending order.

Ex. 2 By adding operators '+' or '-' between the numbers 1 2 3 4 $\cdots$ 1990, What is the least non-negative value.

Ex. 3 Find the sum of $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots\frac{1}{9800}$.

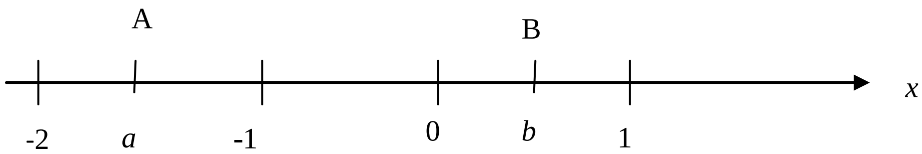
Ex. 4 Evaluate $1+2+2^2+2^3+\cdots+2^{2000}$

Ex. 5 Compare the value of $\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\cdots+\frac{2000}{2^{2000}}$ and 2.

.

**Practices**

1. Refer to the diagram below, two points A and B are on a number line correspond to value *a* and *b*. If $x=\dfrac{a-5b}{a+5b}$, which value, $x$ or $-1$ is larger?




2. Calculate $1+2-3+4+5-6+7+8-9+\cdots+97+98-99$.

3. Derive the formula for $1\times1+2\times2\times1+3\times3\times2\times1+\cdots+n\times n\times(n-1)\times\cdots\times2\times1$.

4. In this number pattern:

$$1;\frac{1}{2},\frac{2}{2},\frac{1}{2};\frac{1}{3},\frac{2}{3},\frac{3}{3},\frac{2}{3},\frac{1}{3};\frac{1}{4},\cdots$$

(a) which position is $\dfrac{7}{10}$?

(b) which number is on the 400th position?

5. Calculate $1+\dfrac{3}{2}+\dfrac{5}{4}+\cdots\dfrac{2\times1991+1}{2^{1991}}$.

6. Given that *O* is the origin on a number line. Point *A* and *B* are positions on 1 and 2. Let $P_1$ be the mid-point on *AB*, $P_2$ be the mid-point on $AP_1$, $\cdots$, $P_{100}$ be the mid-point on $AP_{99}$, find the value of $P_1+P_2+P_3+\cdots+P_{100}$.

7. Evaluate $-1-(-1)^1-(-1)^2-(-1)^3-\ldots-(-1)^{99}-(-1)^{100}$.

8. Find the sum of $\dfrac{1}{10}+\dfrac{1}{40}+\dfrac{1}{88}+\dfrac{1}{154}+\dfrac{1}{238}$.

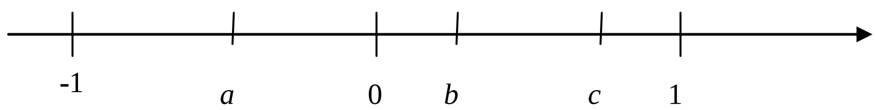
9. Evaluate $1^2-2^2+3^2-4^2+\ldots-2008^2+2009^2$.

10. The sum of $\frac{1}{2\times3\times4}+\frac{1}{3\times4\times5}+\frac{1}{4\times5\times6}+...+\frac{1}{13\times14\times15}+\frac{1}{14\times15\times16}$ is $\frac{m}{n}$ in its lowest terms. Find the value of $m + n$.

***Comparing Rational numbers***

By number line

Number a, b, c are on a number line as shown in the diagram below




Of these 4 numbers, $-\frac{1}{a}$, $-a$, $c-b$, $c+a$, which is the largest?

On the number line, a number on the left is always less than a number on its right.

All positive number is always larger than 0, “zero”.

All negative number is less than 0.

Therefore all positive number is larger than all negative number.

In the diagram above, $-1<a<0$, $0<b<c<1$.

$\therefore -1<c+a<1$, $c-b<1-0=1$

$\because -1<a<0$, $\therefore 0<-a<1$, $-\frac{1}{a}>1$.

$\therefore$ Of these numbers $-\frac{1}{a}$, $-a$, $c-b$, $c+a$, $-\frac{1}{a}$ is the largest.

Alternative Method,

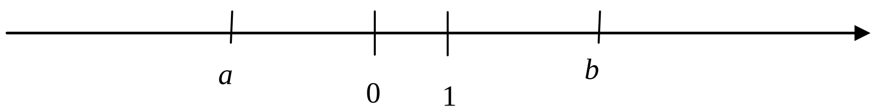
Let $a=-\frac{1}{2}$, $b=\frac{1}{4}$, $c=\frac{3}{4}$,

$-\frac{1}{a}=2$, $-a=\frac{1}{2}$, $c-b=\frac{3}{4}-\frac{1}{4}=\frac{1}{2}$, $c+a=\frac{3}{4}+\left(-\frac{1}{2}\right)=\frac{1}{4}$

Therefore $-\frac{1}{a}$ is the largest.

**Practice**

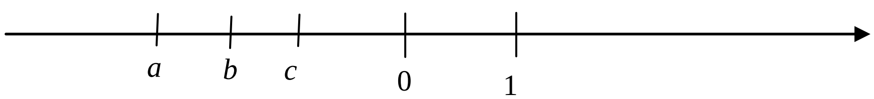
1. $a, b \in Q$, as shown on a number line below



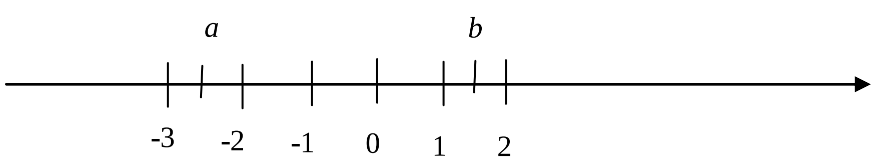

A. $\frac{1}{a} < 1 < \frac{1}{b}$ B. $\frac{1}{a} < \frac{1}{b} < 1$ C. $\frac{1}{b} < \frac{1}{a} < 1$ D. $1 < \frac{1}{b} < \frac{1}{a}$

2. $a, b, c$ are value on the number line below, find the largest of these 3 numbers

$$\frac{1}{a-b}, \frac{1}{c-b}, \frac{1}{a-c}.$$




3. $a, b$ are shown on a number line below, which one of the inequality is incorrect?




A. $|a| > |b|$

B. $a^2 > b^2$

C. $a > -b$

D. $-a > b$

By the difference of the numbers

Ex. If $a$, $b$ are rational, and $b<0$, then what is the relative sizes of $a$, $a-b$, $a+b$?

Given two numbers $x$ and $y$, if $x-y>0$, then $x>y$; if $x-y=0$, then $x=y$; if $x-y<0$, then $x<y$.

$a-(a-b)=a-a+b=b<0$, therefore $a<a-b$.

$a-(a+b)=a-a-b=-b>0$, therefore $a>a+b$.

$\therefore a+b<a<a-b$

**Practice**

1. Comparing $a$ and $\frac{a}{3}$.

2. Comparing $A=\frac{7890123456}{8901234567}$ and $B=\frac{7890123455}{8901234566}$.

3. Comparing $-\frac{2^{2000}+1}{2^{2001}+1}$ and $-\frac{2^{2001}+1}{2^{2002}+1}$.

By division

Comparing $P=\frac{99^9}{9^{99}}$ and $Q=\frac{11^9}{9^{90}}$.

$\frac{a}{b}>1$ and $b>0 \Rightarrow a>b$

$\frac{a}{b}<1$ and $b>0 \Rightarrow a<b$

$\frac{a}{b}=1 \Rightarrow a=b$

$$\frac{P}{Q}=\frac{99^9}{9^{99}}\times\frac{9^{90}}{11^9}=\frac{(11\times 9)^9}{9^{99}}\times\frac{9^{90}}{11^9}=\frac{11^9\times 9^9\times 9^{90}}{9^{99}\times 11^9}=1$$

$$\therefore P=Q$$

**<u>Practice</u>**

1. Comparing $3^{555}$, $4^{444}$, and $5^{333}$.

2. Given that $m<0$, $-1<n<0$, comparing $m$, $mn$, $mn^2$.

3. If $n>1$, comparing $\dfrac{n}{n-1}$, $\dfrac{n-1}{n}$, $\dfrac{n}{n+1}$.

4. If $ab < 0$, then then the relation in sizes of $(a-b)^2$ and $(a+b)^2$ is
   (a) $(a-b)^2 < (a+b)^2$
   (b) $(a-b)^2 = (a+b)^2$
   (c) $(a-b)^2 > (a+b)^2$
   (d) Not determined

# Application of basic Algebra

Find the sum of $\left(1^2+3^2+5^2+7^2+\cdots+99^2\right)-\left(2^2+4^2+6^2+8^2+\cdots+100^2\right)$.

We can re-arrange the above expression into $1^2-2^2+3^2-4^2+5^2-6^2+\cdots+99^2-100^2$.

There is a pattern of $\left(2n-1\right)^2-\left(2n\right)^2=4n^2-4n+1-4n^2=-4n+1$. Therefore,

$$\left(1^2+3^2+5^2+7^2+\cdots+99^2\right)-\left(2^2+4^2+6^2+8^2+\cdots+100^2\right)$$

$$=1^2-2^2+3^2-4^2+5^2-6^2+\cdots+99^2-100^2$$

$$=-4(1)+1-4(2)+1-4(3)+1\cdots-4(50)+1$$

$$=\left\{-4\left(1+2+3+4+\cdots+50\right)\right\}+50$$

$$=-4\left\{\frac{50(50+1)}{2}\right\}+50$$

$$=-5050$$

**Practices**

1. Find the sum of $1^2-2^2+3^2-4^2+5^2-6^2+\cdots-1998^2+1999^2$.

2. Find the product of $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{6^2}\right)$.

3. Find the sum of $(-1)+(-1)^2+(-1)^3+\cdots+(-1)^{99}+(-1)^{100}$.

# *Evaluate simple algebraic expression(I)*

Calculate $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$.

