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13th Edition

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Practical Applications

§ 2a. Equations of Value, Time Value of Money, and Time Diagrams

Equations of Value and the Time Value of Money

Suppose that deposits made today can earn 5% effective over the next year. Which would you rather have, \$1, 000 today or \$1, 050 in a year?

A basic principle of equivalence that we will use in interest theory is that it doesn't matter to us whether we have \$1, 000 today or \$1, 050 in a year (assuming that the effective rate of interest is 5%). In fact, if we are told that the effective rate is 5% for the next 10 years, we will assume that we would be just as happy with \$1, 000 today as we would be with \$1, $000(1.05)^t$ in t years, where t is any time from 0 to 10.

This basic principle of equivalence may seem so obvious that it doesn't even have to be stated. But we must agree to it if we want to play the game in accordance with the rules of interest theory.[1]

Another way to say this is that money has a ***time value***: \$1,000 today is worth more than \$1,000 a year from now. So if person A paid person B \$1,000 now in return for a payment from B in a year, A would want that payment to be more than \$1,000. You will often see the term ***time value of money*** in the financial literature.

The principle of equivalence allows us to solve interest problems by setting up ***equations of value*** as of a common ***comparison date***. As an example, let's say that you want to accumulate \$5,000 in two years by making a deposit of X today and another deposit of X in a year. If the effective rate of interest is 6%, determine X.

To solve for X we will equate the value of the deposits and the accumulated value of \$5,000. In order to do this, we must choose a common date, called the comparison date. Let's say we choose time 2. As of that date, the deposits are worth $X(1.06)^2 + X(1.06)$, by the principle of equivalence. The AV we want at that point is \$5,000. Thus, the equation of value as of time 2 is:

$$X(1.06^2 + 1.06) = 5{,}000$$

$$X = \frac{5{,}000}{1.06^2 + 1.06} = \$2{,}289.80$$

Two points should be noted:

[1] There may be some people who would not be just as happy with \$1, 050 a year from now as they would be with \$1, 000 today, even if 5% were a fair rate of interest. (Can you think of any reasons why?) But we will assume that the principle of equivalence applies when we solve problems in this course.

(i) You ***must*** choose the same date (the comparison date) to evaluate the deposits and the AV. You cannot directly compare X deposited today and X deposited a year from now with \$5,000 received two years from now, unless you determine the equivalent values of these amounts as of the same date.

(ii) ***Any*** date can be used for the comparison date. For example, suppose we had used a ridiculous date like 122 years from now for the comparison date. What would the equation of value be?

Answer: $X(1.06)^{122} + X(1.06)^{121} = 5,000(1.06)^{120}$

This is the same equation of value as before, but with both sides multiplied by 1.06^{120}. Thus, we would get the same solution for X

Certainly, choosing time 122 would unnecessarily complicate the solution. In this example, time 0 or time 2 (and perhaps time 1) are the obvious choices. Choosing a convenient comparison date is something you will learn to do as you get more experience with solving compound interest problems.

Time Diagrams: The Student's Friend

Some people are lucky: They can visualize deposits and withdrawals, payment periods and interest periods, etc., without the aid of a diagram. But for most of us, a time diagram is a very useful aid in solving problems in compound interest.

A time diagram is a statement of a problem in picture form. Once the diagram is drawn, the problem often practically solves itself.

The following examples are basic in nature. If you feel comfortable about setting up time diagrams, or if you don't need them to solve problems, you can skip this section. (But you might want to look at Examples 3 and 4 in any event.)

Stepping Stones

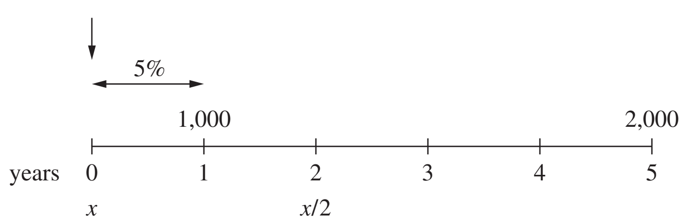
EXAMPLE 1

Draw a time diagram and write an equation of value for the following problem:

A deposit is to be made today and a second deposit, which is one-half the first, is to be made 2 years from now, to provide for withdrawals of \$1,000 one year from now and \$2,000 5 years from now. Interest is at 5% effective. What is the amount of the initial deposit?

