Slope-Deflection Method

Theory of Structures-II

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Slope-Deflection Method

- Previously we have discussed Force/Flexibility methods of analysis of statically indeterminate structures.

- In force method, the unknown redundant forces are determined first by solving the structure's compatibility equations; then other response characteristics of the structure are evaluated by equilibrium equations or superposition.

- An alternative approach can be used for analyzing is termed the displacement or stiffness method.

Slope-Deflection Method

- In displacement method, the unknown displacements are determined first by solving the structure's equilibrium equations; then the other response characteristics are evaluated through compatibility considerations and member force-deformation relationships.

- The displacement methods includes Slope-Deflection Method and Moment-Distribution Method.

- The slope-deflection method was introduced by George A. Maney in 1915.

Slope-Deflection Method

- This method takes into account only the bending deformations.

- This method gives an understanding of the Matrix-Stiffness Method, which forms the basis of most computer software currently used for structural analysis.
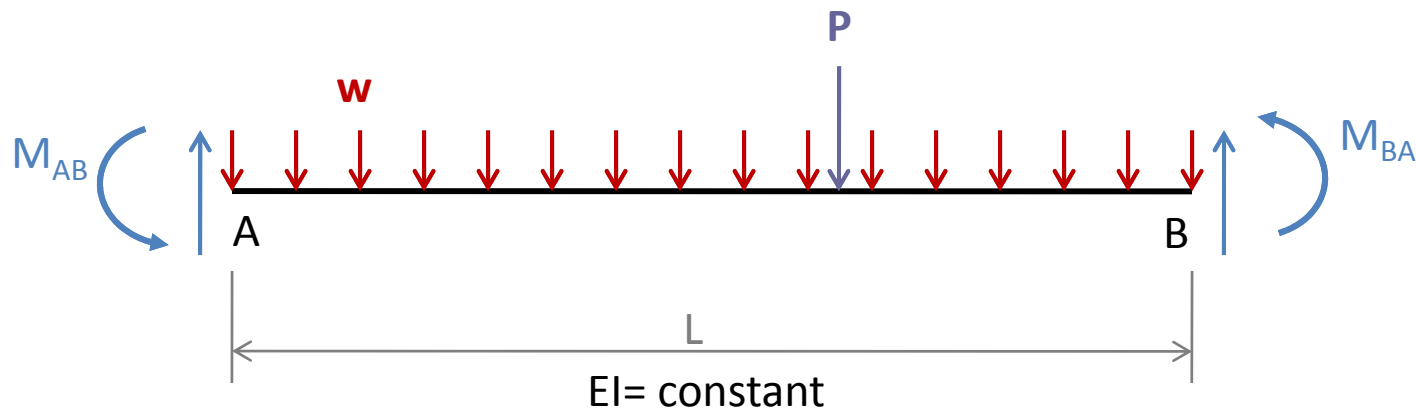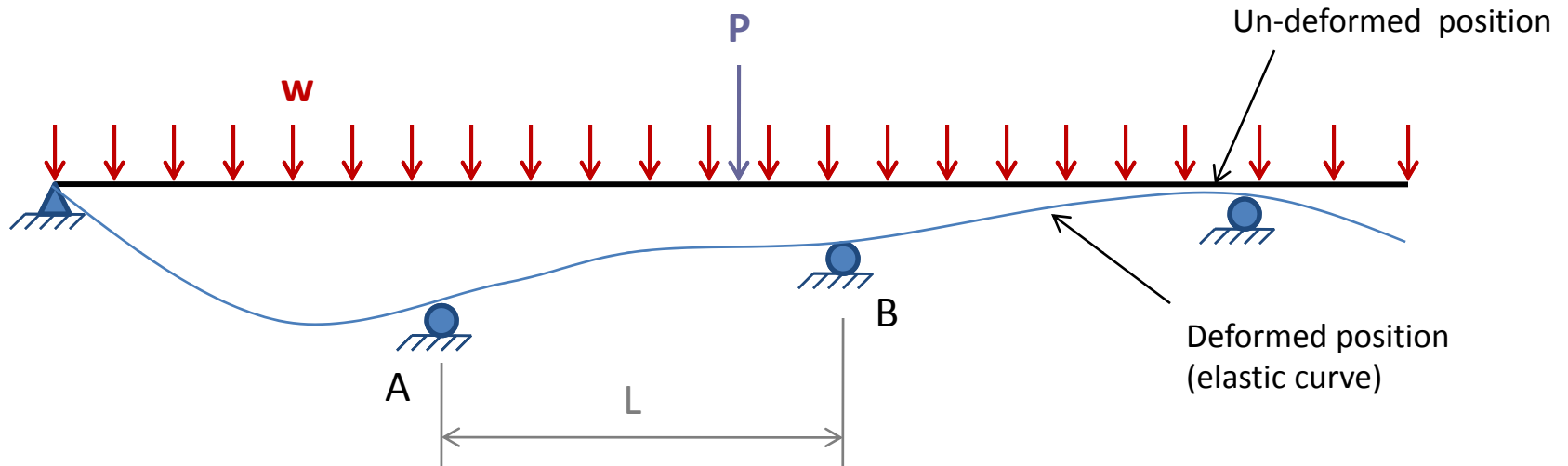
Slope-Deflection Equations

- When a continuous beam or a frame is subjected to external loads, internal moments generally develop at the ends of its individual members.
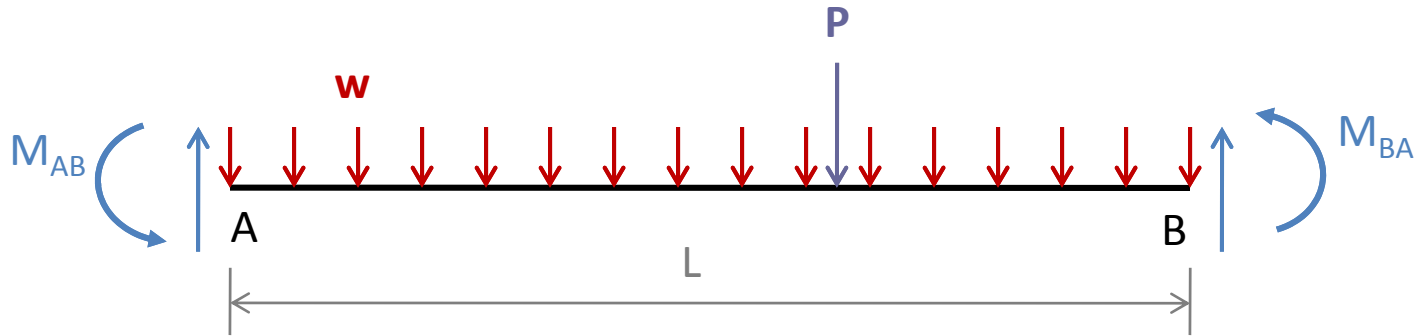
"The slope-deflection equations relate the moments at the ends of the member to the rotations and displacements of its end and the external loads applied to the member."

- Let us consider an arbitrary member AB of the continuous beam.


A   L   B
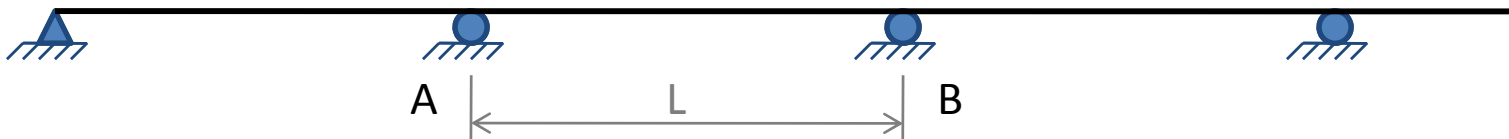
- When the beam is subjected to external loads and support settlements, the member AB deforms as shown (exaggerated), and internal moments are induced at its ends.

P

w

Un-deformed position

A

L

B

Deformed position
(elastic curve)

P

w

M_{AB}

A

B

M_{BA}

L

EI= constant

- Double-subscript notation is used for member end moments, with the first subscript identifying the member end at which the moment acts and the second subscript indicating the other end of the member.

- M_{AB} denotes the moment at end A of the member AB.

- M_{BA} denotes the moment at end B of the member AB.
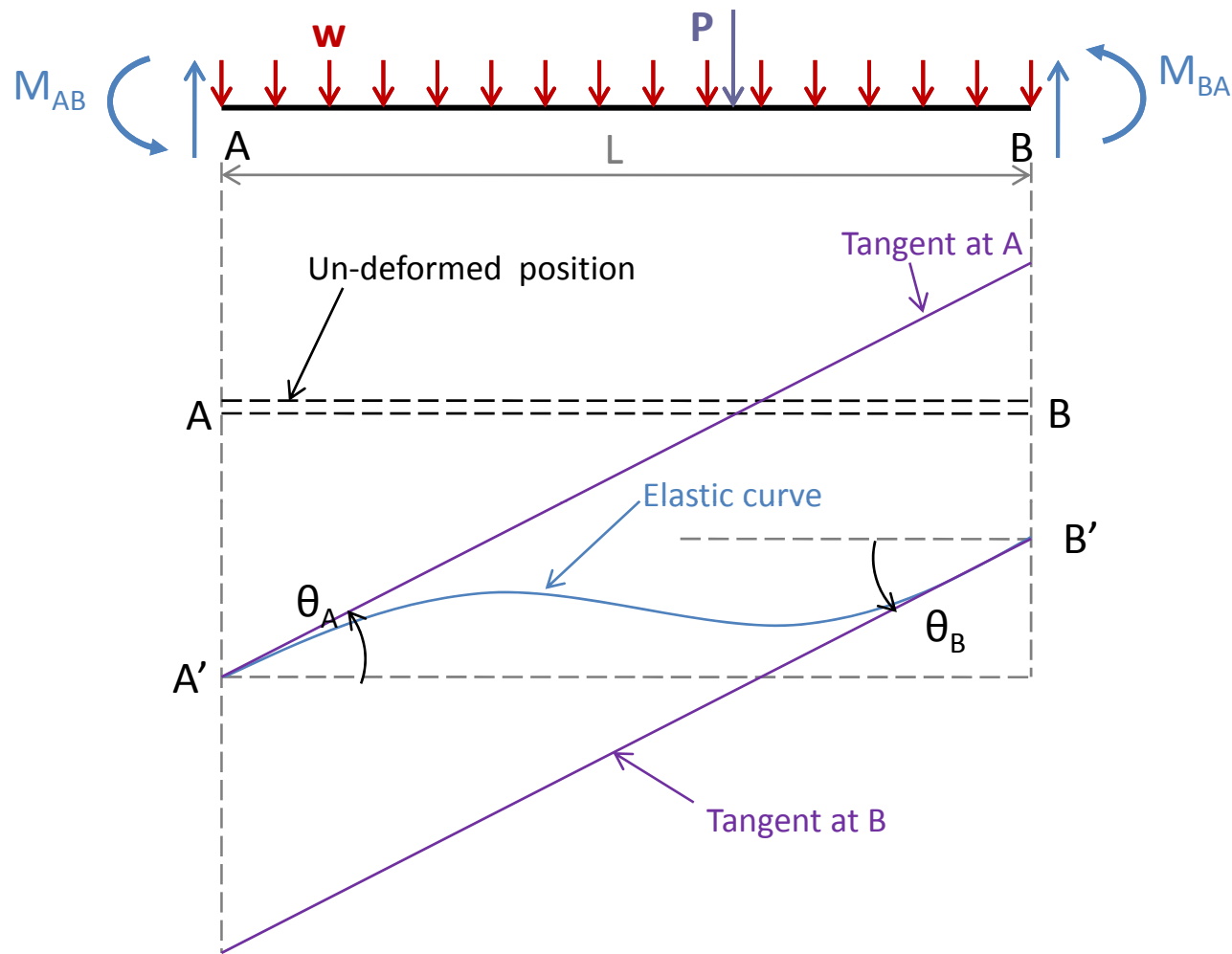
- θ_A & θ_B denote, respectively, the rotations of end A and B with respect to the un-deformed (horizontal) position of the member.
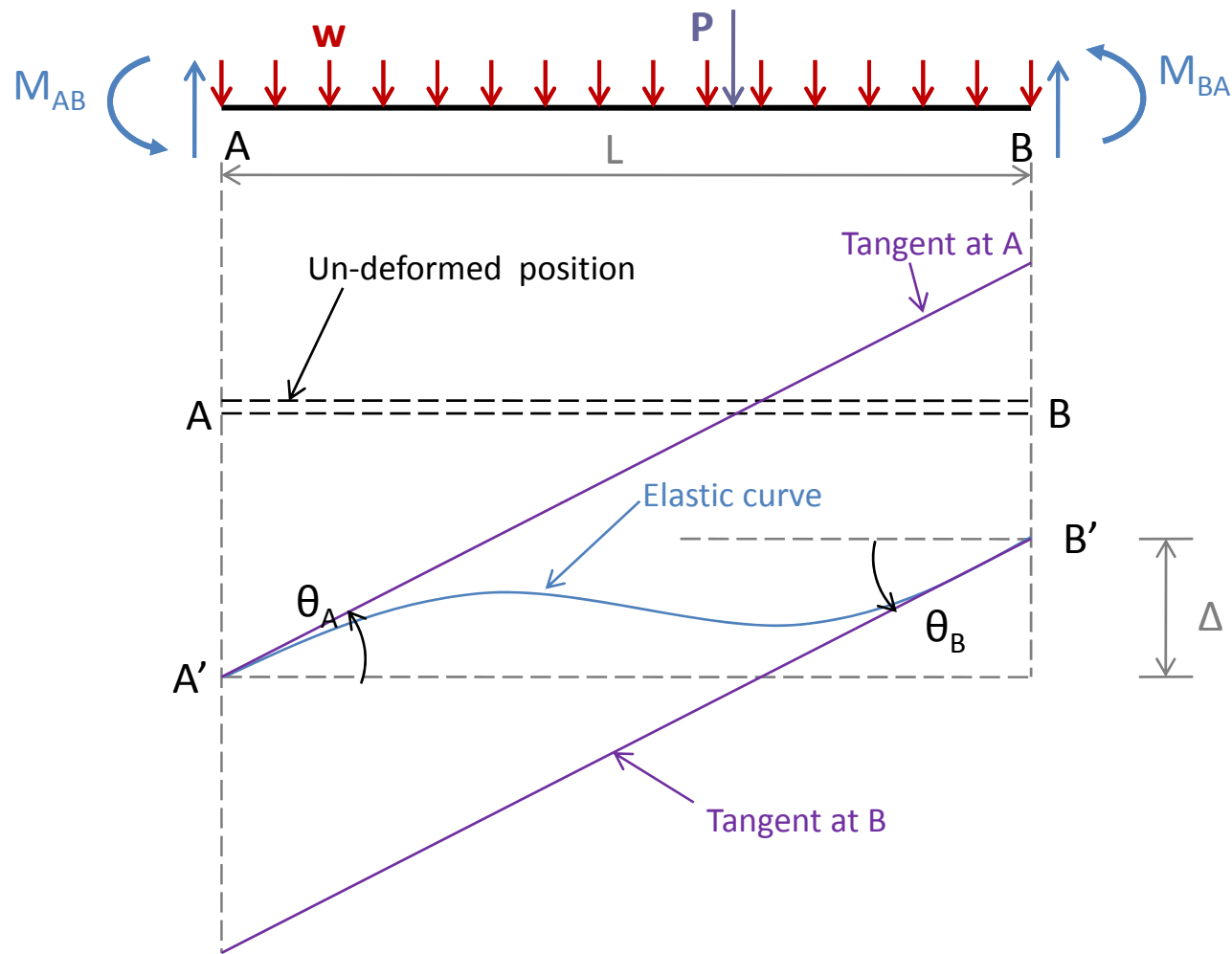
- Δ denotes the relative translation between the two ends of the member in the direction perpendicular to the un-deformed axis of the member.
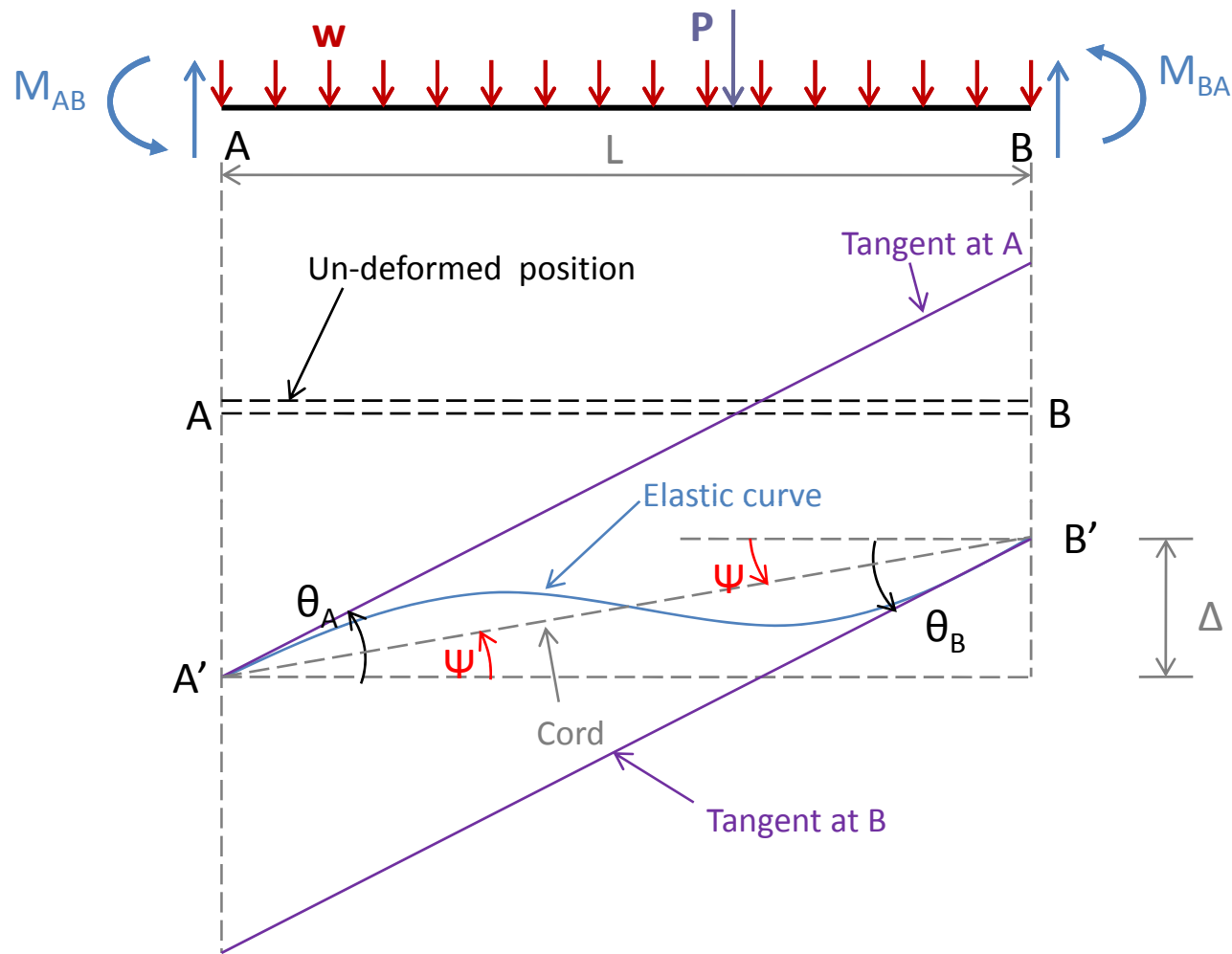
- Ψ denotes the rotation of the member's chord (straight line connecting the deformed positions of the member ends) due to the relative translation Δ.
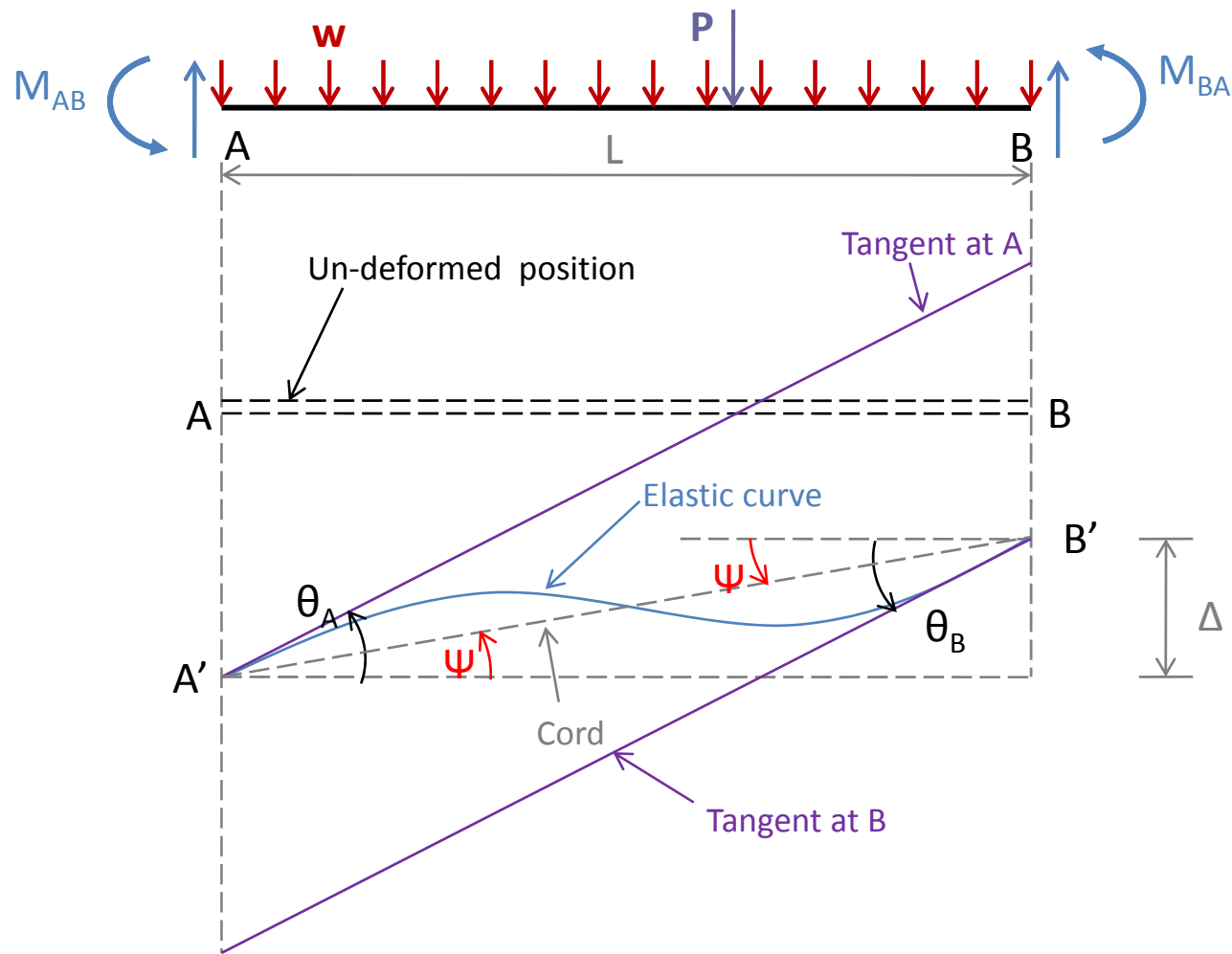
- Since the deformations are assumed to be small, the chord rotation can be expressed as

$$\psi = \frac{\Delta}{L} \qquad (1)$$

- The sign convention used in this chapter is as follows:

  "The member end moments, end rotations, and chord rotation are positive when counterclockwise."

- Note that all the moments and rotations are shown in positive sense in figure on previous slide.