Let $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=a$, and $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)=1+a$; $\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)=1+a+\frac{1}{5}$; then

$$\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\times\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)$$

$$=(1+a)\left(a+\frac{1}{5}\right)-\left(1+a+\frac{1}{5}\right)a$$

$$=a+\frac{1}{5}+a^2+\frac{1}{5}a-a-a^2-\frac{1}{5}a$$

$$=\frac{1}{5}$$

**Try it out!**

**1.** From 2009, subtract half of it at first, then subtract $\frac{1}{3}$ of the remaining number, next subtract $\frac{1}{4}$ of the remaining number, and so on, until $\frac{1}{2009}$ of the remaining number is subtracted. What is the remaining number?

**2.** Evaluate
$\left(\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2009}\right)\left(1+\frac{1}{2}+...+\frac{1}{2008}\right)-\left(1+\frac{1}{3}+\frac{1}{4}+...+\frac{1}{2009}\right)\left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2008}\right)$.

**<u>Practices</u>**

1. Given that $y=ax^4+bx^2+c$, when $x=-5$, $y=3$. Find the value of $y$ when $x=5$.

2. Given that when $x=7$, the value of the expression $ax^5+bx-8$ is 4. Find the value of $\frac{a}{2}x^5+\frac{b}{2}x+3$, when $x=7$.

3. Given the expression $ax^5+bx+c$ has a value of 8 and 1 when $x=-3$ or 0 respectively, find the value of the expression when $x=3$.

## *Evaluate simple algebraic expression(II)*

Given that $\dfrac{a+b}{a-b}=7$, find the value of $\dfrac{2(a+b)}{a-b}-\dfrac{a-b}{3(a+b)}$.

Since $\dfrac{a+b}{a-b}=7$, then we have $\dfrac{a-b}{a+b}=\dfrac{1}{7}$.

$$\therefore \qquad \frac{2(a+b)}{a-b}-\frac{a-b}{3(a+b)}=2\left(\frac{a+b}{a-b}\right)-\frac{1}{3}\left(\frac{a-b}{a+b}\right)$$

$$=2(7)-\frac{1}{3}\left(\frac{1}{7}\right)$$

$$=13\frac{20}{21}$$

**Practice**

1. If the expression $2y^2+3y+7=2$, then find the value of $4y^2+6y-9$.

2. Given that $a=2000x+1999$, $b=2000x+2000$, and $c=2000x+2001$, then find the value of $(a-b)^2+(b-c)^2+(c-a)^2$.

3. Let $ab^2=6$, find the value of $ab\left(ab^3+a^2b^5-b\right)$.

## *Evaluate simple algebraic expression(III)*

If $ab=1$, find the value of $\frac{a}{a+1}+\frac{b}{b+1}$.

$\because \quad ab=1, \therefore \quad a=\frac{1}{b}$

$$\frac{a}{a+1}+\frac{b}{b+1}=\frac{\frac{1}{b}}{\frac{1}{b}+1}+\frac{b}{b+1}$$

$$=\frac{1}{1+b}+\frac{b}{b+1}$$

$$=\frac{b+1}{b+1}$$

$$=1$$

**Practice**

1. If $x+y=2z$, and $x\neq y$, then find the value of $\frac{x}{x-z}+\frac{y}{y-z}$.

2. If $x-y=2$, and $2y^2+y-4=0$, find the value of $\frac{x}{y}-y$.

3. Given that $x-2y=2$, find the value of $\frac{3x-y-6}{4x-y-8}$.

4. If $\frac{1}{x}-\frac{1}{z}=4$, find the value of $\frac{2x+4xz-2z}{x-z-2xz}$.

## *Evaluate simple algebraic expression(IV)*

If $x+y=2z$, $x \neq y$, then find the value of $\frac{x}{x-z}+\frac{y}{y-z}$.

Given that $x+y=2z$, $x \neq y$, we can choose $x=1$, $y=3$, and $z=2$ to find the value of the expression.

$$\frac{x}{x-z}+\frac{y}{y-z}=\frac{1}{1-2}+\frac{3}{3-2}=-1+3=2$$

**Practice**

1. Given that $x-2y=2$, find the value of $\frac{3x-y-6}{4x-y-8}$.

2. Given that $f(x,y)=3x+2y+m$, and $f(2,1)=18$, find the value of $f(3,-1)$.

3. If $a+b+c=0$, and $\frac{b-c}{a}+\frac{c-a}{b}+\frac{a-b}{c}=0$, find the value of $\frac{bc+b-c}{b^2c^2}+\frac{ca+c-a}{c^2a^2}+\frac{ab+a-b}{a^2b^2}$.

## *Evaluate simple algebraic expression(V)*

If $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, and $3x-2y+z=18$, find the value of $x+5y-3z$.

From $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, let $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}=k$.

Then we have $x=3k$, $y=4k$, $z=5k$.

$$3x-2y+z=18$$

$$3(3k)-2(4k)+(5k)=18$$

$$k=3,\ x=9,\ y=12,\ z=15$$

$$\therefore\ x+5y-3z=9+5(12)-3(15)=24$$

**Practice**

1. If $\frac{x}{3}=\frac{y}{4}=\frac{z}{5}$, and $4x-5y+2x=10$, find the value of $2x-5y+z$.

2. If $x=\frac{y}{2}=\frac{z}{3}$, and $x+y+z=12$, find the value of $2x+3y+4z$.

3. If $\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}$, find the value of $x+y+z$.

4. SMO 2009 Junior Paper

   Given that $a+\frac{1}{a+1}=b+\frac{1}{b-1}-2$ and $a-b+2\neq 0$, find the value of $ab-a+b$.

# Odd and Even Integers

Integers can be divided into two sets of number. The set with numbers that can be divided by 2 is called Even; and the rest that cannot be divided by 2 exactly, is called Odd.

An Even integer can be denoted as $2n$, where $n$ is an integer.

An Odd integer can be denoted as $2n+1$, where $n$ is an integer.

An integer must be either Odd or Even, but not both.

Some of the other properties:

(1) Odd $\neq$ Even
(2) Odd + Even = Even + Odd = Odd; Odd – Even = Even – Odd = Odd.
(3) Even + Even = Even; Even – Even = Even.
(4) If $a \times b =$ even, then at least one of the factors is even.
(5) The product of two consecutive integers must be even, $n \times (n+1) =$ even.
(6) If the sum or difference of integers is odd, then there must be at least an Odd integer in the sum or difference.
(7) If the sum or difference of integers is even, then the number of odd integers must be even.
(8) If the sum or difference of integers is odd, then the number of odd integers must be odd.
(9) If the product of integers is odd, then all the numbers in the product must be odd.
(10) If the product of integers id even, then there must be at least one even integer.

Ex. 1 If $a, b, c$ are random integers, then among the three numbers $\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2}$, there are

(A) all non-integers. (B) at least an integer.

(C) at least two integers. (D) all integers.

Ex. 2 Given that the sum of 100 positive integers is 10000, the number of odd integers is more than the number of even integers, what is the most number of even integers?

(A) 49 (B) 48 (C) 47 (D) 46

Ex. 3 Given that $a$ and $b$ are consecutive integers and that $c = ab$, $N^2 = a^2 + b^2 + c^2$, what is the parity of $N^2$?

(A) Odd (B) Even (C) maybe Odd or Even (D) none of the above

Ex. 4 There are *n* number: $x_1$, $x_2$, $x_3$, $\cdots$; $x_n$. Each of these numbers is either a 1 or -1.

If $x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 = 0$, what must be truth about $n$?

(A) Even (B) Odd (C) multiple of 4 (D) cannot tell at all.

Ex. 5 Given that positive integers $p, q, p-q$ are primes, and also that $p+q$ is even,

evaluate the value of $\left(1+\frac{1}{2}\right)^p \left(1-\frac{1}{3}\right)^q$.

Ex. 6 In this number pattern: $1, 2, 5, 13, 34, 89, \cdots$, starting from the second number, the sum of any two adjacent number is equal to three times the middle number. What should be the parity of the 2003rd number?

Ex. 7 Given three integers $x, y, z$ with two odds and one even, prove that $(x+1)(y+2)(z+3)$ must be even.

**Practices**

1. Given 2003 consecutive positive integers: $1, 2, 3, 4, \cdots, 2003$, if either a '+' or '$-$' operator is added in between any two numbers, will the result be odd or even? $[\text{even}]$

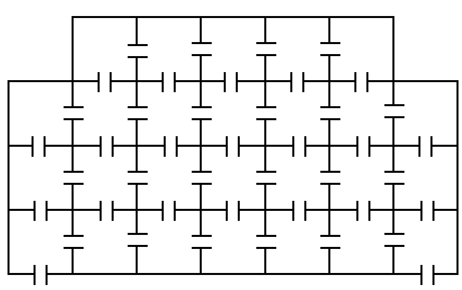
2. In a Math competition, there are 40 questions. A correct answer scores 5 points; a nil return scores 1 point; and a wrong answer deducts 1 point. Should the total score of all the competitors be even or odd. $[\text{even}]$

3. 30 books are to be packed into 5 boxes. Each box must have odd number of books. How can this be done? $[\text{impossible}]$

4. Is it possible to arrange these 10 numbers: 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, on a line such that there is a number in between the two '1's; there are two numbers in between the two '2's; $\cdots$; there are five numbers in between the two '5's? $[\text{impossible}]$

5. Is it possible to visit all the 26 rooms just once without re-entering any one of them?

$[\text{impossible}]$



# Prime and Composite Numbers

Among all the positive integers, 1 is the only one that has only one positive factor, and that is itself.