SOLUTION

Let x = initial deposit. Then the deposit at time 2 is $x/2$.



Using time 0 as the comparison date, the equation of value is:

$$x + (x/2)v^2 = 1{,}000v + 2{,}000v^5 \quad \text{at } 5\%$$

$$x = \frac{1{,}000v + 2{,}000v^5}{1 + v^2/2} = \$1{,}733.34$$

Note the following points:

(i) Deposits and withdrawals are placed on opposite sides of the time line (for example, all deposits below the line and all withdrawals above the line, as in the above diagram). This will assure that we get the correct terms on the left and right-hand sides of the equation of value.

(ii) To the left of the time line, the interest period is noted ("years"). In this problem, where the interest period and payment period are the same, it isn't too important but get into the habit of noting it on your diagrams anyway.

(iii) The effective interest rate for one interest period (5%) is marked off, as a reminder of the rate to be used in calculations.

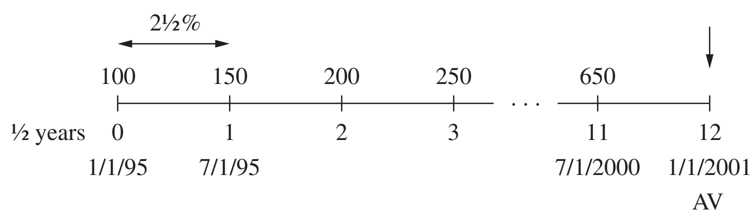
(iv) A vertical arrow is placed at the comparison date chosen; in this solution, time 0 was chosen as the comparison date. Of course, any chosen date would lead to the correct value of x.

EXAMPLE 2

Deposits are made on January 1 and July 1 of every year from 1995 to 2000, inclusive. The initial deposit is \$100 and each subsequent deposit increases by \$50. What is the accumulated value of the deposits on January 1, 2001 at a nominal rate of interest of 5% compounded semiannually?

SOLUTION

In this problem, the effective rate of interest is $2\frac{1}{2}\%$ for a ***half-year*** period and payments are made every ***half-year***, so it's a no-brainer that we should label the diagram in terms of half-year periods.



We have marked off "$\frac{1}{2}$ years" to the left of the time line as a reminder that "time 1" is 6 months from now, "time 2" is a year from now, etc., and that the effective rate of $2\frac{1}{2}\%$ applies to each of these 6-month periods. Also, we have translated the given ***dates*** into ***interest periods***, starting with 1/1/95 as time 0 and ending with 1/1/2001 as time 12 (6 years later).

The equation of value using "brute force" (that is, term-by-term without any fancy annuity symbols) and using time 12 as the comparison date is:

$$100(1.025)^{12} + 150(1.025)^{11} + 200(1.025)^{10} + \cdots + 650(1.025) = AV$$

Of course, the techniques for handling varying annuities that are developed later in this manual would be much more efficient for determining the AV than a term-by-term calculation. But at this point, we are primarily interested in the time diagram as a technique for setting up problems.

EXAMPLE 3

Draw a time diagram and write an equation of value for the following problem:

Find the present value of quarterly payments of \$100 for 10 years, first payment 3 months from now, at a nominal rate of interest of 6% compounded semiannually.

SOLUTION

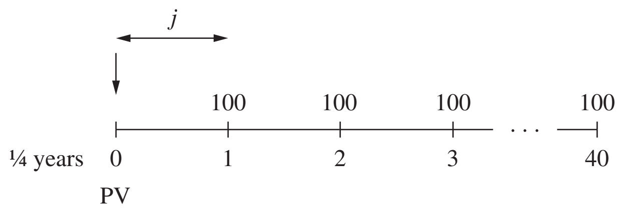
In this problem, the payment period and the interest period are not the same. The payment period is 3 months and the interest period is 6 months (since the given nominal interest rate implies an effective rate of 3% for a half-year period).