- The slope deflection equations can be derived by relating the member end moments to the end rotations and chord rotation by applying the second moment-area theorem.
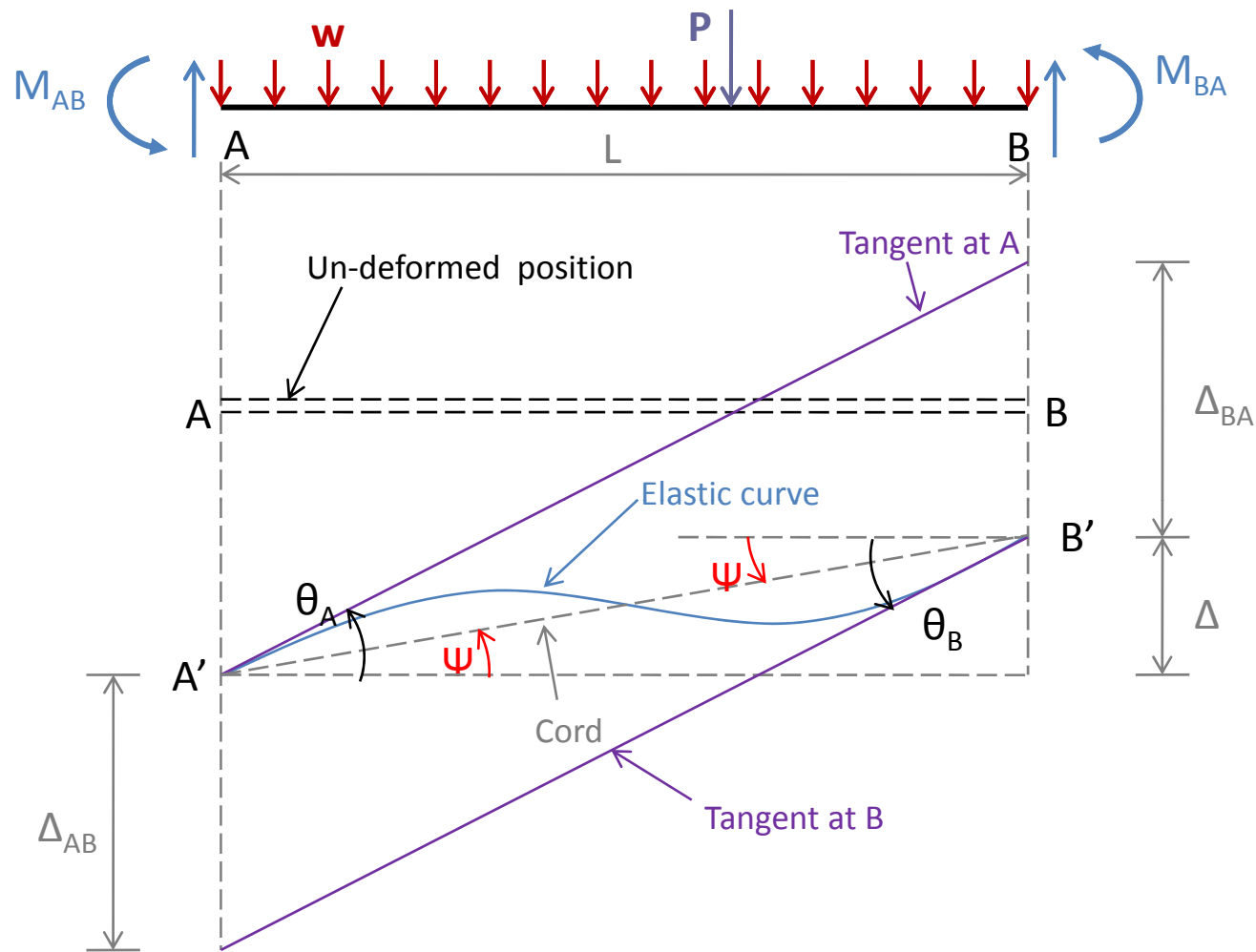
- From figure we can see that

$$\theta_A = \frac{\Delta_{BA} + \Delta}{L} \qquad \theta_B = \frac{\Delta_{AB} + \Delta}{L} \qquad (2)$$

- By substituting *Δ/L=Ψ* into the preceding equation we have,

$$\theta_A - \psi = \frac{\Delta_{BA}}{L} \qquad \theta_B - \psi = \frac{\Delta_{AB}}{L} \tag{3}$$

- Δ_{BA} is tangential deviation of end B from the tangent to the elastic curve at end A and Δ_{AB} is the tangential deviation of end A from the tangent to the elastic curve at end B.

- According to the second-moment area theorem, the expressions for the tangential deviations Δ_{AB} and Δ_{BA} can be obtained by summing the moments about the ends A and B, respectively, of the area under M/EI diagram between the two ends.

- The bending moment diagrams for the member is constructed in parts by applying M_{AB}, M_{BA}, and the external loading separately on the member with simply supported ends.

- The three simple-beam bending moment diagrams thus obtained are shown in Figure.

M_L diagram (simple beam bending moment diagram due to external loads)

M_{BA}

3

2

A

1

B

M_{AB}

Bending Moment Diagram
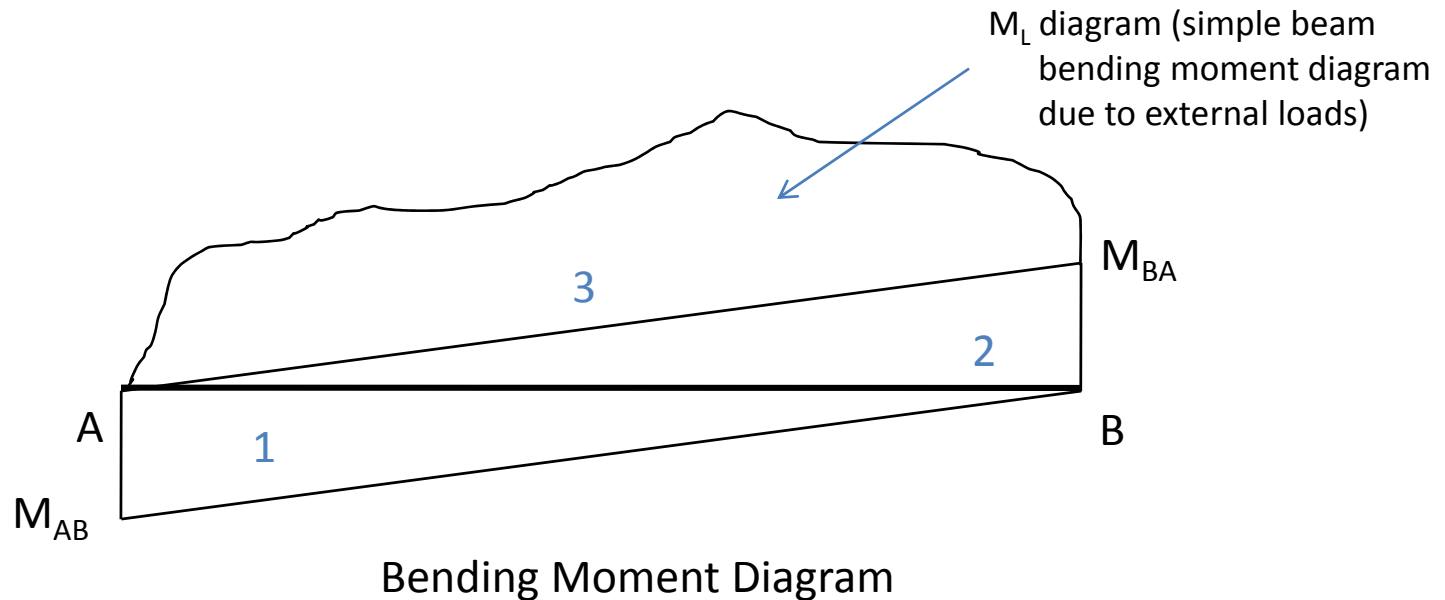
- Assuming that the member is prismatic (EI is constant along the length of the member) we sum the moments of the area under the M/EI diagram about the ends B and A, respectively, to determine the tangential deviations.
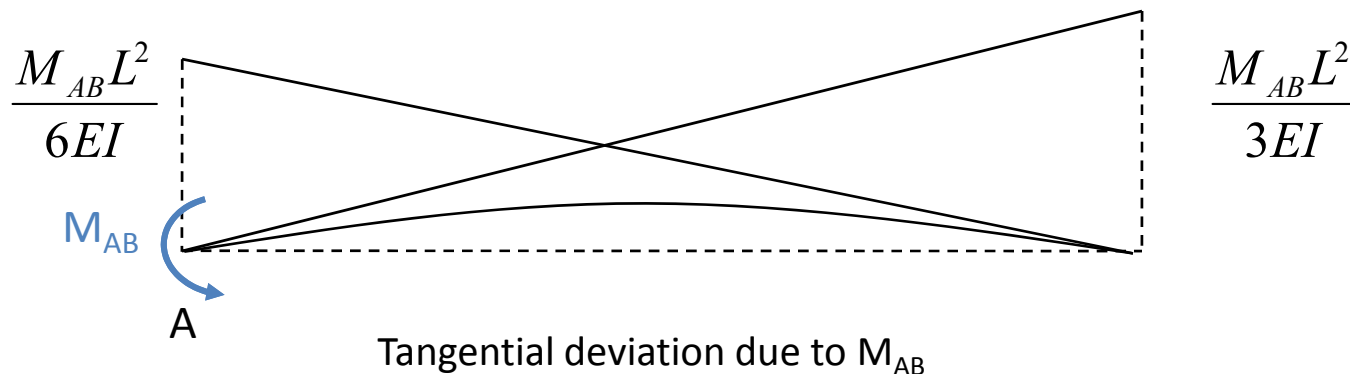
$$\Delta_{BA} = \frac{1}{EI}\left[\left(\frac{M_{AB}L}{2}\right)\left(\frac{2L}{3}\right) - \left(\frac{M_{BA}L}{2}\right)\left(\frac{L}{3}\right) - g_B\right]$$
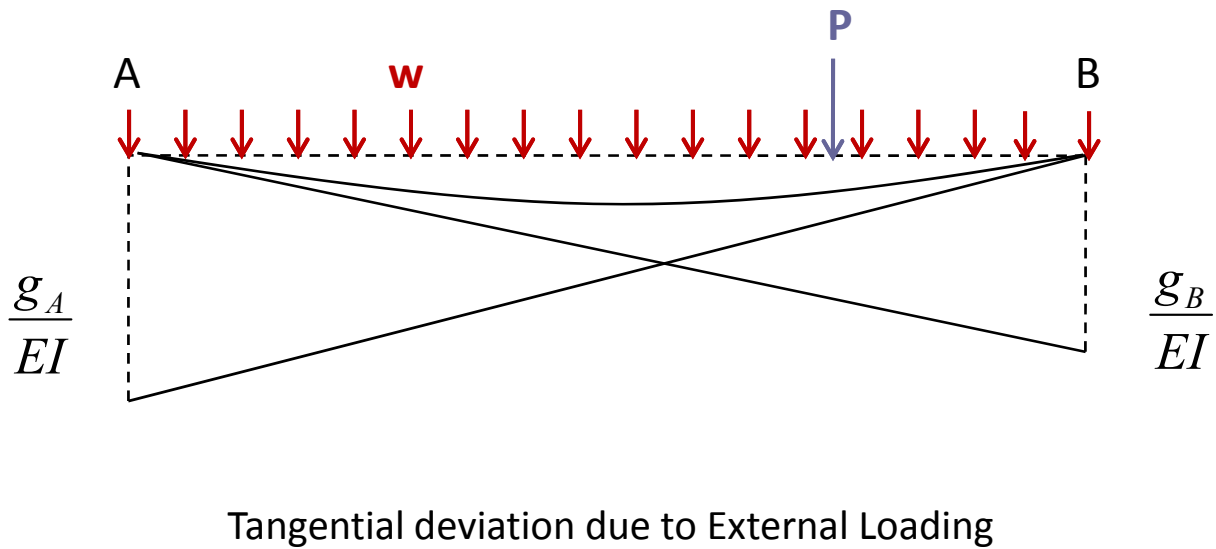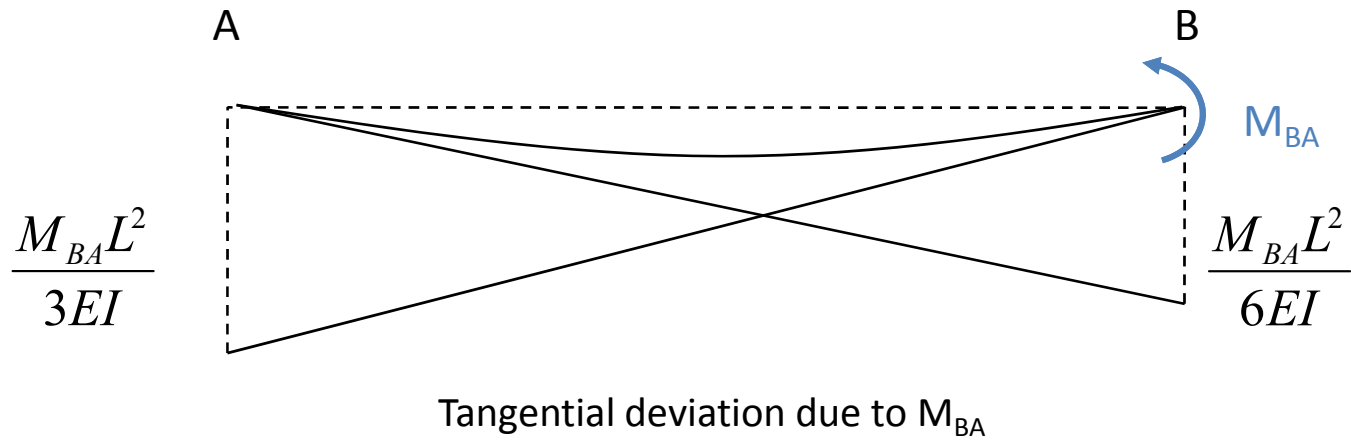
$$\Delta_{BA} = \frac{M_{AB}L^2}{3EI} - \frac{M_{BA}L^2}{6EI} - \frac{g_B}{EI} \qquad (4a)$$

$$\Delta_{AB} = \frac{1}{EI}\left[\left(-\frac{M_{AB}L}{2}\right)\left(\frac{L}{3}\right) + \left(\frac{M_{BA}L}{2}\right)\left(\frac{2L}{3}\right) + g_A\right]$$

$$\Delta_{AB} = -\frac{M_{AB}L^2}{6EI} + \frac{M_{BA}L^2}{3EI} + \frac{g_A}{EI} \qquad (4b)$$

- In which g_B and g_A are the moments about the ends B and A, respectively, of the area under the simple-beam bending moment diagram due to external loading (M_L diagram).

- The three terms in equations (4.a & 4.b) represent the tangential deviations due to M_{AB}, M_{BA}, and the external loading, acting separately on the member, with a negative term indicating that the corresponding tangential deviation is in the direction opposite to that shown on the elastic curve of the member.

$$\frac{M_{AB}L^2}{6EI}$$

$$\frac{M_{AB}L^2}{3EI}$$

M_{AB}

A

Tangential deviation due to M_{AB}

A   B

M_{BA}

$$\frac{M_{BA}L^2}{3EI}$$ $$\frac{M_{BA}L^2}{6EI}$$

Tangential deviation due to M$_{BA}$

P

A   w   B

$$\frac{g_A}{EI}$$ $$\frac{g_B}{EI}$$

Tangential deviation due to External Loading

- By substituting the expressions for Δ_{BA} and Δ_{AB} into Eq. 3, we have

$$\theta_A - \psi = \frac{M_{AB}L}{3EI} - \frac{M_{BA}L}{6EI} - \frac{g_B}{EIL} \qquad (5a)$$

$$\theta_B - \psi = -\frac{M_{AB}L}{6EI} + \frac{M_{BA}L}{3EI} + \frac{g_A}{EIL} \qquad (5b)$$

- To express the member end moments in terms of the end rotations, the chord rotation, and the external loading, we solve Eq. 5 simultaneously for M_{AB} and M_{BA}. Rewriting Eq. 5a as

$$\frac{M_{BA}L}{3EI} = \frac{2M_{AB}L}{3EI} - \frac{2g_B}{EIL} - 2(\theta_A - \psi)$$

- By substituting this equation into Eq. 5b and solving the resulting equation for M$_{AB}$, we have

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi\right) + \frac{2}{L^2}\left(2g_B - g_A\right) \qquad (6a)$$

and by substituting Eq. 6a into either Eq. 5a or 5b, we have

$$M_{BA} = \frac{2EI}{L}\left(\theta_A + 2\theta_B - 3\psi\right) + \frac{2}{L^2}\left(g_B - 2g_A\right) \qquad (6b)$$

- It indicates that the moments develop at the ends of a member depend on the rotations and translations of member's ends as well as on the external loading applied between the ends.

- Now, suppose that the member under consideration, instead being a part of a larger structure, was an isolated beam with both ends completely fixed against rotations and translations, as shown.

$\theta_A = \theta_B = \Psi = 0$
Fixed-End Moments
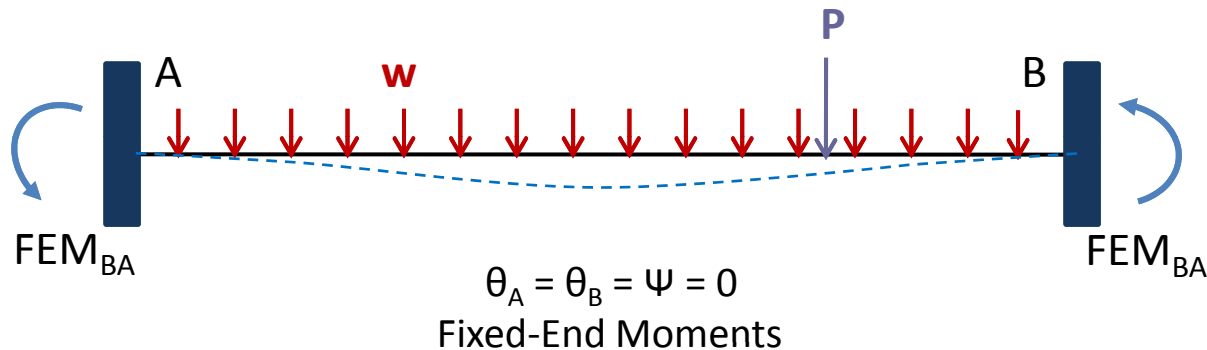
- The moments that would develop at the ends of such a fixed beam are referred to as fixed-end moments and their expression can be obtained by setting $\theta_A = \theta_B = \Psi = 0$; that is,

$$FEM_{AB} = \frac{2}{L^2}\left(2g_B - g_A\right) \qquad (7a)$$

$$FEM_{BA} = \frac{2}{L^2}\left(g_B - 2g_A\right) \qquad (7b)$$

- By comparing Eqs. 6 & 7, we find that the second terms on the right sides of Eqs. 6 are equal to the fixed-end moments.

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi\right) + FEM_{AB} \qquad (8a)$$

$$M_{BA} = \frac{2EI}{L}\left(\theta_A + 2\theta_B - 3\psi\right) + FEM_{BA} \qquad (8b)$$

- Equations (8a &8b), which express the moments at the ends of a member in terms of its end rotations and translations for a specified external loading, are called slope-deflections equations.

- These equations are valid for prismatic members, composed of linearly elastic material and subjected to small deformations.

- The deformations due to axial and shear forces are neglected.

- The two slope-deflection equations have the same form and either end of equations can be obtained from the other simply by switching the subscript A and B.

$$M_{nf} = \frac{2EI}{L}\left(2\theta_n + \theta_f - 3\psi\right) + FEM_{nf} \qquad (9)$$

in which the subscript n refers to the near end of the member where moment M_{nf} acts and the subscript f identifies the far (other) end of the member.

Members with One End Hinged

- The slope deflection equations derived previously are based on the condition that the member is rigidly connected to joints at both ends, so that the member end rotations θ_A and θ_B are equal to the rotations of the adjacent joints.

- When one of the member's ends is connected to the adjacent joint by a hinged connection, the moment at the hinged end must be zero.

- The slope-deflections equations can be easily modified to reflect this condition.

- With reference to the previous Figure of member AB, if the end B of the member AB is hinged, then the moment at B must be zero. By substituting $M_{BA} = 0$ into Equation (8), we write

$$M_{AB} = \frac{2EI}{L}\left(2\theta_A + \theta_B - 3\psi\right) + FEM_{AB} \qquad (10a)$$

$$M_{BA} = 0 = \frac{2EI}{L}\left(\theta_A + 2\theta_B - 3\psi\right) + FEM_{BA} \qquad (10b)$$

- Solving Eq. (10) for θ_B, we obtain

$$\theta_B = -\frac{\theta_A}{2} + \frac{3}{2}\psi - \frac{L}{4EI}\left(FEM_{BA}\right) \qquad (11)$$

- To determine θ_B from the slope deflection equations, we substitute Eq. (11) into Eq. (10a), thus obtaining the modified slope-deflection equations for member AB with a hinge at end B.

$$M_{AB} = \frac{3EI}{L}\left(\theta_A - \psi\right) + \left(FEM_{AB} - \frac{FEM_{BA}}{2}\right) \qquad (12a)$$

$$M_{BA} = 0 \qquad (12b)$$

- Similarly, it can be shown that for a member AB with a hinge at end A, the rotation of the hinged end is given by

$$\theta_A = -\frac{\theta_B}{2} + \frac{3}{2}\psi - \frac{L}{4EI}\left(FEM_{AB}\right) \qquad (13)$$

- And the modified slope-deflection equations can be expressed as

$$M_{BA} = \frac{3EI}{L}\left(\theta_B - \psi\right) + \left(FEM_{BA} - \frac{FEM_{AB}}{2}\right) \qquad (14a)$$

$$M_{AB} = 0 \qquad (14b)$$

- Because the modified slope-deflection equations given by Eqs. (12) and (14) are similar in form, they can be conveniently summarized as

$$M_{rh} = \frac{3EI}{L}\left(\theta_r - \psi\right) + \left(FEM_{rh} - \frac{FEM_{hr}}{2}\right) \qquad (15a)$$

$$M_{hr} = 0 \qquad (15b)$$

In which the subscript r refers to the rigidly connected end of the member where the moment M$_{rh}$ acts and the subscript h identifies the hinged end of the member.

- The rotation of the hinged end can now be written as

$$\theta_h = -\frac{\theta_r}{2} + \frac{3}{2}\psi - \frac{L}{4EI}\left(FEM_{hr}\right) \qquad (16)$$

Basic Concept of the Slope-Deflection Method

* To illustrate the basic concept of the slope-deflection method, consider the three-span continuous beam shown in Figure below.
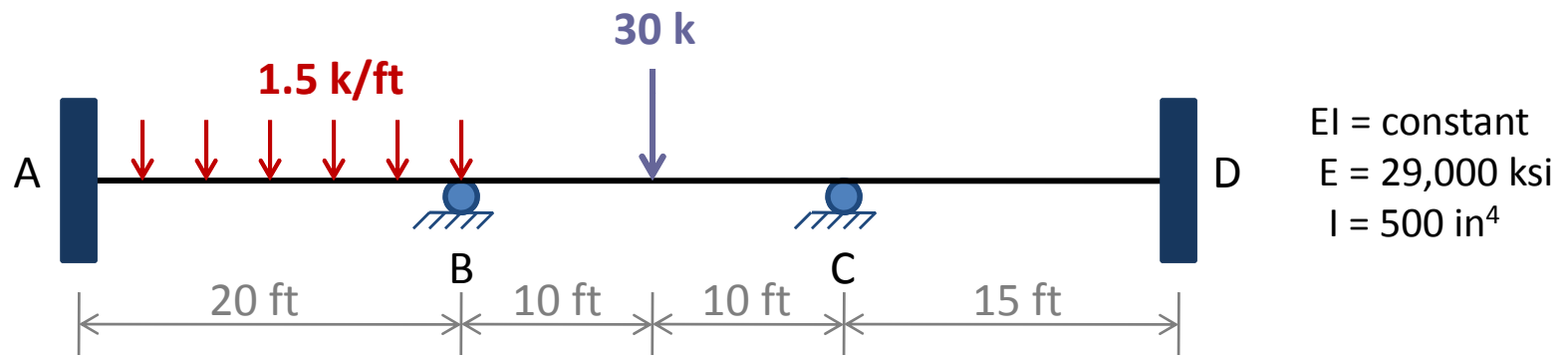
Although the structure actually consists of a single continuous beam between the fixed supports A and D, for the purpose of analysis it is considered to be composed of three members, AB, BC, and CD, rigidly connected at joints A, B, C, and D located at the supports of the structure.

Degrees of Freedom

Identify the unknown independent displacements (translations and rotations) of the joints of the structure. These unknown joint displacements are referred to as the degrees of freedom of the structure.