All positive integers greater than 1 have at least two positive factors.

If an positive integer has only two factors, that is, 1 and the number itself, then this integer is called a Prime number, or Prime in short.

All non-prime integers are collectively called Composite Number, or Composite in short.

By definition, 1 is NOT a prime. There is only one even prime, 2, which is also the smallest prime. The smallest composite is 4.

1 is also NOT a composite.

All positive integer, other than 1, can be prime factorise

$N = p_1^{a_1} p_2^{a_2} p_3^{a_3} \cdots p_k^{a_k}$ where $p_1^{a_1}, p_2^{a_2}, p_3^{a_3}, \cdots, p_k^{a_k}$ are distinct primes.

The number of positive factor of $N$ can be found with this expression:

Number of positive factor of $N = (a_1+1)(a_2+1)(a_3+1)\cdots(a_k+1)$

Ex. 1 If $p$ and $p^3+5$ are prime, what is $p^5+7$?

(A) Prime (B) Prime or Composite (C) Composite

(D) Not prime or composite

Ex. 2 Given three primes $p, q, r$ that satisfy $p+q=r$ and $p<q$, find the value of $p$.

(A) 2 (B) 3 (C) 7 (D) 13

Ex. 3 Given that $n$ is a positive integer such that $n+3$ and $n+7$ are both primes. Find the remainder when $n$ is divided by 3.

Ex. 4 If positive integers $n_1 > n_2$, and $n_1^2 - n_2^2 - 2n_1 - 2n_2 = 19$, find the values of $n_1$, and of $n_2$.

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Ex. 5 If $p, q, r$ are primes, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = \frac{1661}{1986}$, find the value of $p+q+r$.

Ex. 6 If $p$ is a prime not less than 5, and $2p+1$ is also a prime, prove that $4p+1$ is a composite.

**Practices**

1. Given $x, y$ are primes, find the number of ordered pairs of equation $x+y=1999$.

   (A) 1 pair (B) 2 pairs (C) 3 pairs (D) 4 pairs

2. Given that $m, n$ are distinct primes, and $p=m+n+mn$ with $p$ being the minimum value, evaluate the expression $\dfrac{m^2+n^2}{p^2}$.

3. Given that $p, q$ are primes and $p=m+n$, $q=mn$, where $m, n$ are positive integers, evaluate the expression $\dfrac{p^p+q^q}{m^n+n^m}$.

4. If $m, n$ are primes that satisfy $5m+7n=129$, find the values of $m+n$.

5. Given that $p, p+2, p+6, p+8, p+14$ are primes, find the number of $p$.

6. **SMO question 2009**

   Let $p$ and $q$ represent two consecutive prime numbers. For some fixed integer $n$, the set $\{n-1, 3n-19, 38-5n, 7n-45\}$ represents $\{p, 2p, q, 2q\}$, but not necessarily in that order. Find the value of $n$.

# Divisibility

A **divisibility rule** is a method that can be used to determine whether a number is evenly divisible by other numbers. Divisibility rules are a shortcut for testing a number's factors without resorting to division calculations. Although divisibility rules can be created for any base, only rules for decimal are given here.

*The rules given below transform a given number into a generally smaller number while preserving divisibility by the divisor of interest. Therefore, unless otherwise noted, the resulting number should be evaluated for divisibility by the same divisor.*

*For divisors with multiple rules, the rules are generally ordered first for those appropriate for numbers with many digits, then those useful for numbers with fewer digits.*

*If the result is not obvious after applying it once, the rule should be applied again to the result.*

| **Divisor** | **Divisibility Condition** | **Examples** |
|---|---|---|
| **1** | Automatic. | Any integer is divisible by 1. |
| **2** | The last digit is even (0, 2, 4, 6, or 8). | 1,294: 4 is even. |
| **3** | The sum of the digits is divisible by 3. For large numbers, digits may be summed iteratively. | 405: 4 + 0 + 5 = 9, which clearly is divisible by 3. 16,499,205,854,376 sums to 69, 6 + 9 = 15, 1 + 5 = 6, which is clearly divisible by 3. |
| **4** | The number obtained from these examples must be divisible by 4, as follows: | |
| | If the tens digit is even, the last digit is divisible by 4 (0, 4, 8). | 168: 6 is even, and 8 is divisible by 4. |
| | If the tens digit is odd, the last digit plus 2 is divisible by 4 (2, 6). | 5,496: 9 is odd, and 6+2 is divisible by 4. |
| | If the number formed by the last two digits is divisible by 4. | 2,092: 92 is divisible by 4. |
| **5** | The last digit is 0 or 5. | 490: the last digit is 0. |
| **6** | It is divisible by 2 and by 3. | 24: it is divisible by 2 and by 3. |
| | Add the last digit to four times the sum of all other digits. | 198: $(1 + 9) \times 4 + 8 = 48$ |
| **7** | The number obtained from these examples must be divisible by 7, as follows: | |
| | Form the alternating sum of blocks of three from right to left. | 1,369,851: $851 - 369 + 1 = 483 = 7 \times 69$ |

| | | |
|---|---|---|
| | Double the number with the last two digits removed and add the last two digits. | 364: $(3 \times 2) + 64 = 70$. |
| | Add 5 times the last digit to the rest. | 364: $36 + (5 \times 4) = 56$. |
| | Subtract twice the last digit from the rest. | 364: $36 - (2 \times 4) = 28$. |
| | The number obtained from these examples must be divisible by 8, as follows: | |
| | If the hundreds digit is even, examine the number formed by the last two digits. | 624: 24. |
| **8** | If the hundreds digit is odd, examine the number obtained by the last two digits plus 4. | 352: $52 + 4 = 56$. |
| | Add the last digit to twice the rest. | 56: $(5 \times 2) + 6 = 16$. |
| **9** | The sum of the digits is divisible by 9. For larger numbers, digits may be summed iteratively. Result at the final iteration will be 9. | 2,880: $2 + 8 + 8 + 0 = 18$: $1 + 8 = 9$. |
| **10** | The last digit is 0. | 130: the last digit is 0. |
| | The number obtained from these examples must be divisible by 11, as follows: | |
| **11** | Form the alternating sum of the digits. | 918,082: $9 - 1 + 8 - 0 + 8 - 2 = 22$. |
| | Add the digits in blocks of two from right to left. | 627: $6 + 27 = 33$. |
| | Subtract the last digit from the rest. | 627: $62 - 7 = 55$. |
| **12** | It is divisible by 3 and by 4. | 324: it is divisible by 3 and by 4. |
| | Subtract the last digit from twice the rest. | 324: $(32 \times 2) - 4 = 60$. |
| | The number obtained from these examples must be divisible by 13, as follows: | |
| **13** | Add the digits in alternate blocks of three from right to left, then subtract the two sums. | 2,911,272: $-(2 + 272) + 911 = 637$ |
| | Add 4 times the last digit to the rest. | 637: $63 + (7 \times 4) = 91$, $9 + (1 \times 4) = 13$. |
| **14** | It is divisible by 2 and by 7. | 224: it is divisible by 2 and by 7. |
| | Add the last two digits to twice the rest. The answer must be divisible by 7. | 364: $(3 \times 2) + 64 = 70$. |

| | | |
|---|---|---|
| **15** | It is divisible by 3 and by 5. | 390: it is divisible by 3 and by 5. |
| | The number obtained from these examples must be divisible by 16, as follows: | |
| **16** | If the thousands digit is even, examine the number formed by the last three digits. | 254,176: 176. |
| | If the thousands digit is odd, examine the number formed by the last three digits plus 8. | 3,408: 408 + 8 = 416. |
| | Sum the number with the last two digits removed, times 4, plus the last two digits. | 176: $(1 \times 4) + 76 = 80$. |
| | Subtract the last two digits from twice the rest. | 3213: $(2 \times 32) - 13 = 51$. |
| **17** | Alternately add and subtract blocks of two digits from the end, doubling the last block and halving the result of the operation, rounding any decimal end result as necessary. | 20,98,65: $(65 - (98 \times 2)) : 2 + 40 = -25.5 = 255 = 15 \times 17$ |
| | Subtract 5 times the last digit from the rest. | 221: $22 - (1 \times 5) = 17$. |
| **18** | It is divisible by 2 and by 9. | 342: it is divisible by 2 and by 9. |
| **19** | Add twice the last digit to the rest. | 437: $43 + (7 \times 2) = 57$. |
| **20** | It is divisible by 10, and the tens digit is even. | 360: is divisible by 10, and 6 is even. |
| | If the number formed by the last two digits is divisible by 20. | 480: 80 is divisible by 20. |

Ex.1 The greatest integer that can divide the sum of any three consecutive integers is:

(A) 1 (B) 2 (C) 3 (D) 6 .

Ex. 2 A six-digit integer $\overline{568abc}$ is divisible by 3, 4, 5. Find the smallest number $\overline{abc}$.

Ex. 3 Prove that $1999^{2001}+2001^{1999}$ is a multiple of 10.

Ex. 4 Find the remainder when $\overline{1234567891011\cdots 200120022003}$ is divided by 9.

Ex. 5 Given that $A=1\times 3\times 5\times 7\times 9\times 11\times 13\times\cdots\times 1999\times 2001\times 2003$, find the last three digits of A.

Ex. 6 Given natural numbers from 1 to 1995 inclusively, find the number of $a$ from these 1995 numbers that $(a+1995)$ can divide exactly $1995\times a$.