You have a choice:

(i) You can draw the diagram in terms of the ***payment period*** (3 months in this case) and determine the equivalent effective rate for this period; or

(ii) You can draw the diagram in terms of the ***interest period*** given in the problem (6 months in this case).

Following the first alternative (the payment period), the diagram would look like this:



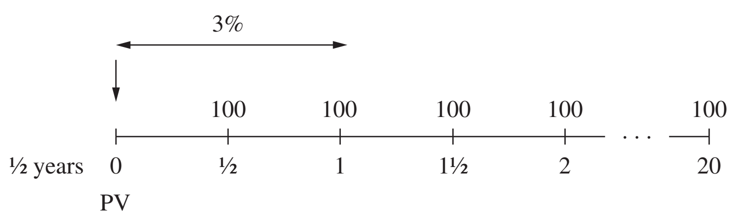
where $j = 1.03^{\frac{1}{2}} - 1 = 1.48892\%$.

(The thinking was: "I want the diagram to be in terms of *payment periods*. This will lead to the simplest form of the equation of value. Since payments are quarterly, the period in the diagram will be quarter-years and the payments will fall at time 1, 2, 3, ... Since there are 40 payments (quarterly payments for 10 years), the last payment is at time 40. The only thing left to do is to find the effective rate of interest for a quarter of a year—call it j—equivalent to the given nominal rate of 6% compounded semiannually. I can do this by accumulating 1 for one year (or for any period of time) at each rate: $1.03^2 = (1+j)^4$, from which $j = 1.03^{\frac{1}{2}} - 1$.")

The equation of value at time 0 is:

$$PV = 100(v + v^2 + \ldots + v^{40}) \text{ at rate } j = 1.03^{\frac{1}{2}} - 1$$

Following the second alternative (the interest period), the diagram looks like this:



(The thinking was: "The effective rate of interest implied by 6% compounded semiannually is 3% for a half-year period, so I will label my diagram in terms of half-year interest periods. The payments are quarterly, or twice each interest period, so I will place the payments at time $\frac{1}{2}$, 1, $1\frac{1}{2}$, etc. This is a 10-year annuity, which consists of 20 interest periods, so the last payment is at time 20.")

The equation of value at time 0 is:

$$PV = 100(v^{\frac{1}{2}} + v^1 + v^{1\frac{1}{2}} + \ldots + v^{20})$$

where $v = 1.03^{-1}$.

Which method is better, using the payment period or using the interest period?

If you only want numerical results, using the payment period is generally the best approach. Interest functions that we will cover later in this manual are at their simplest when the interest period is the same as the payment period. You do have to calculate the effective rate for the payment period and to use it in any equation of value. The fact that this rate will usually not be a round number doesn't matter, since you can handle any rate with the calculator.

On the other hand, the answers to an exam question may be left in symbolic form, although this has happened very infrequently, if at all, in recent exams. If this is the case, chances are that the examiners will use symbols at the "original" effective rate (3% per half-year in the above problem). In that case, your thinking would have to be in terms of interest periods.

The upshot of this is that you really have to know both approaches and have facility in drawing diagrams and writing equations of value in terms of either the interest period or the payment period. This is covered in greater detail in Sections 4a and 4b of this manual.

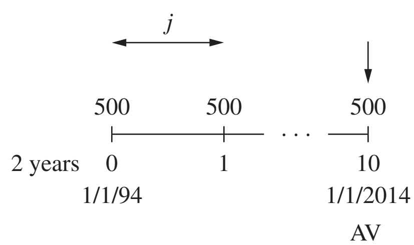
EXAMPLE 4

Draw a time diagram and write an equation of value for the following problem:

Deposits of \$500 are made on January 1 of even years only from 1994 to 2014 inclusive. Find the accumulated value on the date of the last deposit if the nominal rate of interest is 8% compounded quarterly.

SOLUTION

Using the payment period:



where $1+j=1.02^8$; $j=1.02^8-1=17.1659\%$.

