Multi-period Asset Allocation

SmartFolio Theoretical Background

Contents

General Information

In this document we present a brief summary of portfolio theory in **multi-period time setting**. The latter implies that an investor is allowed to rebalance his/her portfolio continuously. Its single-period counterpart was developed by Harry Markovitz in his celebrated Modern Portfolio Theory. Apart from multi-period settings, single-period investment model restricts investor from rebalancing his portfolio during portfolio life. In other words, any portfolio strategy under a single-period setting holds the constant number of units in each asset until the investment horizon has been reached.

While the math behind the continuous-time portfolio theory is far more complicated, its logic substantially stays the same. Fortunately, it appears that the essential properties of efficient portfolios under single-period and multi-period settings are quite similar, which makes it possible to use results obtained in the single-period model in its multi-period counterpart. In particular, in the simplest case structure of respective optimal portfolio strategies is shifted from constant number of units in each asset to constant portfolio weights.

Chapters overview

- Discussion is started with the Correction of Historical Prices algorithm, which is used prior to other analysis procedures.
- Examination of various definitions of **returns** and **rates of return** takes place in Types of Returns.
- General lognormal continuous-time model of asset prices evolution is expounded in Analytical Model of Financial Market.
- Frequently used definitions and formulas related to portfolio and its dynamics are outlined in Portfolio Analytics.
- Definitions of **utility functions** and related measures are presented in Utility Functions and Measures of Risk Aversion respectively.
- Definitions and results related to a notion of **efficient frontier** are discussed in Efficient Frontier.
- Various approaches to **portfolio optimization** are outlined in Optimality Criteria. Also read extremely useful topic of Robust Optimization, which deals with frequently underestimated problem of parameter uncertainty.
- General framework of **factor models** is examined in Factor-based Asset Pricing Models. Particular cases are considered in Capital Asset Pricing Model and Fama-French 3-factor Model.
- The problem of estimating the model parameters (expected returns and covariances) plays the central role in portfolio analysis and optimization. Brief description of sample estimates is given in Sample Estimates for Means and Covariances, while more complicated techniques can be accessed through Advanced Estimates.
- Definitions of **portfolio insurance strategies** are given in Portfolio Insurance.

- Discussion of optimal portfolio strategies in the presence of **proportional transaction costs** takes place in Proportional Transaction Costs and Inaction Region.

- Definitions of **Value-at-Risk** and **Conditional Value-at-Risk**, coupled with several calculation techniques, are presented in Risk Management Tools.

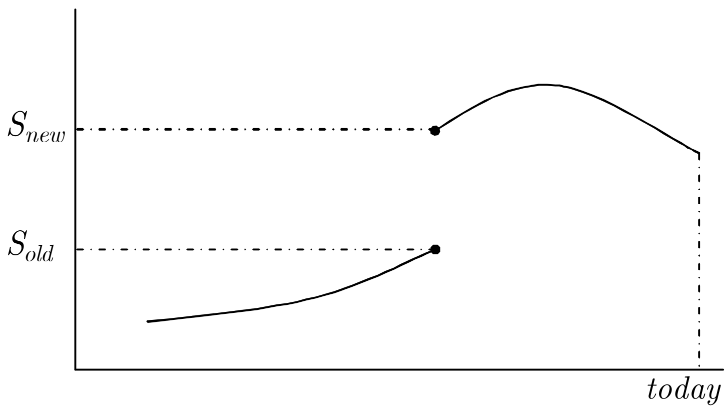
Correction of historical prices

There are several cases when the price of an asset changes while its value for an investor stays the same. In such situations it is necessary to impose corrections to the corresponding time series. Correction is necessary in the case of the following events:

- **Dividend payments** in Stocks and Mutual Funds
- **Splits** and **Reversed Splits** in Stocks and Mutual Funds
- **Rollovers** in Futures

Note. Data downloaded from **Yahoo!Finance** server is already corrected for possible dividends and splits.

The general correction algorithm is described below.
Imagine that there is a gap on the price chart, which is induced by one of the above-listed events. Let S_{old} and S_{new} denote the prices just before and after the break point respectively.



Then the correction coefficient k is calculated by the following formula: $k = \dfrac{S_{new}}{S_{old}}$. All the prices before the break point are then multiplied by k .

In case of n gaps let's start by sorting them from the latest to the earliest. The corresponding correction coefficients denote by $k_1,...,k_n$.

Let's multiply by k_1 all the prices that lie between the second and the first break points. Then multiply the prices between the third and the second break points by $k_1 k_2$. Subsequently performing the analogous operation over all other time intervals bounded by adjacent break points, we will finish at the section that lies to the left of the earliest break point. It is easy to see that its prices must be multiplied by $\prod_{i=1}^{n} k_i$.

Types of returns

Consider some asset S (S is allowed to denote investor's portfolio wealth as well) over the period 0 to T. Time T is measured in years. Let $S(0)$ and $S(T)$ denote prices of S at time 0 and T respectively. Below are presented the definitions actively used in the current document. The corresponding terminology is not settled yet; therefore the divergences with the terms used in other sources might occur.

Types of Returns

Various definitions of return serve to measure the degree of price change overthe given time period.

Definition. Quantity $p_{[0,T]} = \frac{S(T) - S(0)}{S(0)}$ is called **Simple return** or **Arithmetic return** over the period 0 to T.

Definition. Quantity $r_{[0,T]} = \ln\frac{S(T)}{S(0)}$ is called **Log return** or **Geometric return** over the period 0 to T.

Note. Applying Taylor series expansion up to fourth-order term, one can obtain the following approximation of $r_{[0,T]}$ when $\frac{S(T)}{S(0)}$ is close to 1: $r_{[0,T]} \simeq p_{[0,T]} - \frac{1}{2}p_{[0,T]}^2 + \frac{1}{3}p_{[0,T]}^3 - \frac{1}{4}p_{[0,T]}^4$.

Rates of Return

Under **Rates of return** we will further denote returns that are normalized to annual basis. Rates of return may also be called **Annualized returns** or simply **Annual returns**. Another frequently used synonym is **Growth Rate**.

Definition. Quantity $P_{[0,T]} = \frac{1}{T}p_{[0,T]}$ will be called **Simple rate of return** (other notations include **Arithmetic rate of return** or **Rate of return without compounding**) over the period 0 to T.

Definition. Quantity $R_{[0,T]} = \frac{1}{T}r_{[0,T]}$ will be called **Logarithmic rate of return** (**Geometric rate of return** or **Continuously compounded rate of return** respectively) over the period 0 to T.

Important! For simplicity reasons everywhere hereinafter under the terms **rate of return** and **growth rate** a **logarithmic** rate of return will be implied. Then, when one requires using **simple** rates of return, the latter will be indicated explicitly.

Expected Rates of Return

The symbol $\mathbf{E}$ below stands for Mathematical Expectation (averaging over all possible outcomes with weights equal to respective probabilities).

Definition. Quantity $\mu_{[0,T]} = \mathbf{E}P_{[0,T]}$ will be called **Expected simple rate of return** (**Expected arithmetic rate of return** or **Expected rate of return without compounding**) over the period 0 to T.

Note. Maximization of portfolio expected rate of return is equivalent to the maximization of expected portfolio wealth.

Definition. Quantity $\rho_{[0,T]} = \mathbf{E}R_{[0,T]}$ will be called **Expected log rate of return** (**Expected geometric rate of return** or **Expected continuously compounded rate of return**, or simply **Expected growth rate**) over the period 0 to T.

Note. Maximizing portfolio **expected growth rate** is equivalent to maximizing logarithmic utility function from portfolio wealth.

Analytical Model

Analytical Model of Financial Market

Warning! Reading of the subsequent text assumes basic knowledge of probability theory.

The **Analytical Model of Financial Market** (or simply **Analytical Model**), utilized in **SmartFolio**, is based on **Multidimensional Geometric Brownian Motion** – the most common class of stochastic processes used in mathematical finance to model the dynamics of prices.

Consider n risky assets $S_1,...,S_n$ available for the investments. Assume also that the investor has access to some risk-free asset S_0, which yields continuously compounded rate of return r_f that will be referred to as the riskless rate or **risk-free rate**. An investor expresses prices of assets $S_0,...,S_n$ in units of asset S_0. Discounted price of asset S_i at time t is denoted by $S_i(t)$. Upper asterisk $(\,^*)$ in consequent text denotes transposition operation.

The main assumption of analytical model is given by the following system of stochastic differential equations, which describe evolution of discounted prices of assets $S_0,...,S_n$:

$$
\begin{cases}
S_0(t) = 1 \\
\dfrac{dS_1(t)}{S_1(t)} = (\mu_1 - r_f + d_1)dt + \sigma_{11}dW_1(t) + ... + \sigma_{1n}dW_n(t) \\
\bullet \\
\bullet \\
\bullet \\
\dfrac{dS_n(t)}{S_n(t)} = (\mu_n - r_f + d_n)dt + \sigma_{n1}dW_1(t) + ... + \sigma_{nn}dW_n(t)
\end{cases}
$$

In the above expression **drift vector** $\vec{\mu} = (\mu_1,...,\mu_n)^*$ (further referred to as the **Mu** vector), the vector of continuously compounded dividend yields $\vec{d} = (d_1,...,d_n)^*$ and **volatility matrix** $\Sigma = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$ consist of constant values, while elements of $\vec{W} = (W_1,...,W_n)^*$ represent independent Wiener processes.

Definition. For the sake of convenience the vector $\vec{\mu}^e = \vec{\mu} - r_f + \vec{d}$, will be further referred to as the **Excess Mu** vector.

Definition. Matrix $\Omega = \Sigma^*\Sigma$ is called **Covariance Matrix**.

Note. Readers, who are not familiar with stochastic differential equations in continuous time, may interpret dt as very short time range, $\frac{dS_i(t)}{S_i(t)}$ as simple return in i-th asset over $[t,t+dt]$ and $d\vec{W}(t)$ as normally distributed random vector, whose elements are independent of each other (and of components of other vectors $d\vec{W}(s)$, $s \notin [t-dt,t+dt]$), have zero mean and variance dt.

Denote by $Diag(\vec{x})$ diagonal matrix with elements of vector $\vec{x}$ at the main diagonal. At the same time for diagonal matrix A denote column vector $Diag(A)$, whose elements are equal to diagonal elements of A.

In these notations the above system of stochastic differential equations reduces to:

$$d\vec{S}(t) = Diag\left(\vec{S}(t)\right)\left(\vec{\mu}^e dt + \Sigma d\vec{W}(t)\right).$$

A simple solution is presented below:

$$S_1(t) = S_1(0)e^{\left(\mu_1^e - \frac{\sigma_1^2}{2}\right)t + \sigma_{11}W_1(t) + \ldots + \sigma_{1n}W_n(t)}$$

$$\vdots$$

$$S_n(t) = S_n(0)e^{\left(\mu_n^e - \frac{\sigma_n^2}{2}\right)t + \sigma_{n1}W_1(t) + \ldots + \sigma_{nn}W_n(t)}$$

where components of the **volatility vector** $\vec{\sigma} = (\sigma_1, \ldots, \sigma_n)$ are calculated by means of the formula $\sigma_i = \sqrt{\sigma_{i1}^2 + \ldots + \sigma_{in}^2}$.

Using compact notation, this solution is written as

$$\vec{S}(t) = \vec{S}(0)e^{\left(\vec{\mu}^e - \frac{Diag(\Omega)}{2}\right)t + \Sigma\vec{W}(t)}.$$

Definition. Components of vector

$$\vec{\rho}^e = \vec{\mu}^e - \frac{Diag(\Omega)}{2} = \left(\mu_1^e - \frac{\sigma_1^2}{2}, \ldots, \mu_n^e - \frac{\sigma_n^2}{2}\right)^*$$

are called **Expected Excess Growth Rates**.

The above definition arises from the fact that for any $i \le n$

$$\rho_i^e = \frac{1}{T}\mathbf{E}\ln\frac{S_i(T)}{S_i(0)},$$

where symbol $\mathbf{E}$ denotes mathematical expectation.

On the other hand, it can be shown that

$$\mu_i^e = \frac{1}{T}\ln\left(\mathbf{E}\frac{S_i(T)}{S_i(0)}\right).$$

Note. Elements of $\vec{\mu}$ are also called **Expected Instantaneous Rates of Return**, reflecting the notion of expected simple rate of return over an infinitesimally short period of time. Indeed, it is easy to check that for any $i \le n$ the following chain of equalities holds: $\frac{1}{T}\left[\mathbf{E}\frac{S_i(T)}{S_i(0)} - 1\right] = \frac{1}{T}\left(e^{\mu_i^e T} - 1\right) \to \mu_i^e$ when T approaches zero.

Portfolio Analytics

Definitions

Portfolio Weights

As before, assume that the investor is able to invest in a riskless asset S_0 and n risky assets $S_1,...,S_n$.
Definition. By investor's **discounted wealth** we imply his/her wealth, measured in units of S_0.
Having at his disposal at time t the amount of discounted wealth X_t, the investor distributes his wealth among assets $S_0,...,S_n$ according to proportions $\pi_0(t),...,\pi_n(t)$. The above means that in i-th asset the investor puts a share $\pi_i(t)$ of his total wealth, i.e. amount $\pi_i(t)X_t$.

Amounts $\pi_0(t),...,\pi_n(t)$ should satisfy the condition $\sum_{i=0}^{n}\pi_i(t)=1$, therefore for full determination of portfolio structure at time t it is sufficient to provide the amounts $\pi_1(t),...,\pi_n(t)$ only . Then

$$\pi_0(t)=1-\sum_{i=1}^{n}\pi_i(t).$$

Definition. Vector $\vec{\pi}(t)=(\pi_1(t),...,\pi_n(t))^*$ of dimension $n\times 1$ refers to a vector **of proportions** or **weights** of a portfolio at time t.
Definition. Stochastic process $(\vec{\pi}(t))_{t\geq 0}$ is a **Portfolio Strategy**.

Note. In general, SmartFolio operates with portfolio strategies, whose weights are constant in time. At the time of writing the only exceptions are "Inaction Region" Portfolio Strategies, used when proportional transaction costs are present in the market, and Portfolio Insurance Strategies, used to satisfy specific constraints put on wealth dynamics. An economic ground for using portfolio strategies with constant weights is given here.

It will be assumed further, that weights of portfolios are constant in time. Such portfolios will be associated with the vectors of weights $\vec{\pi}$.

Types of Portfolio Weights Structures

Consider the portfolio $\vec{\pi}$.

Long-Only Portfolio

$$\pi_i\geq 0,\ 1\leq i\leq n$$

Fully Invested Portfolio

$\sum_{i=1}^{n}\pi_i=1$ or equally $\pi_0=0$

Long-Only Fully Invested Portfolio

$\pi_0 = 0$ and $0 \le \pi_i \le 1,\ 1 \le i \le n$

Zero-Invested Portfolio

$\sum_{i=1}^{n} \pi_i = 1$ or equally $\pi_0 = 1$

Analytical portfolio

Dynamics of Portfolio Wealth, generated by Portfolio Strategy with Constant Weights

Assume that discounted prices of assets $S_1,...,S_n$ move according to the analytical model with **Mu** vector $\vec{\mu}$, vector of continuously compounded dividend yields $\vec{d}$, **covariance matrix** Ω and risk-free rate r_f.

For simplicity assume that the vector $\vec{\pi}$ is **constant** through time.

Then dynamics of discounted portfolio wealth X^π, corresponding to portfolio vector $\vec{\pi}$, satisfies the following equation:

$$\frac{dX_t^\pi}{X_t^\pi} = \left(\mu_P - r_f + d_P\right)dt + \sigma_P dW_t,$$

where

$(W_t)_{t\ge 0}$ is a Wiener process,

$$\mu_P = (\vec{\pi},\vec{\mu}),$$
$$d_P = \left(\vec{\pi},\vec{d}\right),$$
$$\sigma_P = \sqrt{\vec{\pi}^*\Omega\vec{\pi}}.$$

Note. Readers who are not familiar with stochastic differential equations in continuous time may interpret dt as very short time range, $\frac{dX_t^\pi}{X_t^\pi}$ as a simple return on portfolio over $[t,t+dt]$ and dW_t as a normally distributed random variable with zero mean and variance dt, independent of other variables dW_s, $s \notin [t-dt,t+dt]$.