From the qualitative deflected shape of the continuous beam shown in Figure below, we can see that none of its joints can translate.
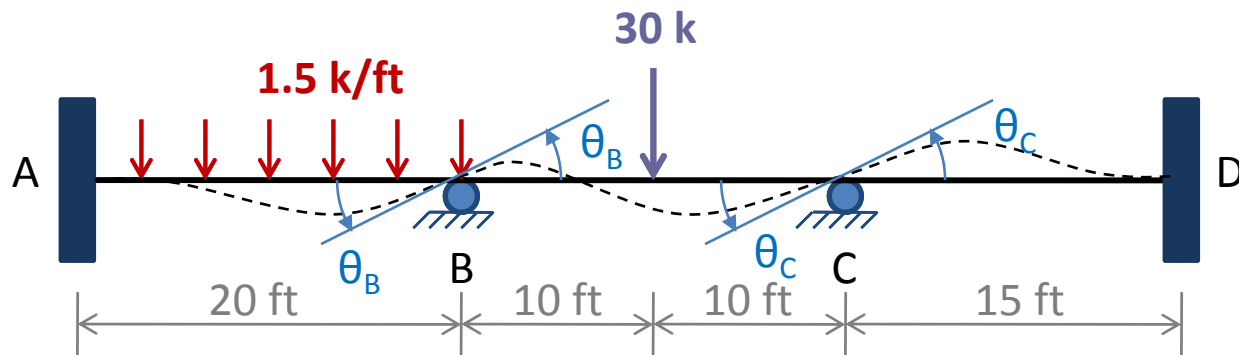
The fixed joints A and D cannot rotate, whereas joints B and C are free to rotate.

Degrees of Freedom
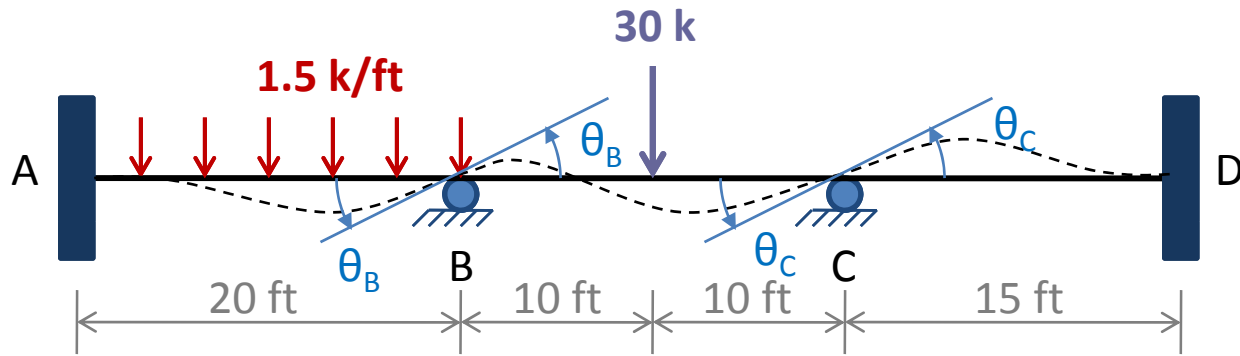
This beam has two degrees of freedom, θ_B and θ_C, which represent the unknown rotations of joints B and C, respectively.

The number of degrees of freedom is sometimes called the degree of kinematic indeterminacy of the structure. This beam is kinematically indeterminate to the second degree.

A structure without any degrees of freedom is termed kinematically determinate.

Equations of Equilibrium

The unknown joint rotations are determined by solving the equations of equilibrium of the joints that are free to rotate. The free body diagrams of the members and joints B and C of the continuous beam are shown.
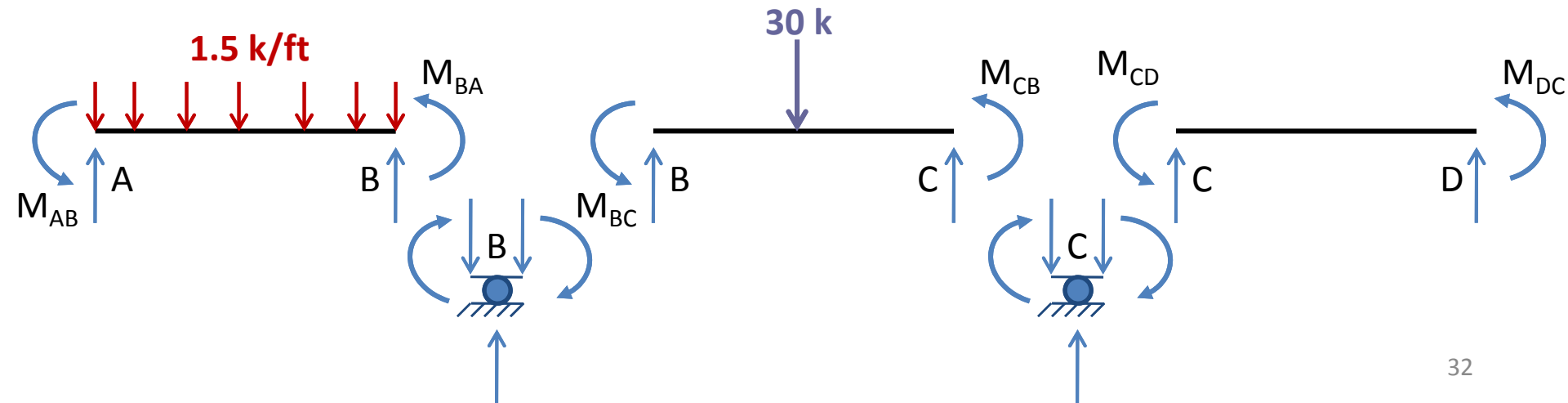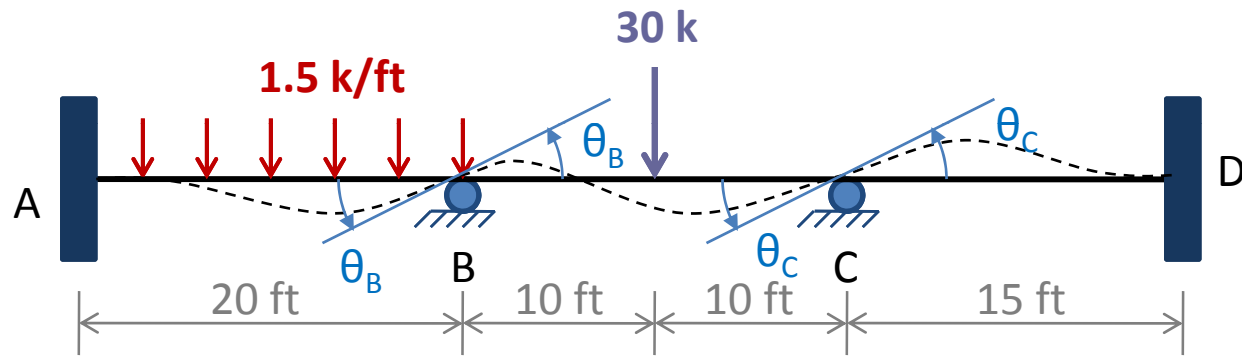
Equations of Equilibrium

In addition to the external loads, each member is subjected to an internal moment at each of its ends.

The correct senses of the member end moments are not yet known, it is assumed that the moments at the ends of all the members are positive (counterclockwise).

The free body diagrams of the joints show the member end moments acting in an opposite (clockwise) direction in accordance with Newton's law of action and reaction.
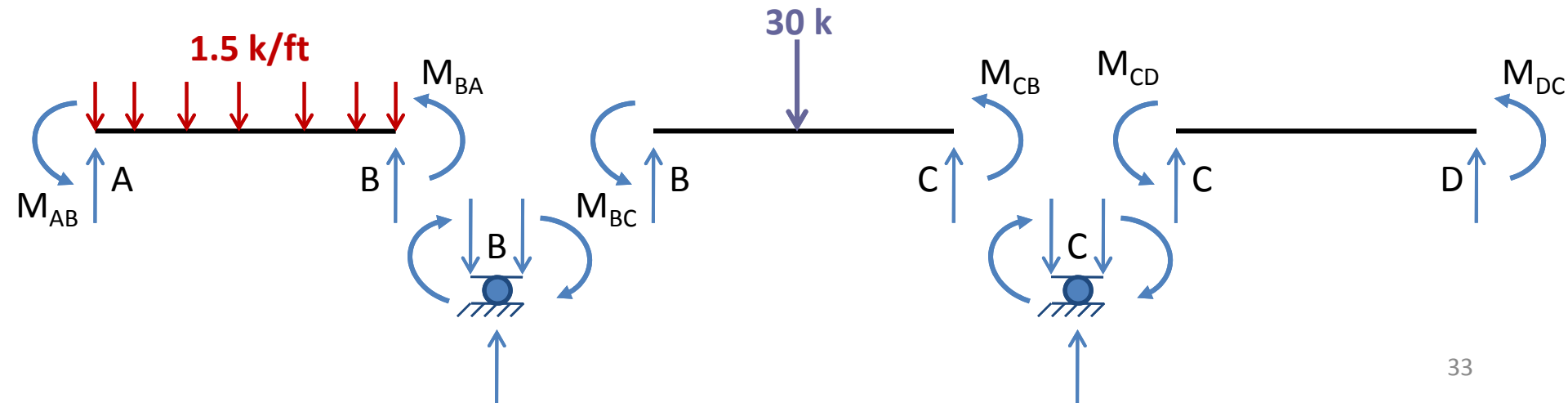
Equations of Equilibrium

Because the entire structure is in equilibrium, each of its members and joints must also be in equilibrium. By applying the moment equilibrium equations $\sum M_B = 0$ and $\sum M_C = 0$, respectively, to the free bodies of joints B and C, we obtain the equilibrium equations

$$M_{BA} + M_{BC} = 0 \qquad\qquad (17a)$$

$$M_{CB} + M_{CD} = 0 \qquad\qquad (17b)$$
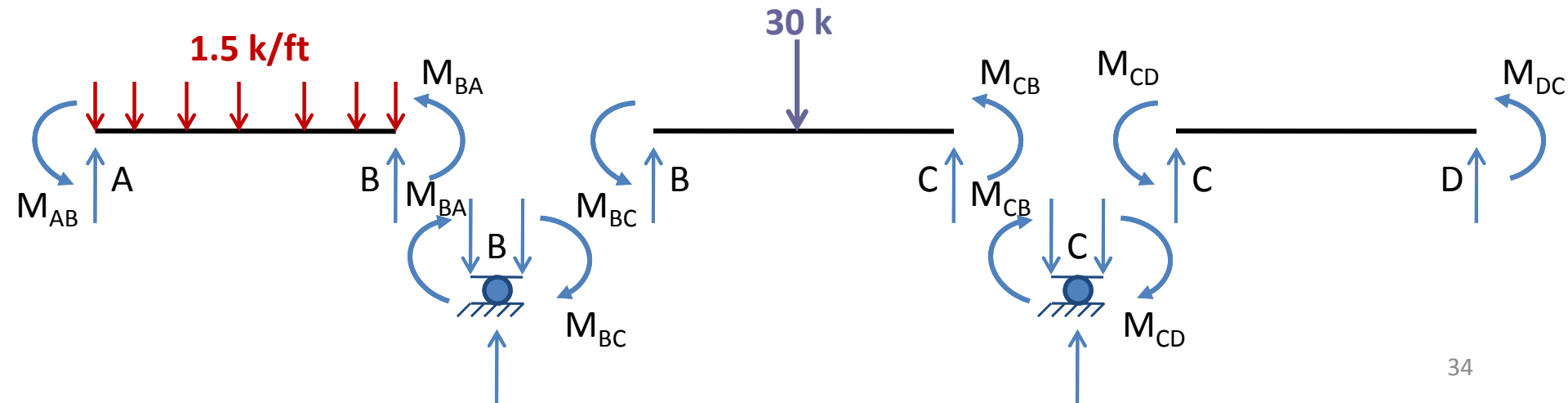
Slope-Deflection Equations

The equilibrium equations Eqs. (17) can be expressed in terms of the unknown joint rotations, θ_B and θ_C, by using slope-deflection equations that relate member end moments to the unknown joint rotations.

Before we can write the slope-deflection equations, we need to compute the fixed-end moments due to the external loads acting on the members of the continuous beam.

To calculate the fixed-end moments, we apply imaginary clamps at joints B and C to prevent them from rotating.

Or we generally provide fixed-supports at the ends of each member to prevent the joint rotations as shown.

Slope-Deflection Equations
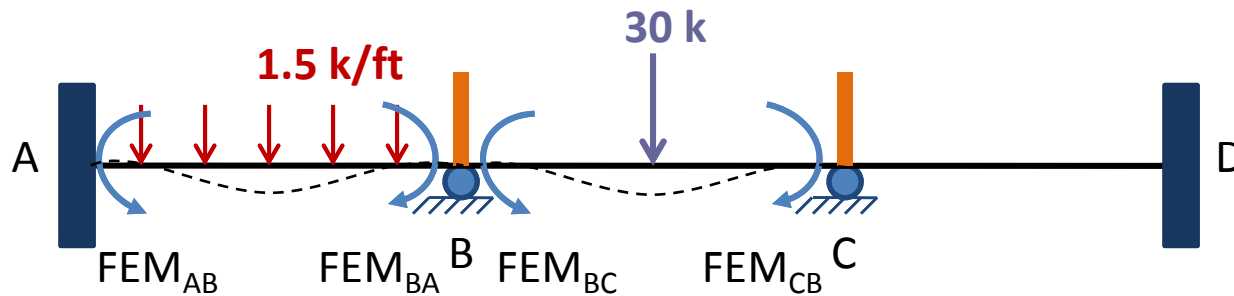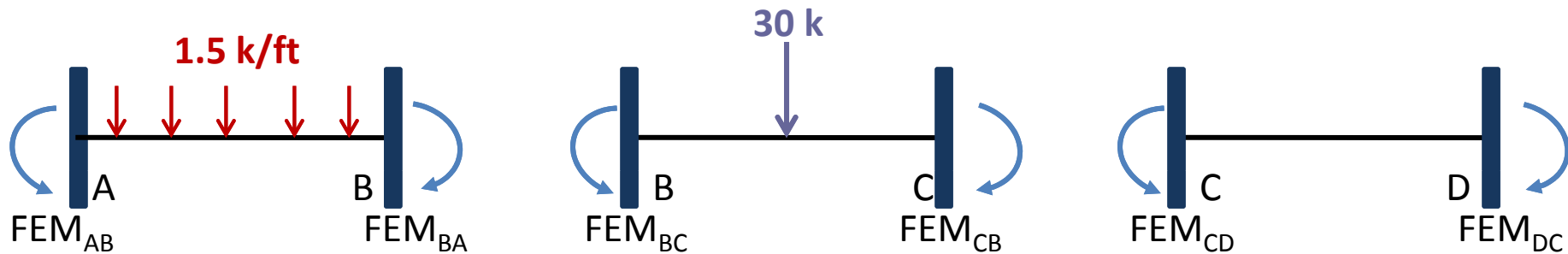
The fixed-end moments that develop at the ends of the members of this fully restrained or kinematically determinate structure can easily be evaluated by using the fixed-end moment expressions given inside the back cover of book.
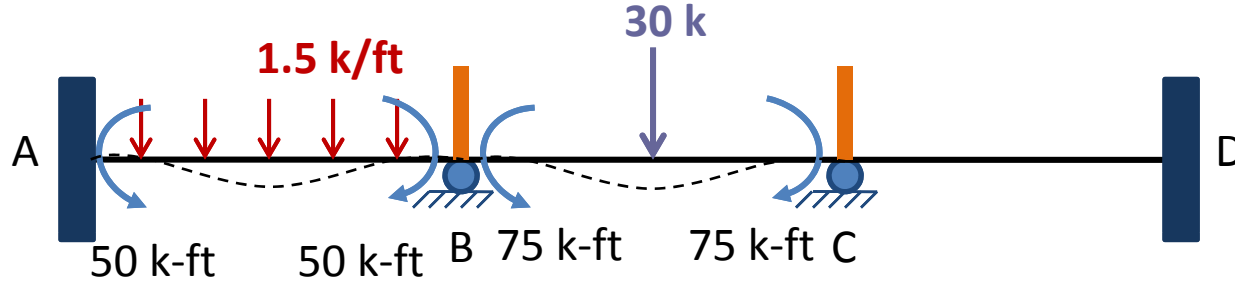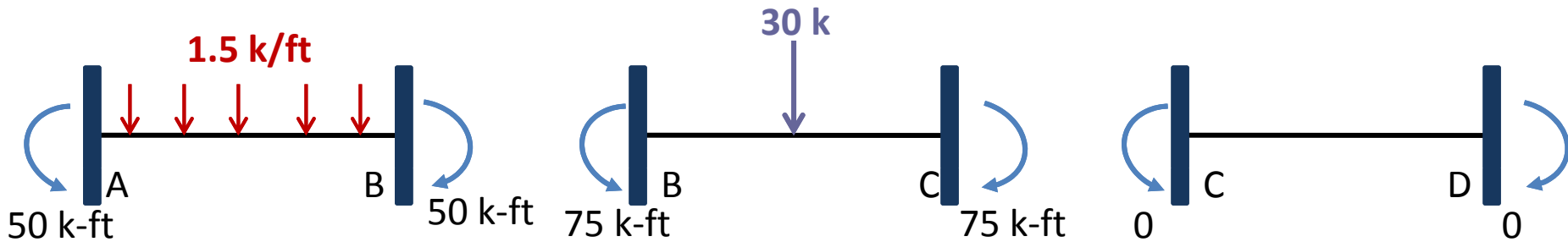
Slope-Deflection Equations
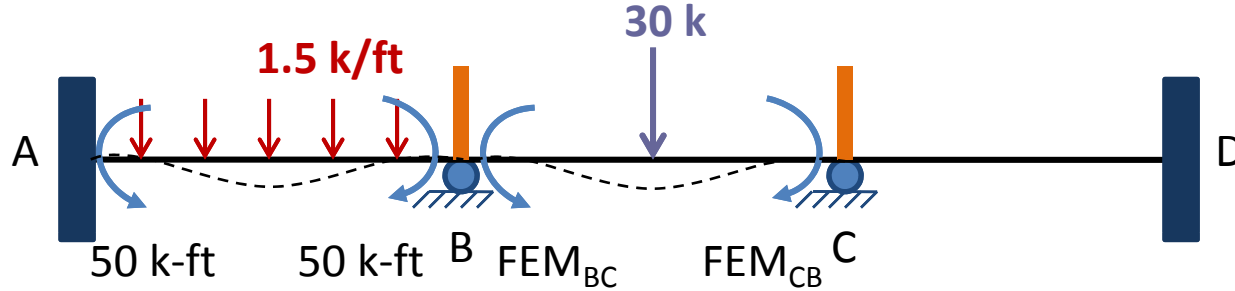
OR
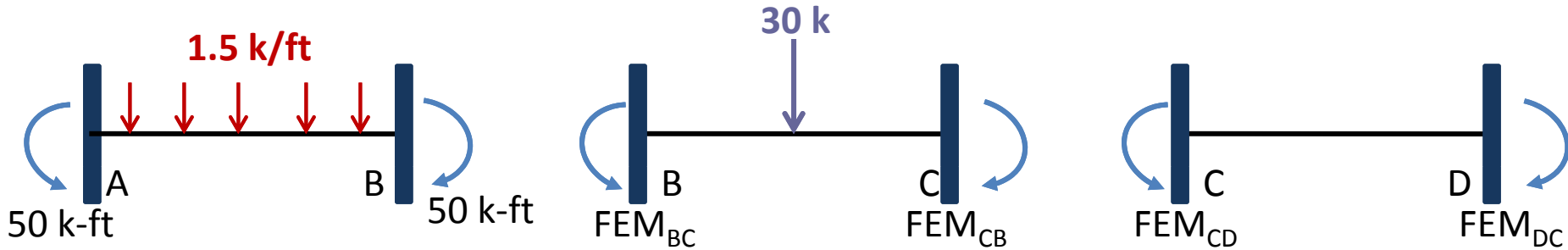
For member AB:

$$FEM_{AB} = \frac{wL^2}{12} = \frac{1.5(20)^2}{12} = 50k - ft$$

$$FEM_{BA} = \frac{wL^2}{12} = \frac{1.5(20)^2}{12} = 50k - ft$$

Slope-Deflection Equations

OR

For member BC:

$$FEM_{BC} = \frac{PL}{8} = \frac{30(20)}{8} = 75k-ft$$

$$FEM_{CB} = \frac{PL}{8} = \frac{30(20)}{8} = 75k-ft$$

Slope-Deflection Equations

The slope-deflection equations for the three members of the continuous beam can now be written by using Eq. (9).

Since none of the supports of the continuous beam translates, the chord rotations of the three members are zero ($\Psi_{AB} = \Psi_{BC} = \Psi_{CD} = 0$).

Also, supports A and D are fixed, the rotations $\theta_A = \theta_D = 0$. By applying Eq. (9) for member AB, with A as the near end and B as the far end, we obtain the slope-deflection equation

$$M_{AB} = \frac{2EI}{20}\left(0 + \theta_B - 0\right) + 50 = 0.1EI\theta_B + 50 \qquad (18a)$$

Next, by considering B as the near end and A as the far end, we write

$$M_{BA} = \frac{2EI}{20}\left(2\theta_B + 0 - 0\right) - 50 = 0.2EI\theta_B - 50 \qquad (18b)$$

Slope-Deflection Equations

Similarly, by applying Eq. (9) for member BC, we obtain

$$M_{BC} = \frac{2EI}{20}\left(2\theta_B + \theta_C\right) + 75 = 0.2EI\theta_B + 0.1EI\theta_C + 75 \qquad (18c)$$

$$M_{CB} = \frac{2EI}{20}\left(2\theta_C + \theta_B\right) - 75 = 0.2EI\theta_C + 0.1EI\theta_B - 75 \qquad (18d)$$

and for member CD,

$$M_{CD} = \frac{2EI}{15}\left(2\theta_C\right) = 0.267EI\theta_C \qquad (18e)$$

$$M_{DC} = \frac{2EI}{15}\left(\theta_C\right) = 0.133EI\theta_C \qquad (18f)$$

Joint Rotations

To determine the unknown joint rotations θ_B & θ_C, we substitute the slope-deflection equations Eqs. (18) into the joint equilibrium equations Eqs. (17) and solve the resulting systems of equations simultaneously for θ_B & θ_C. By substituting Eqs. (18b) and (18c) into Eq. (17a), we obtain

$$\left(0.2EI\theta_B - 50\right) + \left(0.2EI\theta_B + 0.1EI\theta_C + 75\right) = 0$$

or $\qquad 0.4EI\theta_B + 0.1EI\theta_C = -25$ $\qquad\qquad\qquad\qquad\left(19a\right)$

and by substituting Eqs. (18d) and (18e) into Eq. (17b), we get

$$\left(0.2EI\theta_C + 0.1EI\theta_B - 75\right) + 0.267EI\theta_C = 0$$

or $\qquad 0.1EI\theta_B + 0.467EI\theta_C = 75$ $\qquad\qquad\qquad\qquad\left(19b\right)$

Joint Rotations

Solving Eqs. (19a) & (19b) simultaneously for EIθ$_B$ and EIθ$_C$, we obtain

$$EI\theta_B = -108.46 \, k - ft^2$$

$$EI\theta_C = 183.82 \, k - ft^2$$

By substituting the numerical values of E = 29,000 ksi = 29,000(12)^2 ksf and I = 500 in.^{4}, we determine the rotations of joints B and C to be

$$\theta_B = -0.011 \, rad \qquad or \qquad 0.011 \, rad$$

$$\theta_C = 0.0018 \, rad$$

Member End Moments

The moments at the ends of the three members of the continuous beam can now be determined by substituting the numerical values of $EI\theta_B$ and $EI\theta_C$ into the slope-deflection equations (Eqs. 18).

$$M_{AB} = 0.1(-108.46) + 50 = 39.2 \; k\text{-}ft \; \circlearrowright$$

$$M_{BA} = 0.2(-108.46) - 50 = -71.7 \; k\text{-}ft \quad \text{or} \quad 71.7 \; k\text{-}ft \; \circlearrowleft$$

$$M_{BC} = 0.2(-108.46) + 0.1(183.82) + 75 = 71.7 \; k\text{-}ft \; \circlearrowright$$

$$M_{CB} = 0.2(183.82) + 0.1(-108.46) - 75 = -49.1 \; k\text{-}ft \quad \text{or} \quad 49.1 \; k\text{-}ft \; \circlearrowleft$$

$$M_{CD} = 0.267(183.82) = 49.1 \; k\text{-}ft \; \circlearrowright$$

$$M_{DC} = 0.133(183.82) = 24.4 \; k\text{-}ft \; \circlearrowright$$

Member End Moments

To check that the solution of simultaneous equations (Eqs. 19) has been carried out correctly, the numerical values of member end moments should be substituted into the joint equilibrium equations (Eqs. 17). If the solution is correct, then the equilibrium equations should be satisfied.

$$M_{BA} + M_{BC} = -71.7 + 71.7 = 0 \qquad\qquad Checks$$
$$M_{CB} + M_{CD} = -49.1 + 49.1 = 0 \qquad\qquad Checks$$

The member end moments just computed are shown on the free body diagrams of the members and joints in Figure on next slide.