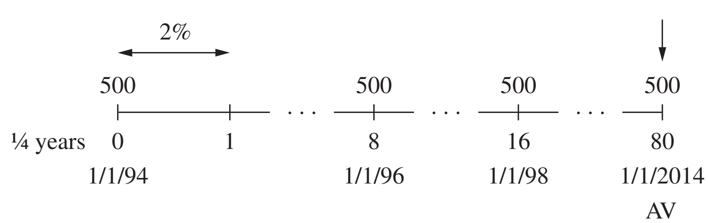
Equation of value:

$$AV = 500[1+(1+j)+\ldots+(1+j)^{10}] \text{ at } j = 17.1659\%$$

(This can be expressed as $500\,s_{\overline{11}|}$ at rate j, using a symbol introduced in the next chapter.)

Note: You can number the time periods in the diagram any way you want to, as long as the payments are spaced correctly. For example, in the above diagram you could have used time 1 for 1/1/94 and time 11 for 1/1/2014. This would have placed the payments at times 1 to 11, inclusive. Verify that if you then choose time 11 (i.e., 1/1/2014) as the comparison date, you would get the same equation of value as the one above.

Using the interest period:



Equation of value:

$$AV = 500(1+1.02^8+1.02^{16}+\cdots+1.02^{80})$$

(In a later chapter, this could be expressed as $500\frac{s_{\overline{88}|}}{s_{\overline{8}|}}$ at $i = 2\%$.)

Note: Once again, you can number the time periods in the diagram any way you want to. For example, in the above diagram, you could have used time 8 for 1/1/94 and time 88 for 1/1/2014. Of course, this would have led to the same equation of value using time 88 as the comparison date.

§ 2b. Unknown Time and Unknown Interest Rate

Unknown Time

Problems involving unknown time can often be solved using the logarithm $\boxed{\text{LN}}$ key or the $\boxed{\text{TVM}}$ keys. (See questions 5 and 6 in Calculator Notes #1.)

Stepping Stones

EXAMPLE 1

500 accumulates to 1,500 in t years at an effective annual interest rate of 4%. Determine t.

SOLUTION

$$(500)(1.04^t) = 1{,}500$$
$$1.04^t = 3$$

Using the $\boxed{\text{LN}}$ key we have $t = \log 3/\log 1.04 = 1.098612/.039221 = 28.01$ years to two decimals.

Using the $\boxed{\text{TVM}}$ keys, we enter: 4 $\boxed{\text{I/Y}}$ 500 $\boxed{\text{PV}}$ 1500 $\boxed{+/-}$ $\boxed{\text{FV}}$ $\boxed{\text{CPT}}$ $\boxed{\text{N}}$ for the same result.

EXAMPLE 2

How long will it take money to double at an effective annual rate of interest of 5%?

SOLUTION

$$1.05^t = 2$$

Using either logarithms or the $\boxed{\text{TVM}}$ keys, you should get $t = 14.2067$ to 4 decimals. Note that this is question 5 in Calculator Notes #1.

In your work or readings, you may come across an approximation for t, sometimes called the Rule of 72: The time it takes money to double at a specified interest rate is equal to 72 divided by the interest rate in percent. At 5%, this results in $72/5 = 14.4$ years. A better approximation is 69.3 divided by the interest rate, plus 0.35. At 5%, this gives 14.21, which is correct to two decimals. (These approximations are based on certain series expansions and the fact that the natural log of 2 is approximately 0.693.)

EXAMPLE 3

Bill will receive two payments of 1,000 each, the first payment in t years and the second in $2t$ years. The present value of these payments is 1,200 at an effective annual rate of 4%. Determine t.

SOLUTION

The equation of value as of time 0 is:

$$1000(v^t + v^{2t}) = 1200 \quad \text{at } 4\%$$
$$v^{2t} + v^t - 1.2 = 0$$

This is a quadratic in v^t. For convenience, we will substitute x for v^t:

$$x^2 + x - 1.2 = 0$$

Taking the positive root, you should get $x = .704159 = v^t$.

Using either the $\boxed{\text{LN}}$ key or the $\boxed{\text{TVM}}$ keys, you should get $t = 8.943$ years to 3 decimals.

EXAMPLE 4

Fund A grows at a force of interest $\delta_t = 1/(1+t)$. Fund B grows at a constant force of interest of 5%. Equal amounts are invested in each fund at time 0. Determine the time at which the excess of Fund A over Fund B is at a maximum.