Definition. Quantity μ_P is called **Portfolio Mu**, or **Portfolio Expected Instantaneous Simple Rate of Return**.

Definition. Quantity d_P is called **Portfolio Dividend Yield**.

Definition. Quantity $\mu_P^e = \mu_P - r_f + d_P = \left(\vec{\pi},\vec{\mu}^e\right)$ is called **Portfolio Excess Mu**.

Definition. Quantity σ_P is called **Portfolio Volatility**.

Definition. Quantity $\rho_P^e = \mu_P^e - \frac{\sigma_P^2}{2}$ is called **Portfolio Expected Excess Growth Rate**

The latter definition reflects the easily derived equality $\rho_P^e = \frac{1}{T}\mathbf{E}\ln\frac{X_T^\pi}{X_0^\pi}$.

Decomposition of Portfolio Variance

The direct consequence of the **portfolio volatility** definition is the following decomposition of **portfolio variance** σ_P^2:

$$\sigma_P^2 = \sum_{i=1}^{n} \upsilon_i \sigma_P^2 ,$$

where $\upsilon_i = \pi_i \frac{(\Omega\vec{\pi})_i}{\sigma_P^2}$. Obviously, $\sum_{i=1}^{n} \upsilon_i = 1$.

Element υ_i denotes proportion of portfolio variance, contributed by i-th asset.

In other words, vector $\vec{\upsilon}$ with components, called **Contributions to Portfolio Risk**, determines alternative representation of portfolio structure, measured in units of portfolio variance rather than wealth, as in case of vector $\vec{\pi}$.

Historical portfolio

Let $S_j(0), S_j(\Delta), ..., S_j(m\Delta)$ denote **discounted prices** of j-th portfolio component at respective times $0, \Delta, ..., T$, where $T = m\Delta$.

Symbols p_{ij} and r_{ij} denote respectively simple return and logarithmic return in j-th portfolio component over the period $(i-1)\Delta$ to $i\Delta$.

Consider a portfolio with constant weights $\vec{\pi}$. It means that at the end of each period such portfolio is rebalanced to state $\vec{\pi}$.

Portfolio Dynamics

Symbols p_{iP} and r_{iP} denote respectively *simple return* and *logarithmic return* in discounted portfolio wealth X over the period $(i-1)\Delta$ to $i\Delta$. Then

$$p_{iP} = \sum_{j=1}^{n} \pi_j \left(\exp^{r_{ij} + \Delta(d_j - r_f)} - 1 \right),$$

where

r_f denotes the risk-free rate;

$\vec{d}$ denotes $n \times 1$ vector of dividend yields.

Accordingly, $r_{iP} = \ln(p_{iP} + 1)$.

Thus, the discounted portfolio wealth at time $i\Delta$ is equal to $X(i\Delta) = X_0 \exp\left(\sum_{k=1}^{i} r_{kP} \right)$, where X_0 denotes initial wealth.

Historical Portfolio Excess Growth Rate

Historical portfolio excess growth rate ρ_P^e is equal to $\rho_P^e = \frac{1}{T}\sum_{i=1}^{m} r_{iP}$.

Note. When using SmartFolio it might appear that for the same portfolio $\vec{\pi}$ value of the **historical portfolio excess growth rate** differ substantially from the **expected excess growth rate**, calculated under the analytical model assumptions. There are three reasons that explain such deviation:

- Parameters Ω and $\vec{\mu}$ used in analytical portfolio don't correspond tothe historical data. It happens when portfolio components have different lengths of historical data; analyzed time period in parameters estimation settings doesn't coincide with historical one; sample estimates are modified by some of more advanced estimation methods.
- Distribution of log returns for some assets in portfolio significantly deviates from normality. This is often the case when hedge funds or derivatives are included in the portfolio.
- Rebalancing period Δ is sufficiently long to violate the approximation of an analytical portfolio, which is rebalanced continually, with a historical simulation, where rebalancing takes place at the end of every Δ period.

Historical Portfolio Volatility

Historical portfolio volatility σ_P is defined as $\sigma_P = \frac{1}{m-1}\sum_{i=1}^{m}\left(r_{iP} - \Delta\rho_P^e\right)^2$

Historical Portfolio Excess Mu

Historical portfolio excess Mu μ_P^e is equal to $\mu_P^e = \frac{1}{\Delta}\ln\left(\frac{1}{m}\sum_{i=1}^{m} e^{r_{iP}}\right)$.

Contribution to Portfolio Risk

For calculation of vector $\vec{v}$ of portfolio components contributions to risk SmartFolio utilizes the following approximate formula:

$$v_j = \pi_j \frac{\mathrm{Cov}\left(\vec{r}_j, \vec{r}_P\right)}{\mathrm{Var}\left(\vec{r}_P\right)},$$

where $\vec{r}_j = \left(r_{1j},\ldots,r_{mj}\right)^*$, $\vec{r}_P = \left(r_{1P},\ldots,r_{mP}\right)^*$. It is easy to see that for sufficiently small Δ quantity $\sum_{i=1}^{n} v_i$ is close to 1.

Portfolio Optimization

Utility functions

The utility function describes the dependence between the amount of wealth obtained by the investor, when reaching the investment horizon, and that "usefulness" the investor can extract from it. Usefulness itself is measured in some abstract units. The shape of the utility function gives a notion of what exactly the investor puts in the "risk" concept. The optimal investment problem consists in finding the portfolio strategy that maximizes the expected value of the utility function at some predetermined moment in the future.

Utility Functions Properties

1. The utility function is an **increasing** function of wealth.
   Obviously, with growth of wealth usefulness that investor can extract from it, should grow also, therefore it is advisable to limit oneself to consideration of increasing utility functions only.

2. The utility function is **convex**.
   It is easy to see that the investor, who is averse to risk, will have a convex function of utility.
   Indeed, let's consider the following game: the wealth of the investor at time 0 is equal to X_0. At time 1 two outcomes are possible:

   $X_1 = X_0 - x$ with the probability $\frac{1}{2}$

   $X_1 = X_0 + x$ with the probability $\frac{1}{2}$

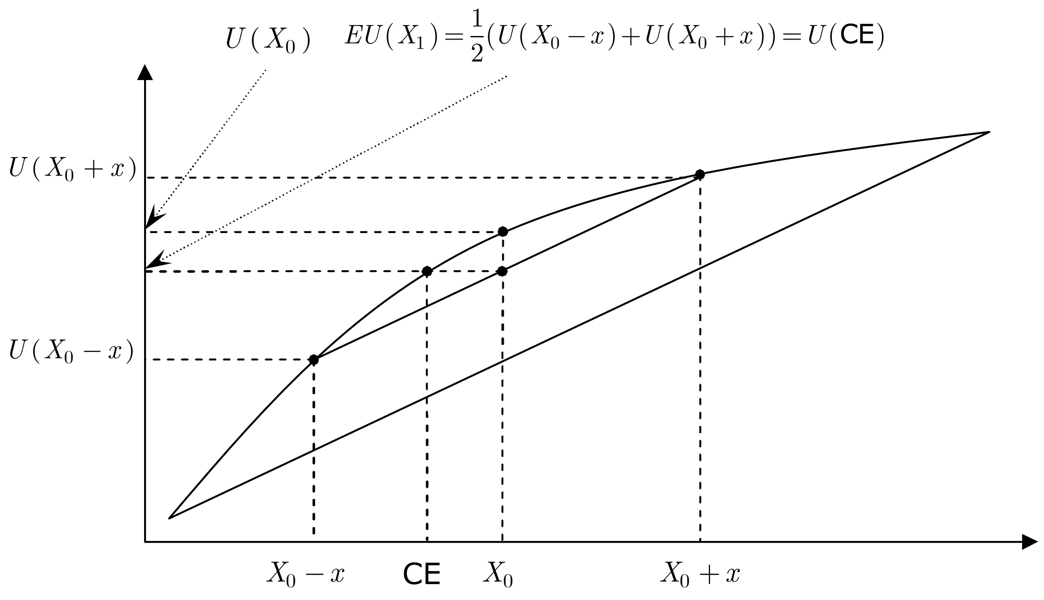
   This game is fair in the sense that the expected game profit is equal to zero: $\mathbf{E}[X_1] = X_0$.
   However for an investor with a convex utility function U such game will appear unfavorable. Indeed, by the definition of convexity

   $$\mathbf{E}U(X_1) = \frac{1}{2}(U(X_0 - x) + U(X_0 + x)) < U(X_0).$$

   Graphical representation of the game is represented in figure 1.
   The quantity called **Certainty Equivalent**, which is defined by the expression
   $\text{CE} = U^{-1}(\mathbf{E}U(X_1))$, is also shown there.

   Definition. **Certainty Equivalent** (**CE**) is a minimum amount of wealth, guaranteed preservation of which allows the investor to decline the proposed game.

   Figure 1. *Graphical representation of a fair game for an investor with a convex utility function.*



The degree of convexity for a given utility function is determined by the so-called Risk Aversion Measures.

Commonly used Utility Functions

1. **Quadratic Utility** $U(x) = x - ax^2$, $a > 0$.
   Used to obtain Mean-Variance optimal portfolios in Harry Markovitz single-period framework.
2. **Power Utility** $U(x) = \frac{x^\beta}{\beta}$, $\beta \in (-\infty,1) \setminus \{0\}$
   The maximization of expected value of such utility function at time T is equivalent to the maximization of the expected rate of return ρ_β, compounded $\frac{1}{\beta T}$ times per annum: $\rho_\beta = \frac{1}{\beta T}\mathbf{E}\left(\left(\frac{X(T)}{X(0)}\right)^\beta - 1\right)$, where $X(0)$ and $X(T)$ denote the initial and terminal wealth respectively. Values $\beta < 0$ correspond to the notion of **discount rate**. With decrease in β the investor's risk tolerance also decreases.

3. **Logarithmic Utility** $U(x) = \ln(x)$
   It can be considered as a limiting case of a power utility function as $\beta \to 0$. It is used for the maximization of expected continuously compounded growth rate: $\rho = \rho_0 = \frac{1}{T}\mathbf{E}\ln\frac{X_T}{X_0}$.
4. **Exponential Utility** $U(x) = -\exp\{-ax\}$, $a > 0$.
5. **Level reaching indicator** $U(x) = \begin{cases} 1, & x \ge a \\ 0, & x < a \end{cases}$.
   It is used for the maximization of the probability of reaching the given amount of capital (in the so-called problems of hedging with probability, smaller than one).

Measures of risk aversion

For the classification of utility functions U it is efficient to use special measures reflecting character and degree of investor's risk aversion. Most common are two types of such measures: Absolute Risk Aversion Coefficient and Relative Risk Aversion Coefficient.

Absolute Risk Aversion and CARA Utility Functions

Illustrative Example

Assume Total investor's Wealth is equal to $1000. Out of this the investor is willing to risk $500.
How much money X investor will be willing to risk when his/her Total Wealth reaches $2000?

$X < \$500$ corresponds to **Increasing Absolute** Risk Aversion
$X = \$500$ corresponds to **Constant Absolute** Risk Aversion
$X > \$500$ corresponds to **Decreasing Absolute** Risk Aversion

Note. Natural assumption is that most investors have **decreasing** absolute risk aversion.

Definition. **Absolute Risk Aversion Coefficient** at point x is defined as $\lambda_A(x) = -\dfrac{U''(x)}{U'(x)}$. Utility functions with **C**onstant **A**bsolute **R**isk **A**version Coefficient are called **CARA Utility Functions**.

Example of CARA utility function

Exponential utility functions: $U(x) = -e^{-ax}$, $a > 0$, $\lambda_A = \lambda_A(x) \equiv \alpha$.

Relative Risk Aversion and CRRA utility functions

Illustrative Example

Assume that we are once again in the framework of the previous example.

$X < \$1000$ corresponds to **Increasing Relative** Risk Aversion
$X = \$1000$ corresponds to **Constant Relative** Risk Aversion
$X > \$1000$ corresponds to **Decreasing Relative** Risk Aversion

Note. Most often investors are assumed to have **Constant** Relative Risk Aversion

Definition. **Relative Risk Aversion Coefficient** at point x is defined as $\lambda_R(x) = -x\dfrac{U''(x)}{U'(x)}$. Utility functions with **C**onstant **R**elative **R**isk **A**version Coefficient are called **CRRA Utility Functions**.

Examples of CRRA Utility Functions

1. **Power** utility functions: $U(x) = \dfrac{x^\beta}{\beta}$, $\beta \in (-\infty, 1) \setminus \{0\}$, $\lambda_R = \lambda_R(x) \equiv 1 - \beta$.
2. **Logarithmic** utility functions: $U(x) = \ln(x)$, $\lambda_R = \lambda_R(x) \equiv 1$.

Note. As a rule, existing methods of λ_R estimation, when applied to real-world situations, produce results in range from 2 to 4.

Key Result

*Maximization of expected value of CRRA utility from portfolio's terminal wealth leads to portfolio strategies with **constant** portfolio weights over time.*

Note. SmartFolio **optimization module** handles portfolio strategies with **constant portfolio weights** only. The above result justifies this approach.

Utility functions approach vs. mean-variance approach

Assume that the analytical model holds in the market with n assets, excess Mu vector $\vec{\mu}^e$ and the covariance matrix Ω. By $\vec{\pi}$ we denote the vector of constant portfolio weights.

Mean-Variance Approach can be formulated in several equivalent ways.

1. Minimization of portfolio **volatility** subject to the lower constraint on portfolio **excess Mu** (over all admissible portfolios $\vec{\pi}$):
   $\sigma_P(\vec{\pi}) \to \min_{\vec{\pi}}$ subject to $\mu_P^e(\vec{\pi}) \geq a$
2. Maximization of portfolio **Excess Mu** subject to the upper constraint on portfolio **volatility** (over all admissible portfolios $\vec{\pi}$):
   $\mu_P^e(\vec{\pi}) \to \max_{\vec{\pi}}$ subject to $\sigma_P(\vec{\pi}) \leq b$, where b is a strictly positive constant.
3. Maximization of the following expression (over all admissible portfolios $\vec{\pi}$):
   $Q_c(\vec{\pi}) = \mu_P^e(\vec{\pi}) - c\sigma_P^2(\vec{\pi}) = \vec{\pi}^*\vec{\mu}^e - c\vec{\pi}^*\Omega\vec{\pi}$, where c is a strictly positive constant.

Utility Function Approach consists in maximizing the expected value $\mathbf{E}U(X_T(\vec{\pi}))$ of utility function U from portfolio terminal wealth X_T over all admissible portfolios $\vec{\pi}$.

Key Result

Under the assumptions of the analytical model the maximization of expected CRRA utility with the relative risk aversion coefficient $\lambda_R = \lambda$ from terminal wealth and the maximization of $Q_{\frac{\lambda}{2}}(\vec{\pi})$ (both maximizations are taken over all admissible portfolio vectors $\vec{\pi}$) result in the same optimal portfolio $\hat{\pi}_\lambda$.

Important! The above statement allows all the results obtained in **single-period framework** to be easily extended to **multi-period** one with CRRA Utility Functions.

Note. $Q_{\frac{\lambda}{2}}(\vec{\pi})$ is also known as the **Risk-Adjusted Expected Excess Rate of Return**, corresponding to the relative risk aversion coefficient λ.

Efficient frontier

Definition. The portfolio $\hat{\pi}$, which maximizes the expected value of CRRA utility function for some value of relative risk aversion coefficient, will be called **CRRA-optimal** or **CRRA-efficient**.

Definition. **The efficient frontier** is the entire set of all CRRA-optimum portfolios.

Note. The definition, mentioned above, differs a little bit from the classical definition of theefficient frontier as the entire set of portfolios, which are optimal according to Mean-Variance criterion. If assumptions of the analytical model hold true, then by virtue ofthe result, formulated in the previous section, these two definitions coincide. In the general case, however, the above sets can differ. Our definition of the efficient frontier may be more suitable here since the maximization of expected utility better corresponds to the purposes of an investor, than does maximization by the means of Mean-Variance criterion.

Graphic representation of the efficient frontier is the corresponding curve on the Risk-Reward plane. Within the framework of the analytical model and convex constraints on the set of admissible portfolios this curve is convex; in general case, however, the above convexity property might be violated.