**Practices**

1. Find the number of 7-digit number $\overline{1287xy6}$ that is multiple of 72.

2. If $n$ is a positive integer, $n+3, n+7$ are primes, find the remainder when $n$ is divided by 3.

3. Find the last digit of $1988^{1989}+1989^{1988}$.

4. Let $a_n$ be the last two digits of $7^n$, find the value of $a_1+a_2+a_3+\cdots+a_{2002}+a_{2003}$.

5. Prove that $53^{53}-33^{33}+3^{1998}+4^{1998}$ can be divided exactly by 5.

6. **SMO Question 1995**

   A natural number greater than 11 gives the same remainder (not zero) when divided b 3, 5, 7 and 11. Find the smallest possible value of this natural number.

7. **SMO Question 1998**

   Find the smallest positive integer *n* sch that $100 \le n \le 1100$ and $1111^n+1222^n+1333^n+1444^n$ are divisible by 10.

8. **SMO Question 1999**

   When the three numbers 1238, 1596 and 2491 are divided by a positive integer *m*, the remainders are all equal to a positive integer *n*. Find *m* + *n*.

9. **SMO Question 2001**

   Find the smallest positive integer *k* sch that $2^{69}+k$ is divisible by 127.

10. **SMO Question 2003**

How many integers $n$ between 1 and 2003 (inclusive) are there such that $1^n + 2^n + 3^n + 4^n + 5^n$ is divisible by 5?

11. **SMO Question 2009**

$m$ and $n$ are two positive integers satisfying $1 \le m \le n \le 40$. Find the number of pairs of ($m$, $n$) such that their product is divisible by 33.

# Ratio and its properties

If John has 75 cents and Mary has 50 cents, the ratio of the amount of money of John to Mary is

75 : 50 or 3 : 2.

You know this very well. If we let the amount of money that John has be J and the amount of money that Mary has be M, then here is how we can express this ratio

$$J : M = 3 : 2 .$$

Alternatively, we can also write the ratio as

$$\frac{J}{M} = \frac{3}{2} .$$

Now consider this: let say that the ratio of the amount of money of John and Mary be 3 : 2, what is the amount of money of each of them?

The ratio is $\frac{J}{M} = \frac{3}{2}$. The amount of money of John is $J = 3m$; the amount of Mary is $M = 2m$, where $m$ is the common unit.

We can see that $m = \frac{J}{3} = \frac{M}{2}$. Therefore this ratio of John and Mary can be expressed as

$$\frac{J}{3} = \frac{M}{2} .$$

These are all equivalent statement of the same thing.

$$J : M = 3 : 2 \quad \Leftrightarrow \quad \frac{J}{M} = \frac{3}{2} \quad \Leftrightarrow \quad \frac{J}{3} = \frac{M}{2}$$

If the ratio of $A : B : C = a : b : c$, it will be easier to express it as $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$, rather than

$$\frac{A}{B} = \frac{a}{b} \quad \text{and} \quad \frac{B}{C} = \frac{b}{c} .$$

Can you prove this relationship, $\frac{A}{a} = \frac{B}{b} = \frac{C}{c}$?

In general, given that $a:b=c:d \Leftrightarrow ad=bc$

(i) $\frac{a}{c}=\frac{b}{d}$

(ii) $\frac{b}{a}=\frac{d}{c}$

(iii) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a+b}{b}=\frac{c+d}{d}$. Can you prove this?

(iv) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a-b}{b}=\frac{c-d}{d}$. Can you prove this?

(v) Given $\frac{a}{b}=\frac{c}{d}$, we have $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. Can you prove this?

(vi) Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots=\frac{n}{m}=\frac{a+c+e\cdots+n}{b+d+f+\cdots+m}$; if $b+d+f+\cdots+m\neq 0$

The above extension of the basic ratio can be helpful in solving ratio related problems.

Ex. 1 Given that $\frac{a}{b}=\frac{2}{3}$, which of the following is/are not a true statement?

(A) $a=2,b=3$ (B) $a=2k,b=3k\left(k\neq 0\right)$ (C) $3a=2b$

(D) $a=\frac{2}{3}b$

Ex. 2 Fill in the blank.

(A) Given that $\frac{a}{b}=\frac{3}{5}$, $\frac{a+2b}{2a-b}=$ __________ .

(B) Given that $\frac{3a-4b}{2a-3b}=\frac{7}{4}$, $\frac{a}{b}=$ __________ .

(C) Given that $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{3}{4}$, $\frac{3a+2c-e}{3b+2d-f}=$ __________ .

(D) Given that $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{4}$, $\dfrac{xy+yz+zx}{x^2+y^2+z^2}=$ ___________ .

Ex. 3 Which of the following is/are true statements,

(A) if $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a}{b}=\dfrac{c+m}{d+m}$;

(B) if $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}$;

(C) if $\dfrac{a}{b}=\dfrac{c}{d}$, then $\dfrac{a+c}{b}=\dfrac{b+c}{d}$;

(D) if $\dfrac{a^2}{b^2}=\dfrac{c^2}{d^2}$, then $\dfrac{a}{b}=\dfrac{c}{d}$.

Ex. 4 (A) Given $\dfrac{x}{y}=2$, find the value of $\dfrac{2x^2-3xy+y^2}{x^2+2y^2}$.

. (B) if $6x^2-5xy+y^2=0$, where $(x, y \neq 0)$, find the value of $\dfrac{2x-3y}{2x+3y}$.

Ex. 5 Given that $x : y = 3 : 5$ and $y : z = 2 : 3$, find the value of $\dfrac{x + y - z}{2x - y + z}$.

Ex. 6 Given that $2x - 3y + z = 0$, $3x - 2y - 6z = 0$ and $xyz \neq 0$, find the value of,

(A) $x : y : z$,

(B) $\dfrac{x^2 + y^2 + z^2}{2x^2 + y^2 - z^2}$.

**Practices**

1. Given that $x = \frac{c}{a+b} = \frac{a}{b+c} = \frac{b}{a+c}$, find the value of $x$.

2. Given that $\frac{1}{x} - \frac{1}{y} = 3$, find the value of $\frac{2x+3xy-2y}{x-2xy-y}$.

3. Given that $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$, find the value of $\frac{a}{b} + \frac{b}{a}$.

4. Given that $\frac{ab}{a+b} = \frac{3}{2}$, $\frac{bc}{b+c} = 3$, $\frac{ca}{c+a} = \frac{1003}{1004}$, find the value of $\frac{abc}{ab+bc+ac}$.

5. Given that $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a^2+b^2}{ab} = \frac{c^2+d^2}{cd}$.

6. Given that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, show that $\frac{x^3}{a^2} + \frac{y^3}{b^2} + \frac{z^3}{c^2} = \frac{(x+y+z)^3}{(a+b+c)^2}$.

7. **SMO Question 1998**

Three boys, Tom, John and Ken, agreed to share some marbles in the ratio of 9: 8 : 7 respectively. John then suggested that the should share the marbles in the ratio 8: 7: 6 instead. Who should then get more marbles than before and who world get less than before if the ratio is changed?

8. **SMO Question 2009**

There are two models of LCD television on sale. One is a 20 inch standard model while the other is a 20 inch widescreen model. The ratio of the length to the height of the standard model is 4: 3, while that of the widescreen model is 16: 9. Television screens are measured b the length of their diagonals, so both models have the same diagonal length of 20 inches. If the ratio of the area of the standard model to that of the widescreen model is $A$: 300, find the value of $A$.

# Perfect Squares

Perfect squares are natural numbers: $1^2$, $2^2, 3^2, 4^2$, …., $n^2$. In general, we call these numbers – Square numbers.

The last digit of a square cannot be 2, 3, 7, or 8. It can only be 0, 1, 4, 5, 6, 9.

Here is the list of the last two digits of a square numbers:

| | | | | | |
|---|---|---|---|---|---|
| 00 | 01 | 21 | 41 | 61 | 81 |
| 25 | 09 | 29 | 49 | 69 | 89 |
| | 04 | 24 | 44 | 64 | 84 |
| 16 | 36 | 56 | 76 | 96 | |

These are the 22 possibilities of a square numbers.

Notice that the tens-digit of an odd square number must be even.

The tens-digit of a square number ends in 6, must be odd.

The tens-digit of a square number ends in 4, must be even.

We can make use of these rules to deduce whether a number is square or not square.

In addition, here are the other properties of a square:

1. Each even square can be expressed as a multiple of 4 (eg. $4k$ ).
2. Each odd square can be expressed as a multiple of 4 plus 1 (eg $4k+1$).
3. Each odd square can be expressed as a multiple of 8 plus 1 (eg $8k+1$).
4. Each $3n+2$ integer cannot be a square.
5. Each $5n+2$ or $5n+3$ integer cannot be a square.
6. Each $8n+2$, $8n+3$, $8n+5$, $8n+6$, $8n+7$ integer cannot be a square.
7. Each $9n+2$, $9n+3$, $9n+5$, $9n+6$, $9n+8$ integer cannot be a square.
8. If the tens digit and ones digit of an integer are odd, it is not a square.
9. If the tens digit is even and ones digit is 6 of an integer, it is not a square.

Lastly, if a natural number $N$, such that $n^2 < N < (n+1)^2$, then $N$ is not a square.

Ex.1 A number is formed using 300 digits of 2 and some digit 0, can it be arrange into a square?

Ex. 2 Prove that there is no square number in this number pattern: 11, 111, 1111 , 11111, … .

Ex. 3 Find a 4-digit square number in the form of $\overline{aabb}$ .

Ex. 4 How many square numbers are there from 1 to 1000 inclusively?

Ex. 5 Given a 2-digit number $N$ , by adding another 2 digits on its left, becomes the square of $N$ . Find all such positive integers, $N$ .