SOLUTION

In general, the accumulation function is equal to the base of the natural logarithms, e, raised to the power of the integral of the force of interest. Determine the accumulation functions for Fund A and Fund B, take the difference, take the derivative of this difference, and set it equal to 0 to determine a relative minimum or maximum:

$$\text{Accumulation function for Fund A} = e^{\int_0^t \frac{1}{1+r}dr} = e^{[\ln(1+r)]_0^t} = 1 + t$$

$$\text{Accumulation function for Fund B} = e^{.05t}$$

Excess of A over B (call it E(t)):

$$\text{E}(t) = 1 + t - e^{.05t}$$

Take the derivative:

$$\text{E}'(t) = 1 - .05e^{.05t}$$

Setting this equal to 0, we get $e^{.05t} = 20$ and $t = 20 \ln 20 = 59.91$ years to two decimals. (Note that the second derivative of E(t) is negative, which verifies that a relative maximum occurs at 59.91 years.)

Method of Equated Time

A type of problem involving unknown time is one where a series of payments is to be replaced by a single payment equal to the sum of the series. For example, suppose someone is scheduled to make three payments to you, as follows:

Time Due	**Payment**
1	5
3	1
10	15
Total Payments =	21

Now, suppose that in lieu of the scheduled payments, you and the other party agree that a ***single*** payment will be made to you equal to the ***sum*** of the scheduled payments. In this example, the single payment would be 21.

When should this single payment be made?

Clearly, this would depend on the interest rate we agree on, say, 5%. Let the time of the single payment be t. Then the following equation of value as of time 0 must be satisfied:

$$21v^t = 5v + v^3 + 15v^{10}$$

You can see that the solution for t involves taking logarithms of both sides, etc. This procedure will give the exact value of t. You can verify that if $i = 5\%$, $t = 7.12$ to 2 decimals.

The "method of equated time" is a method for determining an ***approximate*** value of t using simple arithmetic. The general idea is to find the arithmetic mean of the times at which the payments are due. But it wouldn't be a very good approximation if we just averaged the times at which each payment is due without regard to the ***amount*** due on each date. So to find t by the method of equated time, we compute the ***weighted*** average time. We will use the symbol $\bar{t}$ for the approximation:

$$\bar{t} = \frac{5 \times 1 + 1 \times 3 + 15 \times 10}{21} = \frac{158}{21} = 7.52$$

Note that the time at which each payment is due is weighted by the amount due: 5 is due at time 1 (5×1); 1 is due at time 3 (1×3); 15 is due at time 10 (15×10). Then the sum of these products (158) is divided by the sum of the weights (21) to give the weighted average time of 7.52.

This is certainly an approximation, since we didn't even use the interest rate in calculating it. In fact, the method of equated time gives the same answer regardless of the interest rate.

(Note that in this example, the method of equated time gives a result which is greater than the exact answer at 5%: $7.52 > 7.12$. This is not an accident. It can be shown that the approximation always exceeds the exact answer for a positive interest rate.)

EXAMPLE 5

An annuity provides an infinite series of annual payments of d, $\frac{d^2}{2}$, $\frac{d^3}{3}$, . . . , first payment one year from now, where d is the effective rate of discount. In lieu of these payments, a single payment equal to their sum is to be made at time t. Determine t using the method of equated time.

(A) 1 (B) i/δ (C) d/δ (D) $e^{\frac{i}{d}}$ (E) $e^{\frac{i}{d}} - 1$

SOLUTION

This question tests you on a series that we covered in Section 1f, as well as testing you on the method of equated time.

To determine the approximate time using the method of equated time, we calculate the weighted average time. In this problem, there are an infinite number of payments, so the weighted average involves infinite series.

The sum of the products of the weights (payments) and the times is as follows:

$$\begin{aligned}&(d)(1) + (\frac{d^2}{2})(2) + (\frac{d^3}{3})(3) + \cdots \\ &= d + d^2 + d^3 + \cdots \\ &= \frac{d}{1-d} \text{ (sum of an infinite geometric progression)}\end{aligned}$$

You may recognize this as i or you can substitute $d = iv$ and $1 - d = v$ to obtain i.