Depending on the purposes of the analysis, definitions of Risk and Reward measures, which correspond to the axes on the chart, can vary. Below we present variations of such measures that are realized in SmartFolio package.

Portfolio Risk measures

- Volatility σ_P.
- Beta (for portfolios with one factor only)
- Value-at-Risk
- Semi-volatility (for historical portfolio only)

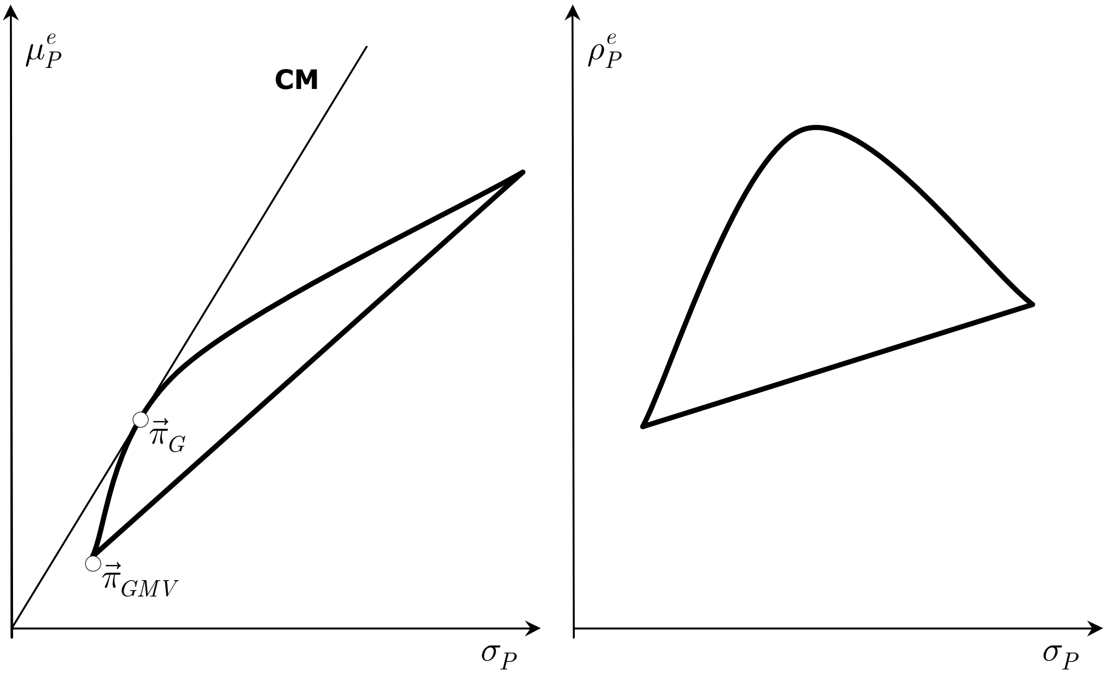
Portfolio Reward measures

- Excess Mu μ_P^e
- Expected excess growth rate $\rho_P^e = \mu_P^e - \dfrac{\sigma_P^2}{2}$

Note. In a **single-period** framework it is common to restrict the admissible portfolios tofully-invested portfolios only. In a **multi-period** framework it often makes sense to omit this restriction, in particular ifthe **portfolio expected excess growth rate** is chosen as a **Portfolio Reward** measure.

Definition. If the set of admissible portfolios is restricted to fully-invested portfolios only, then the corresponding efficient frontier will be denoted as **FI-efficient frontier**.

Examples of **FI-efficient frontier** graphs for each of the two previously defined measures of portfolio **reward** and σ_P chosen as the portfolio **risk**, are shown below. Additional elements on the first graph refer to the subsequent topics.



The rest of the chapter is devoted to the analytical model case, where the notions of CRRA-efficiency and Mean-Variance efficiency coincide.

Analytical model: Efficient Frontier generation

Consider Risk-Adjusted Expected Excess Rate of Return $Q_{\frac{\lambda}{2}}(\vec{\pi}) = \mu_P^e(\vec{\pi}) - \frac{\lambda}{2}\sigma_P^2(\vec{\pi})$, corresponding to the relative risk aversion coefficient $\lambda_R = \lambda$.

The **efficient frontier** can be parameterized by $\lambda > 0$ in the following way:

$$\hat{\pi}(\lambda) = \arg\max_{\vec{\pi}\in\Pi} Q_{\frac{\lambda}{2}}(\vec{\pi}),$$

where Π denotes the admissible portfolios set.

Note. If portfolio $\vec{\pi}$ lies on **FI-efficient frontier**, and no other constraints are imposed on Π, then implied value of λ that makes $\vec{\pi}$ optimal for $Q_{\frac{\lambda}{2}}(\vec{\pi})$ is $\lambda = \frac{\mu_P^e}{\sigma_P^2}$.

Analytical model: Global Minimum Variance portfolio

Definition. The **Global Minimum Variance** (**GMV**) **portfolio** is a fully-invested portfolio with the **minimum** volatility value σ_P.

The GMV portfolio belongs to FI-efficient frontier and is located on its left end. If no constraints are imposed on Π apart from the full-investment condition, then the GMV portfolio allows for the analytical representation:

$$\vec{\pi}_{GMV} = \frac{\Omega^{-1}\mathbb{1}}{\mathbb{1}^*\Omega^{-1}\mathbb{1}},$$

where $\mathbb{1}$ is $n \times 1$ vector of ones.

The corresponding values of μ^e_{GMV} and σ_{GMV} are calculated according to the following expressions:

$$\mu^e_{GMV} = \frac{\vec{\mu}^e\Omega^{-1}\mathbb{1}}{\mathbb{1}^*\Omega^{-1}\mathbb{1}}, \sigma_{GMV} = \frac{1}{\sqrt{\mathbb{1}^*\Omega^{-1}\mathbb{1}}}.$$

Analytical model: Tangency portfolio

Definition. The **tangency portfolio** is a fully-invested portfolio with maximum value of Instantaneous Sharpe Ratio.

Definition. The straight line on the graph $\left(\sigma_P, \mu^e_P\right)$, passing through the origin and being a tangent to FI-efficient frontier, is called the **Capital Market Line** (**CML**).

The tangency portfolio corresponds to the point, where **CML** touches the FI-efficient frontier.

If no constraints are imposed on Π apart from the full-investment condition, then:

1. Tangency portfolio $\vec{\pi}_G$ admits the analytical representation:

$$\vec{\pi}_G = \frac{\Omega^{-1}\vec{\mu}^e}{\mathbb{1}^*\Omega^{-1}\vec{\mu}^e},$$

where $\mathbb{1}$ is $n \times 1$ vector of ones.

2. Formulas for μ^e_G and σ^e_G have the following form:

$$\mu^e_G = \frac{\left(\vec{\mu}^e\right)^*\Omega^{-1}\vec{\mu}^e}{\mathbb{1}^*\Omega^{-1}\vec{\mu}^e}, \qquad \sigma^e_G = \frac{\sqrt{\left(\vec{\mu}^e\right)^*\Omega^{-1}\vec{\mu}^e}}{\left|\mathbb{1}^*\Omega^{-1}\vec{\mu}^e\right|}.$$

3. Any portfolio on the FI-efficient frontier can be obtained as a linear combination of the **GMV portfolio** and the **tangency portfolio**.

4. If the full-investment condition is omitted, then the renewed efficient frontier coincides with Capital Market Line. Any portfolio, belonging to CML, can be represented as a linear combination of a riskless asset and a tangency portfolio. The latter statement has a title of the **Two-Fund Separation Theorem**.

Two-Fund Separation Theorem

Theorem. Assume the following conditions hold:

1. There are no constraints imposed on admissible portfolios
2. Riskless asset is the same for all investors
3. Risk-free rates for **lending** and **borrowing** are equal

4. There are no transaction costs and **taxes**

Then any portfolio belonging to the efficient frontier is a combination of the **tangency portfolio** and a riskless asset.

Two-Fund Separation Theorem serves as a theoretical basis for index funds activity. Indeed, if Two-Fund Separation Theorem holds, then all rational investors regardless of their risk profile hold the same mix of risky securities. Therefore, the market share of each asset is equal to its weight in the tangency portfolio. In other words, any rational investor who isn't faced with portfolio constraints would hold all of his funds in a riskless asset and in a mutual fund that replicates the market portfolio.

Optimality criteria

The structure of investor's optimal portfolio depends on objective factors (such as budget and administrative constraints on portfolio structure), as well as on the subjective preferences of the investor. The above preferences depend on both the investor's attitude to risk and the character of the investor's goals.

The formalization of investor's preferences results in the formation of some **optimality criterion**.

The optimality criterion is any function dependent from portfolio weights. The structure of portfolio that is optimal relative to the selected criterion corresponds to the maximum (or minimum) value of such function (taking into account possible constraints).

Below are presented the optimality criteria that are contained in SmartFolio:

1. CRRA Utility Function Criterion
2. Target Shortfall Probability Criterion
3. Benchmark Tracking Criterion

CRRA Utility Function Criterion

Assume that utility function of the investor belongs to the CRRA class. Assume also that the relative risk aversion coefficient value is known and is equal to λ. Consider the following functions:

1. Polynomial utility function $U_\lambda(x) = \dfrac{x^{1-\lambda}}{1-\lambda}$, if $\lambda \in (0,\infty) \setminus \{1\}$
2. Logarithmic utility function $U_1(x) = \ln(x)$, if $\lambda = 1$.

The portfolio that is optimal relative to a CRRA utility function is obtained by the maximization (over the set of all admissible portfolios) of one of the above mentioned functions from the portfolio terminal wealth. According to the result stated above (which connects the Utility Function Approach and the Mean-Variance Approach) under the assumptions of the analytical model it is equivalent to the maximization of the quantity

$$Q_{\frac{\lambda}{2}}(\vec{\pi}) = \vec{\mu}^e \vec{\pi}^* - \frac{\lambda}{2} \vec{\pi}^* \Omega \vec{\pi}.$$

Analytically tractable optimal portfolios

Merton Portfolio

If the assumptions of the analytical model hold and there are no constraints on portfolio structure, the portfolio $\vec{\pi}_M$ that is optimal relative to a CRRA utility function allows for the analytical representation and is referred to as **Merton Portfolio**.

$$\vec{\pi}_M = \frac{1}{\lambda}\Omega^{-1}\vec{\mu}^e$$

Let us consider some arbitrary portfolio $\vec{\pi}$. It might be informative to find out values of excess Mu that make the portfolio $\vec{\pi}$ optimal given the covariance matrix Ω and the relative risk aversion λ. From the above formula it follows that

$$\vec{\mu}^e_{imp} = \lambda\Omega\vec{\pi}\,.$$

Vector $\vec{\mu}^e_{imp}$, calculated above, is called **Implied Excess Mu**. If one wants to measure the distance between the existing and the optimal portfolios, one way to do it is to look at the difference between $\vec{\mu}^e_{imp}$ and the available estimate $\hat{\mu}^e$ of excess Mu.

Merton Portfolio with higher Interest Rate for Borrowing

The presented material is based on the results, obtained in [Cvitanic, Karatzas; 1992]. Assume that analytical model holds, and the role of a riskless asset is played by cash. Consider situation, when the **borrowing rate** in cash r_f^b is higher than the **lending rate** r_f^l.

Denote $A = \mathbb{1}^*\Omega^{-1}\mathbb{1}$, $B = \mathbb{1}^*\Omega^{-1}\vec{\mu}^e$, where $\mathbb{1}$ is $n \times 1$ vector of ones.

λ denotes relative risk aversion coefficient. Then weights $\vec{\pi}_M$ of the **Merton portfolio** are calculated according to the following rule:

$$\vec{\pi}_M = \begin{cases} \frac{1}{\lambda}\Omega^{-1}\left(\vec{\mu} + \vec{d} - \left(r_f^l + \frac{B-\lambda}{A}\right)\mathbb{1}\right), & 0 < B-\lambda < A(r_f^b - r_f^l) \\ \frac{1}{\lambda}\Omega^{-1}\left(\vec{\mu} + \vec{d} - r_f^l\mathbb{1}\right), & B \le \lambda \\ \frac{1}{\lambda}\Omega^{-1}\left(\vec{\mu} + \vec{d} - r_f^b\mathbb{1}\right), & B-\lambda \ge A(r_f^b - r_f^l) \end{cases}$$

Three-Fund Portfolio

The **Three-Fund portfolio** rule was proposed in [Kan, Zhow; 2005] as an effective way to deal with model parameters **uncertainty** (it is assumed that the analytical model holds). Three-fund portfolio consists of three parts: riskless asset, tangency portfolio and global minimum variance portfolio. Corresponding weights are chosen to minimize the expected loss in utility due to the presence of an **estimation error**. In such case, the three-fund portfolio provides significant improvement in utility over the traditional Merton portfolio (which is effectively a two-fund portfolio), where the sample estimates are "plugged" in place of the population values.

Three-fund portfolio rule has the following form:

$$\vec{\pi}_{3f} = \vec{\pi}_{3f}(c,d) = \frac{1}{\lambda}\left(c\hat{\Omega}^{-1}\hat{\mu}^{e} + d\hat{\Omega}^{-1}\mathbb{1}\right),$$

where

λ denotes relative risk aversion coefficient;

$\hat{\mu}^{e}$ and $\hat{\Omega}$ denote sample excess Mu and sample covariance matrix respectively;

$\mathbb{1}$ is $n \times 1$ vector of ones;

c, d are appropriate constants (see [Kan, Zhow; (2005)] for exact formulas).

Target Shortfall Probability Criterion

The criterion based on the maximization of CRRA utility functions has one defect. The point is that the true value of the relative risk aversion coefficient λ_R is extremely difficult to estimate. Moreover, experiments exist that show that λ_R of an investor can vary depending on specific circumstances. These reasons form the sufficient ground for the search of more direct optimality criteria.

It seems that one of the most successful attempts in this direction was made by Michael Stutzer (see [Stutzer; 2003]).

The essence of Stutzer's criterion is that an investor chooses the critical level for the excess growth rate he/she would like to beat in the future, whereupon minimizes the probability of falling short of the selected goal.

Note. It is worth mentioning that Stutzer's work develops ideas, first proposed by A. D. Roy in 1952 for the single-period model. (see [Roy; 1952])

Criterion algorithm

1. Investor selects his/her investment horizon T.
2. Investor selects his/her **target excess growth rate** $R^{e}_{\min}$ over the risk-free rate.

Note. If investor's goal is to beat by rate $R^{e}_{\min}$ some **benchmark** B, then portfolio wealth X_T must be denominated in units of B. See riskless asset.

3. Investor constructs the portfolio strategy with constant weights $\vec{\pi}\left(R^{e}_{\min}, T\right)$ that maximize the probability of portfolio realized excess growth rate $R^{e}_{[0,T]}$ on $[0,T]$ exceeding $R^{e}_{\min}$.

Note. Portfolio strategy with **constant** portfolio weights is suboptimal in terms of minimizing the probability of $R^{e}_{[0,T]} < R^{e}_{\min}$ event only. For example, if for some $t < T$ portfolio wealth X_t exceeds $X_0 \exp\left(R^{e}_{\min} T\right)$, then target excess growth rate $R^{e}_{\min}$ on $[0,T]$ is achieved with 100% probability by investing all funds for remaining time $[t,T]$ to riskless asset.
To make optimal portfolio structure constant through time, target shortfall probability criterion should be defined a little bit more precisely.

Criterion definition. At every point in time t investor minimizes the probability of $R^{e}_{[t,T]} < R^{e}_{\min}$.

Optimal portfolio strategy, which corresponds to this more accurate definition, no longer depends onthe current portfolio wealth and coincides withthe portfolio strategy $\vec{\pi}\left(R_{\min}^{e}, T\right)$ obtained in step 3.

Features of Optimal Portfolio $\vec{\pi}\left(R_{\min}^{e}, T\right)$

- Optimal portfolio $\vec{\pi}\left(R_{\min}^{e}, T\right)=\vec{\pi}\left(R_{\min}^{e}\right)$ doesn't depend on T.
- Volatility of optimal portfolio $\vec{\pi}\left(R_{\min}^{e}\right)$ increases with $R_{\min}^{e}$.