Member End Moments
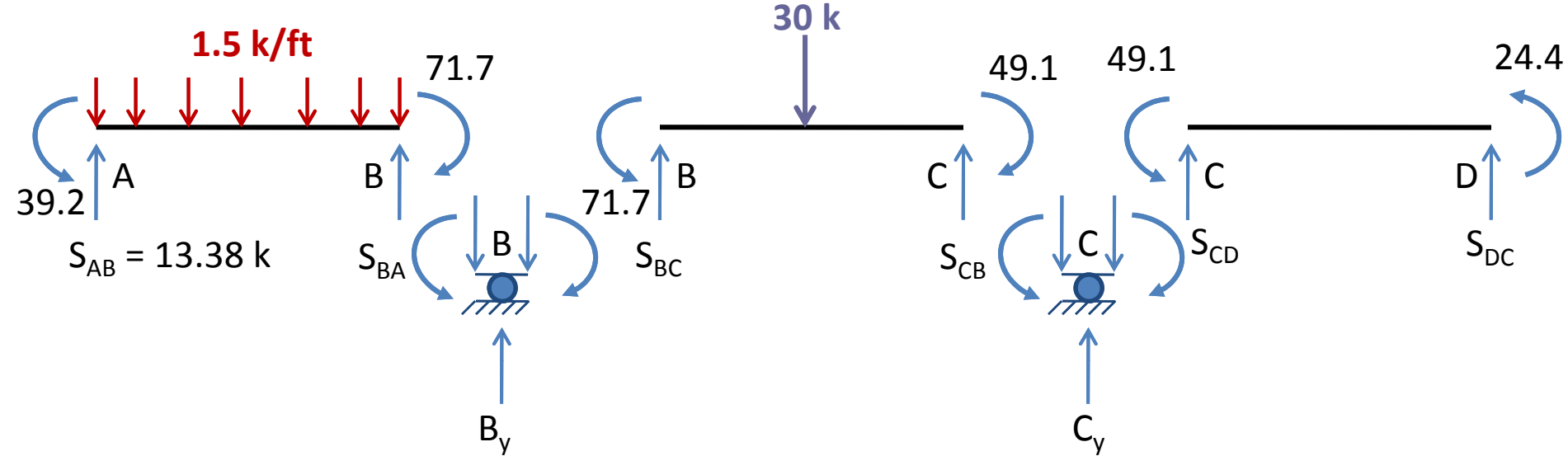
Member End Shears

The shear forces at the ends of members can now be determined by applying the equations of equilibrium to the free bodies of members. For member AB,

$$+\circlearrowleft \sum M_B = 0 \qquad 39.2 - S_{AB}(20) + 1.5(20)(10) - 71.7 = 0$$

$$S_{AB} = 13.38\, k \uparrow$$

Member End Shears

1.5 k/ft

30 k

71.7 49.1 49.1 24.4

A B B C C D

39.2

S_{BA} = 16.62 k

S_{AB} = 13.38 k

71.7 S_{BC} S_{CB} S_{CD} S_{DC}

B C

B_y C_y
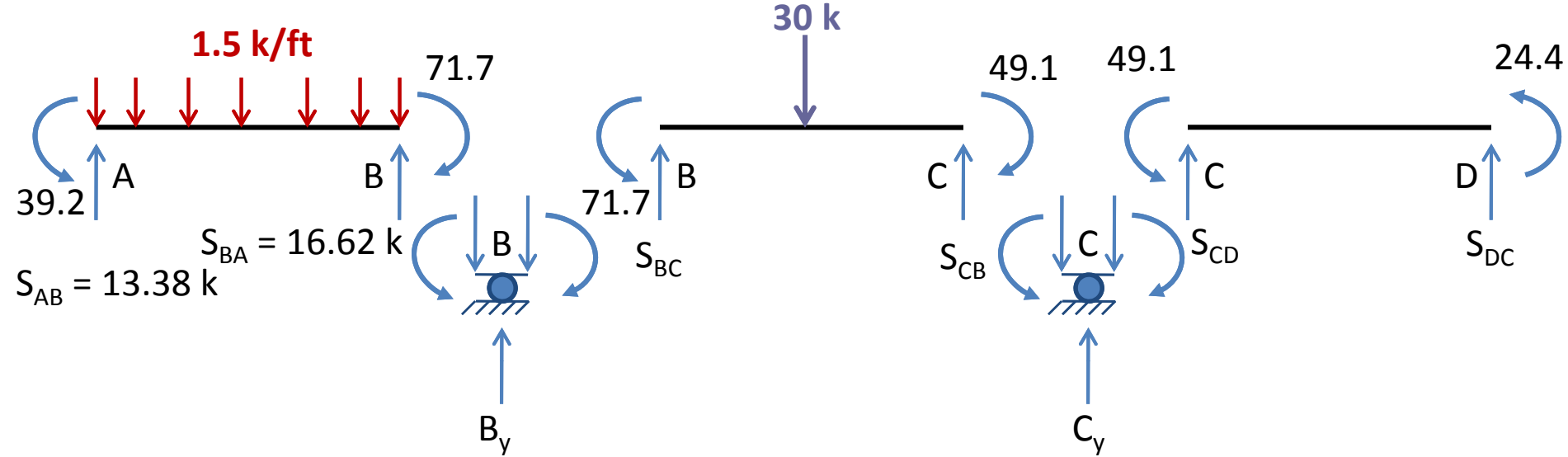
For member AB,

$$+\uparrow \sum F_y = 0$$

$$13.38 - 1.5(20) + S_{BA} = 0$$

$$S_{BA} = 16.62 \, k \uparrow$$

Member End Shears

1.5 k/ft

30 k

71.7   49.1   49.1   24.4

A   B   B   C   C   D

39.2

$S_{BA} = 16.62$ k   71.7 $S_{CB} = 13.87$ k S_{CD} S_{DC}

$S_{AB} = 13.38$ k   B $S_{BC} = 16.13$ k   C

B_y C_y
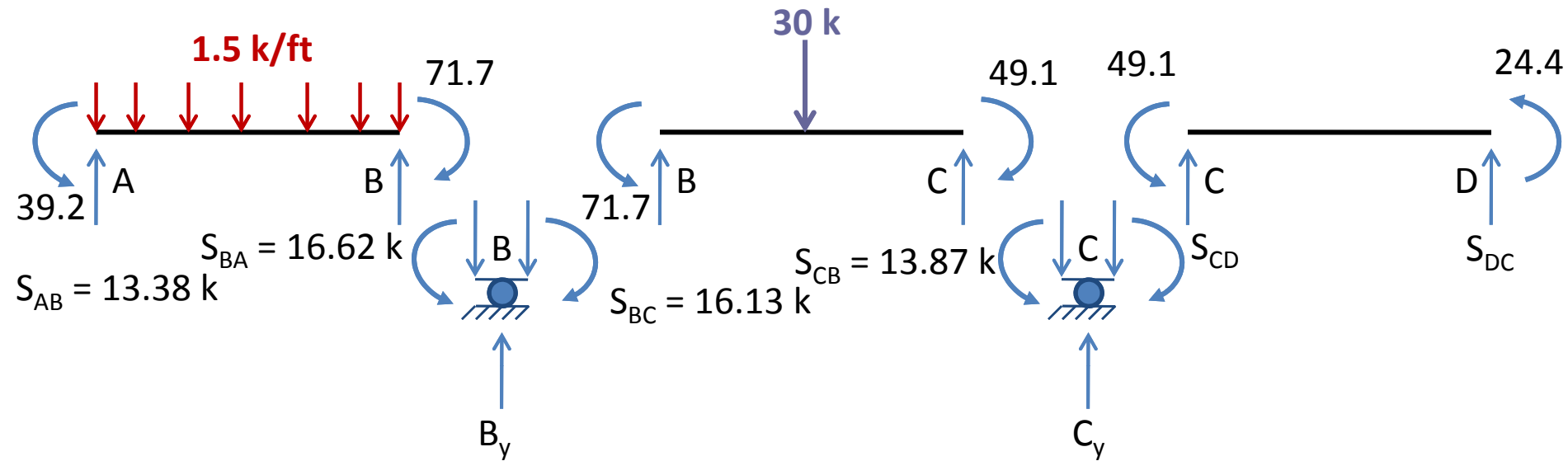
For member BC,

$$+\!\!\!\left(\;\sum M_C = 0 \qquad 71.7 - S_{BC}(20) + 30(10) - 49.1 = 0\right.$$

$$S_{BC} = 16.13\,k\uparrow$$

$$+\uparrow \sum F_y = 0 \qquad\qquad 16.13 - 30 + S_{CB} = 0$$

$$S_{CB} = 13.87\,k\uparrow$$

47

Member End Shears

1.5 k/ft

30 k

71.7

49.1   49.1

24.4

A   B

B   C

C   D

39.2

71.7

S_{BA} = 16.62 k

S_{CB} = 13.87 k

S_{CD} = 4.9 k S_{DC} = 4.9 k

S_{AB} = 13.38 k

S_{BC} = 16.13 k

B_y

C_y
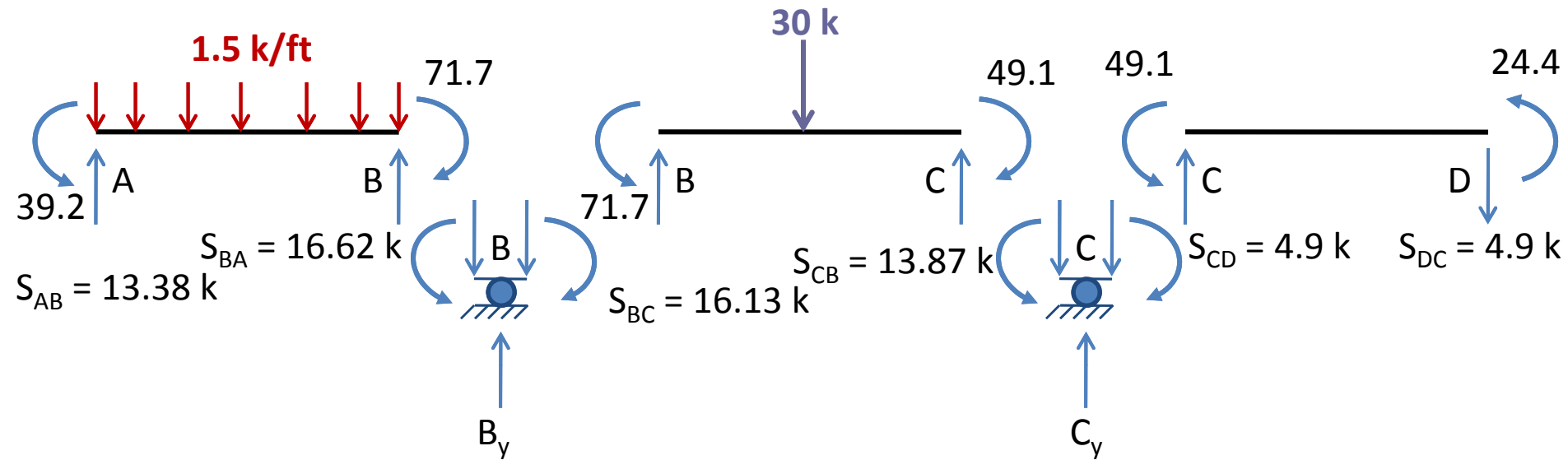
For member CD,

$$+\left(\sum M_D = 0\right.$$

$$49.1 - S_{CD}(15) + 24.4 = 0$$

$$S_{CD} = 4.9\,k \uparrow$$

$$+\uparrow \sum F_y = 0$$

$$4.9 + S_{DC} = 0$$

$$S_{DC} = 4.9\,k \downarrow$$

48

Support Reactions

1.5 k/ft

30 k

71.7

49.1 49.1

24.4

A B

B C

C D

39.2

$S_{BA} = 16.62$ k 16.62 k 16.13 k 71.7

49.1 49.1

24.4

$S_{AB} = 13.38$ k

$S_{BC} = 16.13$ k

$S_{CB} = 13.87$ k

$S_{CD} = 4.9$ k $S_{DC} = 4.9$ k

B

C

$B_y = 32.75$ k

C_y
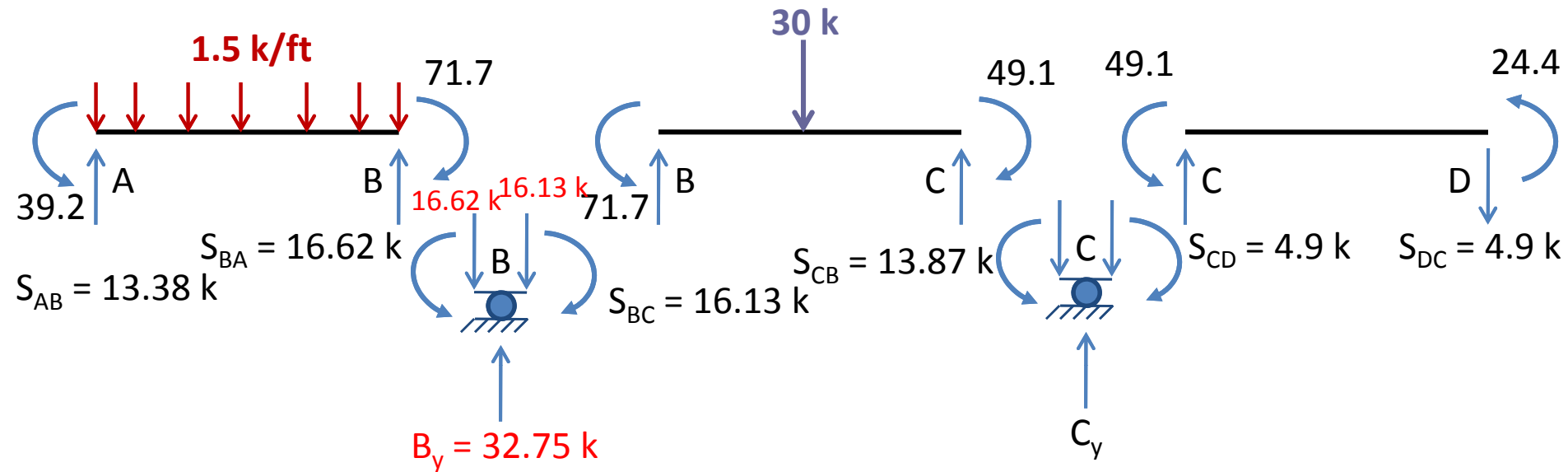
From the free body diagram of joint B, we can see that the vertical reaction at the roller support B is equal to the sum of the shears at ends B of member AB and BC; that is

$$B_y = S_{BA} + S_{BC} = 16.62 + 16.13 = 32.75 \ k \uparrow$$

49
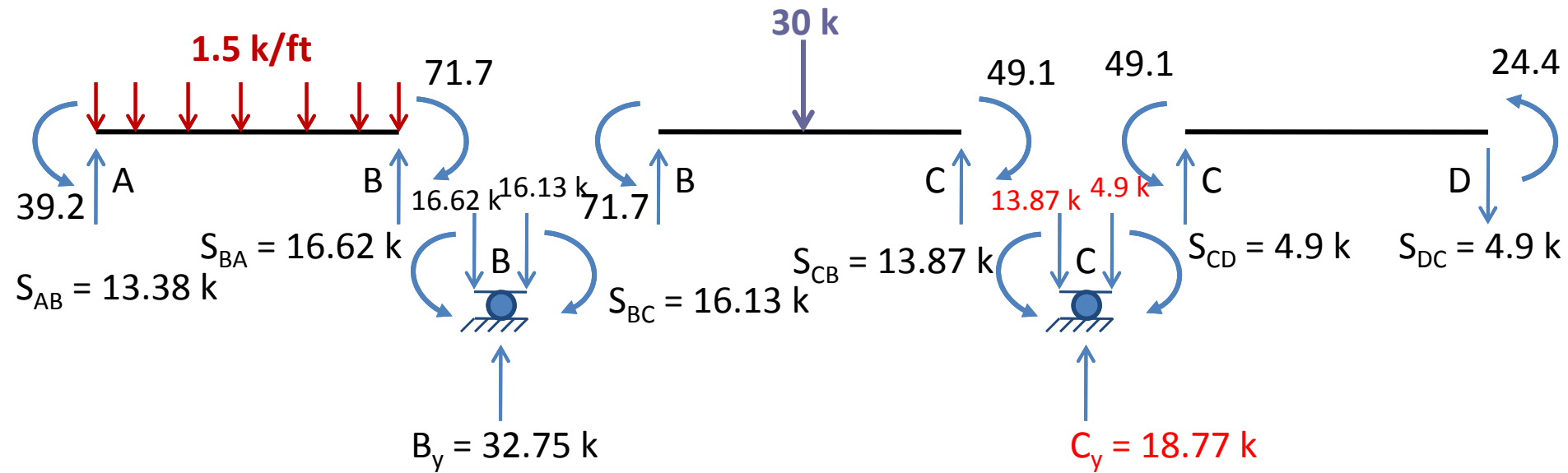
Support Reactions

The vertical reaction at the roller support C equals the sum of shears at ends C of members BC and CD.

$$C_y = S_{CB} + S_{CD} = 13.87 + 4.9 = 18.77\ k \uparrow$$

Support Reactions

1.5 k/ft

30 k

A

B

C

D

32.75 k

18.77 k
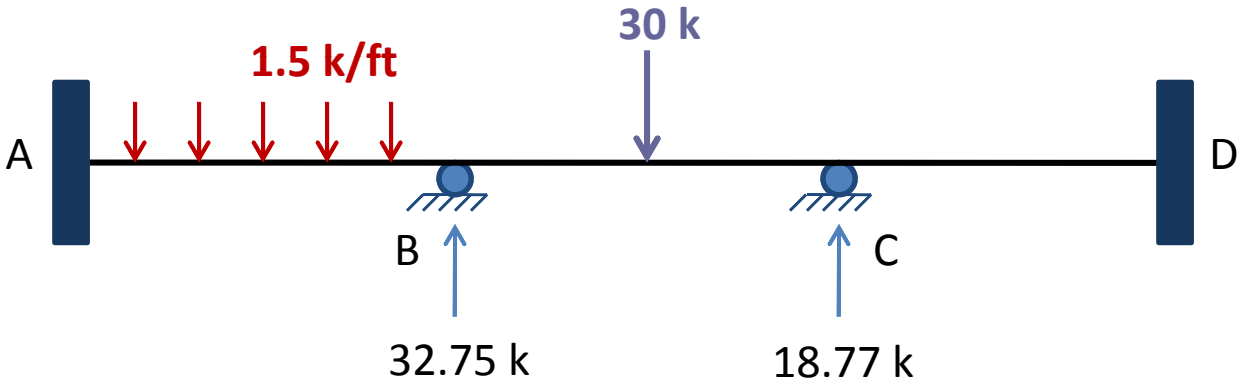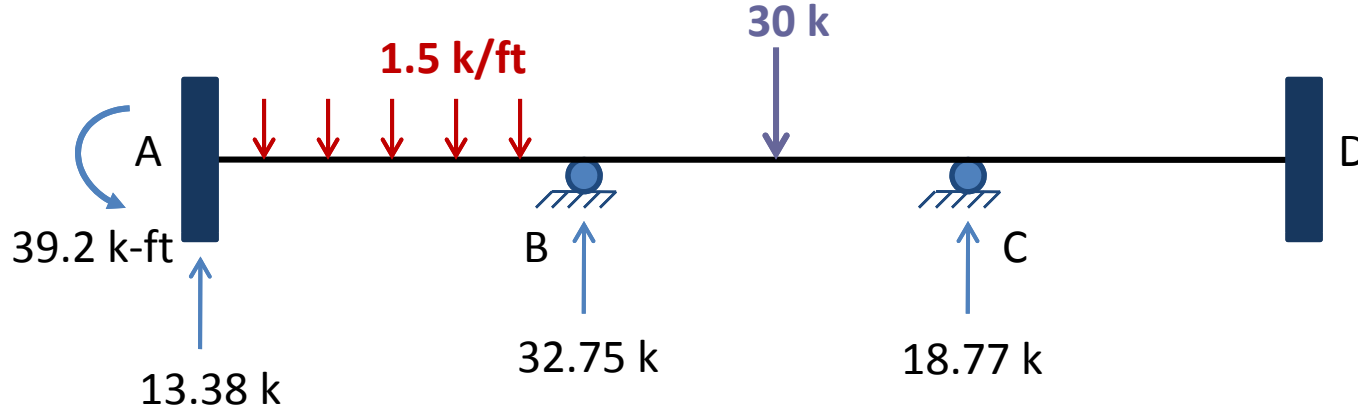
Support Reactions

The reactions at the fixed support A are equal to the shear and moment at the end A of member AB.

$$A_y = S_{AB} = 13.38\,k \uparrow$$

$$M_A = M_{AB} = 39.2\,k - ft$$

Support Reactions
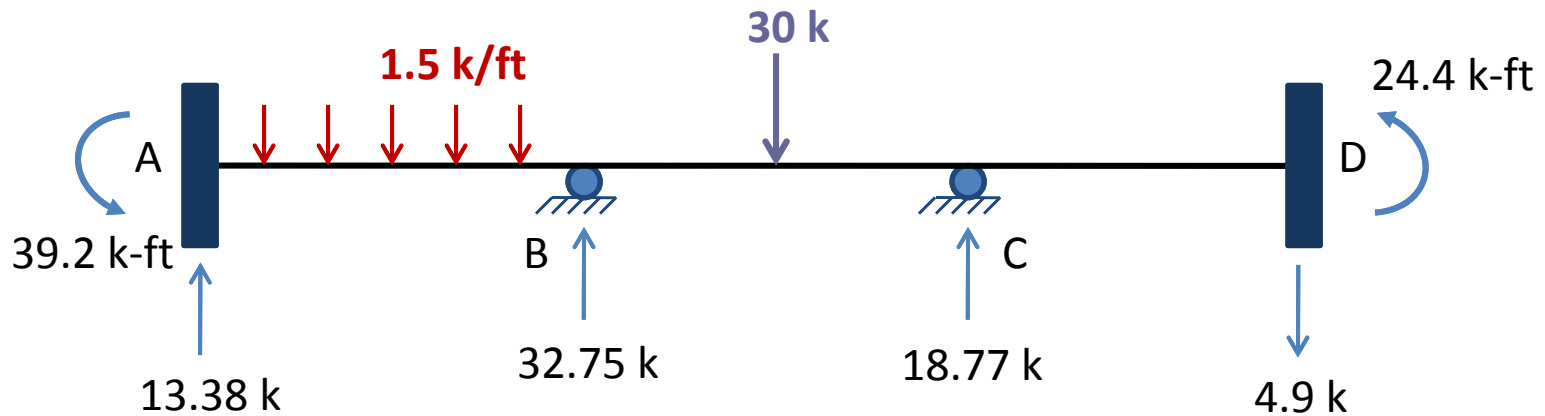
The reactions at the fixed support D equal the shear and moment at end D of the member CD.