To get the weighted average time, we have to divide this result (i) by the sum of the weights (payments), which is:

$$d + \frac{d^2}{2} + \frac{d^3}{3} + \cdots$$

If your memory has held out from Section 1f, you will recognize this as $-\ln(1-d)$, or δ. Therefore, the weighted average time is i/δ. ANS. **(B)**

EXAMPLE 6
Hannah is scheduled to receive the following payments at the end of the years shown:

End of Year	Scheduled Payment
4	100
10	200
12	300

Hannah agrees to accept a single payment equal to the sum of the scheduled payments at the time determined by the method of equated time. The effective annual interest rate is 6%. Let X be the excess of the present value of Hannah's scheduled payments over the present value of her single payment under the method of equated time. Determine X.

SOLUTION
The time of Hannah's single payment under the method of equated time is:

$$\bar{t} = \frac{(4)(100) + (10)(200) + (12)(300)}{100 + 200 + 300} = \frac{6{,}000}{600} = 10$$

The present value of the single payment is $600v^{10} = 335.04$.

The present value of the scheduled payments is $100v^4 + 200v^{10} + 300v^{12} = 339.98$.

$$X = 339.98 - 335.04 = 4.94$$

Unknown Interest Rate

Problems involving an unknown rate of interest are among the most difficult to solve. This is because the unknown rate may be the root (or roots) of a polynomial (or even of a more complex function).

EXAMPLE 7
At what rate of interest will a payment of 1 now and 2 in one year accumulate to 4 in 2 years?

SOLUTION
The equation of value as of time 2 is:

$$(1+i)^2 + 2(1+i) = 4$$

For convenience, we will let $(1+i) = x$. We have:

$$x^2 + 2x - 4 = 0$$

Taking the positive root:

$$\begin{aligned} x &= \frac{-2 + \sqrt{2^2 - (4)(1)(-4)}}{2} \\ &= \frac{\sqrt{20} - 2}{2} = \frac{2\sqrt{5} - 2}{2} \\ &= \sqrt{5} - 1 = 1.236068 \end{aligned}$$

Thus, $x = 1 + i = 1.236068$ and $i = .236068$, or 23.61%.

In this case, it was easy to solve for i, since it was the root of a quadratic. But with polynomials of higher degree, or for more complex functions, it could be very difficult to determine a numerical value for i without a financial calculator. (In fact, formulas do not exist for finding the roots of a polynomial of degree 5 or higher.) We might have to resort to an approximation or iterative technique, like interpolation, successive bisection, Newton-Raphson iteration, etc. But as we will see in later sections, a financial calculator such as the BA II Plus can be used to compute an unknown interest rate in many cases.

EXAMPLE 8

At what effective annual rate of interest $i > 0$ is the present value of 100 due at time 5 plus the present value of 200 due at time 15 equal to the present value of 300 due at time 10?

SOLUTION

The equation of value as of time 0 is:

$$100v^5 + 200v^{15} = 300v^{10}$$

Dividing both sides by $100v^5$ and rearranging:

$$2v^{10} - 3v^5 + 1 = 0$$

Let $x = v^5$:

$$2x^2 - 3x + 1 = 0$$

The left-hand side can be factored as follows:

$$(2x - 1)(x - 1) = 0$$

Taking the first root:

$$2x = 1, \quad x = 0.5 = v^5 \text{ or } (1+i)^5 = 2$$
$$i = 14.87\% \quad \text{to 2 decimals}$$

(The other root is $x = 1 = v^5$, which means that $i = 0$, but we were asked to find $i > 0$.)

Summary of Concepts and Formulas in Sections 2a and 2b

(1) We say that money has a time value, which means that $1.00 paid today is equivalent to $(1+i)^t$ paid at time t (assuming that the effective rate is a constant i).

(2) Using the principle of the time value of money, we can set up ***equations of value*** by evaluating all payments as of a common date called the ***comparison date***.

(3) Time diagrams can be very helpful in solving problems. (See examples in Section 2a.)

(4) The "method of equated time" is the weighted average time of a series of scheduled payments. It is a simple approximation to the exact time at which a single payment equal to the sum of the scheduled payments should be made in lieu of the scheduled payments.