Calculation Techniques

Optimal Portfolio $\vec{\pi}\left(R_{\min}^{e}\right)$ is obtained by solving the following problem:

$$\min_{\vec{\pi}\in\Pi}\min_{\lambda\geq 1}\mathbf{E}e^{\left(R_{[t,T]}^{e}-R_{\min}^{e}\right)(1-\lambda)T}.$$

If the analytical model holds, then the stated problem is equivalent to

$$\min_{\vec{\pi}\in\Pi}\min_{\lambda\geq 1}(1-\lambda)\left(Q_{\frac{\lambda}{2}}(\vec{\pi})-R_{\min}^{e}\right),$$

where $Q_{\frac{\lambda}{2}}(\vec{\pi})$ corresponds to risk-adjusted expected excess rate of return.

Note. Solution $\vec{\pi}\left(R_{\min}^{e}\right)$ to the above problem under the analytical model assumptions can be obtained alternatively by maximizing the so-called information ratio $I_{R_{\min}^{e}}=\dfrac{\rho_{P}^{e}-R_{\min}^{e}}{\sigma_{P}}$, where ρ_{P}^{e} is the portfolio expected excess growth rate. Such calculation technique is slightly more correct because it leads to the right answer even if $R_{\min}^{e}$ surpasses $\max_{\vec{\pi}\in\Pi}\rho_{P}^{e}(\vec{\pi})$. On the contrary, in the former case for all such $R_{\min}^{e}$ one obtains the same portfolio that maximizes the expected excess growth rate. However, the mentioned divergence is hardly significant in practice, since the choice of $R_{\min}^{e}$, surpassing $\max_{\vec{\pi}\in\Pi}\rho_{P}^{e}(\vec{\pi})$, is almost always not economically justified: the resulted optimal portfolio corresponds to a value of λ that is strictly less than 1.

In the general case the information ratio may be replaced with Sortino ratio or STARR ratio.

Benchmark Tracking Criterion

As a rule, **benchmark tracking criterion** serves for the construction of portfolios that contain a modest number of assets and replicate dynamics of more diversified portfolio, such as market portfolio.

To apply benchmark tracking criterion two steps are required:

1. Select the **benchmark asset** and consider it further as a riskless asset.
2. Construct a fully-invested portfolio that minimizes some measure of **Tracking Error** between a **portfolio** and the **benchmark** subject to possible constraints.

In the analytical model framework the most appropriate measure of tracking error is **portfolio volatility**. Often it is coupled together with the lower constraint on **portfolio expected excess growth rate**.

In the general case of non-normal distribution of returns portfolio downside volatility might appear a more plausible measure.

Minimum Acceptance Rate $R^e_{\min}$ from the downside volatility definition may serve the same purpose as the lower constraint on portfolio expected excess growth rate.

Worst-case scenario optimization

The traditional portfolio optimization framework assumes that parameters of the analytical model are known precisely. In practice, however, statistical estimates of parameters may have significant estimation error. In particular it is relevant for the sample estimate $\hat{\mu}^e$ of vector $\vec{\mu}^e$ that represents the excess part of expected instantaneous rates of return. Ignorance of estimation error in $\hat{\mu}^e$ usually leads to unwarrantably extreme values of portfolio weights and dramatic shifts in portfolio structure when previous estimates are modified with recent historical data.

One way to avoid the mentioned above undesirable properties of an optimal portfolio is to include the estimation error explicitly in optimization process. Most practically sound results in this direction were obtained in [Garlappi, Uppal, Wang; 2005]. The authors implement the following algorithm, which they call **Multi-Prior Approach**:

1. The set M of possible values for the true vector $\vec{\mu}^e$ is defined.

2. The optimal portfolio, corresponding to relative risk aversion coefficient $\lambda_R = \lambda$, is obtained by solving the following Min-Max problem:

$$\max_{\vec{\pi}\in\Pi}\min_{\vec{\mu}^e\in\mathrm{M}} Q_{\frac{\lambda}{2}}\left(\vec{\pi},\vec{\mu}^e\right) = \max_{\vec{\pi}\in\Pi}\min_{\vec{\mu}^e\in\mathrm{M}}\left(\vec{\pi}^*\vec{\mu}^e - \frac{\lambda}{2}\vec{\pi}^*\hat{\Omega}\vec{\pi}\right)$$

Note. To consider the robust version of the target shortfall probability criterion the above Min-Max problem must be modified in the following way:

$$\min_{\vec{\pi}\in\Pi}\min_{\lambda\geq 1}\min_{\vec{\mu}^e\in\mathrm{M}}\left(1-\lambda\right)\left(Q_{\frac{\lambda}{2}}\left(\vec{\pi}\right) - R^e_{\min}\right) = \min_{\vec{\pi}\in\Pi}\min_{\lambda\geq 1}\min_{\vec{\mu}^e\in\mathrm{M}}\left(1-\lambda\right)\left(\vec{\pi}^*\vec{\mu}^e - \frac{\lambda}{2}\vec{\pi}^*\hat{\Omega}\vec{\pi} - R^e_{\min}\right)$$

In other words, the **investor seeks for portfolio with the best performance under the worst-case scenario**. Inner minimization over expected instantaneous excess returns in the above expressions can be regarded as reflection of investor's **uncertainty aversion**. A convenient measure for uncertainty aversion degree is **confidence level** α, used below to define M. As α approaches **1**, investor's aversion to uncertainty increases to infinity.

Two versions of set M are considered below.

Separate Confidence Intervals

Let n be the number of portfolio components and $\hat{\mu}^e$ be sample counterpart of $\vec{\mu}^e$. For every $j=\overline{1,n}$ corresponding confidence interval for μ^e_j in the form of $\left[\hat{\mu}^e_j - \delta^j, \hat{\mu}^e_j + \delta^j\right]$ is obtained by solving the following equation:

$$\mathrm{P}\left(\left|\mu^e_j - \hat{\mu}^e_j\right| \leq \delta^j_\alpha\right) = \alpha\,.$$

The above equation has the following solution:

$$\delta_\alpha^j = \hat{\sigma}_j \sqrt{\frac{l}{m_j}} t^{-1}\left(1-\alpha, m_j - 1\right),$$

where
$\hat{\sigma}_j$ – sample volatility in j-th asset
l – average number of observations in one year
m_j – available number of observations in j-th asset
$t^{-1}(p,k)$ – inverse function of Student's t-distribution with k degrees of freedom: it returns value z such that $\mathrm{P}(|\xi_k| \geq z) = p$, where ξ_k is t-distributed random variable with k degrees of freedom.
Set M_α^S, corresponding to **Separate Confidence Intervals** for μ_j^e, is defined as n-dimensional rectangular parallelepiped $\prod_{j=1}^{n}\left[\hat{\mu}_j^e - \delta_\alpha^j, \hat{\mu}_j^e + \delta_\alpha^j\right]$.
Then, the stated Min-Max problem, corresponding to relative risk aversion coefficient λ, reduces to

$$\max_{\vec{\pi}}\left(\vec{\pi}^*\hat{\mu}^e - \frac{\lambda}{2}\vec{\pi}^*\Omega\vec{\pi} - \sum_{j=1}^{n}\left|\pi_j\right|\delta_\alpha^j\right) = \max_{\vec{\pi}}\left(\hat{\mu}_P^{e,W_S} - \frac{\lambda}{2}\hat{\sigma}_P^2\right),$$

where **portfolio worst-case excess Mu** $\hat{\mu}_P^{e,W_S}$ is defined in the following way:

$$\hat{\mu}_P^{e,W_S} = \vec{\pi}^*\hat{\mu}^e - \sum_{j=1}^{n}\left|\pi_j\right|\delta_\alpha^j$$

Analogously, the Min-Max problem, corresponding to the target shortfall probability criterion, simplifies to following:

$$\min_{\vec{\pi}\in\Pi}\min_{\lambda\geq 1}(1-\lambda)\left(\hat{\mu}_P^{e,W_S} - \frac{\lambda}{2}\vec{\pi}^*\hat{\Omega}\vec{\pi} - R_{\min}^e\right)$$

In case of uncorrelated assets the probability of $\vec{\mu}$ falling into M_α^S is α^n. In the general case calculation of the probability of falling into M_α^S is more complicated. It could be desirable to explicitly state the probability of falling into M rather than defining it implicitly by means of confidence intervals for individual assets. This is done in the next section.

Joint Confidence Interval

Suppose all portfolio components possess the same number of observations m. Consider the quantity $\psi^2 = \left(\vec{\mu}^e - \hat{\mu}^e\right)^*\hat{\Omega}^{-1}\left(\vec{\mu}^e - \hat{\mu}^e\right)$, which is a sample estimate of squared instantaneous Sharpe ratio for tangency portfolio, corresponding to **portfolio excess Mu** equal to $\vec{\mu}^e - \hat{\mu}^e$. This quantity is a convenient measure for the distance between $\vec{\mu}^e$ and $\hat{\mu}^e$.

Joint Confidence Interval $[0,\delta]$ for $\psi^2\left(\vec{\mu}^e\right)$ is defined in the following way:

$$\mathrm{P}\left(\psi^2\left(\vec{\mu}^e\right) \leq \delta_\alpha\right) = \alpha$$

Solution for δ_α has the following form:

$$\delta_\alpha = \hat{\sigma}_P^2 \frac{l(m-1)n}{m(m-n)} F^{-1}(1-\alpha, n, m-n),$$

where $F^{-1}(p, k_1, k_2)$ is an inverse function of F-distribution with k_1 and k_2 degrees of freedom: it returns value z such that $\mathrm{P}(|\xi_{k_1,k_2}| \geq z) = p$, where ξ_{k_1,k_2} is F-distributed random variable with k_1 and k_2 degrees of freedom.

Set M_α^J, corresponding to the above **joint confidence interval** for $\psi^2(\vec{\mu}^e)$, is defined as follows:

$$\mathrm{M}_\alpha^J = \{\vec{\mu}^e : \psi^2(\vec{\mu}^e) \leq \delta_\alpha\}.$$

The Min-Max problem, corresponding to risk aversion coefficient λ, simplifies to

$$\max_{\vec{\pi}} \left(\vec{\pi}^* \hat{\mu}^e - \frac{\lambda}{2} \vec{\pi}^* \Omega \vec{\pi} - \sqrt{\delta_\alpha} \right) = \max_{\vec{\pi}} \left(\hat{\mu}_P^{e,W_J} - \frac{\lambda}{2} \hat{\sigma}_P^2 \right),$$

where portfolio worst-case excess Mu $\hat{\mu}_P^{e,W_J}$ is defined as $\hat{\mu}_P^{e,W_J} = \vec{\pi}^* \hat{\mu}^e - \sqrt{\delta_\alpha}$.

Correspondingly, the Min-Max problem, corresponding to the target shortfall probability criterion, simplifies to the following:

$$\min_{\vec{\pi} \in \Pi} \min_{\lambda \geq 1} (1-\lambda) \left(\hat{\mu}_P^{e,W_J} - \frac{\lambda}{2} \vec{\pi}^* \hat{\Omega} \vec{\pi} - R_{\min}^e \right).$$

Walk-forward optimization

Walk-forward optimization provides necessary framework to test for stability in optimal portfolios corresponding to different historical periods. This scheme is also close to real-world practice consisting in frequent recalculation of optimal portfolio weights based on newly arrived information. Well-established portfolio optimization sheme from this standpoint is characterized not only by positive portfolio performance, but also by low deviations in portfolio weights for adjacent time periods.

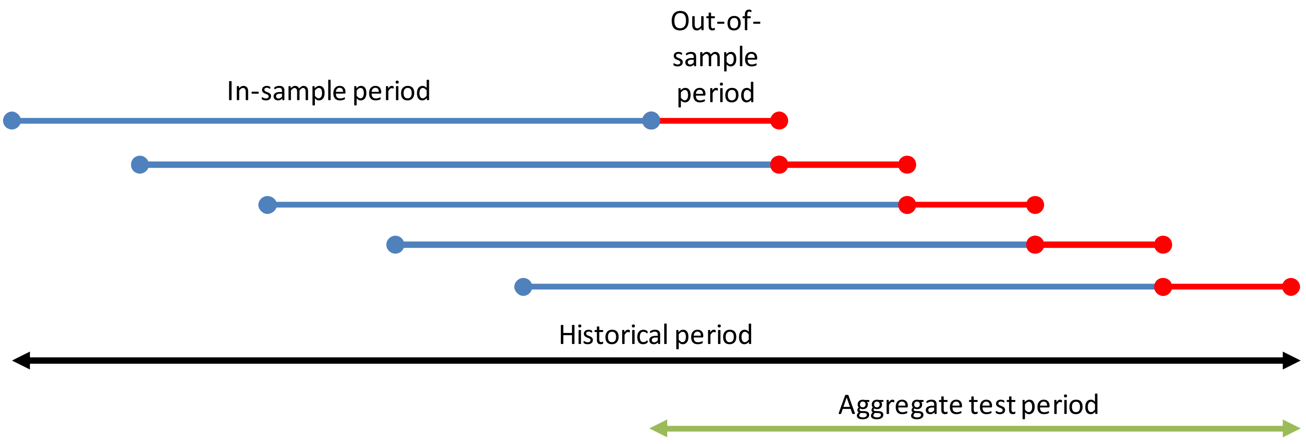
Walk-forward optimization is also used to check whether theoretically optimal portfolios have any value in practice. There are two reasons why this might not be the case:

- Portfolio optimization may result in overfitting the data, especially when performing optimization on relatively short time intervals and/or with too many portfolio components;
- Historical data might be subject to pronounced non-stationarity.

The algorithm

1. One chooses the lengths of in-sample and out-of-sample periods;

2. At each optimization round the in-sample and out-of-sample intervals are shifted forward by the length of the out-of-sample period as shown in the picture below.



3. At each stage portfolio optimization is performed based on the data from the corresponding in-sample period. The results are then calculated by applying the obtained optimal weights to the corresponding out-of-sample period.

Factor-based Asset Pricing Models

General approach

Factor-based Asset Pricing Models (**asset pricing models** for short) help imposing the additional structure on parameters $\vec{\mu}$ and Σ of the analytical model. Restrictions, imposed by an asset pricing model, reduce the number of model parameters that need to be estimated and force "rational" investors to allocate wealth among riskless asset and selected factors only, ignoring the rest of the risky assets. Particular cases of asset pricing models are CAPM and Fama-French 3-Factor Model.

Linear Regression Equation

Assume that the financial market consists of n financial instruments, first n_a of which we denote as **assets**, the others $n_f = n - n_a$ we denote as **factors**[1]. It is supposed that the dynamics of the market satisfies the conditions of the analytical model with $n \times 1$ excess Mu vector $\vec{\mu}^e$ and $n \times n$ covariance matrix Ω. The given parameters can be decomposed according to n_a assets and n_f factors:

$$\vec{\mu}^e = \begin{pmatrix} \vec{\mu}_a^e \\ \vec{\mu}_f^e \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \Omega_{aa} & \Omega_{af} \\ \Omega_{fa} & \Omega_{ff} \end{pmatrix}.$$

Let us denote through $\vec{P}_t^a$ и $\vec{P}_t^f$ vectors of simple rates of return for assets and factors respectively on the interval $[t, t+\Delta]$, where Δ is small enough.

Dependence of $\vec{P}_t^a$ from $\vec{P}_t^f$ can be expressed by the linear regression equation

$$\vec{P}_t^a = \vec{\alpha} + \mathrm{B}\vec{P}_t^f + \vec{\varepsilon}_t,$$

where

$\vec{\alpha}$ is the $n_a \times 1$ vector of mispricing terms (**Jensen's instantaneous alphas**);

B is the $n_a \times n_f$ matrix of factor sensitivities (**Betas**);

$\vec{\varepsilon}_t$ is random $n_a \times 1$ vector of residuals with zero average $\mathbf{E}\vec{\varepsilon}_t = 0$ and such a covariance matrix $\Psi = \mathbf{E}\left[\vec{\varepsilon}_t \vec{\varepsilon}_t^{\,*}\right]$ that components of vectors $\vec{\varepsilon}_t$ and $\vec{P}_t^f$ are uncorrelated.

The elements ψ_{ii} of matrix Ψ are known as **residual variances**. Their roots are referred to as **standard errors of forecast**, implied by the regression.