$$D_y = S_{DC} = 4.9 \, k \downarrow$$

$$M_D = M_{DC} = 24.4 \, k - ft \, \circlearrowright$$

Equilibrium Check
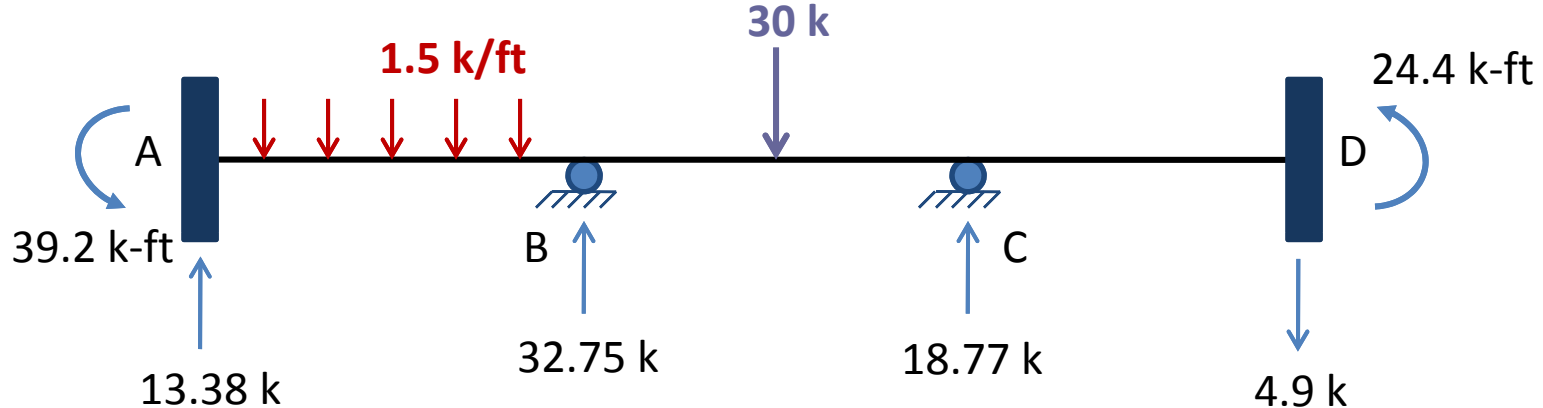
To check out computations of member end shears and support reactions, we apply the equations of equilibrium to the free body of the entire structure.

$$+\uparrow \Sigma F_y = 0$$

$$13.38 - 1.5(20) + 32.75 - 30 + 18.77 - 4.9 = 0 \qquad \text{Checks}$$

$$+\curvearrowleft \Sigma M_D = 0$$

$$39.2 - 13.38(55) + 1.5(20)(45) - 32.75(35) + 30(25)$$
$$-18.77(15) + 24.4 = -0.1 \approx 0 \qquad \text{Checks}$$

Shear Diagram
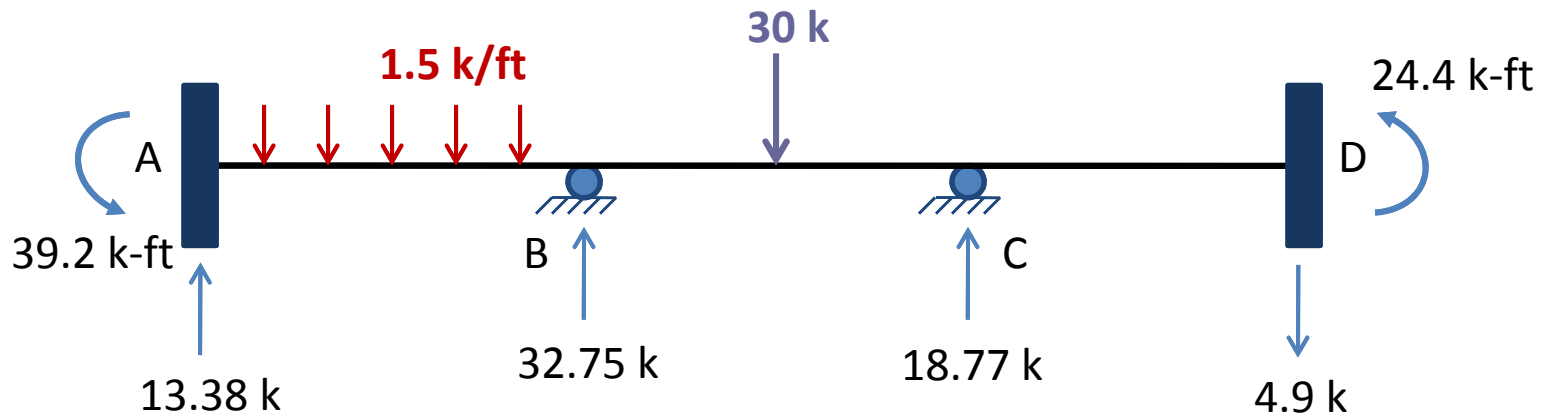
30 k

1.5 k/ft

24.4 k-ft

A   D

39.2 k-ft

B   C

13.38 k

32.75 k   18.77 k   4.9 k

Using General sign conventions

16.13

13.38

A   F   B   E   C   D

8.92 ft

-16.62

-13.87

4.9

Moment Diagram

1.5 k/ft

30 k

24.4 k-ft

A

D

39.2 k-ft

13.38 k

B

32.75 k

C

18.77 k

4.9 k

Using General sign conventions

89.7

20.5

24.4

A

B

C

D

F

E

-39.2

-49.1

-71.7

30 k

1.5 k/ft

24.4 k-ft

A

D

39.2 k-ft

B

C

13.38 k

32.75 k

18.77 k

4.9 k

16.13

13.38

A   F   B   E   C   D

4.9

8.92 ft

-13.87

-16.62

89.7

20.5

24.4

A   B   C   D

F   E

-39.2

-49.1

-71.7

57
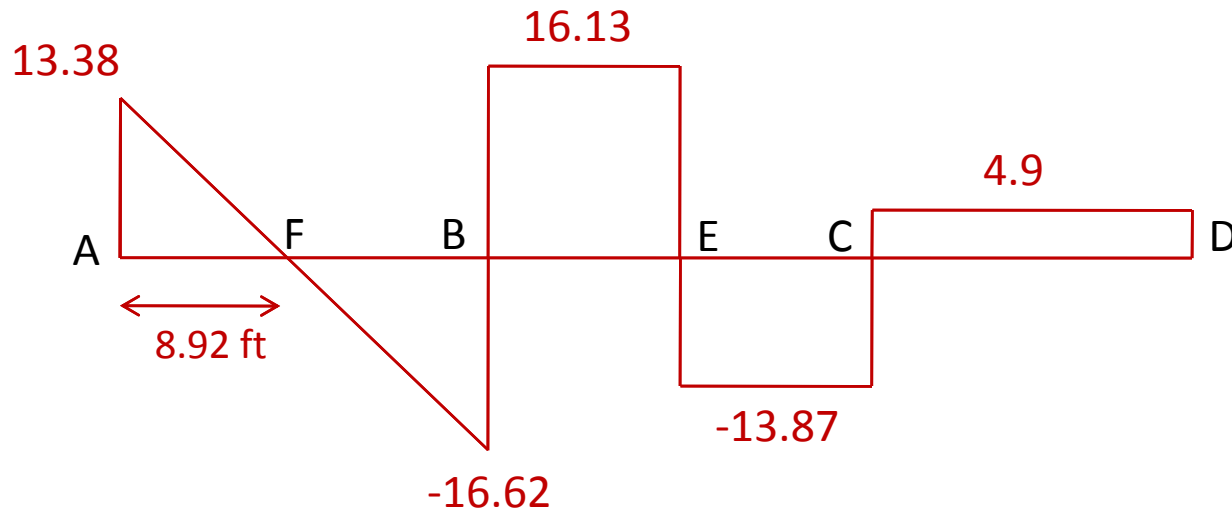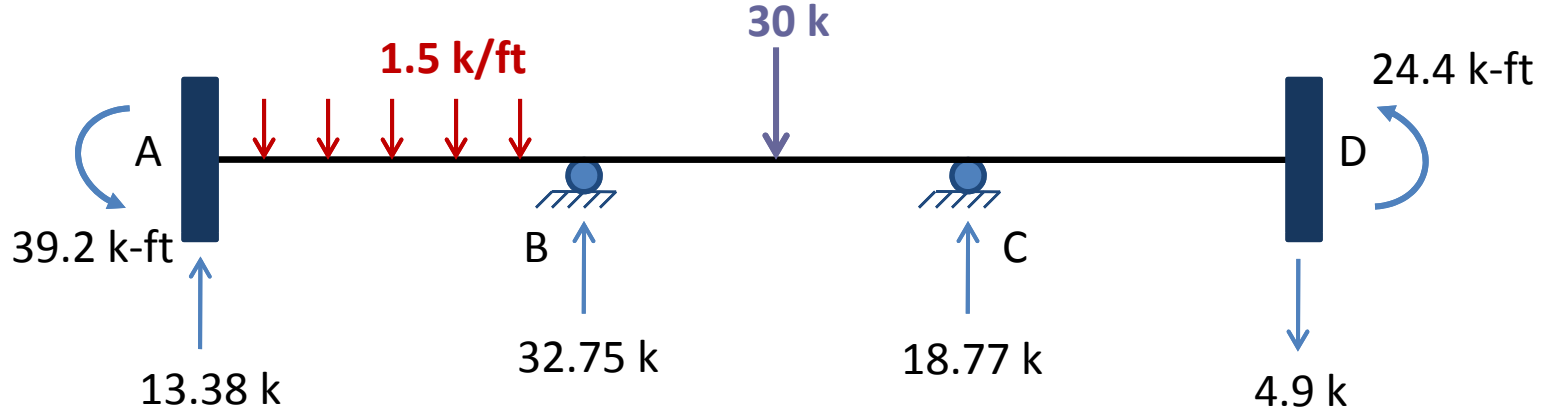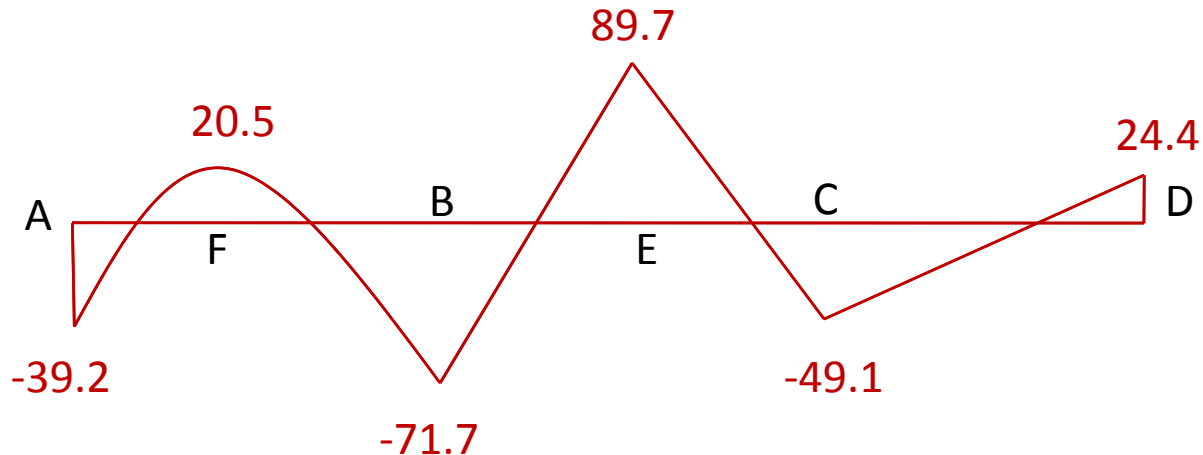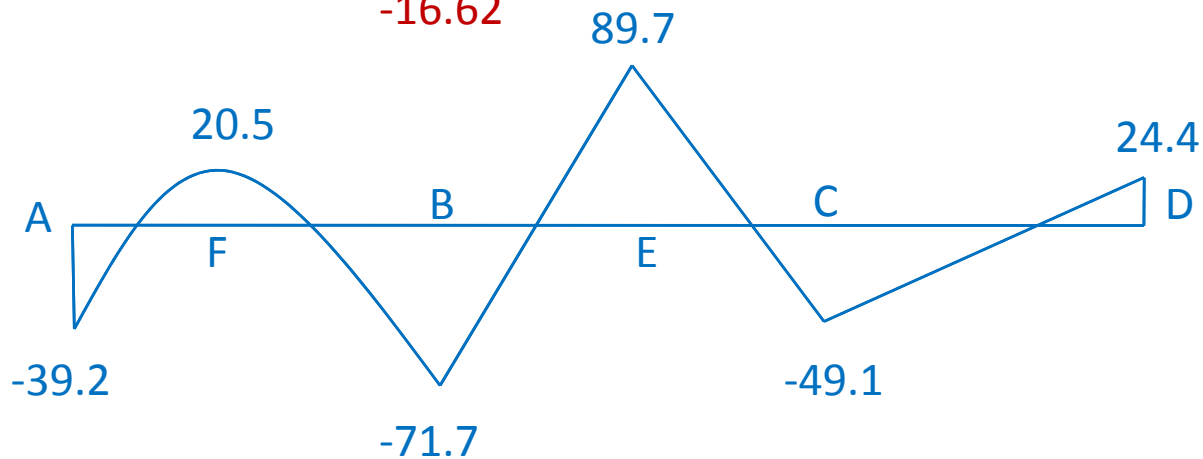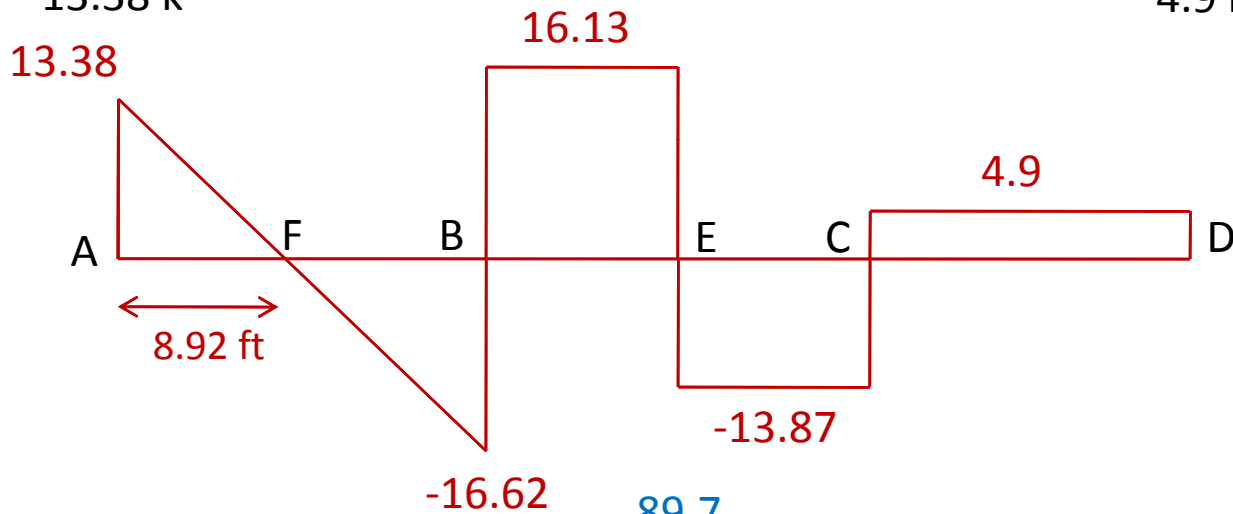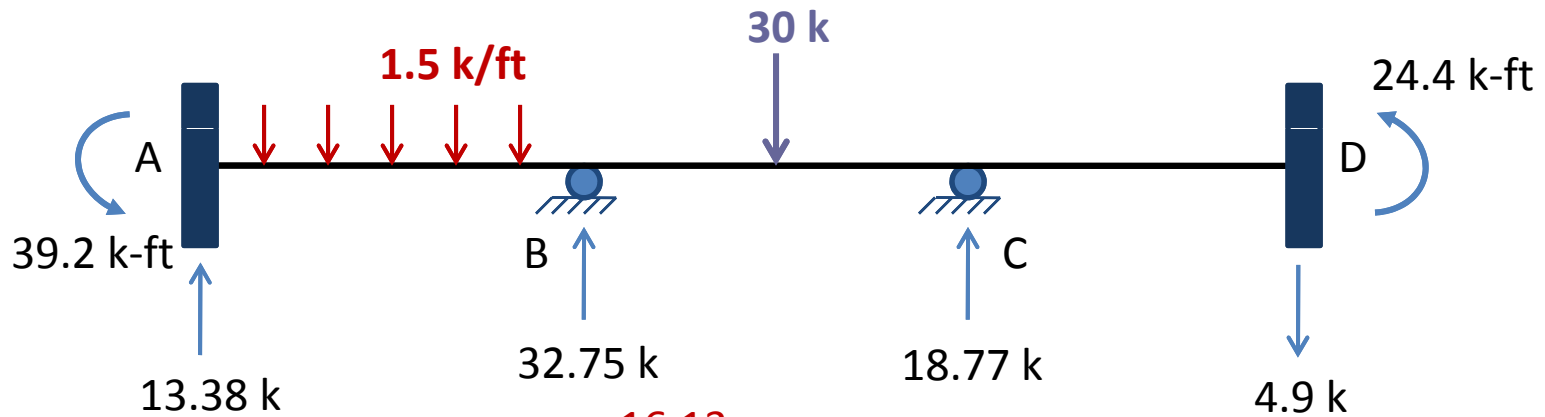
Analysis of Continuous Beam

Based on the discussion above, the procedure for the analysis of continuous beams can be summarized as follows:

1. Identify the degrees of freedom of structure.
2. Compute fixed-end moments.
3. In case of support settlements, determine the chord rotations Ψ.
4. Write slope deflection equations.
5. Write equilibrium equations for each joint.
6. Determine the unknown joint rotations.
7. Calculate member end moments by substituting the numerical values of joint rotations determined in step 6 into the slope deflection equations.
8. Satisfy the equilibrium equations for joints in step 5.
9. Compute member end shears.
10. Determine the support reactions by considering the equilibrium of joints.
11. Satisfy the equilibrium equations for end shears and support reactions.
12. Draw shear and bending moment diagrams using the beam sign convention.

Structures with Cantilever Overhangs

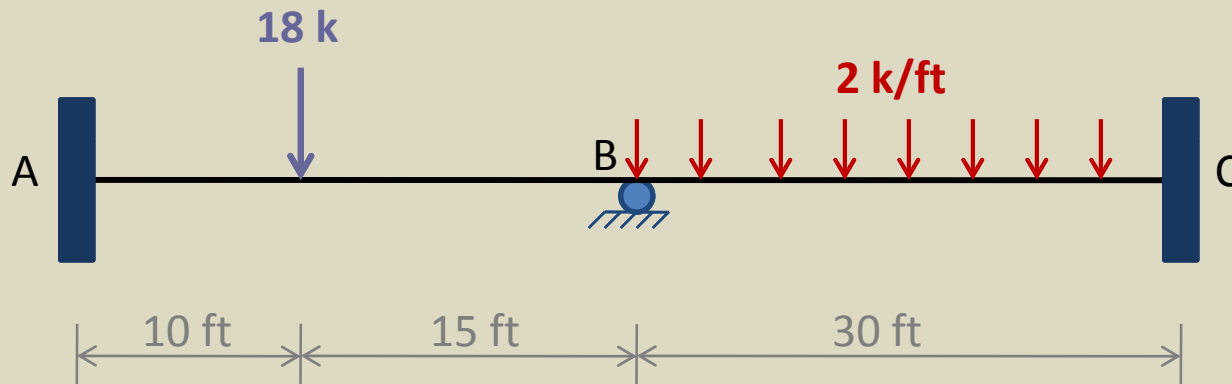Consider a continuous beam with a cantilever overhang, as shown in the figure.

w

A   D   Actual Beam

B   C

a
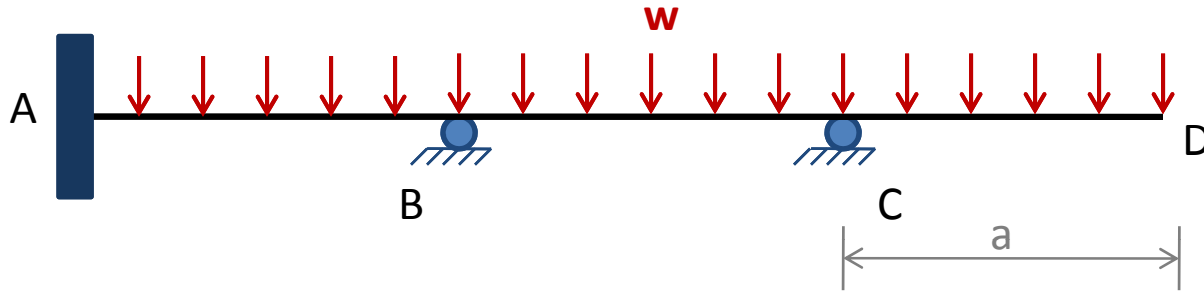
Statically Determinate Cantilever Portion

$M_{CD} = wa^2/2$ **w**

D

C

$S_{CD} = wa$

wa

$wa^2/2$

w   Actual Beam

A

B   C
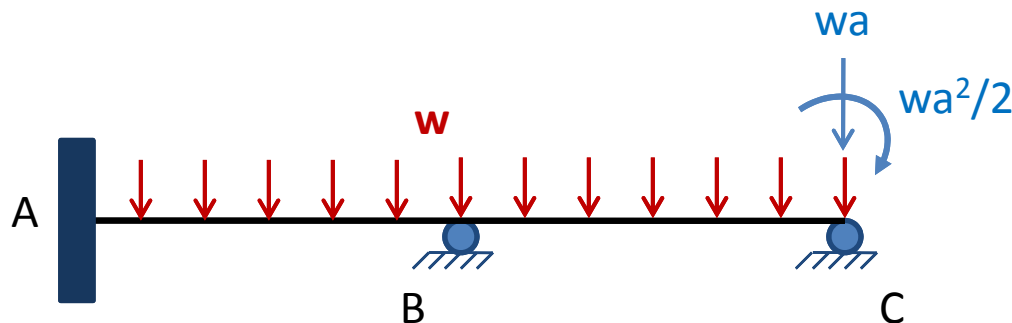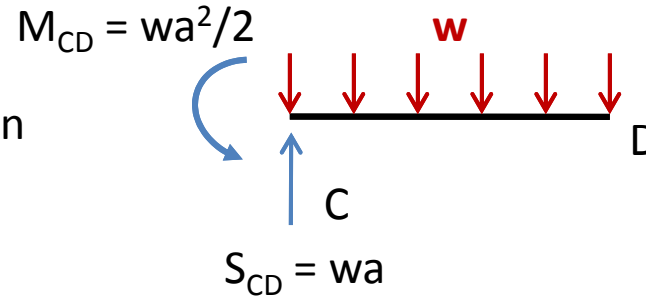
Example 1

- Determine the reactions and draw the shear and bending moment diagrams for the two-span continuous beam shown in Figure.
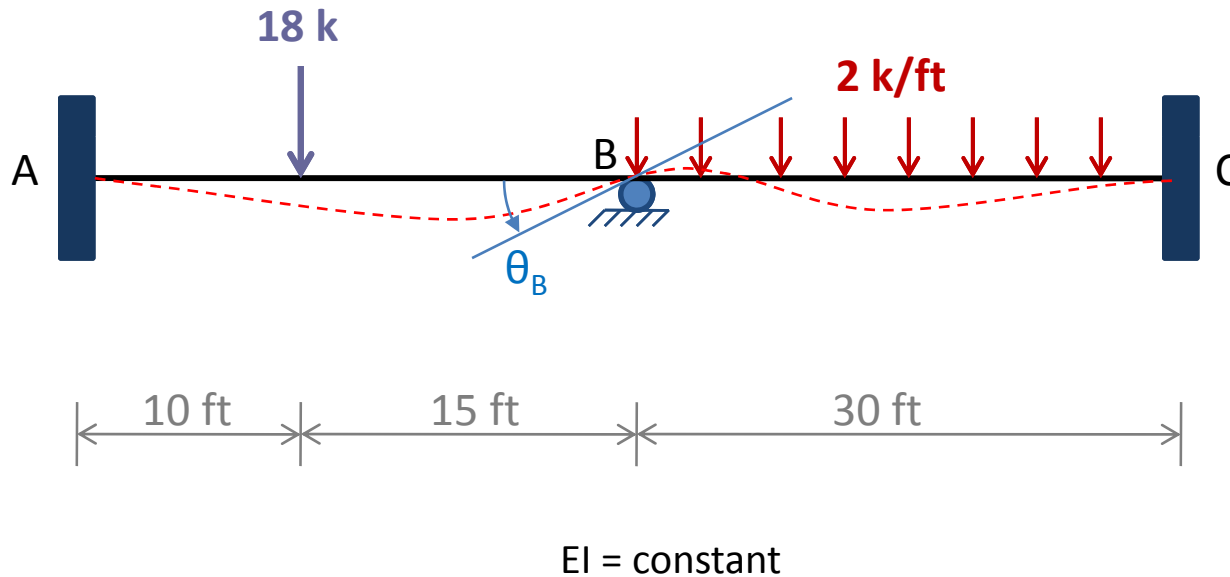

18 k

2 k/ft

A

B

C

10 ft

15 ft

30 ft

EI = constant

Solution

1. Degree of Freedom

We can see that only joint B of the beam is free to rotate. Thus, the structure has only one degree of freedom, which is the unknown joint rotation, θ_B.