Past Exam Questions on Sections 2a and 2b

1. David can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n, and 300 at time $2n$
(ii) 600 at time 10

At an annual effective interest rate of i, the present values of the two streams are equal. Given $v^n = 0.75941$, determine i. [11/01 #24]

(A) 3.5% (B) 4.0% (C) 4.5% (D) 5.0% (E) 5.5%

2. Joe deposits 10 today and another 30 in five years into a fund paying simple interest of 11% per year.

Tina will make the same two deposits, but the 10 will be deposited n years from today and the 30 will be deposited $2n$ years from today. Tina's deposits earn an annual effective rate of 9.15%.

At the end of 10 years, the accumulated amount of Tina's deposits equals the accumulated amount of Joe's deposits.

Calculate n. [5/00 #1]

(A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

3. An investment of $1 will double in 20 years at a force of interest δ.

Determine the number of years required for an investment of $1 to triple at a nominal rate of interest, convertible 3 times per year, and which is numerically equivalent to δ. [CAS 5/98 #7]

(A) Less than 31 years
(B) At least 31 years, but less than 33 years
(C) At least 33 years, but less than 35 years
(D) At least 35 years, but less than 37 years
(E) 37 years or more

4. John is 30 years old. He will receive 2 payments of $2,500 each. The first payment will be an unknown number of years in the future. The second payment will be five years after the first payment. At an annual effective interest rate of $i = 5\%$, the present value of the two payments is $2,607.

Determine at what age John will receive the *second* payment. [CAS 5/94 #1]

(A) Less than 40
(B) At least 40 but less than 45
(C) At least 45 but less than 50
(D) At least 50 but less than 55
(E) At least 55

5. Jim borrows $5,000 from a bank now, an additional $3,000 one year from now and an additional $2,000 five years from now. At what point in time, t, would a single payment of $10,000 be equivalent at a nominal rate of interest of 12% convertible monthly? [CAS 5/91 #4]

(A) 0.0 years $\le t <$ 0.9 years
(B) 0.9 years $\le t <$ 1.0 years
(C) 1.0 years $\le t <$ 1.1 years
(D) 1.1 years $\le t <$ 1.2 years
(E) 1.2 years $\le t$

6. You are given two loans, with each loan to be repaid by a single payment in the future. Each payment includes both principal and interest.

The first loan is repaid by a 3,000 payment at the end of four years. The interest is accrued at 10% per annum compounded semiannually.

The second loan is repaid by a 4,000 payment at the end of five years. The interest is accrued at 8% per annum compounded semiannually.

These two loans are to be consolidated. The consolidated loan is to be repaid by two equal installments of X, with interest at 12% per annum compounded semiannually. The first payment is due immediately and the second payment is due one year from now.

Calculate X. [SOA 11/89 #1]

(A) 2,459 (B) 2,485 (C) 2,504 (D) 2,521 (E) 2,537

7. Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made.

Carl withdraws K at the end of each of years 4, 5, 6 and 7. The balance in the account at the end of year 10 is 10,000.

Calculate K. [SOA 5/89 #5]

(A) 929 (B) 958 (C) 980 (D) 1,005 (E) 1,031

8. Payments of 300, 500 and 700 are made at the end of years five, six and eight, respectively. Interest is accumulated at an annual effective rate of 4%.

You are to find the point in time at which a single payment of 1,500 is equivalent to the above series of payments. You are given:

(i) X is the point in time calculated by the method of equated time.
(ii) Y is the exact point in time.

Calculate $X + Y$. [SOA 11/88 #5]

(A) 13.44 (B) 13.50 (C) 13.55 (D) 14.61 (E) 14.99

9. You are given the following data on three series of payments:

	Payment at end of year			**Accumulated value**
	6	**12**	**18**	**at end of year 18**
Series A	240	200	300	X
Series B	0	360	700	$X + 100$
Series C	Y	600	0	X

Assume interest is compounded annually.

Calculate Y. [SOA 5/88 #4]

(A) 93 (B) 99 (C) 102 (D) 107 (E) 111

10. The present value of a payment of 1,004 at the end of T months is equal to the present value of the following payments:

At the End of	**Amount**
1 month	314
18 months	271
24 months	419

The effective annual interest rate is 5%.