Let us denote sample estimates of $\vec{\mu}^e$ and Ω through $\hat{\mu}^e = \begin{pmatrix} \hat{\mu}_a^e \\ \hat{\mu}_f^e \end{pmatrix}$ and $\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_{aa} & \hat{\Omega}_{af} \\ \hat{\Omega}_{fa} & \hat{\Omega}_{ff} \end{pmatrix}$.

Estimates of parameters α and B are found by means of ordinary least-squares using the following formulae:

[1] Readers shall not confuse the subscript f used in this section with the riskless rate symbol r_f.

$$\hat{\mathrm{B}}_{LS} = \hat{\Omega}_{af} \cdot \hat{\Omega}_{ff}^{-1}$$
$$\hat{\alpha}_{LS} = \hat{\mu}_a^e - \hat{\mathrm{B}}_{LS}\hat{\mu}_f^e.$$

Model Assumptions

The asset pricing model imposes the following restrictions, which must be satisfied by the above regression equation:

1. $\vec{\alpha} = 0$.
   All mispricing terms in regression equation must be equal to zero.

2. Ψ is **diagonal**.
   Regression residuals must be uncorrelated.

Practical Implications

If selected asset pricing model holds true, then the main implication for a CRRA investor (an investor with CRRA utility function) is that there is no sense to invest in individual assets. Corresponding optimal portfolio would contain only riskless asset and portfolio factors as its components. Of course, this is true only when selected factors are available for an investment (i.e. they are at least tradable financial instruments).

Model-implied Estimates of $\vec{\mu}$ and Ω

Assuming that the asset pricing model holds true, leads to the modified estimates for $\vec{\mu}^e$ and Ω. Model-implied estimates $\hat{\mu}_m^e$ and $\hat{\Omega}_m$ are given by the following formulae:

$$\hat{\mu}_m^e = \begin{pmatrix} \hat{\mathrm{B}}_{LS}\hat{\mu}_f^e \\ \hat{\mu}_f^e \end{pmatrix}, \quad \hat{\Omega}_m = \begin{pmatrix} \hat{\mathrm{B}}_{LS}\hat{\Omega}_{ff}\hat{\mathrm{B}}_{LS}^* + \hat{\Psi} & \hat{\mathrm{B}}_{LS}\hat{\Omega}_{ff} \\ \hat{\Omega}_{ff}\hat{\mathrm{B}}_{LS}^* & \hat{\Omega}_{ff} \end{pmatrix},$$

where the matrix $\hat{\Psi} = \left\{\hat{\psi}_{ij}\right\}_{1\le i,j\le n_a}$ is diagonal and $\hat{\psi}_{ii} = \left(\hat{\Omega}_{aa} - \hat{\mathrm{B}}_{LS}\hat{\Omega}_{fa}\right)_{ii}$.

Portfolio Statistics

Portfolio Beta

Consider portfolio $\vec{\pi}$. Vector $\vec{\beta}_P$ of **portfolio betas** can be found in the following way:

$$\vec{\beta}_P = \mathrm{B}^*\vec{\pi}.$$

Portfolio Variance

Under the selected set of factors the portfolio variance σ_P^2 admits the following decomposition

$$\sigma_P^2 = \vec{\beta}_P^*\Omega_{ff}\vec{\beta}_P + \vec{\pi}^*\Psi\vec{\pi}$$

into **systematic risk** (first item) and **diversifiable** (**nonsystematic**) **risk** (second one).

Other Statistics

Other useful formulae related to asset pricing models include:

$\mathcal{R}^2$ Statistics

$\mathcal{R}^2 = \left(\mathcal{R}_1^2, \ldots, \mathcal{R}_{n_a}^2\right)^*$ is $n_a \times 1$ vector, whose elements take values in $[0,1]$. It is called **determination coefficient**. The more the value $\mathcal{R}_i^2$ is closer to unity, the better the selected asset pricing model describes the dynamics of i-th asset. Calculation formula:

$$\mathcal{R}_i^2 = 1 - \hat{\psi}_{ii} / \left(\hat{\Omega}_{aa}\right)_{ii}.$$

Student's t-statistics for vector α and matrix B

t-statistics are used in linear regression for testing of hypotheses about regression coefficients being equal to zero. Formulas for the calculation of t-statistics $t^{\alpha} = \left\{t_i^{\alpha}\right\}_{1 \le i \le n_a}$ and $t^{\mathrm{B}} = \left\{t_{ij}^{\mathrm{B}}\right\}_{\substack{1 \le i \le n_a \\ 1 \le j \le n_f}}$ are presented below.

Assume that m is the sample size on which the regression coefficients were estimated, and T is a number of years contained in sample. Then

$$t_i^{\alpha} = \hat{\alpha}_i \frac{T}{\sqrt{\hat{\psi}_{ii}}}, \quad t_{ij}^{\mathrm{B}} = \hat{\mathrm{B}}_{ij} \sqrt{\frac{(m - n_a - 1)}{\hat{\psi}_{ii}\left(\hat{\Omega}_{ff}^{-1}\right)_{ii}}}.$$

The critical two-sided probabilities for Student's t-distribution with $m - n_a - 1$ degrees of freedom are obtained based on the absolute values of the calculated statistics. If $m - n_a - 1 \ge 30$, then one may use normal distribution instead of t-distribution. The values obtained in this way are probabilities of the corresponding regression coefficients being equal to zero.

Note. The following rule of thumb can be proposed: if the absolute value of t- statistics is more than 3, and the number of degrees of freedom is not less than 10, then with high probability (about 99%) the corresponding regression coefficient is significant.

Capital Asset Pricing model

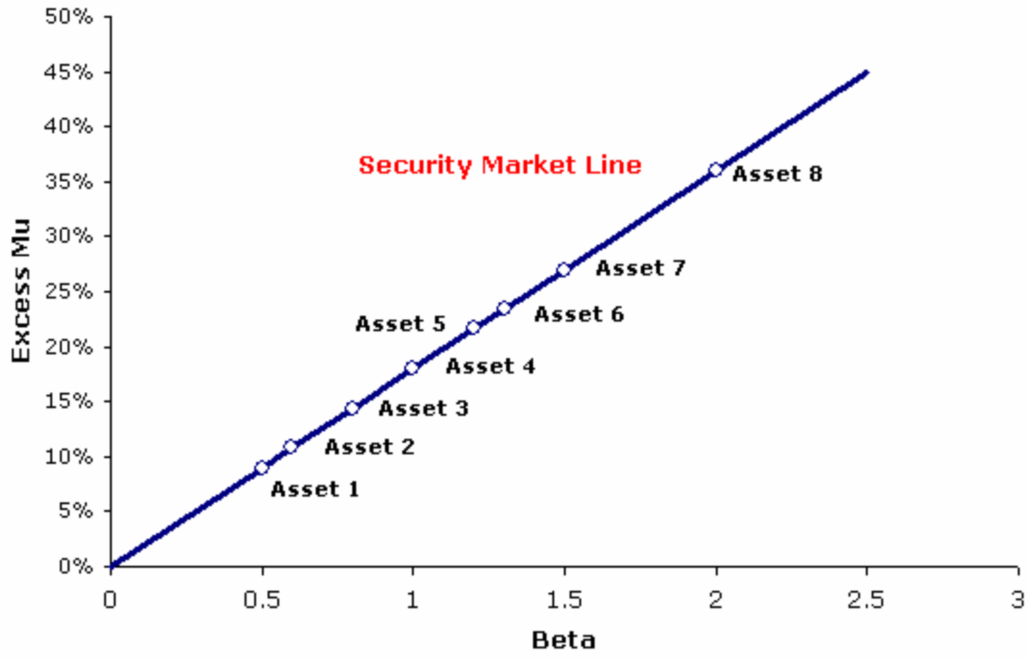
Capital Asset Pricing Model (**CAPM**) is virtually a one-factor asset pricing model with market portfolio as its only factor. The most common choice for market portfolio is **broad market index** such as S&P 500 for U.S. Stock Market.

The assumptions underlying CAPM extensively use notions of an efficient market and "rational" investor, who invests in market portfolio and riskless asset only. Under this assumptions market equilibrium is achieved, that leads to the CAPM framework.

One of CAPM consequences is well-known CAPM equation. This equation sets relationship between assets excess Mu and their sensitivities to market portfolio, called **Beta** (for further notations look in Factor-based Asset Pricing Models):

$$\vec{\mu}_a^e = \mu_f^e \vec{\beta}, \text{ where } \vec{\beta} = \frac{\mathrm{Cov}\left(\vec{P}^a, P^f\right)}{\mathrm{Var}\left(P^f\right)}$$

Essential CAPM advantage is its simplicity, which admits graphical illustration.



Straight line with slope μ_f^e that relates assets betas and their respective excess Mu values is called **Security Market Line** (**SML**).

Portfolio Beta

Beta of portfolio $\vec{\pi}$ is calculated as a linear combination of individual betas:

$$\beta_P = \sum_{i=1}^{n} \pi_i \beta_i$$

As expected, point $\left(\beta_P, \mu_P^e\right)$ also belongs to **SML**: $\mu_P^e = \mu_f^e \beta_P$.

Portfolio Variance

Under the CAPM assumptions portfolio variance σ_P^2 admits the following decomposition:

$$\sigma_P^2 = \beta_P^2 \sigma_f^2 + \sum_{i=1}^{n} \pi_i \psi_{ii}$$, where $\sigma_f^2 = \Omega_{ff}$ denotes variance of the market portfolio.

As before, the first item on the right side of the above expression is called portfolio **systematic risk**, while second one is called portfolio **diversifiable risk**.

Fama-French 3-factor asset pricing model

The asset pricing model, developed by Eugene Fama and Kenneth French, is widely accepted as one of the most successful Factor-based Asset-Pricing Models ever created. Derived with empirical arguments in mind, Fama-French model provides much better fit to real data then popular CAPM.

Fama-French 3-factor asset-pricing model corresponds to the following 3-factor regression (for further notations look in Factor-based Asset-Pricing Models):

$$\vec{P}_t^a = \vec{\alpha} + \beta^M P_t^M + \beta^{SMB} P_t^{SMB} + \beta^{HML} P_t^{HML} + \vec{\varepsilon}_t$$

Three factor portfolios that enter the above equation have the following financial meaning:

1. **M** represents market portfolio — the same factor that appears in CAPM.

2. **SMB** (**S**mall **M**inus **B**ig) portfolio represents zero-investment portfolio that is long in **small-cap** stocks and short in **big-cap** stocks.

3. **HML** (**H**igh **M**inus **L**ow) portfolio represents zero-investment portfolio that is long in **high book-to-market** stocks (so-called "**Value**" stocks) and short in **low book-to-market** stocks (so-called "**Growth**" stocks).

Fama-French model is based on the observation that small cap stocks and "Value" stocks historically tend to do better than market as a whole. A very natural way to formalize this empirical fact is to write down the above regression equation. While $\mathcal{R}^2$-statistics for CAPM usually takes values of around 0.85, Fama-French model is capable of accounting for almost all variation in individual assets.

Note. The reason why Fama-French model is so successful in fitting stock data is far form being obvious. One of intuitively appealing explanations is that **SMB** and **HML** portfolios serve as "correction factors" for a broad-based index, commonly used as market portfolio. Since broad index puts more weight in big-cap and "growth" stocks rather than in small-caps and "value" stocks respectively, it may lead to some bias between broad-based index and practically unobservable market portfolio. It is quite possible that SMB and HML portfolios simply "correct" the broad index for the mentioned effect.

The historical data for SMB and HML portfolios can be downloaded from Kenneth French's website.

Model Parameter Estimation

Maximum likelihood estimates of means and covariances

Consider segment $[0,T]$ with $m+1$ observations of the discounted prices: $\vec{S}(0),\vec{S}(\Delta),...,\vec{S}(m\Delta)$, where $\Delta=\frac{T}{m}$. Proceeding from practical reasons, we shall assume that the above prices don't take into account the dividend yields and the risk-free rate. Based on the prices $\vec{S}(0),...,\vec{S}(T)$ one forms the range of logarithmic returns $\vec{r}_1,...,\vec{r}_m$ of length m by means of the formula

$$r_{ij}=\ln\frac{S_j(i\Delta)}{S_j((i-1)\Delta)},\ 1\le i\le m,\ 1\le j\le n.$$

Sample Expected Growth Rate $\hat{\rho}$

The MLE for **expected growth rate** $\hat{\rho}=(\hat{\rho}_1,...,\hat{\rho}_n)$ is the annualized average of $\vec{r}_1,...,\vec{r}_m$.

$$\hat{\rho}_j=\frac{1}{T}\sum_{i=1}^{m}r_{ij}.$$

Sample estimate for expected excess growth rate $\hat{\rho}^e$ is

$$\hat{\rho}^e=\hat{\rho}-r_f\mathbb{1}+\vec{d},$$

where
r_f denotes risk-free rate;
$\vec{d}$ denotes $n\times 1$ vector of dividend yields;
$\mathbb{1}$ stands for $n\times 1$ vector of ones.

Sample Covariance matrix $\hat{\Omega}$

The **Covariance matrix** estimate $\hat{\Omega}$, calculated in SmartFolio, is an annualized sample covariance matrix of the range $\vec{r}_1,...,\vec{r}_m$.

$$\hat{\Omega}=\frac{1}{\Delta}\frac{1}{m-1}\sum_{i=1}^{m}(\vec{r}_i-\Delta\hat{\rho}\mathbb{1})(\vec{r}_i-\Delta\hat{\rho}\mathbb{1})^*$$

Sample Mu vector $\hat{\mu}$

To estimate $\vec{\mu}$ one should resort to the following expression:

$$\hat{\mu}_j=\frac{1}{\Delta}\ln\left[\frac{1}{m}\sum_{i=1}^{m}e^{r_{ij}}\right].$$

Note. If price process satisfies the analytical model assumptions, then sample Mu vector $\hat{\mu}$ can be calculated in alternative way from a **sample expected growth rate** $\hat{\rho}$ and a **sample covariance matrix** $\hat{\Omega}$:

$$\hat{\mu} = \hat{\rho} + \frac{1}{2} Diag(\hat{\Omega}).$$

Corresponding sample estimate $\hat{\mu}^e$ for **excess Mu** has the following form:

$$\hat{\mu}^e = \hat{\mu} - r_f \mathbb{1} + \vec{d}\,.$$

Advanced estimates

Stambaugh combined-sample estimates for means and covariances

This technique consists in obtaining maximum-likelihood estimates of the model parameters for assets with different start dates and common final date.

In the event of portfolio components having different depths of historical data standard parameters estimates, considered in the previous section, result in a loss of information. The covariance matrix of portfolio components, calculated based on a historical dataset with different start dates is not necessarily positive-definite. Hence, the analyzed dataset is truncated traditionally to the date corresponding to the asset with the shortest history. With such an approach, the loss of the information contained in the initial dataset set is inevitable.

The method suggested in [Stambaugh; 1997], consists in the consecutive use of the regression of assets with a shorter history on assets with a longer history. This time all the information contained in the data is used.

Note. It is worth mentioning that the use of the entire dataset improves estimates for longer-history assets as well as estimates for shorter-history assets.

For more detailed information on the discussed method see [Stambaugh; 1997].

About shrinkage estimators

The shrinkage estimator is a statistical tradeoff between the bias and the estimation error. For the first time the shrinkage estimator appeared in [Stein; 1956], where it was shown that “shrinking” sample means of multivariate normal distribution to an appropriate common constant improves estimation accuracy.

In general, the shrinkage estimate is obtained via a “shrinkage” of sample unbiased estimate towards some biased target with lower estimation error.

Shrinking sample excess Mu towards excess Mu of GMV portfolio
Shrinking sample covariance matrix towards constant correlations covariance matrix
Shrinking sample estimates towards values, implied by asset pricing model

Shrinking sample excess Mu towards excess Mu of GMV portfolio

Shrinkage estimator for $\hat{\mu}_J^e$ was obtained in [Jorion; 1986]. Based on Bayes approach, estimate $\hat{\mu}_J^e$ implies shrinking of sample excess Mu $\hat{\mu}^e$ towards a common constant, equal to sample excess Mu $\hat{\mu}_{GMV}^e$ for global minimum variance portfolio. Calculation formulae for $\hat{\mu}_J^e$ are presented below:

$$\hat{\mu}_J^e = (1-\phi)\hat{\mu}^e + \phi\hat{\mu}_{GMV}^e,$$

$$\phi = \frac{n+2}{n+2+\dfrac{m(m-1)}{m-n-2}\left(\hat{\mu}^e - \hat{\mu}_{GMV}^e\right)^* \hat{\Omega}^{-1}\left(\hat{\mu}^e - \hat{\mu}_{GMV}^e\right)},$$

$$\hat{\mu}_{GMV}^e = \frac{1^*\hat{\Omega}^{-1}\hat{\mu}^e}{1^*\hat{\Omega}^{-1}1},$$

where 1 is $n \times 1$ vector of ones;
n is number of assets;
m is number of observations.