**Practices**

1. Prove that for any given natural number $n$, $n(n+1)$ is not a square.

2. Find the number of 4-digit number, after added 400, it becomes a square.

3. Prove that $\underbrace{11\cdots1}_{m\ digits}\times\underbrace{10\cdots5}_{m+1\ digits}+1$ is a perfect square.

4. Find all square numbers with common last 4 digits.

5. Given that $x$ and $y$ are positive integer, find all the order pair of $(x, y)$ that satisfy the equation $x^2+y^2=7$

6. Given that $n$ is a positive integer, find the smallest $n$ so that $2008n$ is a square.

7. **China 1992**

   If $x$ and $z$ are positive integers, prove that the vales of $x^2+z+1$ and $z^2+4x+3$ cannot be both perfect squares at the same time.

8. **SMO Question 2000**

   Evalate $\sqrt{\underbrace{11\cdots1}_{2000 digits}-\underbrace{222...22}_{1000 digits}}$.

2. **SMO Question 2000**

   Let $n$ be a positive integer such that $n + 88$ and $n – 28$ are both perfect squares. Find all possible values of $n$.

# Modular Arithmetic

## CONGRUENCE

**What is Congruence?**

Two integers are congruent modulo *m* if and only if they have the same remainders after division by *m*.

Let *m* be a fixed positive integer. If $a, b, \in Z$, we say that "*a* is **congruent** to *b* **modulo** *m*" and write

$a \equiv b \pmod{m}$ whenever $m \mid (a-b)$.

If $m \nmid (a-b)$, we write $a \not\equiv b \pmod{m}$.

**Another way to look at Congruency**

The condition for $a$ to be congruent to $b$ modulo *m* is equivalent to the condition that

$a = b + km$ for some $k \in Z$.

For example, $7 \equiv 3 \pmod{4}$, $-6 \equiv 14 \pmod{10}$, $121 \equiv 273 \pmod{2}$, but $5 \not\equiv 4 \pmod{3}$ and $21 \not\equiv 10 \pmod{2}$.

**<u>True or False?</u>**

1. $88 \equiv 54 \pmod{17}$ ( )
2. $185 \equiv 392 \pmod{23}$ ( )
3. $12 \equiv 23 \pmod{7}$ ( )
4. $101 \equiv 72 \pmod{3}$ ( )

Properties of congruency

(i) $a \equiv a \pmod{m}$ (reflexive property)

(ii) $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$ (symmetric property)

(ii) $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$ (transitive property)

Can you prove these properties?

## Modular Arithmetic

If $a \equiv a' \pmod{m}$ and $b \equiv b' \pmod{m}$, then

(1) $a+b \equiv a'+b' \pmod{m}$

(2) $a-b \equiv a'-b' \pmod{m}$

(3) $ab \equiv a'b' \pmod{m}$

(4)If $ac \equiv bc \pmod{m}$ and $\gcd(c,m)=1$, then it follows that $a \equiv b \pmod{m}$

Can you provide proof for these operations?

Congruences occur in everyday life. The short hand of a clock indicates the hour modulo 12, while the long hand indicates the minute modulo 60. For example, 20 hours after midnight, the clock indicates 8 o'clock because $20 \equiv 8 \pmod{12}$. In determining which day of the week a particular date falls, we apply congruence modulo 7. Two integers are congruent modulo 2 if and only if they have the same parity; that is if and only if they are both odd or both even.

The idea of congruence is not radically different from divisibility, but its usefulness lies in its notation, and the fact that congruence, with respect to a fixed modulus, has many of the properties of ordinary equality.

**Example 1**

What is the remainder when $2^{37}$ is divided by 7?

**Example 2**

What is the remainder when $4^{10} \times 7^{7}$ is divided by $5$?

**Example 3**

Find the remainder when $56976$ is divided by $4$.

## Tests for Divisibility

Congruences can be used to prove some of the familiar tests for divisibility by certain integers.

The test of divisibility by 4 works because $100 \equiv 0 \pmod{4}$. Given an integer $n \times 100 + \overline{ab}$, where $n$ is an integer and $a$, $b$ are the last two digits of this number, $100n + \overline{ab} \equiv \overline{ab} \pmod{4}$. Therefore the remainder when $n \times 100 + \overline{ab}$ is divided by $4$ is the same as that of $\overline{ab}$ when divided by $4$.

**Example 4**

Prove that a number is divisible by 9 if and only if the sum of its digits is divisible by 9.

**Example 5**

Prove that a number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11.

## Fermat's Little Theorem

If $p$ is a prime number that does not divide the integer $a$, then

$$a^{p-1} \equiv 1 \pmod{p}$$

In addition, for any integer $a$ and prime $p$,

$$a^{p} \equiv a \pmod{p}$$

Exercise

1. If $a$ and $b$ are divided by 5, it gives a remainder 1 and 4 respectively. Find the remainder when $3a-b$ is divided by 5.

2. If today is a Saturday, what is the day of the week after $10^{2000}$ days?

3. Find the remainder when $1993^{1994}$ is divided by 7.

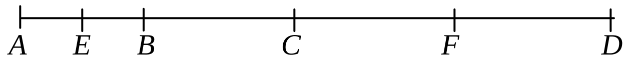
4. Prove that $3^{2000}+4^{1993} \equiv 0 \pmod{5}$

5. **Moscow 1982**

   Find all the positive integers $n$ such that $n.2^{n}+1$ is divisible by 3.

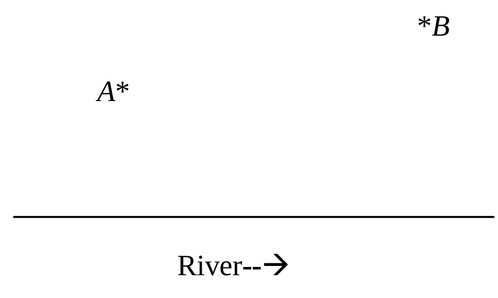
# Lines and Angles

Ex.1 Given that $AB : BC : CD = 2 : 3 : 4$ with $E$ and $F$ being the mid points on $AB$ and $CD$ respectively, if $EF = 12$ cm, find the length of $AD$.




Ex. 2 Given that $AB = 10$ cm on a straight line, $C$ is a point on the same line that $AC = 2$ cm. If $M$ and $N$ are the midpoints on $AB$ and $AC$ respectively, find the length of $MN$.

Ex. 3 If Jenny is to travel from point $A$ to point $B$ but need to reach the river bank to wash her hands, what is the shortest path for her?




Ex. 4 A wire of length 10 cm is to be bent to form a pentagon, what is the longest side of this shape?

.

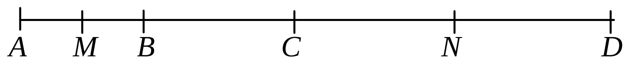
Ex. 5 For an analog clock, if the time is from 0230 to 0250, how much does each hour hand and minute hand have to move, in angles?

Ex. 6 For an angle *A*, if the ratio of its complementary angle to its supplementary angle is 2 : 7, find the measure of angle *A*.

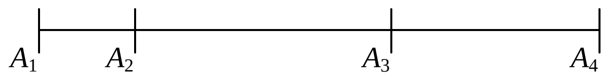
Ex 7. For an analog clock starting at 5 O'clock, how much will the hour hand move before the minute hand overlap it?

**Practices**

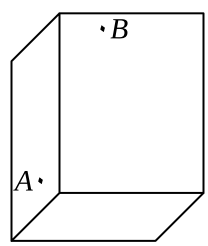
1. For the given diagram, $B$ and $C$ are two points on $AD$. $M$ and $N$ are the midpoints on $AB$ and $CD$ respectively. If $MN = a$ and $BC = b$, find the length $AD$.




2. For the given diagram, $A_2$ and $A_3$ are two points on the line $A_1A_4$. If $A_1A_2 = a$, $A_1A_3 = a^2$, $A_1A_4 = a^3$, find the sum of length of all the line segment between $A_1$ and $A_4$.




3. Two points $A$ and $B$ are on two sides of a box as show in the diagram, find the shortest line to draw from $A$ to $B$.



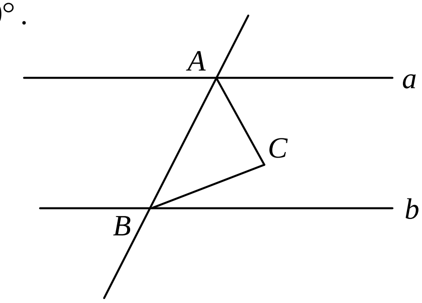

5. If the difference in two supplementary angles is 28°, find the complementary angle of the smaller angle.

6. From 6 O'clock to 7 O'clock, when will the hour and minute hands are at 90° apart?

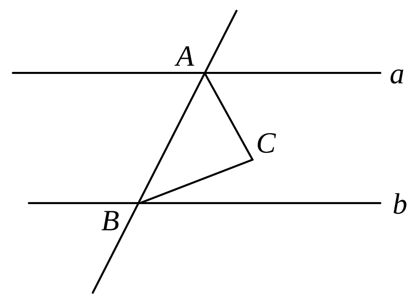
7. From 4 O'clock to 6 O'clock, when will the hour and minute hands are at 120° apart?

# Parallel Lines

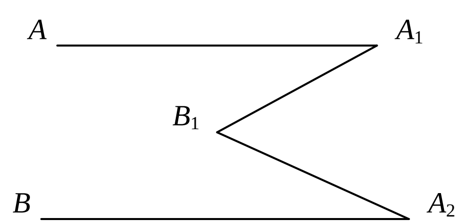
Ex.1 Given the diagram below, $a // b$, line $AB$ intersects $a$ and $b$ at $A$ and $B$ .IF $CA$ bisects $\angle BAa$ and $BC$ bisects $\angle ABb$, prove that $\angle C = 90°$ .