EI = constant

2. Fixed-End Moments

By using the fixed-end moment expressions given inside the back cover of the book, we evaluate the fixed-end moments due to the external loads for each member.

$$FEM_{AB} = \frac{Pab^2}{L^2} = \frac{18(10)(15)^2}{25^2} = 64.8\,k-ft \,\circlearrowright \qquad\qquad or \ +64.8\ k\text{-}ft$$

$$FEM_{BA} = \frac{Pa^2b}{L^2} = \frac{18(10)^2(15)}{25^2} = 43.2\,k-ft \,\circlearrowleft \qquad\qquad or \ -64.8\ k\text{-}ft$$

$$FEM_{BC} = \frac{wL^2}{12} = \frac{2(30)^2}{12} = 150\,k-ft \,\circlearrowleft \qquad\qquad or \ +150\ k\text{-}ft$$

$$FEM_{CB} = 150\,k-ft \,\circlearrowleft \qquad\qquad or \ -150\ k\text{-}ft$$

Counterclockwise FEM are positive, whereas clockwise FEM are negative.

3. Chord Rotations

Since no support settlements occur, the chord rotations of both members are zero; that is, $\Psi_{AB} = \Psi_{BC} = 0$.

4. Slope-Deflection Equations

To relate the member end moments to the unknown joint rotation, θ_B, we write the slope deflection equation for the two members of the structure by applying Eq. (9).

$$M_{nf} = \frac{2EI}{L}\left(2\theta_n + \theta_f - 3\psi\right) + FEM_{nf} \qquad (9)$$

since the supports A and C are fixed, the rotations $\theta_A = \theta_C = 0$.

4. Slope-Deflection Equations

Slope-Deflection Equation for Member AB

$$M_{AB} = \frac{2EI}{25}(\theta_B) + 64.8 = 0.08EI\theta_B + 64.8 \qquad (1)$$

$$M_{BA} = \frac{2EI}{25}(2\theta_B) - 43.2 = 0.16EI\theta_B - 43.2 \qquad (2)$$

Slope-Deflection Equation for Member BC

$$M_{BC} = \frac{2EI}{30}(2\theta_B) + 150 = 0.133EI\theta_B + 150 \qquad (3)$$

$$M_{CB} = \frac{2EI}{30}(\theta_B) - 150 = 0.0667EI\theta_B - 150 \qquad (4)$$

5. Equilibrium Equations

The free body diagram of joint B is shown in Figure.
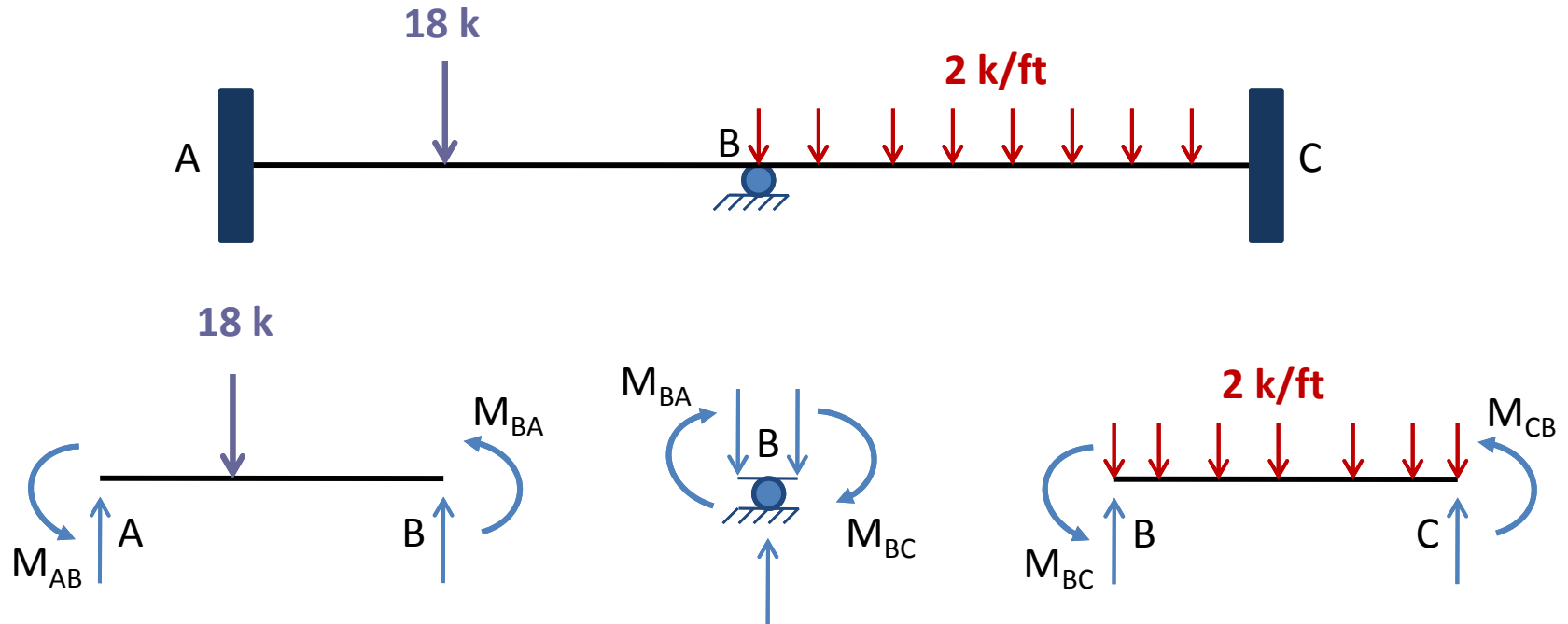
Member end moments, which are assumed to be in counterclockwise direction on the ends of members, must be applied in (opposite) clockwise direction on the free body of the joint in accordance with Newton's Third Law.

The free body diagram of joint B is shown in Figure.
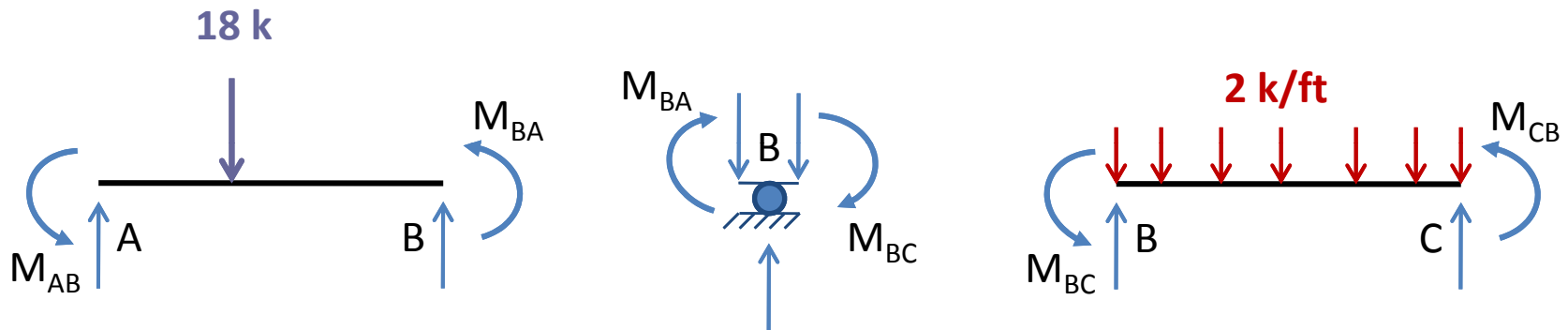
By applying the moment equilibrium equation ∑M$_B$ = 0 to the free body of the joint B, we obtain

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

6. Joint Rotations

To determine the unknown joint rotations, θ_B, substitute the slope deflection equations (Eqs. 2 & 3) into the equilibrium equation (Eq. 5).

$$\left(0.16EI\theta_B - 43.2\right) + \left(0.133EI\theta_B + 150\right) = 0$$

or

$$0.293EI\theta_B = -106.8$$

from which

$$EI\theta_B = -364.5\,k - ft^2$$

7. Member End Moments

The member end moments can now be computed by substituting the numerical value of $EI\theta_B$ back into the slope-deflection equation (Eqs. 1 to 4).

$$M_{AB} = 0.08(-364.5) + 64.8 = 35.6\ k-ft \curvearrowright$$

$$M_{BA} = 0.16(-364.5) - 43.2 = -101.5\ k-ft \qquad or \qquad 101.5\ k\text{-}ft \curvearrowright$$

$$M_{BC} = 0.133(-364.5) + 150 = 101.5\ k-ft \curvearrowright$$

$$M_{CB} = 0.0667(-364.5) - 150 = -174.3\ k-ft \qquad or \qquad 174.3\ k\text{-}ft \curvearrowright$$

Positive answer for an end moment indicates that its sense is counterclockwise, whereas a negative answer implies a clockwise sense. As M_{BA} and M_{BC} are equal in magnitude but opposite in sense, the equilibrium equation $M_{BA} + M_{BC} = 0$ is satisfied.
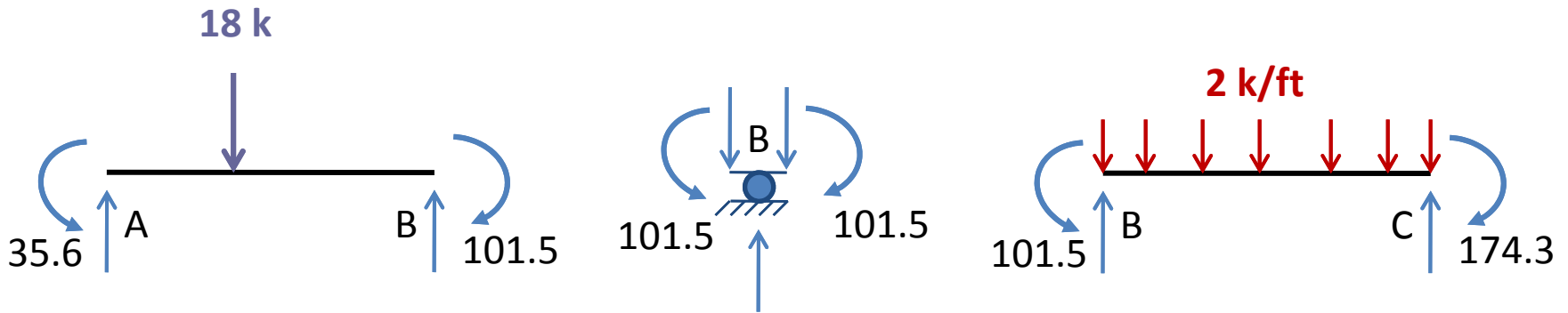
8. Member End Shears

The member end shears, obtained by considering the equilibrium of each member, are shown in figure below
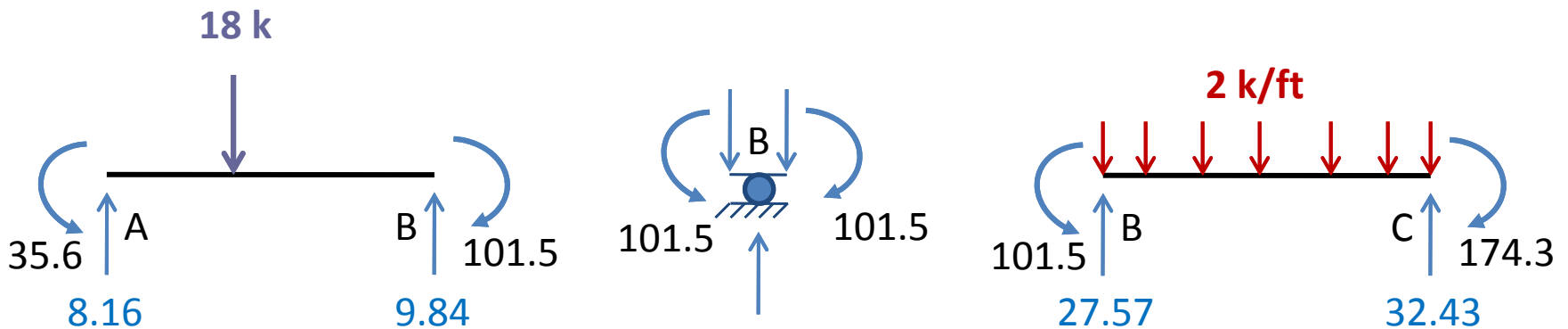
9. Support Reactions

The reactions at the fixed support A and C are equal to the forces and moments at the ends of the members connected to these joints. To determine the reaction at roller support B, consider the equilibrium of the free body of joint B in the vertical direction.
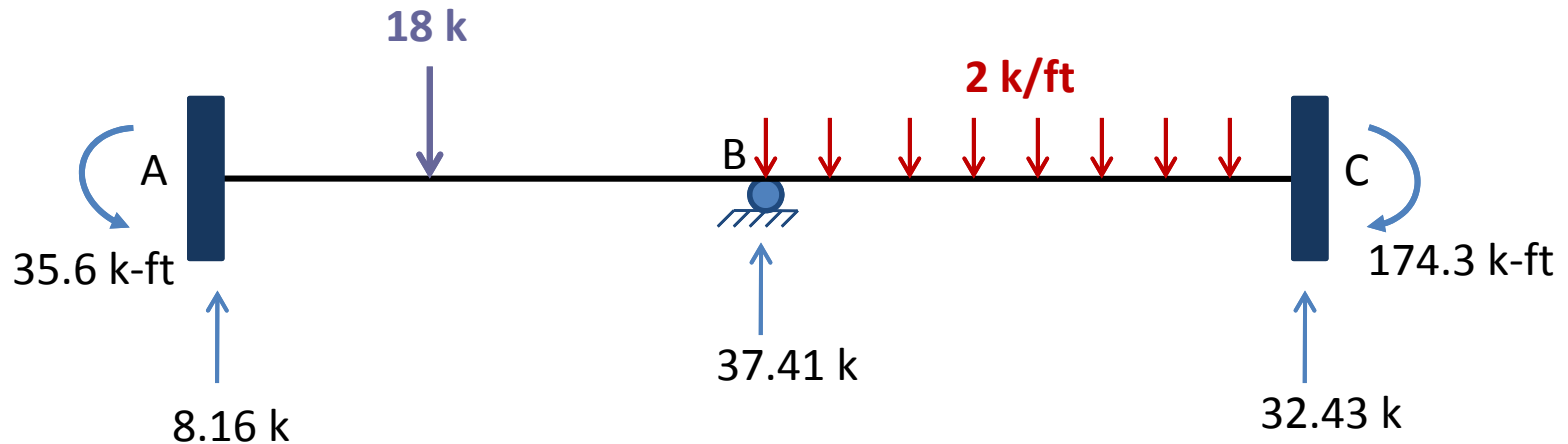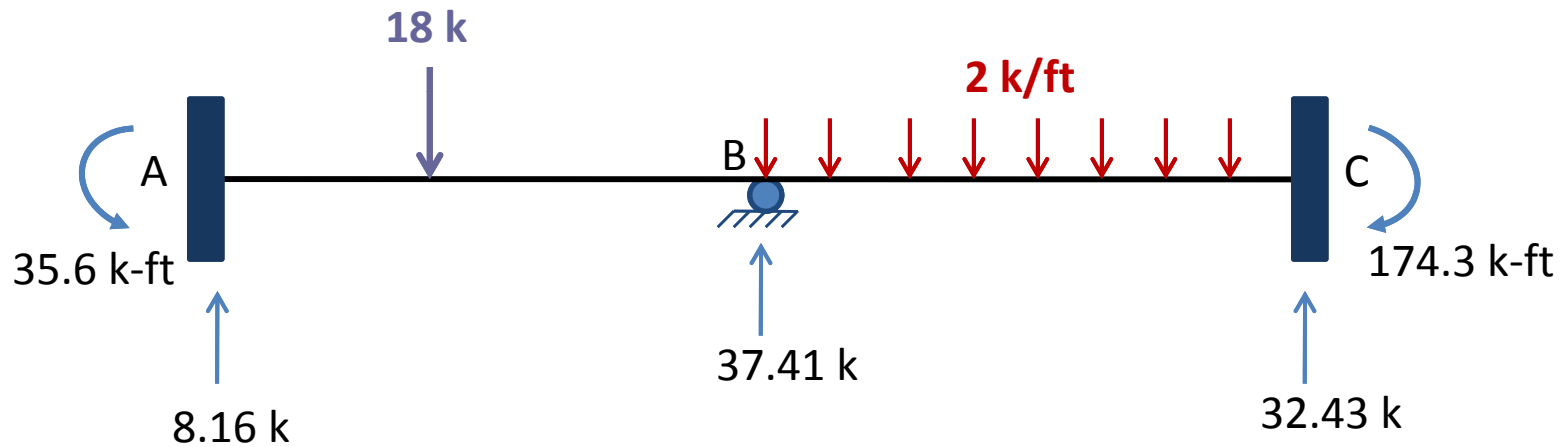
$$B_y = S_{BA} + S_{BC} = 9.84 + 27.57 = 37.41\,k \uparrow \qquad \text{ANS}$$
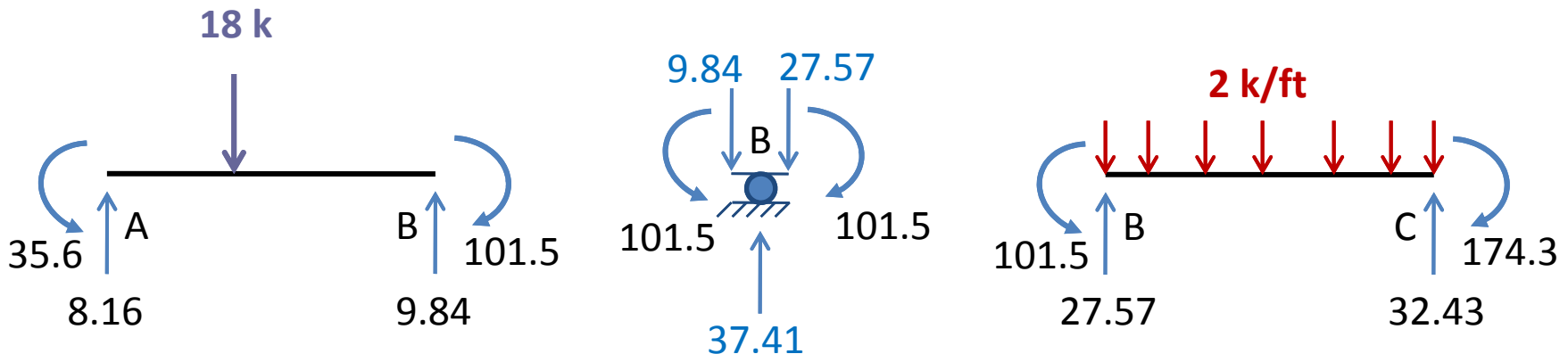
9. Support Reactions

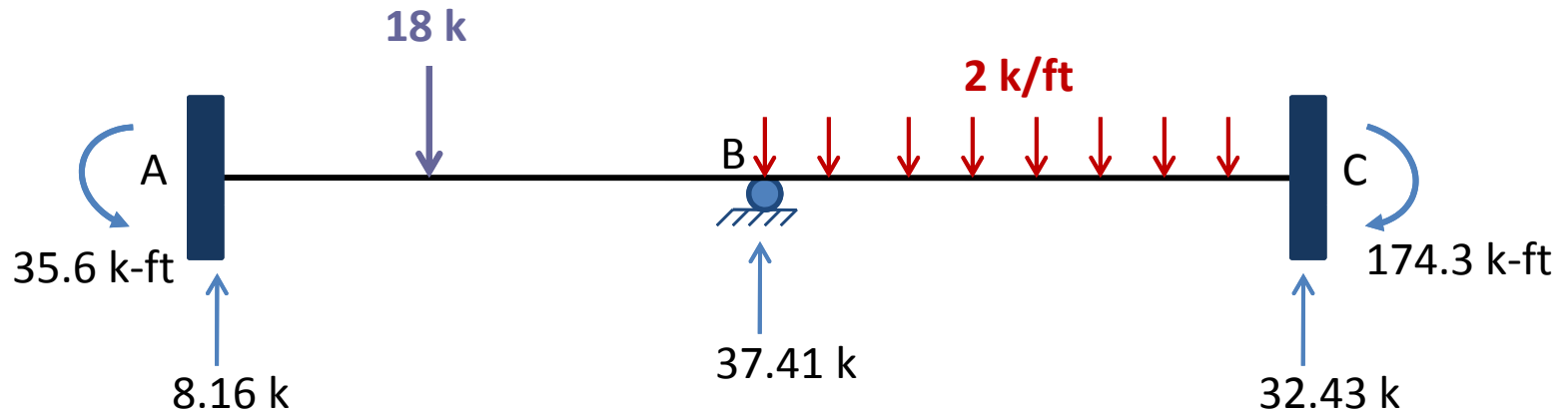The support reactions are shown in figure below.