Shrinking sample covariance matrix towards constant correlations covariance matrix

This estimate is a shrinkage estimator for covariance matrix. It was proposed in [Ledoit, Wolf; 2004]. Final estimate $\hat{\Omega}_{Shrink}$ is obtained via shrinkage of the sample covariance matrix towards the covariance matrix, produced by averaging correlations across asset pairs.

$$\hat{\Omega}_{Shrink} = \delta\hat{\Omega}_{Const} + (1-\delta)\hat{\Omega},$$

where

$\hat{\Omega}$ – sample covariance matrix,

$\hat{\Omega}_{Const}$ – constant correlations covariance matrix.

Details, including the formulas for the optimal shrinkage intensity δ can be found in [Ledoit, Wolf; 2004].

Shrinking sample estimates towards values, implied by asset pricing model

The given method of the estimation of parameters Ω and μ^e, first suggested in [Pastor, Stambaugh; 1999], consists in applying Bayes approach, which assumes a certain degree of investor's confidence in that the market satisfies the selected factor-based asset pricing model (such as CAPM or Fama-French 3-factor model).

The degree of investor's confidence in the chosen factor model is determined by the parameter ω, taking values in the range form 0 to 1. Value $\omega = 0$ means that the investor completely ignores predictions of the factor model, using as estimates for Ω and μ^e their sample counterparts, while $\omega = 1$ means, that the estimates for Ω and μ^e coincide with model-implied estimates $\hat{\mu}_m^e$ and $\hat{\Omega}_m$.

Analytical formulas in general case $0 \le \omega \le 1$ are presented in [Wang; 2005].

Joint estimator for means and covariances based on Missing Factor approach

This method described in [MacKinlay, Pastor; 2000] can be applied in a situation when there are reasons to believe that the dynamics of the portfolio assets is determined by influence of one factor only, however the factor itself is "unobservable", i.e. the structure of the latter is unknown.

Note. By an unobservable factor one may imply the unknown structure of the market portfolio.

Let vector $\vec{P}_t^a$ denote simple rates of return for assets on the interval $[t, t+\Delta t]$, where Δt is small enough. By P_t^f we shall designate corresponding simple rate of return of an unobserved factor.

Suppose that the dynamics of the portfolio components satisfies the one-factor model

$$\vec{P}_t^a = \vec{\beta} P_t^f + \vec{\varepsilon}_t$$
$$\mathbf{E}\vec{\varepsilon}_t = 0, \quad \mathbf{E}\vec{\varepsilon}_t\vec{\varepsilon}_t^* = \Psi, \quad Cov(P_t^f, \vec{\varepsilon}_t) = 0,$$

where $\vec{\beta}$ denotes vector of **betas** and vector $\vec{\varepsilon}_t$ contains **regression residuals**.

The additional assumption made in [MacKinlay, Pastor; 2000] is that the covariance's matrix of the residuals Ψ is proportional to the identity matrix:

$$\Psi = \sigma^2 I, \; I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The above assumption, basically, is not burdensome if portfolio components belong to the same **asset class** (big-cap stocks, small-cap stocks, bonds etc.).

With the assumptions made it is possible to establish a connection between a covariance matrix Ω and a vector $\vec{\mu}$, as a result one obtains estimates $\hat{\Omega}_{miss}$ and $\hat{\mu}_{miss}$. Analytical formulas used in calculation of $\hat{\Omega}_{miss}$ and $\hat{\mu}_{miss}$ are given in [Kan, Zhou, 2005].

The Black-Litterman model

The Black-Litterman model blends together the equilibrium-implied expected returns with expected returns extracted from investor's subjective views. There are several variations of the model implementation; the one realized in SmartFolio is mostly based on the approach taken in [Idzorek; 2004]. Instead of expounding the model here, the reader is suggested to read this very intuitive and well-written paper. Two other extremely useful sources of information are [Walters; 2008] and [Meucci; 2005].

Here we just set forth the main distinguishing features peculiar to the SmartFolio realization of the model.

1. The market equilibrium portfolio is calculated from market capitalizations of portfolio components and market risk aversion as inputs.

2. The posterior covariance matrix is set equal to the prior covariance matrix. This significantly simplifies the computations, while keeping the final results essentially intact.

3. Uncertainty in investor's subjective views can be expressed in two alternative ways:

    a. As a global confidence level varying from 0% to 100%. This is the method proposed in [Meucci; 2005] (see formula (9.42)).
    b. As separate confidence levels (varying from 0% to 100%) for each view. The original approach is described in [Idzorek; 2004].

    In case of one view only these two methods produce the same results. The choice (b) is particularly attractive as it combines transparent intuition with high level of flexibility.

Dynamic portfolio strategies

In the present chapter we consider two practically important portfolio strategies. What makes them different from our previous focus is that their weights vary through time.

Portfolio Insurance

Portfolio Insurance refers to portfolio strategies that take into account the constraints, put onto portfolio wealth dynamics.

SmartFolio combines two types of portfolio insurance strategies into one:

1. A portfolio strategy that consists in preventing discounted portfolio wealth from loosing a prespecified portion of its initial value.
2. A portfolio strategy that guarantees preservation of a prespecified portion of accumulated profits.

Let $\alpha \in [0,1]$ denote the portion of initial wealth that investor wishes to secure.
Let $\beta \in [0,1]$ denote the secured portion of accumulated income.

Note. If $\alpha = \beta$, then the corresponding portfolio insurance strategy is equivalent to securing α portion of the maximum-to-date value of discounted wealth. In other words, an investor does not allow the maximum drawdown of his discounted wealth ever to exceed the given constant α.

Construction of Portfolio Insurance Strategy

Imagine that an investor wishes to apply (α,β)-portfolio insurance to some underlying portfolio strategy $(\vec{\pi}(t))_{t\geq 0}$. Let X_t denote discounted wealth of a final strategy at time t. For simplicity lets assume that trades occur at discrete times $t = 0,1,\ldots$.

Portfolio insurance rules are presented below:

1. At $t = 0$ the initial wealth X_0 is divided in two parts $X_0^S = \alpha X_0$ and $X_0^R = (1-\alpha)X_0$. The former denotes initial value of **secured wealth**, while the latter denotes initial value of **risk wealth**. The sum of secured wealth and risk wealth will be further referred to as **aggregate wealth**.
2. Strategy $(\vec{\pi}(t))_{t\geq 0}$ is applied to risk wealth, while secured wealth is kept in a riskless asset.
3. Let $X_t^{\max}$ denote maximum-to-date value of aggregate wealth. Every time the aggregate wealth X_t reaches its new historical maximum, the amount $\beta\left(X_t^{\max} - X_{t-1}^{\max}\right)$ of extra profits is transferred to secured wealth and put into a riskless asset.

It is obvious, that formulated portfolio insurance rules satisfy wealth constraints, stated above. The reason why these very rules were chosen to determine portfolio insurance is explained by the following fact (similar problem is discussed in [Cvitanic, Karatzas; 1995]):

Key Result

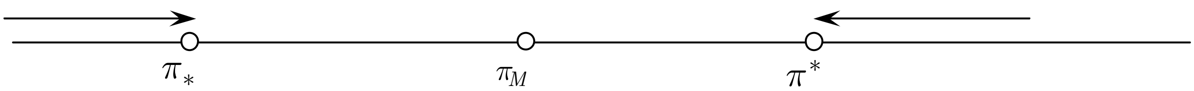
Let $\vec{\pi}_M$ denote Merton portfolio for an investor with CRRA utility function. Then the stated above portfolio insurance rules, applied to $\vec{\pi}_M$, define a portfolio strategy that is optimal for the same investor in presence of the corresponding portfolio wealth constraints.

Proportional Transaction Costs and Inaction Region

Transactions in the financial market are often accompanied by essential costs. As a rule, an investor comes across the transaction costs proportional to the total volume of transactions made. Such costs are more often adhered to the value of **bid-ask spread** and the broker commissions. Alternative kind of costs, also investigated in literature, are called **fixed costs.** Fixed costs do not depend on the volume of the transaction. Below we shall concentrate on the **proportional transaction costs**.
The portfolio strategy which consists in continuous maintenance of given portfolio weights in such a situation appears unacceptable. Indeed, as the turnover of such strategy can be very high, the associated transaction costs arising during portfolio rebalancing also becomes unreasonably high.

The qualitative form of theoretically optimal strategy in the presence of proportional transaction costs is illustrated below by the examples of portfolios which consist of one and two risky assets.

One Asset Example



Let π_M be the optimal weight in risky asset in absence of transaction costs (the only component of the Merton portfolio). Let π_t denote current weight in risky asset at time t.
The optimal strategy in presence of transaction costs is described by the following rules:

1. Appropriate critical weights π_* and π^* such that $\pi_* < \pi_M < \pi^*$ are calculated.
2. While $\pi_* \le \pi_t \le \pi^*$, no transactions take place.
3. Once π_t reaches π_*, investor must transact the minimal amount required to keep $\pi_t \ge \pi_*$.
4. Once π_t reaches π^*, investor must transact the minimal amount required to keep $\pi_t \le \pi^*$.

In other words, $\left[\pi_*, \pi^*\right]$ defines the **inaction region**, and portfolio weight π_t is constantly kept inside this region once (due to price fluctuations) π_t is driven to one of its borders.

General Case

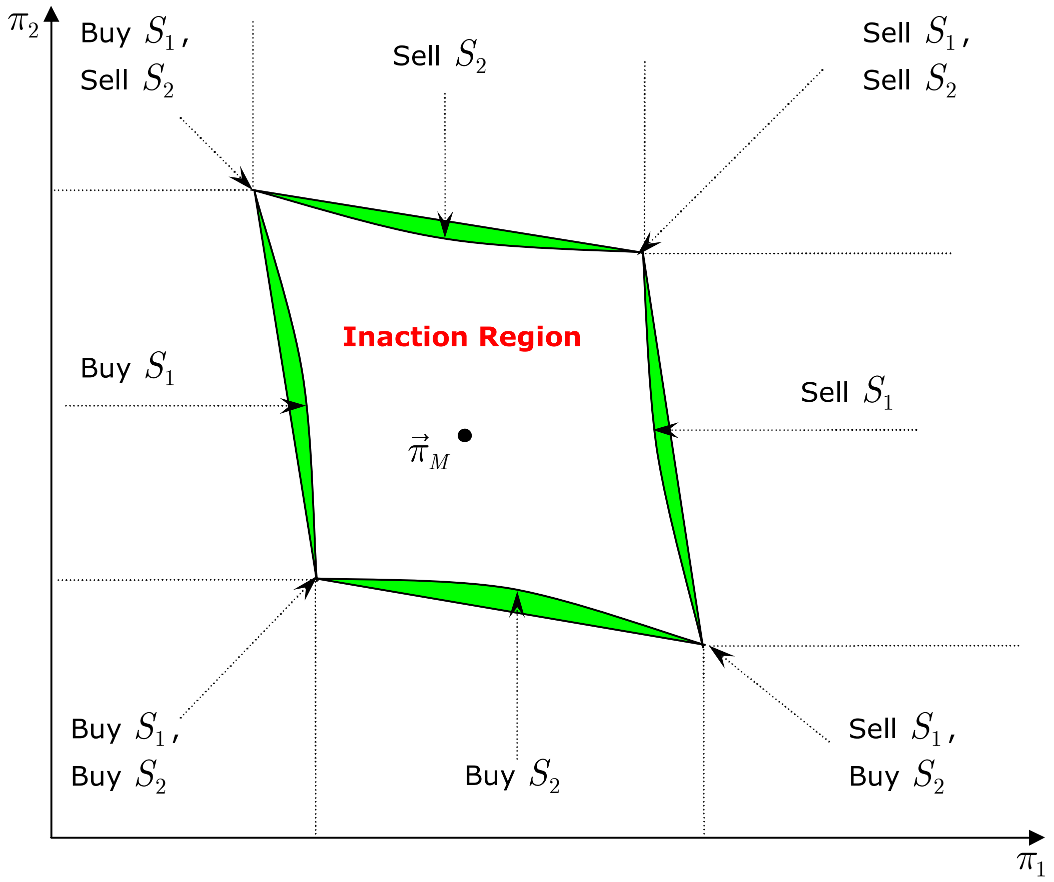
When there is more then one risky asset in portfolio, the situation becomes far more complicated, but the logic of the optimal behavior essentially stays the same.

1. There exists the so-called **inaction region** (coherent set, situated in n-dimensional space of portfolio weights), where no transactions take place while portfolio weights belong to the above set.
2. Immediately after the portfolio vector abandons the inaction region, an investor must transact the minimal amount in appropriate assets, which is required to keep portfolio weights from leaving the inaction region. In other words, the inaction region frontier plays role of **reflecting boundary**.

Note. As a rule, point in n-dimensional space, defined by the Merton portfolio, lies inside the inaction region. The latter point is called the **Merton point**.

Two Assets Example

Lets apply the above n-dimensional optimal rule to two-dimensional case of portfolio assets S_1 and S_2. The picture below shows the inaction region with Merton point $\vec{\pi}_M$ inside it. All possible types of transactions that are required to send portfolio weights back to the inaction region are shown with arrows.



Calculating the Inaction Region

It follows from the above that optimal portfolio strategy under proportional transaction costs is completely defined by its inaction region. Unfortunately, it seems that the obstacles arising when the attempts are made to find analytical solution for the exact form of inaction region are insurmountable.

There are many publications devoted to the problem of finding the reasonable approximation for the inaction region, most of which contain some numerical procedures (see, for example, [Muthuraman, Kumar; 2006]). On the contrary, we will focus on the result first obtained in [Davis, Norman; 1990] and extended to the multidimensional case by D. Kramkov and S. Volkov in 1996.[1] In special settings they managed to find a closed analytical solution for the inaction region approximation, which works quite satisfactory in practice.

[1] To our knowledge, no results of theirs were published in English-speaking journals.

Assumptions of the Model

1. The authors work in the framework of the analytical model, modified to include proportional transaction costs, represented by vector $\vec{\delta} = (\delta_1,\ldots,\delta_n)$, where δ_i denotes the costs size in the i-th asset. For example, $\delta_i = 1\%$ means that 1% of the total amount traded in the i-th asset is contributed to costs. Usually δ_i is close to the Bid-Ask Spread, expressed in % of the asset price.
2. The CRRA utility functions, determined by relative risk aversion coefficient λ, are considered.
3. All possible structures of the inaction region are restricted to **cuboids** (**rectangular parallelepipeds** in n-dimensional space) with the Merton point in the center.

> **Note**. The specified shapes of the inaction region contain the optimal one only when all portfolio assets are uncorrelated. But there are two reasons why their use is justified. On the one hand, portfolio strategies with cuboid region of inaction are significantly easier to implement. On the other hand, practical benefits received from utilizing cuboid region of inaction and from inaction regions of more sophisticated shapes are insignificant.

4. Since obtained results are expressed in asymptotic form, their usage is limited to sufficiently small transaction costs (around 5% or less).

Formulae for the inaction region

Let $\vec{\pi}_M^\lambda$ denote $n\times 1$-vector, corresponding to the Merton portfolio with relative risk aversion λ.
Define an *Inaction Region* G_x in the following way:

$$G_x = \left\{ \left(\vec{\pi}_M^\lambda\right)_i - x_i \le \pi_i \le \left(\vec{\pi}_M^\lambda\right)_i + x_i,\ i = \overline{1,n} \right\},$$

where $\left(\vec{\pi}_M^\lambda\right)_i$, $i = \overline{1,n}$, correspond to i-th component of Merton portfolio.