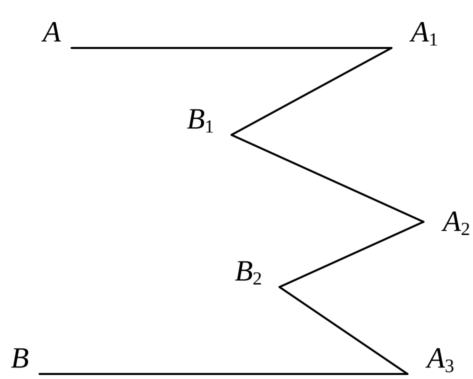

What about straight lines $a$, $b$ have been intersected by another straight line $AB$. $CA$ bisects $\angle BAa$ and $BC$ bisects $\angle ABb$. If $\angle C = 90°$, are the lines $a$, $b$ parallel to one another?



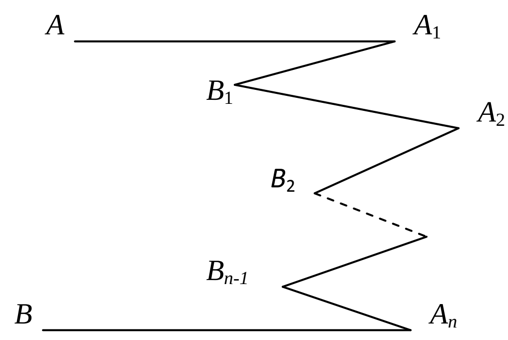

Ex. 2 Given the diagram below, $AA_1 // BA_2$ find the measure of $\angle A_1 - \angle B_1 + \angle A_2$.



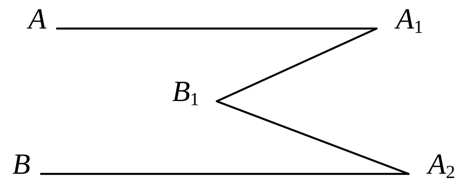

What about this if $AA_1 // BA_3$?



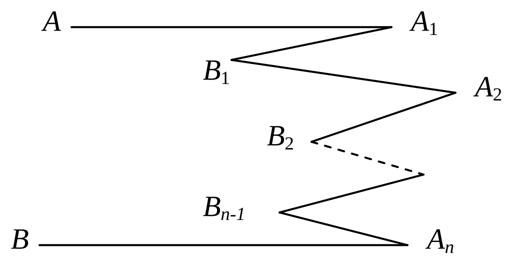
And how about this if $AA_1 // BA_n$ ?



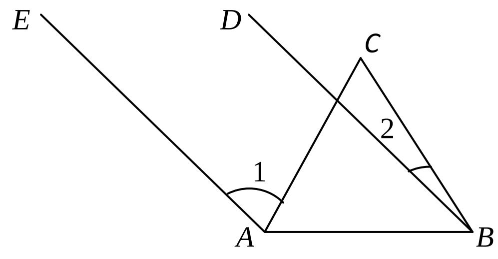

Question : If $\angle A_1 - \angle B_1 + \angle A_2 = 0$, is $AA_1 // BA_2$ ?



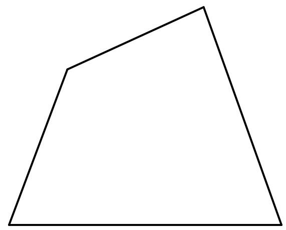

Question : If $\angle A_1 + \angle A_2 + \cdots + \angle A_n = \angle B_1 + \cdots + \angle B_{n-1}$, is $AA_1 // BA_n$ ?




Ex. 3 For the given diagram below, $AE // BD$, $\angle 1 = 3 \times \angle 2$, and $\angle 2 = 25°$. Find $\angle C$.

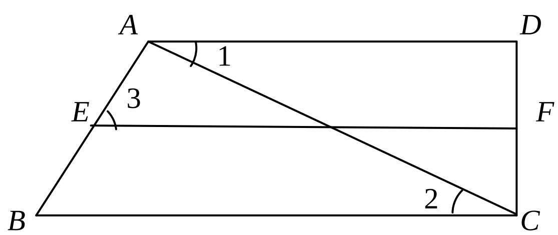


Ex. 4 Prove that the sum of interior angles of the quadrilateral is $360°$.



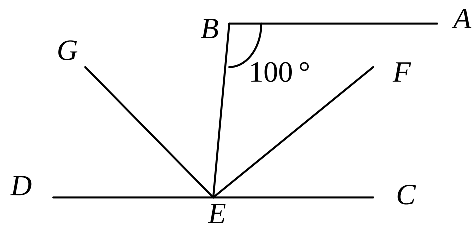
Question : What about the sum of all interior angles of a polygon of $n$ sides?

Ex. 5 For the diagram below, $\angle 1 = \angle 2$, $\angle D = 90°$, $EF \perp CD$, show that $\angle 3 = \angle B$.

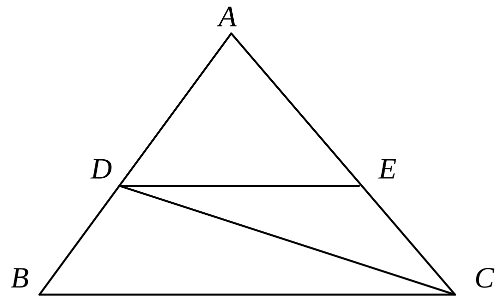


**Practices**

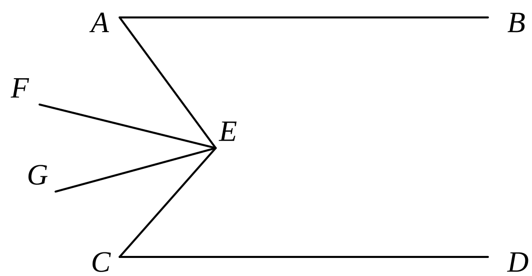
1. For the given diagram, $AB//CD$, $\angle GEF = 90°$, $\angle B = 100°$, $EF$ bisects $\angle BEC$, and $EG \perp EF$. Find $\angle BEG$ and $\angle DEG$.




2. For the given diagram, $CD$ bisects $\angle ACB$. If $\angle ACB = 40°$, $\angle B = 70°$ and $DE//BC$, find $\angle EDC$ and $\angle BDC$.

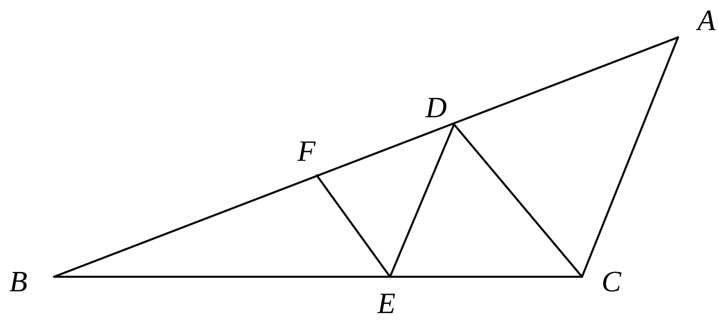



3. For the given diagram, $AB//CD$, $\angle BAE = 30°$, $\angle DCE = 60°$, and $EF$, $EG$ trisects $\angle AEC$ equally. Are $EF$ or $EG$ parallel to $AB$? And why?



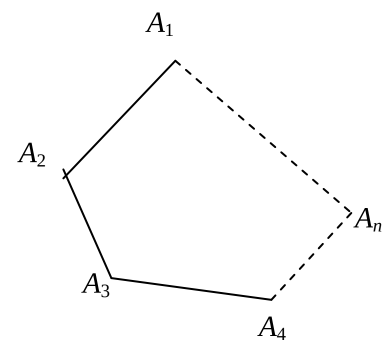

7. Prove that the sum of internal angle of a pentagon is 540 °.

8. For the given diagram, $CD$ bisects $\angle ACB$, $DE // AC$ and $CD // EF$. Prove that $EF$ bisects $\angle DEB$.

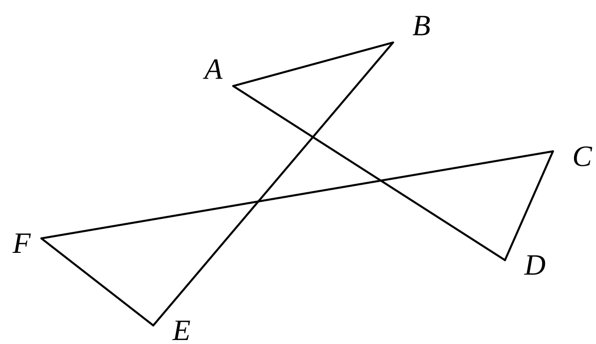


# Triangles

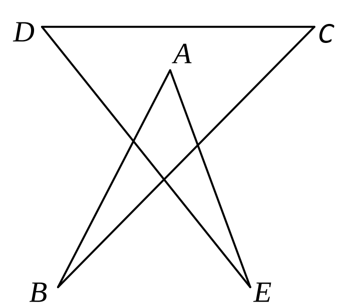
Ex.1 Show that the sum of interior angles of a $n$-gon is given as $180(n-2)°$.



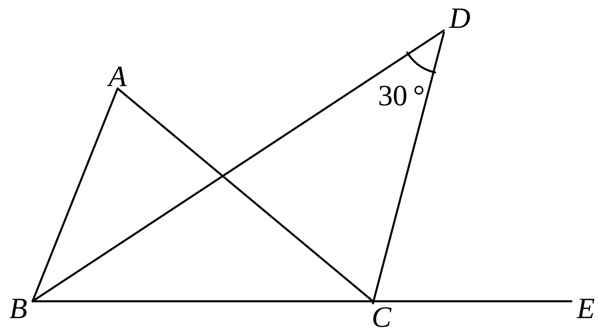

Ex. 2 Find the sum of angles, $\angle A+\angle B+\angle C+\angle D+\angle E+\angle F$.