10. Equilibrium Check

To check our calculations of member end shears and support reactions, we apply the equations of equilibrium to the free body of the entire structure.
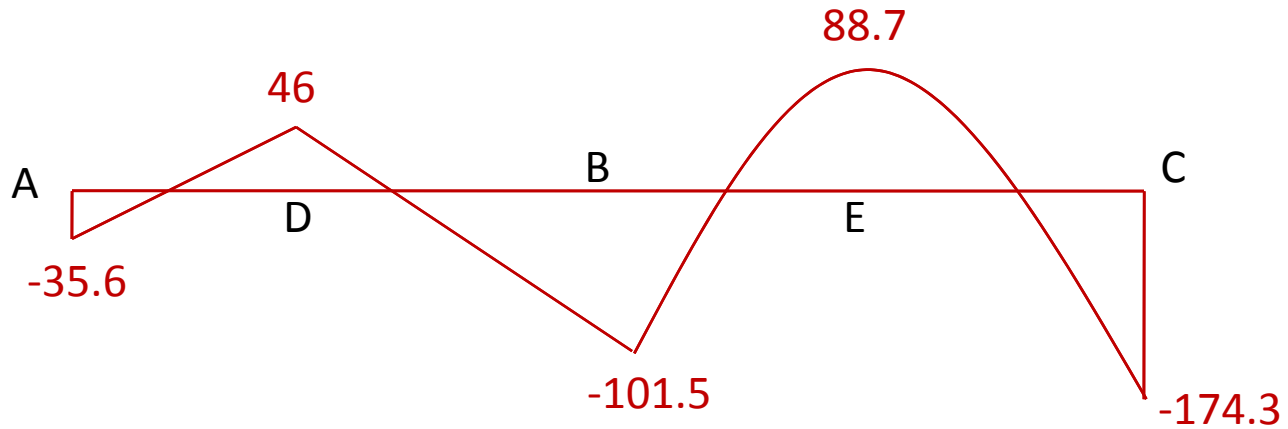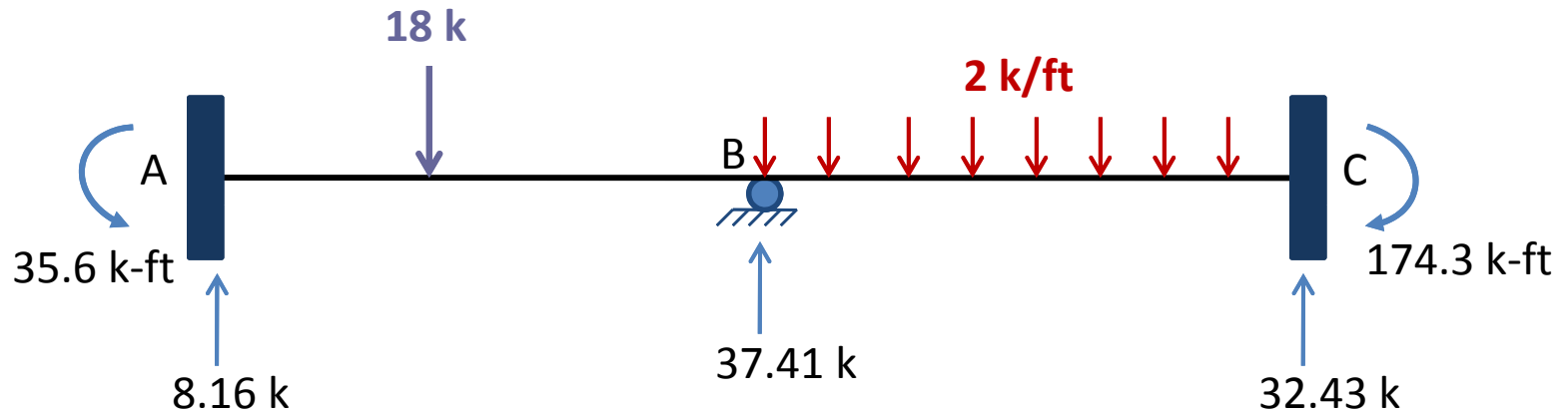

$$+\uparrow \Sigma F_y = 0$$

$$8.16 - 18 + 37.41 - 2(30) + 32.43 = 0 \qquad\qquad \text{Checks}$$

10. Equilibrium Check

To check our calculations of member end shears and support reactions, we apply the equations of equilibrium to the free body of the entire structure.
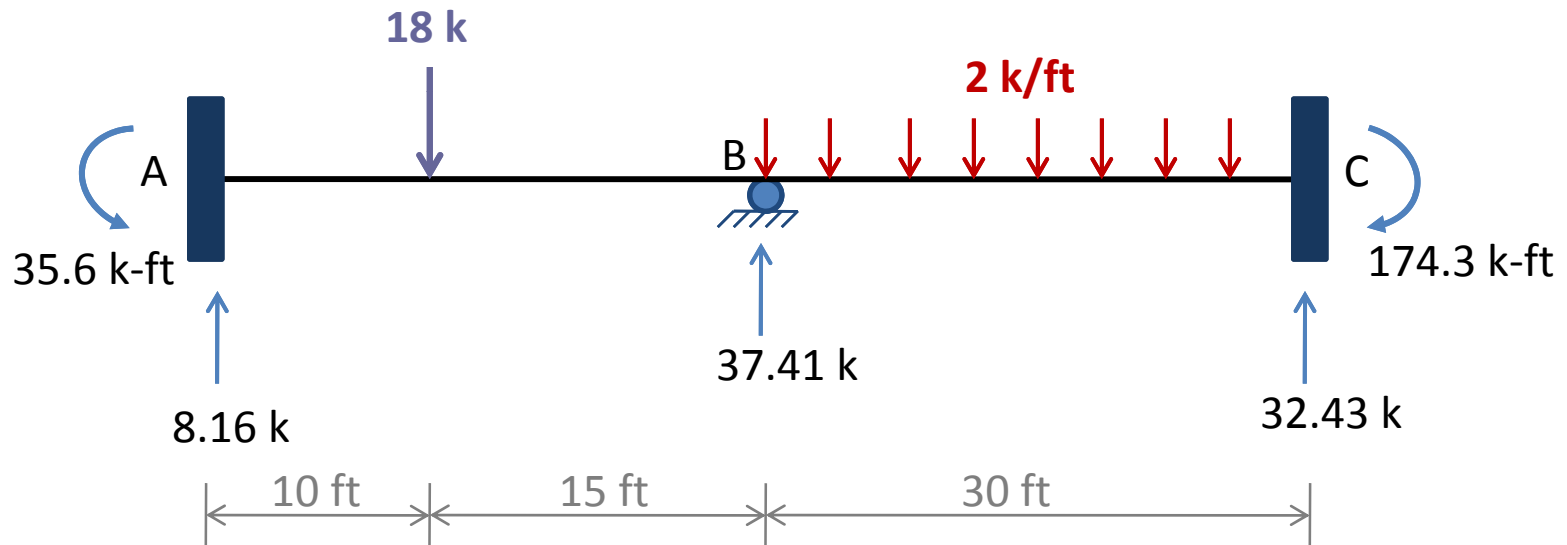
$$+\circlearrowleft \sum M_C = 0$$

$$35.6 - 8.16(55) + 18(45) - 37.41(30) + 2(30)(15) - 174.3 = 0.2 \approx 0 \text{ Checks}$$

11. Shear Force Diagram

18 k

2 k/ft

A

B

C

35.6 k-ft

174.3 k-ft

8.16 k

37.41 k

32.43 k

27.57
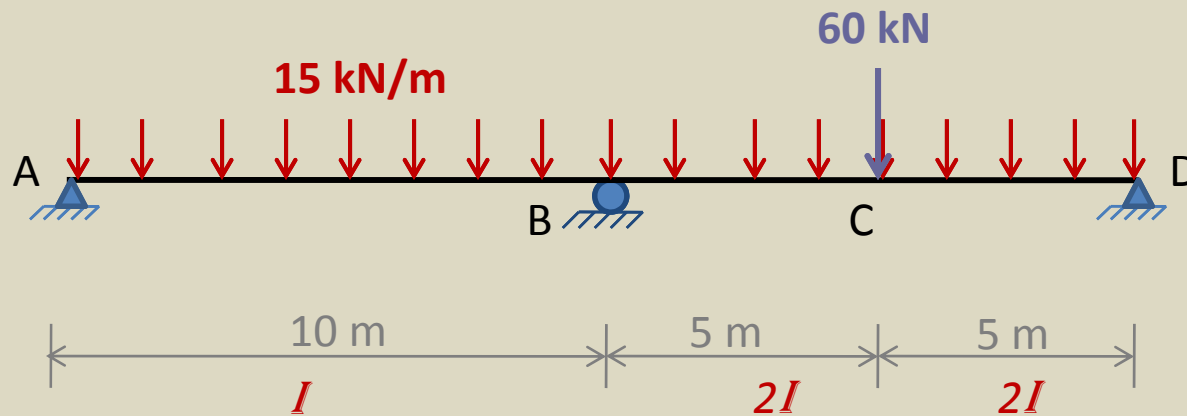
8.16

A

D

B

16.2 ft

C

E

-9.84

-32.43
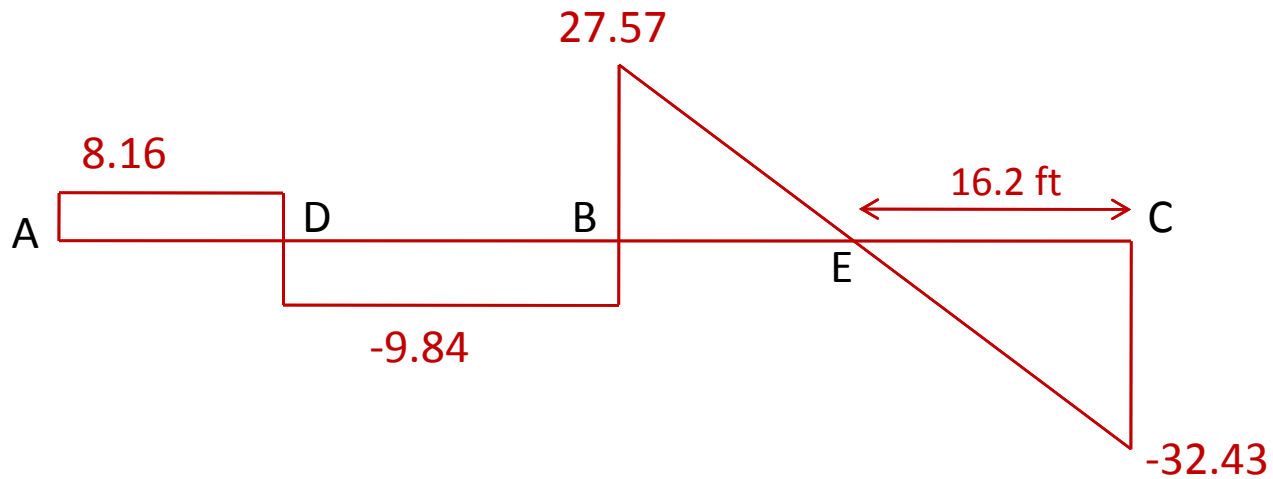
11. Bending Moment Diagram

Example 2

- Determine the reactions and draw the shear and bending moment diagrams for the continuous beam shown in Figure.

60 kN

15 kN/m

A   B   C   D

10 m   5 m   5 m

I $2I$ $2I$

EI = 200 GPa
I = 700 (10⁶) mm⁴

Solution

- From figure we can see that all three joints of the beam are free to rotate. Thus the beam have 3 degrees of freedom, θ_A, θ_B, θ_D.

- The end supports A and D of the beam are simple supports at which no external moment is applied, the moments at the end A of the member AB and at the end D of the member BD must be zero.
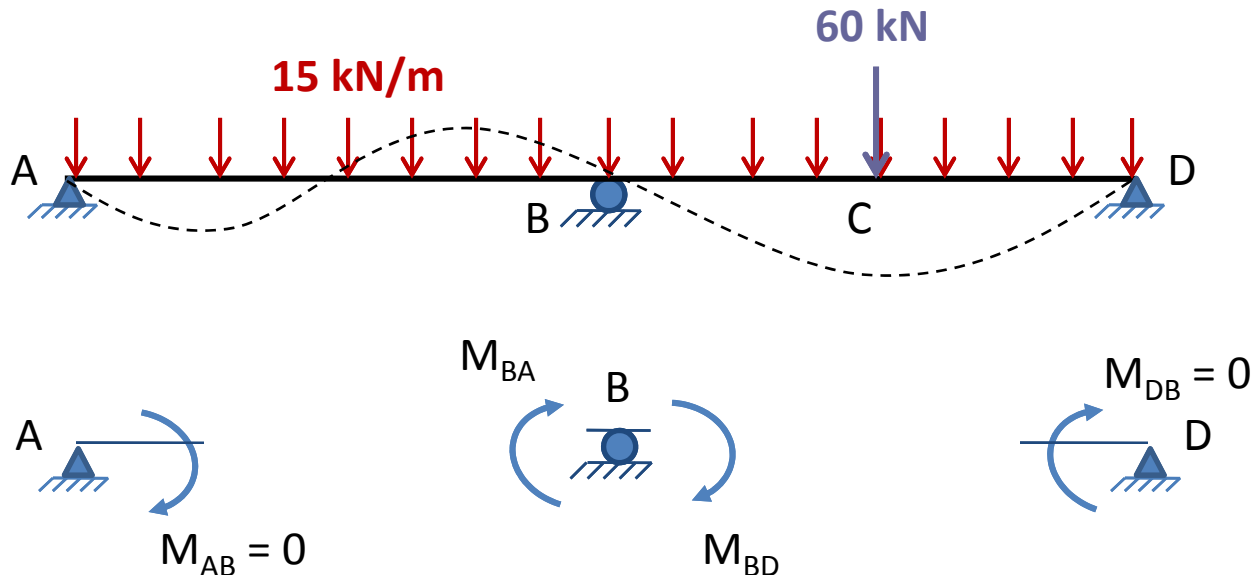
Solution

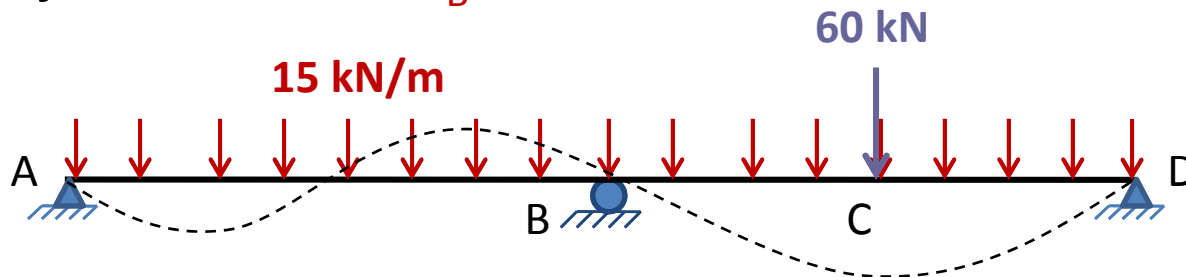- The ends A and D can be considered as hinged ends and the modified slope-deflection equations can be used.

$$M_{rh} = \frac{3EI}{L}(\theta_r - \psi) + \left(FEM_{rh} - \frac{FEM_{hr}}{2} \right) \qquad (15a)$$

$$M_{hr} = 0 \qquad (15b)$$

- The modified SDE do not contain the rotations of the hinged ends, by using these equations the rotations θ_A, and θ_D of the simple supports can be eliminated, which will then involve only one unknown joint rotation, θ_B.

1. Degree of Freedom

$$\theta_B$$

2. Fixed-End Moments

$$FEM_{AB} = \frac{15(10)^2}{12} = 125\, kN - m \quad\curvearrowright \qquad or\ +125\ kN\text{-}m$$

$$FEM_{BA} = 125\, kN - m \quad\curvearrowright \qquad or\ -125\ kN\text{-}m$$

$$FEM_{BD} = \frac{60(10)}{8} + \frac{15(10)^2}{12} = 200\, kN - m \quad\curvearrowright \qquad or\ +200\ kN\text{-}m$$

$$FEM_{DB} = 200\, kN - m \quad\curvearrowright \qquad or\ -200\ kN\text{-}m$$

60 kN

15 kN/m

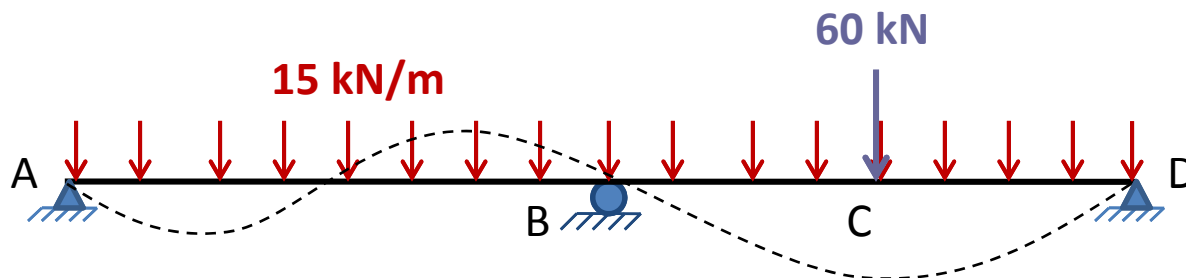A · · · · · · · · · · · · · · · · D

B   C
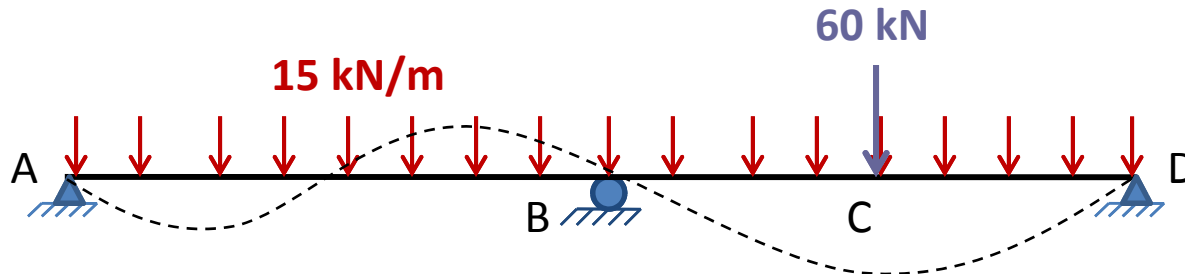
3. Slope-Deflection Equations

Since both members of the beam have one end hinged, we use Eqs. 15 to obtain the slope-deflection equations for both members.

$$M_{AB} = 0 \qquad \qquad \text{ANS}$$

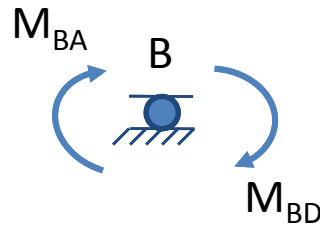$$M_{BA} = \frac{3EI}{10}(\theta_B) + \left(-125 - \frac{125}{2}\right) = 0.3EI\theta_B - 187.5 \qquad (1)$$

$$M_{BD} = \frac{3E(2I)}{10}(\theta_B) + \left(200 + \frac{200}{2}\right) = 0.6EI\theta_B + 300 \qquad (2)$$

$$M_{DB} = 0 \qquad \qquad \text{ANS}$$

4. Equilibrium Equations

By considering the moment equilibrium of the free body of joint B, we obtain the equilibrium equation

$$M_{BA} + M_{BD} = 0 \qquad\qquad (3)$$

5. Joint Rotation

To determine the unknown joint rotation θ_B we substitute the SDE (Eqs. 1 &2) into the equilibrium equations Eq. 3 to obtain

6. Joint Rotation

$$\left(0.3EI\theta_B - 187.5\right) + \left(0.6EI\theta_B + 300\right) = 0$$

or

$$0.9EI\theta_B = -112.5$$

from which

$$EI\theta_B = -125\, kN - m^2$$
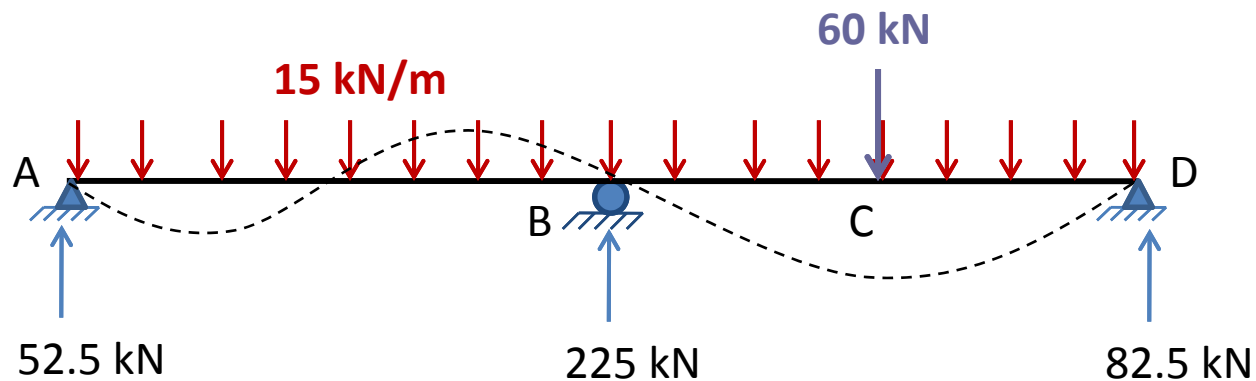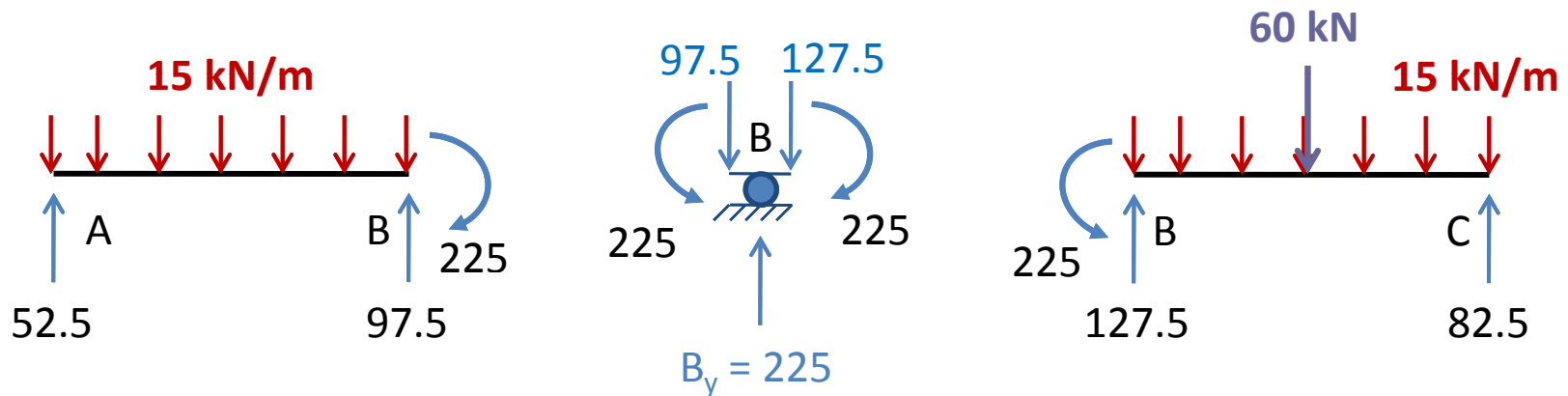
7. Member End Moments

The member end moments can now be computed by substituting the numerical value of $EI\theta_B$ into the slope-deflection equations (Eqs. 1 & 2).
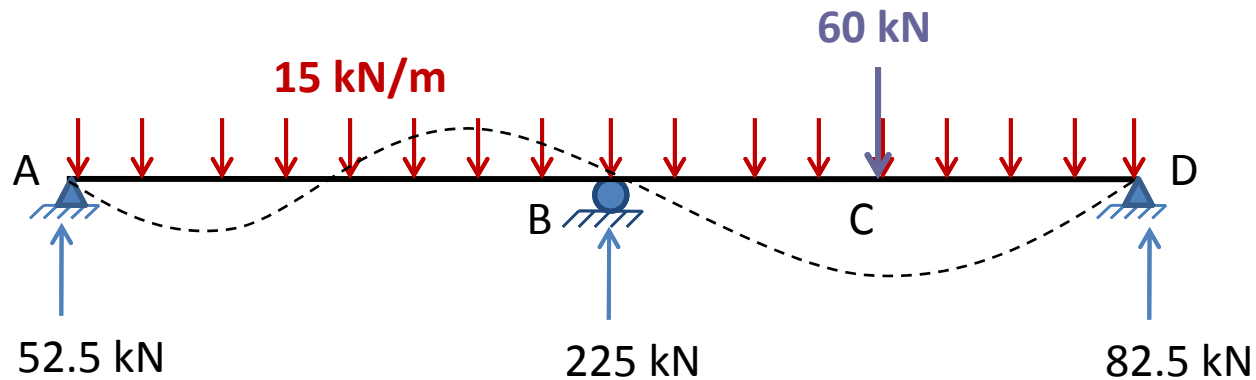
8. Member End Moments

$$M_{BA} = 0.3(-125) - 187.5 = -225 \, kN - m \qquad\qquad or \qquad 225 \, kN\text{-}m \,\text{\Large\rotatebox{-90}{$\curvearrowright$}} \quad \text{ANS}$$

$$M_{BD} = 0.6(-125) + 300 = 225 \, kN - m \,\text{\Large\rotatebox{-90}{$\curvearrowright$}} \qquad\qquad\qquad\qquad \text{ANS}$$