Denote the following vectors and matrices:

$\Omega = (a_{ij})_{i,j\le n}$ – $(n\times n)$-covariance matrix

E – $(n\times n)$-identity matrix

I – $(n\times n)$-matrix of ones

$B = (b_{ij})_{i,j\le n}$ - $(n\times n)$-matrix, defined by the following equality:

$$B = \mathrm{Diag}(\vec{\pi}_M^\lambda)(E - I\,\mathrm{Diag}(\vec{\pi}_M^\lambda))\Omega(E - \mathrm{Diag}(\vec{\pi}_M^\lambda)I)\mathrm{Diag}(\vec{\pi}_M^\lambda),$$

where $\mathrm{Diag}(\vec{\alpha})$ denotes a diagonal matrix with elements of $\vec{\alpha}$ at the main diagonal.
Also denote

$$\zeta_i = \left(\frac{3b_{ii}\delta_i}{2a_{ii}\lambda}\right)^{1/3}.$$

Let S-set denote such a set that is symmetrical relative to all coordinate axes, translated to the Merton point.

Key Result

Vector $\vec{\zeta} = (\zeta_1, \ldots, \zeta_n)^*$*, defined above, determines the inaction region* G_ζ*, which is asymptotically (for* $\left\|\vec{\delta}\right\| \to 0$*) optimal among all* S*-sets. Obtained order of asymptotic convergence is* $o\left(\left\|\vec{\delta}\right\|^{2/3}\right)$*.*

Risk Management Tools

Definitions

Value-at-Risk (**VaR**) and **Conditional Value-at-Risk** (**CVaR**, other notations include **Expected Shortfall**, **Expected Tail Loss**, **Tail VaR**) are indispensable tools of portfolio risk management.

VaR and CVaR measures become increasingly useful when the distribution of portfolio logarithmic returns substantially deviates from normal. In this case, volatility as a common risk measure appears inappropriate.

Portfolio VaR is defined as a maximum portfolio loss (measured in % of initial wealth) over a given time interval at a given level of statistical confidence.

Definition. $\mathbf{VaR}_\alpha^T$ is implicitly defined by the following expression (for simplicity we assume that portfolio returns have continuous distribution):

$$\mathrm{P}\left(p_{[0,T]}^e < -\mathbf{VaR}_\alpha^T \right) = 1 - \alpha ,$$

where $p_{[0,T]}^e$ is portfolio excess simple return over $[0,T]$ period.

To put it more formally, $-\mathbf{VaR}_\alpha^T$ corresponds to the $(1-\alpha)$-quantile of the distribution of portfolio excess simple return over $[0,T]$ period.

Portfolio CVaR is conditional expectation of losses beyond VaR.

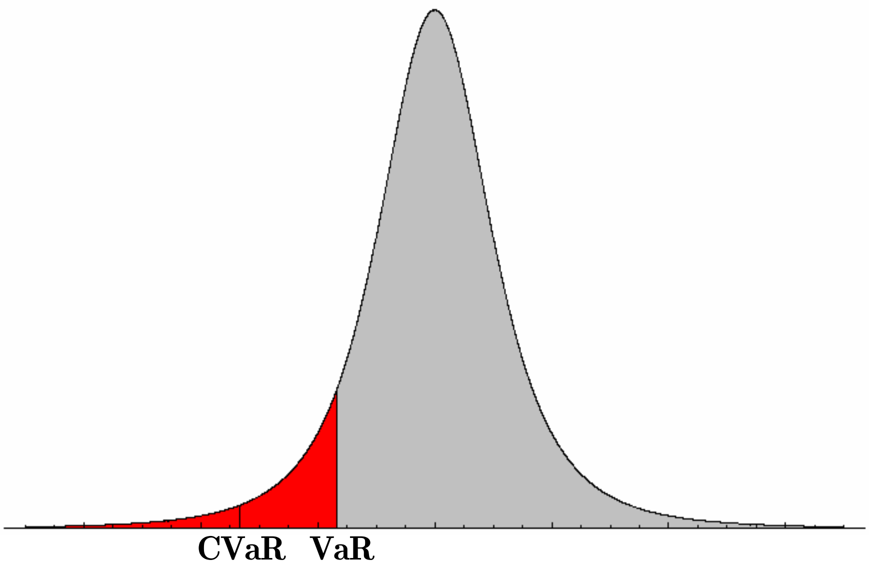
Definition. $\mathbf{CVaR}_\alpha^T$ is an expected value (with opposite sign) of portfolio excess simple return $p_{[0,T]}^e$ under the condition $p_{[0,T]}^e \le -\mathbf{VaR}_\alpha^T$:

$$\mathbf{CVaR}_\alpha^T = -\mathbf{E}\left(p_{[0,T]}^e \middle| p_{[0,T]}^e \le -\mathbf{VaR}_\alpha^T \right)$$

In other words, $-\mathbf{CVaR}_\alpha^T$ is an average value of $(1-\alpha)\cdot 100\%$ of highest losses.

Common values for α are 0.9, 0.95, 0.975 and 0.99. The time horizon T usually corresponds to a period required for complete liquidation of all portfolio positions.

Figure 1. Distribution of logarithmic portfolio returns over the period $[0,T]$.



Important! Although CVaR isn't so widespread a measure of risk as VaR, in contrast to VaR it possesses important property of **subadditivity**. Subadditivity of CVaR means that CVaR of simultaneously holding two portfolios is less or equal to the sum of CVaRs for individual portfolios, considered separately. That's why CVaR is more suitable quantity to be included in portfolio optimization (as constraint or part of an optimization criterion) than VaR is. Essential shortfalls of VaR as a measure of risk due to a lack of subadditivity are discussed in detail in [Artzner, Delbaen, Eber, Heath; 1999].

Calculation techniques

There are plenty of approaches to the calculation of VaR and related measures. The methods that are implemented in SmartFolio are described below.

- Delta-Normal Method
- Empirical Distribution
- Implied Normal Distribution
- Implied Student's t-Distribution
- Cornish-Fisher Expansion

The last four methods utilize block bootstrapping algorithm.

Delta-Normal Method

This method is simplest and the most common in application. Based on the analytical model assumptions, it calculates $(1-\alpha)$-quantile z_α of the normal distribution with parameters $m_T = T\rho_P^e$ and $\sigma_T = \sqrt{T}\sigma_P$. It corresponds to the distribution of portfolio excess growth rate equal to $\frac{1}{T}\ln\frac{X_T}{X_0}$, where X_t denotes the discounted portfolio wealth at time t. $\mathbf{VaR}_\alpha^T$ is then calculated as $\mathbf{VaR}_\alpha^T = 1 - e^{z_\alpha}$.

Accordingly,

$$\mathbf{CVaR}_\alpha^T = 1 - \exp\left(m_T - \frac{\sigma_T}{1-\alpha} f\left(\frac{z_\alpha - m_T}{\sigma_T}\right)\right),$$

where $f(\cdot)$ denotes the density function of the standard normal distribution.

Unfortunately, **D**elta-**N**ormal **M**ethod (**DNM**) is far form being precise. The main drawback of DNM is that it doesn't take into account higher moments of the portfolio returns distribution including fat tails, which are very common in practice and have critical impact on VaR-CVaR values.

Another possible shortcoming of DNM is the assumption that portfolio returns are *independent* through time. It leads to the so-called square-root scaling law for standard deviation, which means that $\sigma_{T_2} = \sqrt{\frac{T_2}{T_1}}\sigma_{T_1}$. In practice it is often the case that instead of square root degree, $a < \frac{1}{2}$ should be used.
Final DNM weakness is its inability to account for non-linear relationships between portfolio components, which arise when options are included in portfolio.

As a consequence, quite often DNM seriously underestimates true values for VaR and CVaR, particularly for the extreme values of α, exceeding 0.95.

Empirical Distribution

Empirical Distribution approach involves the following steps:

1. The array of historical portfolio excess logarithmic returns over the period 0 to T is formed. If T contains more then one period, then prior implementation of block bootstrapping algorithm must increase accuracy.
2. The obtained array is sorted and the worst $(1-\alpha)\cdot 100\%$ values are extracted.
3. The *best* and the *average* values of the selected worst part are calculated.
4. VaR and CVaR are then obtained by transformation of respective values to represent simple returns with the opposite sign using $p = 1 - e^r$ relationship.

If homogeneous historical data of virtually unlimited length was available, then the empirical distribution approach would be ideally suited for the calculation of VaR and CVaR. It accounts for both higher moments of portfolio returns distribution and non-linear interdependencies. In reality its use is limited to portfolios, whose components are traded for time long enough (at least, 5-7 years for daily database).

Implied Normal Distribution

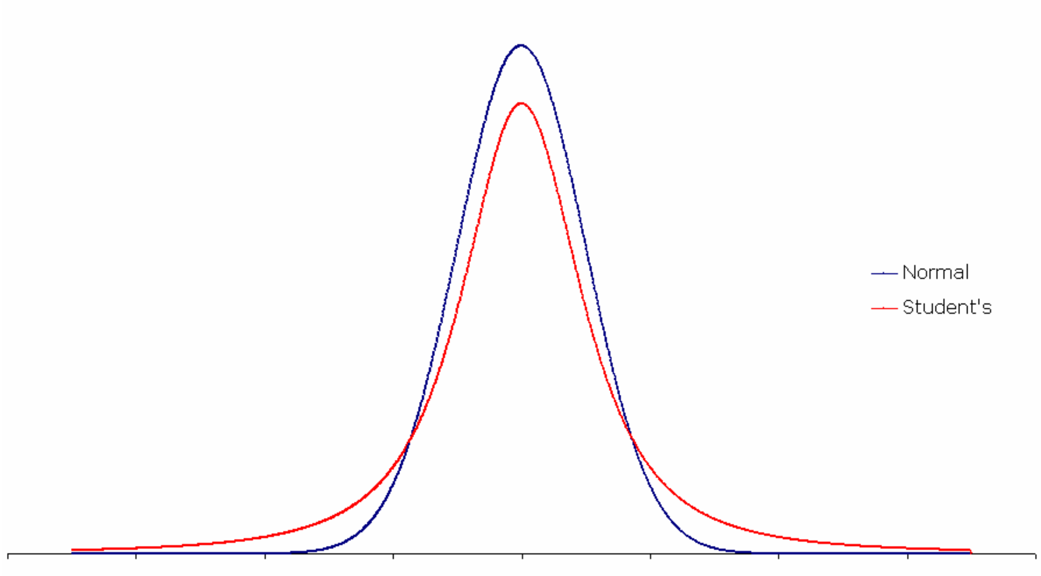
Delta-normal method utilizes the assumption of square-root growth in portfolio standard deviation σ_T as a function of T. On the contrary, **implied normal distribution** approach uses the unique estimate of σ_T for each value of T. For this purpose, analogously to empirical distribution approach, the array of historical excess logarithmic returns over the period $[0, T]$ is formed using block bootstrapping algorithm. Then the sample estimate $\hat{\sigma}_T$ is obtained and inserted in formulas for delta-normal method.

While the implied normal distribution approach doesn't assume the square-root scaling law in standard deviation, it still suffers from two residuary drawbacks, peculiar to delta-normal method: the inability to account for higher moments of portfolio returns distribution and the non-linear relationships among portfolio components.

Student's t-Distribution

There is much evidence, coming from recent publications in financial math, that Student's t-distribution delivers quite satisfactory fit to a wide range of financial assets including stocks, commodities and currencies. Its attractive feature is a power law of tails behavior, which makes t-distribution an appealing alternative to normal distribution thanks to positive *kurtosis excess*.

Figure 2. Student's density vs. Normal density



As before, the VaR-CVaR calculation is anticipated with formation of an array of historical excess logarithmic returns over the period $[0,T]$ by means of block bootstrapping algorithm.

Analytical formulas for VaR and CVaR under the assumption of non-central Student's t-distribution with possibly non-integer degrees of freedom are obtained in [Andreev, Kanto; 2004]. Corresponding density function is defined by the following expression:

$$f(x)=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\beta\nu}}\left(1+\frac{(x-m)^2}{\beta\nu}\right)^{-\frac{(1+\nu)}{2}},$$

where m is location parameter, β is dispersion parameter and ν denotes degrees of freedom. In this case quantity ν is directly related to kurtosis. Corresponding estimate $\hat{\nu}_T$ has the following form:

$$\hat{\nu}_T=4+6/\hat{K}_T,$$

where $\hat{K}_T$ is the sample estimate for kurtosis excess (it is assumed that $\hat{K}_T>0$). Corresponding estimates for m and β are $\hat{m}_T=T\hat{\rho}_P^e$ and $\hat{\beta}_T=\frac{\hat{\nu}_T-2}{\hat{\nu}_T}\hat{\sigma}_T^2$ respectively.

Then $\mathbf{VaR}_\alpha^T=1-\exp(r_\alpha)$, where $r_\alpha=t^{-1}_{\hat{\nu}_T;\hat{m}_T;\hat{\beta}_T}(1-\alpha)$ and $t^{-1}_{\hat{\nu}_T;\hat{m}_T;\hat{\beta}_T}$ stands for inverse Student's t-distribution function with parameters $\hat{\nu}_T$, $\hat{m}_T$ and $\hat{\beta}_T$. Accordingly $\mathbf{CVaR}_\alpha^T=1-\exp(r_\alpha^*)$, where

$$r_\alpha^*=\hat{m}_T-\left((1-\omega)\hat{\sigma}_T^2+\omega(r_\alpha-\hat{m}_T)^2\right)\frac{f_{\hat{\nu}_T;0;\hat{\beta}_T}(r_\alpha-\hat{m}_T)}{1-\alpha}.$$

In the latter expression f denotes Student's t-distribution density function and $\omega=\frac{\hat{K}_T}{6+3\hat{K}_T}$.

Cornish-Fisher Expansion

Cornish-Fisher Expansion approximates the quantiles of an arbitrary distribution with known moments in terms of quantiles of the standard normal distribution.[1] The main advantages of applying Cornish-Fisher expansion in calculation of VaR-CVaR are speed and ability to account not only for fat tails as Student's t-distribution does, but also for asymmetry in returns, measured with *skewness*.

Algorithm

1. An array of historical excess logarithmic returns over the period $[0,T]$ is created by means of block bootstrapping algorithm.
2. Based on the obtained array four moments are estimated: sample mean $\hat{m}_T$, sample standard deviation $\hat{\sigma}_T$, sample skewness $\hat{s}_T$ and sample kurtosis excess $\hat{K}_T$.
3. Let z_α denote $(1-\alpha)$-quantile of standard normal distribution. $(1-\alpha)$-quantile $\tilde{z}_\alpha$, corrected for kurtosis and skewness, is established by means of Cornish-Fisher expansion up to 4-th member (for details see [Zangari; 1996]) :
$$\tilde{z}_\alpha = z_\alpha + \frac{1}{6}(z_\alpha^2 - 1)\hat{s}_T + \frac{1}{24}(z_\alpha^3 - 3z_\alpha)\hat{K}_T - \frac{1}{36}(2z_\alpha^3 - 5z_\alpha)\hat{s}_T^2.$$
4. $\mathbf{VaR}_\alpha^T = 1 - \exp(\hat{m}_T - \tilde{z}_\alpha \hat{\sigma}_T)$
5. $\mathbf{CVaR}_\alpha^T = 1 - \exp\left(\hat{m}_T - \frac{f(\tilde{z}_\alpha)}{F(\tilde{z}_\alpha)}\hat{\sigma}_T\right)$, where $f(\cdot)$ and $F(\cdot)$ are the standard normal density and distribution function respectively.

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[1] For more details about Cornish-Fisher expansion visit http://www.riskglossary.com/link/cornish_fisher.htm

Appendices

Appendix A. Block Bootstrapping Algorithm

Statistical **bootstrapping** algorithms generate artificial data series from an original sample via *random resampling with replacement*. Applied to financial time series, bootstrapping procedures help to construct the distribution of returns over long investment horizons, the problem that can hardly be solved without any model assumptions using original sample only.

Block Bootstrapping is a variation of bootstrapping that randomly selects (potentially overlapping) blocks of contiguous observations, as opposed to individual observations. The nice property of block bootstrapping is that it helps preserving serial dependence in the dataset.

In the following example block bootstrapping algorithm is described:

1. Suppose, 4 years of daily portfolio returns are available (1000 observations total). The goal is to construct the distribution of portfolio returns over 5 year horizon.
2. Let's group original sample into 950 overlapping blocks of 50 days in each.
3. To generate 100 artificial samples of 5 year length, it is necessary to randomly resample with replacement (i.e. the same block is allowed to be used for several times) 2500 (250*5*100/50) blocks, obtained from the original sample.
4. The produced samples are then used to construct the empirical distribution of portfolio returns over 5 year horizon.