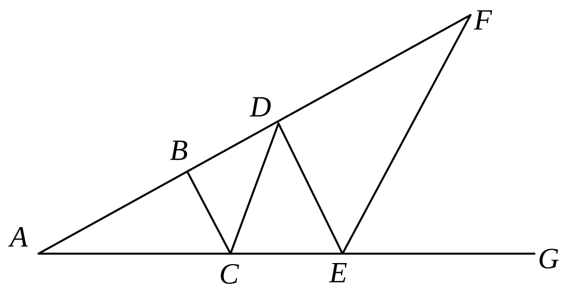

Ex. 3 Find the sum of angles, $\angle A+\angle B+\angle C+\angle D+\angle E$.



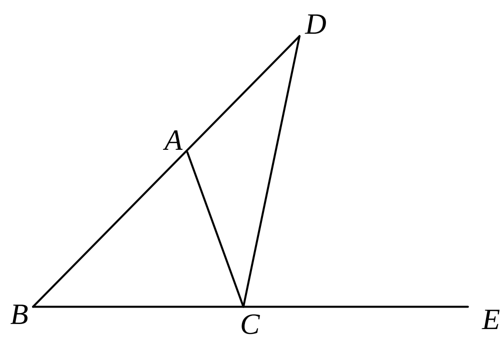
Ex. 4 In this $\Delta ABC$, the angle bisector of $\angle B$ and angle bisector of $\angle ACE$ intersects at $D$. If $\angle D = 30°$, find $\angle A$.




Ex. 5 For the diagram below, $\angle A = 10°$, $\angle ABC = 90°$, $\angle ACB = \angle DCE$, $\angle ADC = \angle EDF$, $\angle CED = \angle FEG$, find $\angle F$.

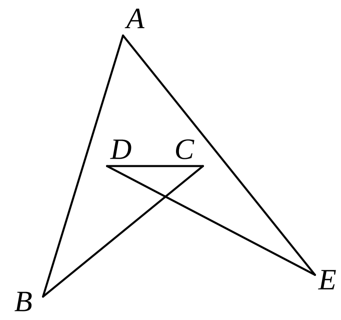



Ex. 6 For the diagram $\Delta ABC$, $BA$ extend to intersect the angle bisector of $\angle ACE$ at $D$. Show that $\angle BAC > \angle B$.

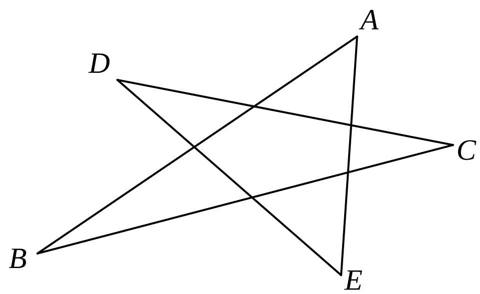


**Practices**

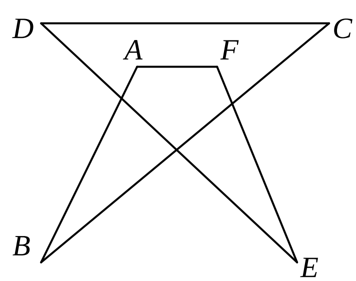
1. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E$.



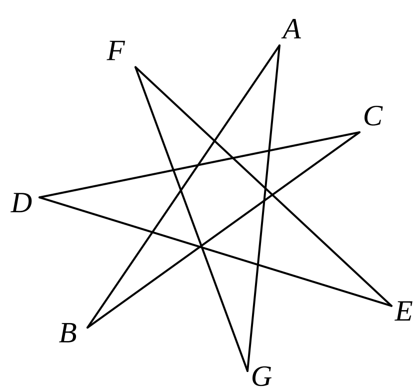

2. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E$.




3. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F$.




4. For the given diagram, find the value of $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + \angle G$.



5. If the sum of all interior angles of a polygon is as following respectively, find the number of sides of each polygon.
(a) $1260°$ (b) $2160°$

6. Show that the sum of all exterior angles of a polygon is $360°$.

7. **China 1998**

In a right angle triangle $ABC$, $\angle ACB = 90^{o}$, $E$, $F$ are on $AB$ such that $AE = AC$, $BF = BC$, find $\angle ECF$ in degrees.

8. **AHSME 1996**

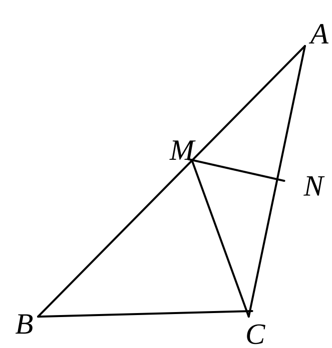
Triangles $ABC$ and $ABD$ are isosceles with $AB = AC = BD$, and $AC$ intersects $BD$ at $E$. If $AC$ is perpendicular to $BD$, find the value of $\angle C + \angle D$.

9. **SMO Question 1995**

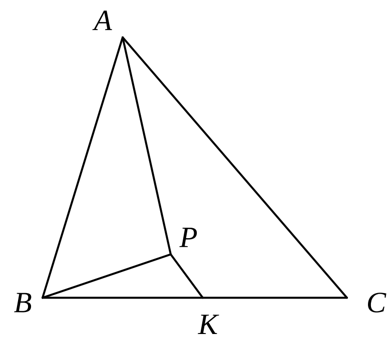
In a right angled triangle, the lengths of the adjacent sides are 550 and 1320. What is the length of the hypotenuse?

10. **SMO Question 1996**

In the figure below, $ABC$, $M$ and $N$ are points on $AB$ and $AC$ respectively such that $AM : MB = 1: 3$ and $AN : NC = 3: 5$. What is the ratio of $[MNC] : [ABC]$?



11. **SMO Question 1998** In the figure below, $AP$ is the bisector of $\angle BAC$ and $BP$ is perpendicular to $AP$. Also, $K$ is the midpoint of $BC$. Suppose that $AB = 8$ cm and $AC = 26$ cm. Find the length of $PK$ in cm.




12. **SMO Question 2000**

Determine the number of acute angled triangles having consecutive integer sides and perimeter not more than 2000.

13. **SMO Question 2001**

Two of the three altitudes of a right-angled triangle are of lengths 12 and 15. Find the largest possible length of the third altitude of the triangle.

# Areas

Area of triangle, $S_{\Delta} = \frac{1}{2}HB$ where $H$ = height of a triangle, $B$ = the base that is $\perp$ to its height.

Area of a parallelogram, $S_{\Diamond} = HB$

Area of a trapezium, $S = \frac{1}{2}(A+B)H$ where $A$ and $B$ are the lengths of the parallel sides

Some properties of area of triangles:

(1) common base and common height of two triangles have the same area.

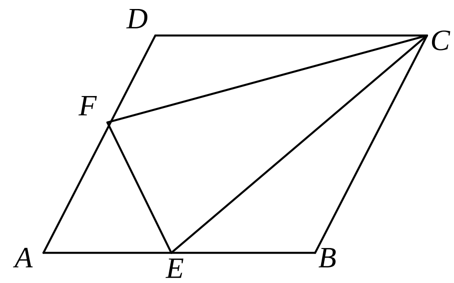
(2) the ratio of the areas of two triangle is the ratio of the products of the base and height of each triangle.

(3) the ratio of the areas of two common base triangles is the same as the ratio of its heights.

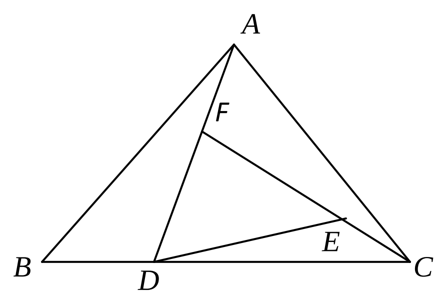
(4) the ratio of the areas of two common height triangles is the same as the ratio of its bases.

Ex.1 Given a $\Delta ABC$, the three sides of $a$, $b$, $c$ corresponds to heights 4, 5, and 3. Find the ratio of sides $a : b : c$.

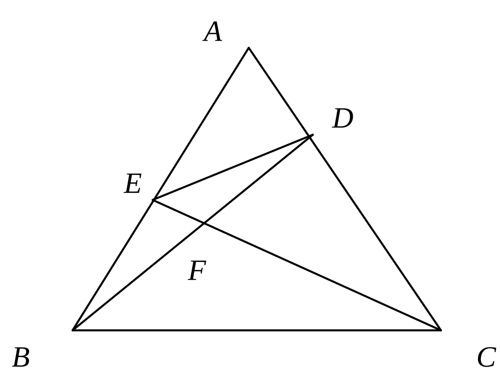
Ex. 2 Given the parallelogram $ABCD$ of area 64 square cm, $E$ and $F$ are midpoints of $AB$ and $AD$ respectively. Find the area of $\Delta CEF$ .