9. Member End Shears and Support reactions

10. Equilibrium Checks

$$+\uparrow \sum F_y = 0$$
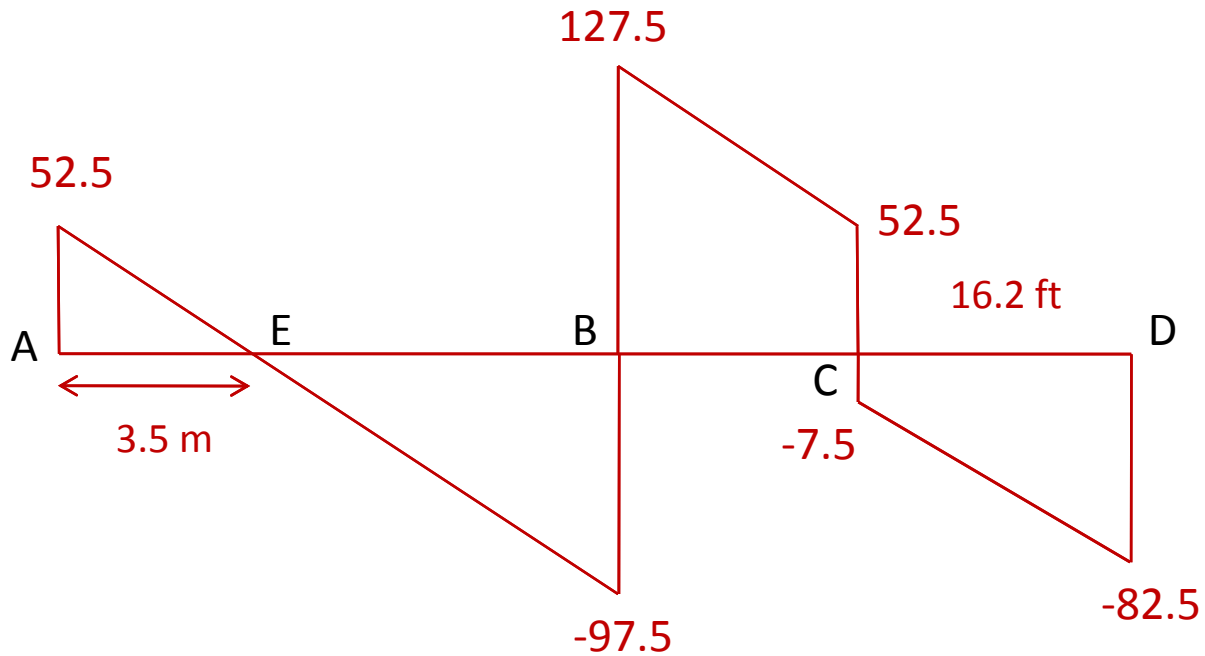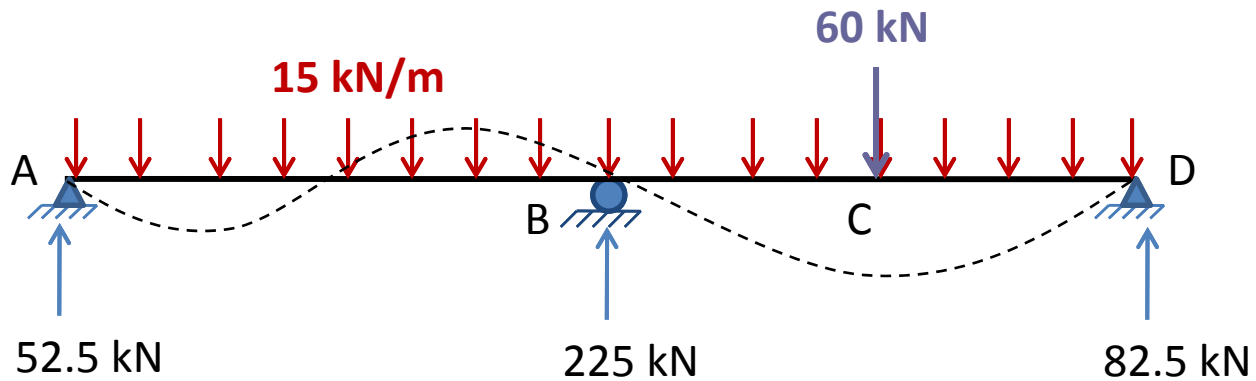
$$52.5 - 15(20) + 225 - 60 + 82.5 = 0 \qquad\qquad \text{Checks}$$

$$+\!\!\curvearrowleft \sum M_C = 0$$

$$-52.5(20) + 15(20)(10) - 225(10) + 60(5) = 0 \qquad\qquad \text{Checks}$$

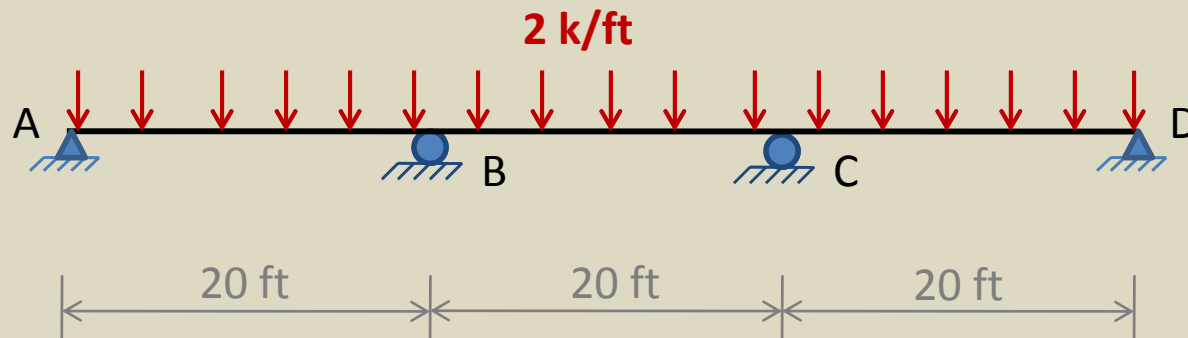# 11. Shear Force & Bending Moment Diagrams

11. Shear Force & Bending Moment Diagrams
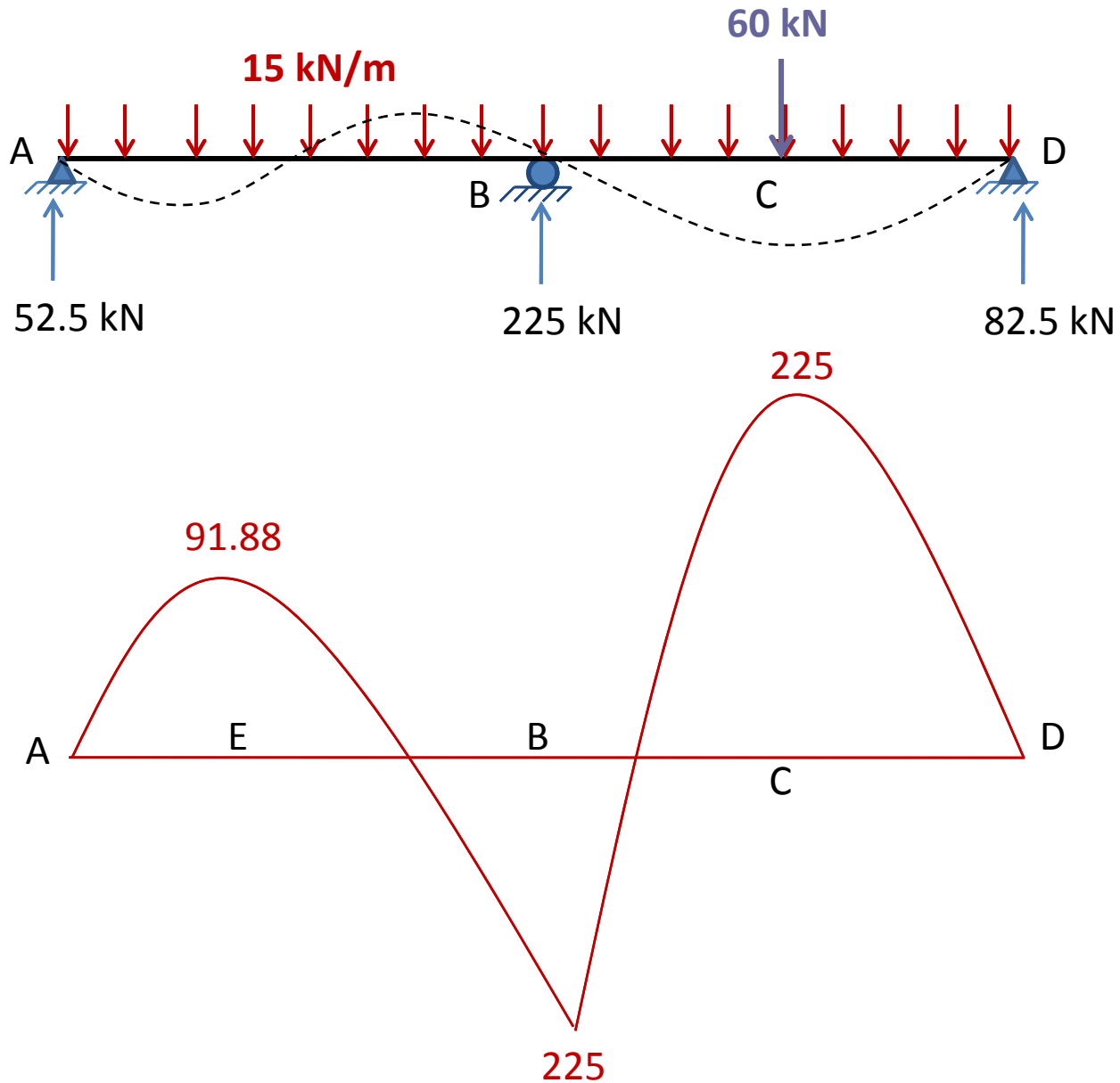
Example 3

- Determine the member end moments and reactions for the three-span continuous beam shown, due to the uniformly distributed load and due to the support settlements of 5/8 in. at B, and 1.5 in. at C, and ¾ in. at D.
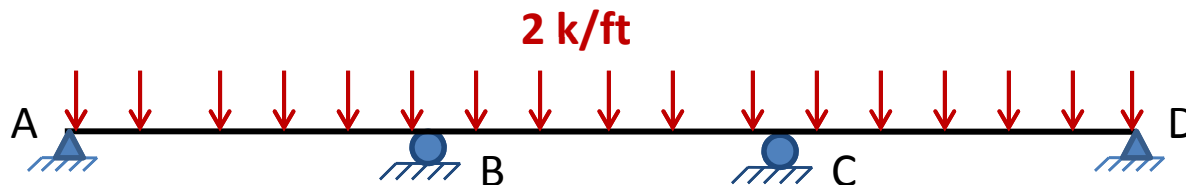

2 k/ft

A   B   C   D

20 ft   20 ft   20 ft

EI = 29,000 ksi
I = 7,800 in.⁴

Solution

1. Degree of Freedom

Four joints of the beam are free to rotate, we will eliminate the rotations of simple supports at ends A and D and use the modified SDE for member AB and CD respectively.

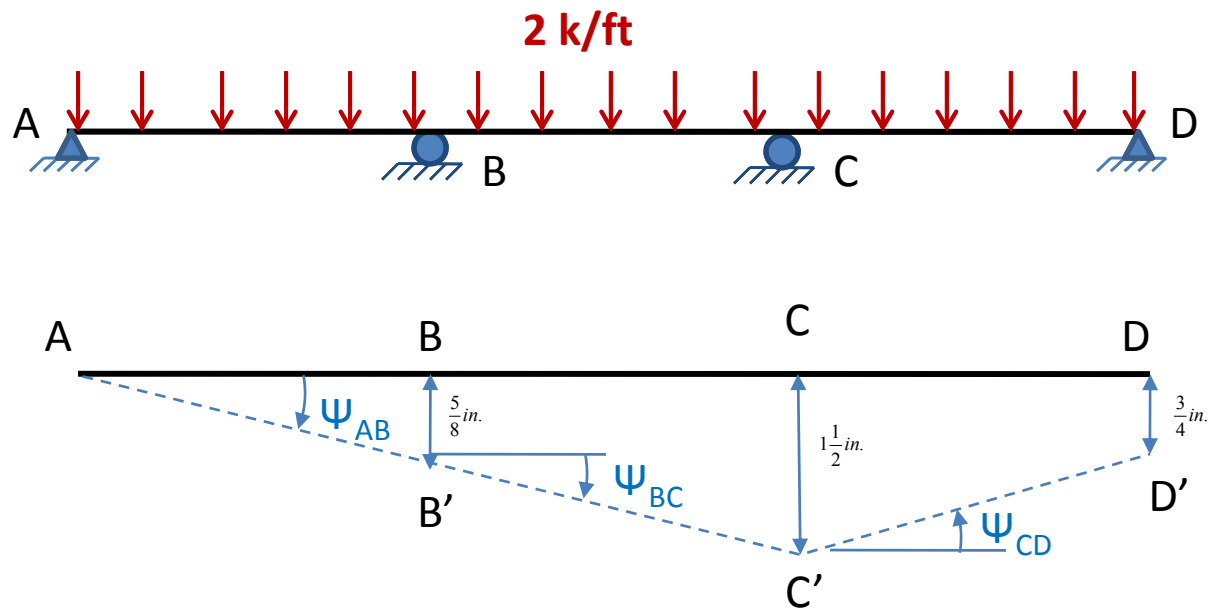The analysis will involve only two unknown joint rotations, θ_B and θ_C.

2. Fixed End Moments

$$FEM_{AB} = FEM_{BC} = FEM_{CD} = \frac{2(20)^2}{12} = 66.7\,k-ft \ \circlearrowright \qquad\qquad or\ +66.7\ k\text{-}ft$$

$$FEM_{BA} = FEM_{CB} = FEM_{DC} = 66.7\,k - ft\ \circlearrowright \qquad\qquad or\ -66.7\ k\text{-}ft$$

3. Chord Rotations

The specified support settlements are shown on a exaggerated scale.

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3. Chord Rotations
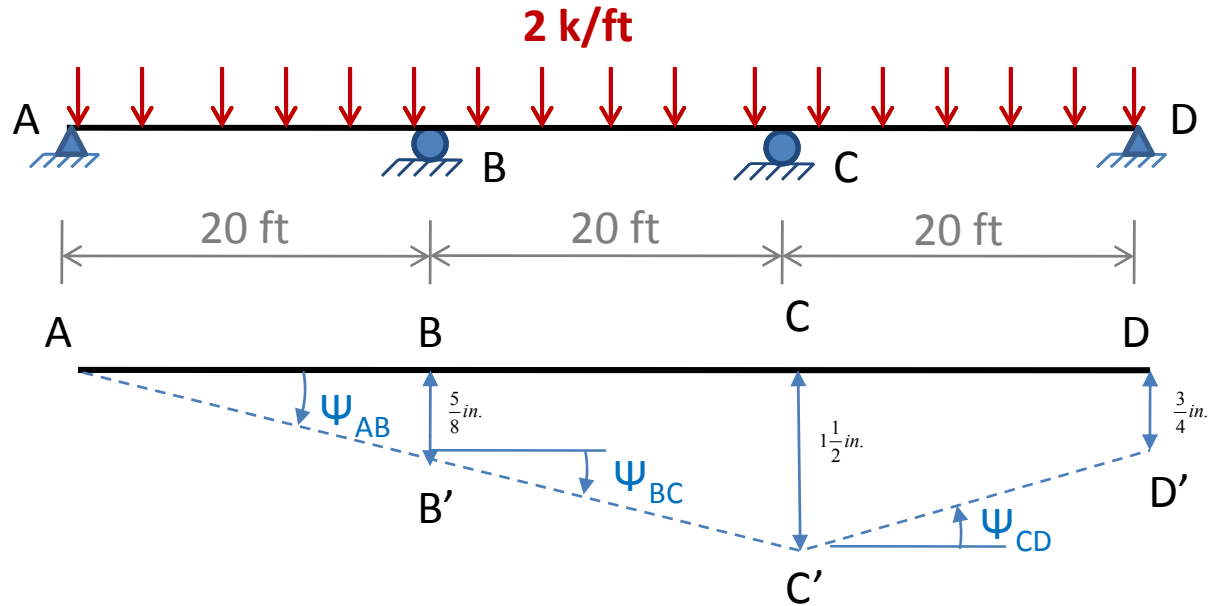
$$\psi_{AB} = -\frac{0.0521}{20} = -0.0026$$

$$\psi_{BC} = -\frac{0.0729}{20} = -0.00365$$

$$\psi_{CD} = \frac{1.5 - 0.75}{(12)20} = 0.00313$$

4. Slope-deflection Equations

$$M_{AB} = 0 \qquad\qquad\qquad\qquad \text{ANS}$$

$$M_{BA} = \frac{3EI}{10}\left(\theta_B + 0.0026\right) - 100 = 0.15EI\theta_B + 0.00039EI - 100 \qquad (1)$$

$$M_{BC} = \frac{2EI}{20}\left[2\theta_B + \theta_C - 3(-0.00365)\right] + 66.7$$

$$= 0.2EI\theta_B + 0.1EI\theta_C + 0.0011EI + 66.7 \qquad\qquad (2)$$

$$M_{CB} = \frac{2EI}{20}\left[2\theta_C + \theta_B - 3(-0.00365)\right] - 66.7$$

$$= 0.1EI\theta_B + 0.2EI\theta_C + 0.0011EI - 66.7 \qquad\qquad (3)$$

$$M_{CD} = \frac{3EI}{20}\left(\theta_C - 0.00313\right) + 100 = 0.15EI\theta_C - 0.00047EI + 100 \qquad (4)$$
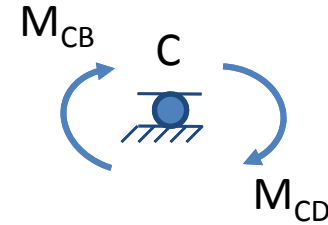
$$M_{DC} = 0 \qquad\qquad\qquad\qquad \text{ANS}$$

5. Equilibrium Equations

M_{BA} B

M_{BC}

$$M_{BA} + M_{BC} = 0 \qquad (5)$$

$$M_{CB} + M_{CD} = 0 \qquad (6)$$

M_{CB} C

M_{CD}
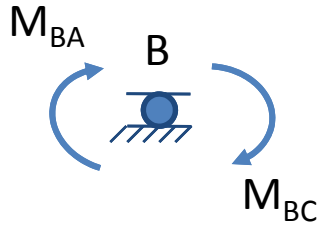
6. Joint Rotations

By substituting the slope-deflection equations (Eqs. 1 – 4) into the equilibrium equations (Eqs. 5 & 6), we obtain

$$0.35EI\theta_B + 0.1EI\theta_C = -0.00149EI + 33.3$$

$$0.1EI\theta_B + 0.35EI\theta_C = -0.00063EI - 33.3$$

substituting EI = (29,000)(7,800)/(12)^2 k-ft^2 into the right sides of the above equations yields

6. Joint Rotations

$$0.35EI\theta_B + 0.1EI\theta_C = -2{,}307.24 \tag{7}$$

$$0.1EI\theta_B + 0.35EI\theta_C = -1{,}022.93 \tag{8}$$

By solving Eqs. (7) and (8) simultaneously, we determine the values of EIθ_B and EIθ_B to be

$$EI\theta_B = -6{,}268.81\, k-ft^2$$

$$EI\theta_C = -1{,}131.57.81\, k-ft^2$$

7. Member End Moments

To compute the member end moments, substitute the numerical values of EIθ_B and EIθ_C back into the slope-deflection equations (Eqs. 1 – 4) to obtain
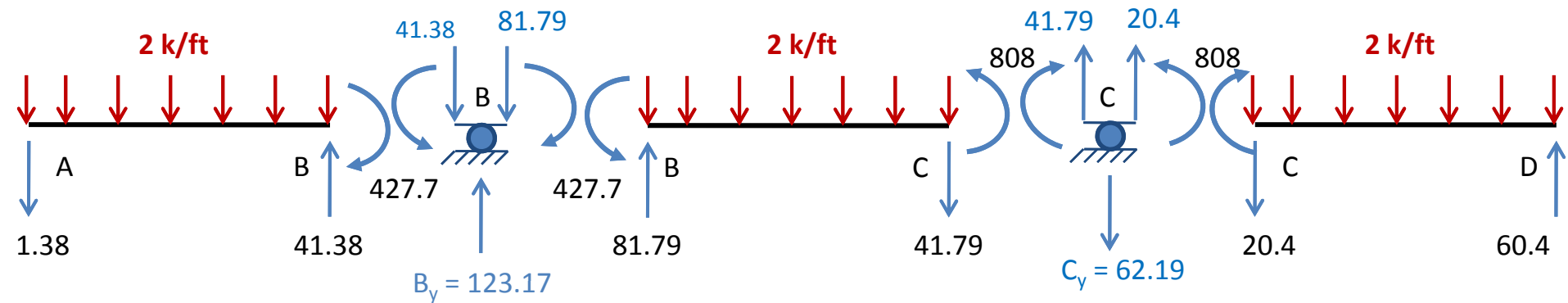
7. Member End Moments

$$M_{BA} = -427.7 \, k - ft \qquad or \qquad 427 \, k\text{-}ft \, \curvearrowright \qquad\qquad\qquad \text{ANS}$$

$$M_{BC} = 427 \, k - ft \, \curvearrowleft \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ANS}$$

$$M_{CB} = 808 \, k - ft \, \curvearrowleft \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{ANS}$$

$$M_{CD} = -808 \, k - ft \, \curvearrowright \qquad or \qquad 808 \, k\text{-}ft \qquad\qquad\qquad \text{ANS}$$

8. Member End Shears and Support Reactions

8. Member End Shears and Support Reactions

2 k/ft

41.38   81.79

B

A

B   427.7   427.7   B

1.38   41.38   81.79

$B_y = 123.17$

2 k/ft

41.79   20.4

808   C   808

C   C   D

41.79   20.4   60.4

$C_y = 62.19$

2 k/ft

A   D

B   C

1.38 k   123.17 k   62.19 k   60.4 k
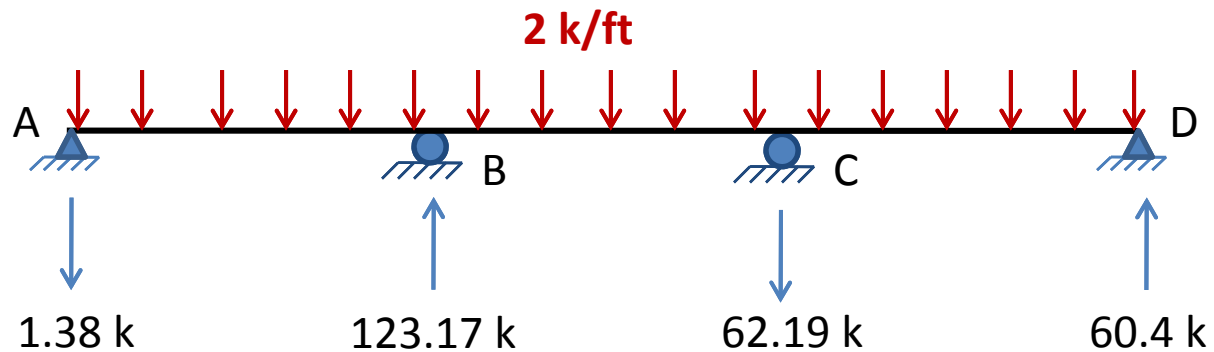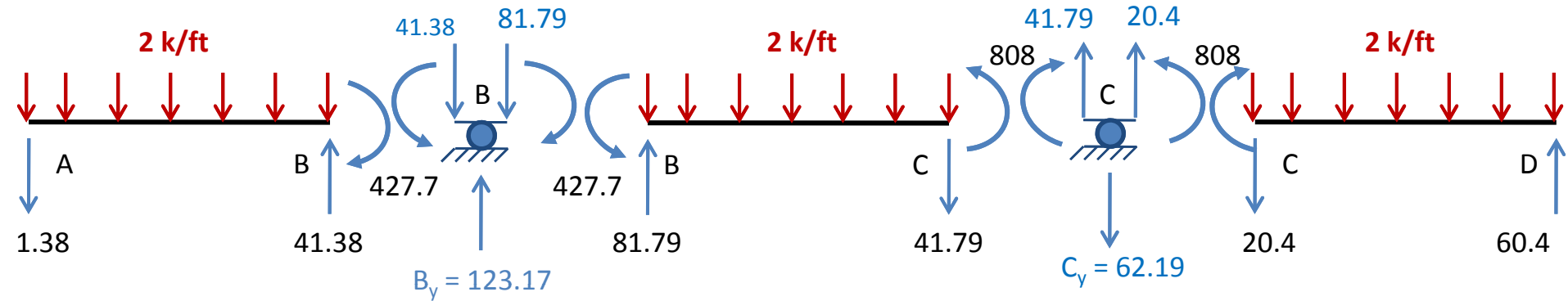
9. Shear and Bending Moment Diagrams

9. Shear and Bending Moment Diagrams

2 k/ft

A B C D

1.38 k   123.17 k   62.19 k   60.4 k

808.2

A   B   C   D

-427.6