Bootstrapping at Wikipedia

Appendix B. Downside Volatility

Let's start by selecting a value for the continuously compounded **minimum acceptance excess rate** (**MAR**). Downside volatility takes into account only those values of observed excess rates of return that lie below MAR. In other words, downside volatility is a measure of risk, defined as volatility below MAR.

Definition. Downside volatility ϕ , corresponding to selected value $R^e_{\min}$ of MAR, is defined as

$$\phi_{R^e_{\min}} = \sqrt{\mathbf{E}\left[\min\left(R^e_{[0,T]} - R^e_{\min}, 0\right)\right]^2},$$

where T denotes chosen investment horizon.

Note. In the case of $R^e_{\min} = \rho^e_{[0,T]}$ the corresponding downside volatility measure is called **semi-volatility**.

Note. Compare semi-volatility definition to an expression for volatility σ :

$$\sigma = \sqrt{\mathbf{E}\left[R^e_{[0,T]} - \rho^e_{[0,T]}\right]^2}.$$

Obviously, if $R^e_{[0,T]}$ is symmetrically distributed, then volatility is equal to semi-volatility multiplied by 2.

If a distribution of logarithmic returns deviates from normality, then downside volatility is a more preferable risk measure than often more common volatility measure. Most evident advantage of downside volatility is its ability to give a proper weight for possible asymmetry in distribution of returns measured with skewness. But with the proper choice of MAR the downside volatility can also reflect possibly positive kurtosis excess in price increments.

From the practical point of view, downside volatility is better suited for portfolio risks measurement , since virtually all investors are tolerant to sudden upside movements in wealth, while tending to avoid corresponding downside movements.

Normalized Downside Volatility

In practice, it is more convenient to use an adjusted measure of downside volatility, called normalized downside volatility. Under the assumptions of the analytical model normalized downside volatility coincides with σ .

Definition. Normalized downside volatility ϕ^{norm} , corresponding to value $R^e_{\min}$ of MAR, is defined as

$$\phi^{norm}_{R^e_{\min}} = \frac{\phi_{R^e_{\min}}}{xf(x)+(1+x^2)F(x)},$$

where $x = \dfrac{R^e_{\min} - \rho^e_{[0,T]}}{\sigma}$, and $f(\cdot)$ and $F(\cdot)$ stand for standard normal density and distribution function respectively.

Appendix C. Investment ranking and Performance measures

Most of the existing measures, destined to evaluate performance of a portfolio or an individual asset, have the form of **Risk-to-Reward** ratio. They differentiate depending on particular definitions of **Risk** and **Reward**. Choices of risk and reward, utilized in SmartFolio, are summarized in the following table.

	Reward	
Risk	Expected excess growth rate	Excess Mu
Volatility	Information Ratio	Instantaneous Information Ratio
Downside Volatility	Sortino Ratio	Instantaneous Sortino Ratio
Normalized Downside Volatility	Normalized Sortino Ratio	Normalized Instantaneous Sortino Ratio
Conditional Value-at-Risk	STARR Ratio	Instantaneous STARR Ratio
Normalized Conditional Value-at-Risk	Normalized STARR Ratio	Normalized Instantaneous STARR Ratio

Fix some minimum acceptance excess rate $R^e_{\min}$. In the following context the latter is also is also called **Target Excess Rate**.

Information Ratio

Definition. Information Ratio I, corresponding to $R^e_{\min}$, is equal to the difference between expected excess growth rate and $R^e_{\min}$, divided by volatility:

$$I_{R^e_{\min}} = \frac{\rho^e_{[0,T]} - R^e_{\min}}{\sigma}.$$

Note. Widely recognized **Sharpe Ratio** is a particular case of information ratio corresponding to $R^e_{\min} = 0$.

Sortino Ratio

Definition. Sortino Ratio s, corresponding to $R^e_{\min}$, is equal to the difference between expected excess growth rate and $R^e_{\min}$, divided by downside volatility with MAR equal to $R^e_{\min}$:

$$s_{R^e_{\min}} = \frac{\rho^e_{[0,T]} - R^e_{\min}}{\phi_{R^e_{\min}}}.$$

Normalized Sortino Ratio

Definition. Normalized Sortino Ratio, corresponding to $R^e_{\min}$, is similar to Sortino ratio, but with normalized downside volatility in denominator:

$$s^{norm}_{R^e_{\min}} = \frac{\rho^e_{[0,T]} - R^e_{\min}}{\phi^{norm}_{R^e_{\min}}}.$$

Under the assumptions of the analytical model, the normalized Sortino ratio coincides with information ratio.

STARR Ratio

Definition. STARR Ratio is equal to the difference between expected excess growth rate $\rho^e_{[0,T]}$ and $R^e_{\min}$, divided by the Conditional Value-at-Risk, transformed to logarithmic return:

$$\mathbf{STARR}^T_\alpha = \frac{\rho^e_{[0,T]} - R^e_{\min}}{-\ln\left(1 - \mathrm{CVaR}^T_\alpha\right)}.$$

Normalized STARR Ratio

Definition. Normalized STARR Ratio (**NSTARR**) is the STARR ratio, corrected in such a way that in case of normally distributed logarithmic returns it coincides with information ratio.

$$\mathbf{NSTARR}^T_\alpha = \frac{\rho^e_{[0,T]} - R^e_{\min}}{\mathrm{NCVaR}^T_\alpha},$$

where NCVaR^T_α denotes **Normalized CVaR**.

Definition. Normalized CVaR is a measure, based on CVaR, corrected in such a way that under the assumptions of the analytical model it coincides with volatility measure.

$$\mathbf{NCVaR}_{\alpha}^{T} = \frac{1-\alpha}{f(F(1-\alpha))\sqrt{T}}\left[T\rho_{[0,T]}^{e} - \ln\left(1-\mathrm{CVaR}_{\alpha}^{T}\right)\right],$$

where $f(\cdot)$ and $F(\cdot)$ are standard normal density and distribution function respectively.

Under the assumptions of the analytical model, all above performance measures are equivalent when used to sort the list of available portfolios according to their investment attractiveness. Otherwise, because of the properties of downside volatility and CVaR respectively, the Sortino Ratio and the STARR ratio might become more relevant measures of performance.

Standard performance measures vs. their instantaneous counterparts

Below we focus on the **Sharpe ratio**, but the same logic holds true for all other performance measures, presented above.

Definition. Sharpe Ratio S is equal to the expected excess growth rate divided by volatility:

$$S = \frac{\rho^{e}}{\sigma}.$$

Definition. Instantaneous Sharpe Ratio S_{inst} is equal to excess Mu divided by volatility:

$$S_{inst} = \frac{\mu^{e}}{\sigma} = S + \frac{\sigma}{2}.$$

Both the *Sharpe ratio* and the *instantaneous Sharpe ratio* sort assets according to their relative performance in the past. However, there is an essential distinction in the information the corresponding rankings reflect.

- If compared assets are supposed to be used as components of a continuously rebalanced portfolio, then the *instantaneous Sharpe ratio* becomes more appropriate performance measure.

- If one compares already formed portfolios, rather than separate assets, then their relative investment appeal should be measured with the *Sharpe ratio*.

In the former case, when selecting portfolio components, the investor has an opportunity of combining them with each other and with the riskless asset. On the contrary, in the latter case one considers the portfolios as separate alternative investments, thus depriving the investor of the continuous rebalancing advantages. For further details see [Nielsen, Vassalou; 2004].

Appendix D. Selected definitions

Broad-based Index

Broad-based Index at InvestorWords
Broad-based Index at Investopedia

Cost of Carry

Cost of Carry at InvestorWords

Dividend Yield

Dividend Yield, as used in current review, determines continuously compounded rate of return paid on an asset.
Selecting an appropriate value for dividend yield allows correct evaluation of such financial instruments, included in portfolio, as **currency rates**, **futures** and **coupon bonds**.

- **Currency Exchange Rates**
  Dividend yield must be set to the difference between continuously compounded *foreign* interest rate and *domestic* interest rate.

- **Futures**
  Dividend yield must be equal to cost of carry (expressed in the form of continuously compounded rate of return) in the contract under consideration.

- **Coupon Bonds**
  Dividend yield must be equal to coupon rate (once again, its continuously compounded counterpart must be taken).

Efficient Market

Efficient Market at InvestorWords

Index Fund

Index Fund at InvestorWords

Investment Horizon

The **Investment Horizon** is a critical date for the investor: when reaching it he/she evaluates success made by the investments. For private persons such date often corresponds to the moment up to which they postpone their consumption. So, for example, it can be scheduled date of a large purchase, an expected birth of a child or the moment of retire.[1] For portfolio manager the investment horizon is equal, as a rule, to one or two years: after this time the management estimates his work and on the basis of results the manager receives bonus for the specified period.

It should be noted that the gap, formed thus between an investment horizon of the investor and of his portfolio manager is one of the problems that the managment company comes across and which rarely attracts sufficient attention. (see [Cvitanic, Lazrak, Wang; 2006]).

Investment Horizon at InvestorWords

Market Portfolio

Market Portfolio at InvestorWords

[1] The latter example of an investment horizon is slightly incorrect. When investing the retirement savings one should select as his investment horizon the date corresponding to the expected life span of the family members rather than the moment of retirement itself. The above observation comes from the fact that the pensionary consumption, as a rule, is uniformly extended in time, so the average length of the investment period in this case significantly exceeds time left up to the retirement.

Risk-free Rate

The **Risk-free Rate** (or **Riskless Rate**) is continuously compounded rate r_f earned on riskless asset. If Cash (Bank Account) is used as riskless asset, then risk-free rate corresponds to bank account interest rate. Otherwise, if the riskless asset differs from cash, then risk-free rate is equal to dividend yield for that asset.

Riskless Asset

Riskless Asset or **Risk-free Asset** (other common notation is **Numeraire**) – it is an asset, in units of which the investor measures his welfare. In an ideal the investor should be indifferent to changes in value of the riskless asset.

As a rule, the choice of riskless asset is adhered to investor's expenses. For example, if the investor carries the most part of expenses in US dollars, then the most natural choice of welfare measure would also correspond to US dollars.
Common choices for riskless asset:

1. **Cash** (bank account, denominated in domestic currency) – better suits for short-term investments since it doesn't account for the inflation risk.
2. **Fixed-Coupon Bond**
3. **Inflation-Linked Bond** – the best choice for the long-term investments of pension funds
4. **Foreign Currency** or Currency Basket
5. **Broad-based Index** – natural choice for many fund managers, whose task consists in maximizing the growth of investments relative to the selected index.

Wiener Process

Wiener Process at Wikipedia
Brownian motion RiskGlossary

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Useful links

Recommended books

Risk and Asset Allocation (Springer Finance) by Attilio Meucci (**Hardcover** - Jan 11, 2008)

Robust Portfolio Optimization and Management (Frank J Fabozzi Series) by Frank J. Fabozzi, Petter N. Kolm, Dessislava Pachamanova, and Sergio M. Focardi (**Hardcover** - Mar 2007)

Quantitative Equity Portfolio Management (McGraw-Hill Library of Investment and Finance) by Ludwig B Chincarini and Daehwan Kim (**Hardcover** - Jul 27, 2006)

Financial Modeling of the Equity Market: From CAPM to Cointegration (Frank J. Fabozzi Series) by Frank J. Fabozzi, Sergio M. Focardi, and Petter N. Kolm (**Hardcover** - Jan 3, 2006)

Active Portfolio Management: A Quantitative Approach for Producing Superior Returns and Controlling Risk by Richard C. Grinold and Ronald N. Kahn (**Hardcover** - Oct 26, 1999)

Investment Dictionaries

RiskGlossary.com

InvestorWords.com

Investopedia.com

AndreasSteiner.net

Brilliant review of the Modern Portfolio Theory

moneychimp.com

Web resource dedicated to the Black-Litterman model

blacklitterman.org

Table of symbols

$\mathbb{1}$	Vector of ones
α	Confidence level
$\vec{\alpha}$	Vector of instantaneous alphas
$\hat{\alpha}_{LS}$	Vector of ordinary least-squares estimates for instantaneous alphas
$\vec{\beta}_P$	Vector of portfolio betas
B	Matrix of betas
$\hat{\mathrm{B}}_{LS}$	Ordinary least-squares estimate for matrix of betas
CE	Certainty equivalent
$\mathbf{CVaR}_{\alpha}^{T}$	Conditional Value-at-Risk
$\vec{d}$	Vector of dividend yields
d_P	Portfolio dividend yield
$Diag(\vec{x})$	Diagonal matrix with elements of $\vec{x}$ at the main diagonal.
$Diag(A)$	Vector, whose elements are equal to diagonal elements of A
$\vec{\delta}$	Vector of transaction costs
$\vec{\varepsilon}_t$	Vector of regression residuals
$\mathbf{E}$	Mathematical expectation symbol
$\phi_{R^e_{\min}}$	Downside volatility
$\phi^{norm}_{R^e_{\min}}$	Normalized downside volatility
G_x	Inaction region
$I_{R^e_{\min}}$	Information ratio
λ_A	Absolute risk aversion coefficient
λ_R	Relative risk aversion coefficient
$\mu_{[0,T]}$	Expected simple rate of return
$\vec{\mu}$	Mu vector
$\vec{\mu}^e$	Excess Mu vector
$\vec{\mu}^e_{imp}$	Implied excess Mu vector
$\hat{\mu}^e$	Vector of sample estimates for excess Mu
$\hat{\mu}^e_m$	Model-implied estimate for excess Mu vector
μ_P	Portfolio Mu
μ^e_P	Portfolio excess Mu
μ^e_{GMV}	Excess Mu for GMV portfolio
μ^e_G	Excess Mu for tangency portfolio
M	Set of possible values for excess Mu vector
$\mathbf{NSTARR}_{\alpha}^{T}$	Normalized STARR ratio
$p_{[0,T]}$	Simple return

$P_{[0,T]}$	Simple annual rate of return
π_0	Weight in riskless asset
$\vec{\pi}$	Vector of portfolio weights, corresponding to risky assets
$\vec{\pi}_{GMV}$	Vector of portfolio weights for GMV portfolio
$\vec{\pi}_G$	Vector of portfolio weights for tangency portfolio
$\vec{\pi}_M$	Vector of portfolio weights for Merton portfolio
$\vec{\pi}_{3f}$	Vector of portfolio weights for "three-fund" portfolio
Π	Set of admissible portfolios
$Q_c(\vec{\pi})$	Risk-adjusted expected excess rate of return
$r_{[0,T]}$	Logarithmic return, calculated on $[0,T]$ period
$R_{[0,T]}$	Simple annual rate of return, calculated on $[0,T]$ period
$\rho_{[0,T]}$	Expected logarithmic rate of return, calculated on $[0,T]$ period
ρ_P^e	Portfolio expected excess growth rate in the analytical model
r_f	Risk-free rate
r_f^b	Borrowing rate
r_f^l	Lending rate
$\mathcal{R}^2$	Determination coefficient in regression
$R_{\min}^e$	Target excess growth rate, minimum acceptance excess rate (MAR)
$\vec{\sigma}$	Volatility vector
σ_P	Portfolio volatility
$\vec{\rho}^e$	Vector of expected excess growth rates in the analytical model
σ_{GMV}	Volatility of GMV portfolio
σ_G^e	Volatility of tangency portfolio
$s_{R_{\min}^e}$	Sortino ratio
$s_{R_{\min}^e}^{norm}$	Normalized Sortino ratio
S	Sharpe ratio
S_{inst}	Instantaneous Sharpe ratio
$\mathbf{STARR}_\alpha^T$	STARR ratio
Σ	Volatility matrix
t^α	t-statistics vector for instantaneous alphas
t^{B}	t-statistics matrix for betas
T	Investment horizon
$U(x)$	Utility function
$\vec{\upsilon}$	Vector of assets relative contributions to portfolio variance
$\mathbf{VaR}_\alpha^T$	Value-at-Risk
Ω	Covariance matrix
$\hat{\Omega}$	Sample covariance matrix
$\hat{\Omega}_m$	Model-implied estimate of covariance matrix
Ψ	Covariance matrix for regression residuals