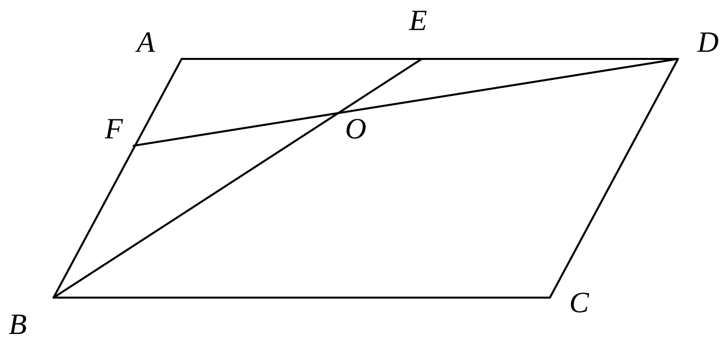

Ex. 3 In this $\Delta ABC$ of area 1 unit, $BD=\frac{1}{2}DC$ , $AF=\frac{1}{2}FD$ , $CE=\frac{1}{2}EF$ , find the area of $\Delta DEF$ .




Ex. 4 In this $\Delta ABC$ , $E$ is a mid point on $AB$, $D$ is a point on $AC$ such that $AC$: $DC$ = 2 : 3, and $BD$ and $CE$ intersect at $F$. If the area of triangle $ABC$ is 40 units, find the area of $AEFD$.

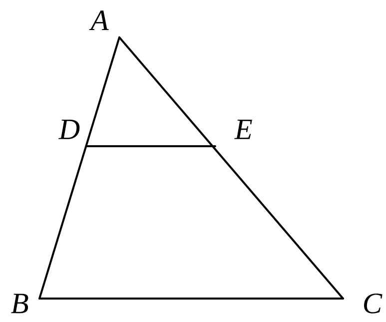


Ex. 5 For the diagram below, *ABCD* is parallelogram. *E* and *F* are a point on *AD* and *AB* respectively. If $BE = DF$ and *BE* and *DF* intersects at *O*, prove that *C* is equip-distant from *BE* and *DF*.

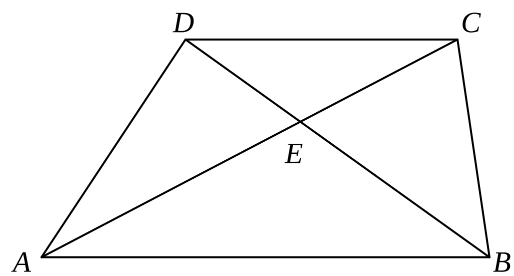


**Practices**

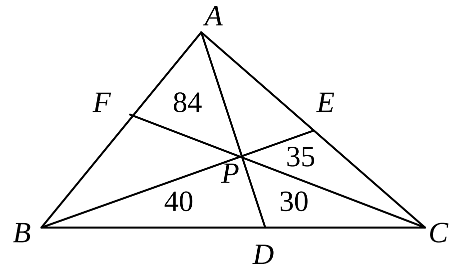
1. For the given diagram, in this $\Delta ABC$, $EF//BC$, $AE:EB=m:1$, show that $AF:FC=m:1$.



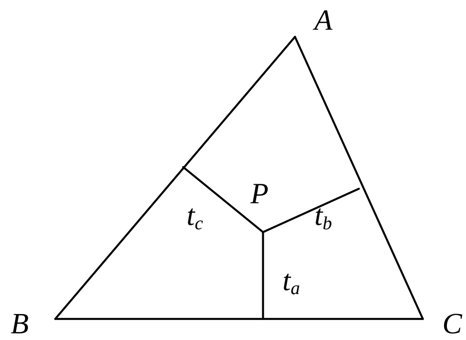

2. For the given diagram, ABCD is a trapezium. $AB//CD$. If the area of $\Delta DCE$ is a quarter of the area of $\Delta DCB$, find the fraction of the area of $\Delta DCE$ to that of the area of $\Delta ABD$.



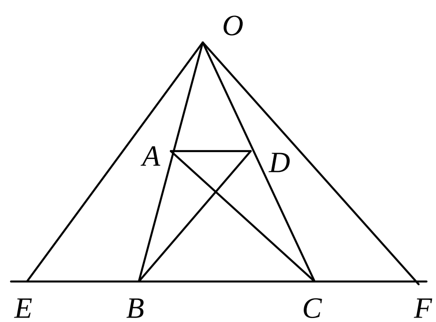

3. For the given diagram, in this $\Delta ABC$, *P* is a point within this triangle. *AP*, *BP* and *CP* intersect on the opposite sides at *D*, *E* and *F* respectively. Given the area of 4 of the six smaller triangles, find the area of $\Delta ABC$.



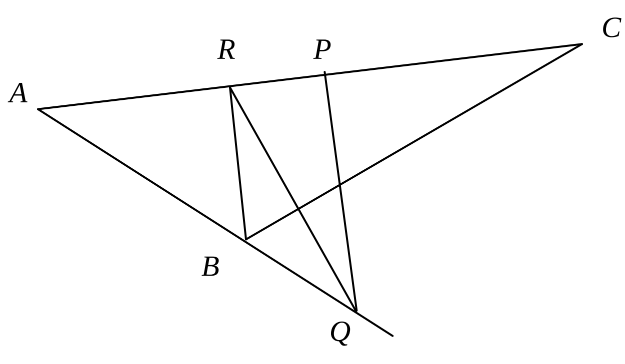
4. For the given diagram, $P$ is a point within this triangle. Given that the altitudes from the vertices $A$, $B$ and $C$ are $h_a$, $h_b$ and $h_c$ respectively, and the perpendicular distances from $P$ to each sides as $t_a$, $t_b$ and $t_c$, show that $\frac{t_a}{h_a}+\frac{t_b}{h_b}+\frac{t_c}{h_c}=1$.




5. For the given diagram, $ABCD$ is a trapezium. $BA$ and $CD$ produce a intersection at $O$. $OE//DB$, $OF//AC$. $E$, $B$, $C$ and $F$ are on a straight line. Show that $BE=CF$.

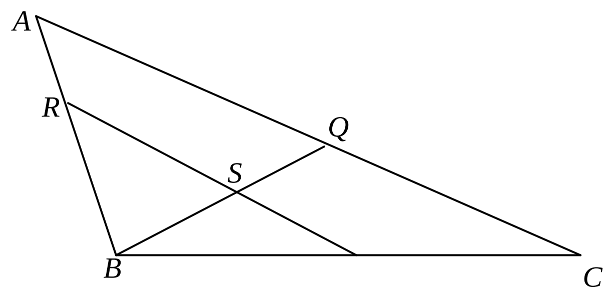



6. For the given diagram, $P$ is a mid-point on $AC$ in the $\Delta ABC$. $PQ \perp AC$ and intersects at the $AB$ produce at $Q$. $BR \perp AC$. Show that $S_{\Delta ARQ}=\frac{1}{2}S_{\Delta ABC}$.



7. **SMO Question 1998**

In the figure below, triangle $ABC$ is a right-angles triangle with $\angle B = 90^o$. Suppose that $\dfrac{BP}{CP} = \dfrac{AQ}{CQ} = 2$ and $AC$ is parallel to $RP$. If the area of triangle $BSP$ is 4 square nits, find the area of triangle $ABC$ in square units.




8. **SMO Question 1999**

In triangle $ABC$, $D$, $E$, $F$ are points on the sides $BC$, $AC$ and $AB$ respectively such that $BC = 4CD$, $AC = 5AE$ and $AB = 6BF$. Suppose the area of triangle $ABC$ is 120 cm$^2$. What is the area of triangle $DEF$?

9. **SMO Question 2004**

In a triangle $ABC$, $AB = 2BC$ and $P$ lies within the triangle $ABC$ such that $\angle APB = \angle BPC = \angle CPA$. Given that $\angle ABC = 60^o$ and that $[ABC] = 70$ cm$^2$, find $[APC]$.

# Solving Diophantine Equations

## What is a Diophantine equation?

A Diophantine equation is concerned with only integer solutions. In some cases, it concerns only with positive integer solutions. What make solving Diophantine equation interesting and challenging is that there are more unknowns than available equations. For example, you may have 2 equations but three unknowns. The usual method of solving system of equations will then need further development. You will also need to have a strong understanding in Number Properties. So, enjoy the lesson.

Ex.1 Jane spent five dollars on a number of erasers and pencils. An eraser cost 30 cents, a pencil cost one dollar. How many erasers and pencils did Jane buy?

Ex. 2 Find the integer solutions of $11x + 15y = 7$.

Ex. 3 Find the non-negative integer solutions of $6x+22y=90$.

Ex. 4 Find the positive integer solutions of $7x+19y=213$.

Ex. 5 Find the integer solutions of $37x+107y=25$.

Ex. 6 Find the number of ways of using numerous 5 or 7 only to sum up to exactly 142.

Ex. 7 Find the integer solutions to $9x+24y-5z=1000$.

**Practices**

1. Find the integer solutions to:

   (a) $72x+157y=1$, (b) $9x+21y=144$, (c) $103x-91y=5$

2. Find the positive integer solutions to:

   (a) $3x-5y=19$, (b) $12x+5y=125$,

3. Find the integer solutions to:

   (a) $5x+8y+19z=50$, (b) $39x-24y+9z=78$.

4. Find the integer solutions to $2x+5y+7z+3t=10$.

5. Find the positive integer solutions to $5x+7y+2z=24$ and $3x-y-4z=4$.

6. **China**

   *m* and *n* are integers satisfying $3m + 2 = 5n + 3$ and $30 < 3m + 2 < 40$. find the vale of *mn*.

7. **AHSME 1992**

   If *k* is a positive integer such that the equation in *x* $kx-12=3k$ has an integer root, then what is *k*?

8. **RSMO 1983**

   Given that a pile of 100 small weights have a total weight of 500g, and the weight of a small weight is 1g, 10g or 50g. Find the number of each kind of weights in the pile.

9. **China 2001**

   How man pairs of ($x$, $z$) of two integers that satisfy the eqation $x^2 - z^2 = 12$?