Calculate T. [SOA 11/87 #8]

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

11. A loan of 1000 is made at an interest rate of 12% compounded quarterly. The loan is to be repaid with three payments: 400 at the end of the first year, 800 at the end of the fifth year, and the balance at the end of the tenth year.

Calculate the amount of the final payment. [SOA 5/87 #1]

(A) 587 (B) 658 (C) 737 (D) 777 (E) 812

12. You are given:

(i) $\delta_t = \frac{2t^3+8t}{t^4+8t^2+16}$, $0 \le t \le 1$.
(ii) i = the effective annual interest rate equivalent to δ_t.
(iii) Fund X accumulates with simple interest at the rate i.
(iv) Fund Y accumulates at δ_t.
(v) An amount of 1 is deposited in each of Fund X and Fund Y at time $t = 0$.

At what time, t, is (Fund X $-$ Fund Y) a maximum? [SOA 5/87 #14]

(A) 0.250 (B) 0.375 (C) 0.500 (D) 0.625 (E) 0.750

13. Fund F accumulates at the rate $\delta_t = \frac{1}{1+t}$. Fund G accumulates at the rate $\delta_t = \frac{4t}{1+2t^2}$.

You are given:

(i) $F(t) =$ Amount in Fund F at time t
(ii) $G(t) =$ Amount in Fund G at time t
(iii) $H(t) = F(t) - G(t)$
(iv) $F(0) = G(0)$
(v) T is the time t when $H(t)$ is a maximum.

Calculate T. [SOA 11/86 #2]

(A) 1/4 (B) 1/2 (C) 3/4 (D) 1 (E) 5/4

14. Jones agrees to pay an amount of $2X$ at the end of 3 years and an amount of X at the end of 6 years. In return he will receive \$2,000 at the end of 4 years and \$3,000 at the end of 8 years.

At an 8% effective annual interest rate, what is the size of Jones' second payment? [CAS 5/86 #4]

(A) Less than \$1,250
(B) At least \$1,250, but less than \$1,300
(C) At least \$1,300, but less than \$1,350
(D) At least \$1,350, but less than \$1,400
(E) \$1,400 or more

15. At a certain interest rate the present value of the following two payment patterns are equal:

(i) 200 at the end of 5 years plus 500 at the end of 10 years
(ii) 400.94 at the end of 5 years

At the same interest rate, 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years.

Calculate P. [SOA 5/86 #1]

(A) 901 (B) 918 (C) 942 (D) 967 (E) 992

16. (This question actually belongs in Section 3 of the manual, although you could answer it by summing a geometric series.)

On January 1, 1985, Marc has the following two options for repaying a loan:

(i) Sixty monthly payments of 100 commencing February 1, 1985.

(ii) A single payment of 6000 at the end of K months.

Interest is at a nominal annual rate of 12% compounded monthly. The two options have the same present value.

Determine K. [SOA 11/85 #1]

(A) 29.0 (B) 29.5 (C) 30.0 (D) 30.5 (E) 31.0

17. The XYZ Casualty Insurance Company has found that for a particular type of insurance policy it makes the following payments for insurance claims:

(i) on 10% of the policies, XYZ Company pays \$1,000 exactly one year after the effective date of the policy;

(ii) on 3% of the policies, XYZ Company pays \$10,000 exactly three years after the effective date of the policy;

(iii) on the remaining policies, XYZ Company makes no payment for claims.

In addition to the above payments, XYZ Company pays \$20 for the expenses of administering the policy: \$10 is paid on the effective date of the policy and the remaining \$10 is paid six months after the effective date of the policy.

The annual interest rate is 8%, compounded semiannually.

The premium for this type of insurance policy is due six months after the effective date of the policy.

If the present value of the premium is set equal to the present value of the claim payments and expenses, what is the premium? [CAS 11/82 #5]

(A) Less than \$355
(B) At least \$355 but less than \$380
(C) At least \$380 but less than \$415
(D) At least \$415 but less than \$440
(E) At least \$440

Solutions to Past Exam Questions on Sections 2a and 2b

1.
$$100 + 200v^n + 300v^{2n} = 600v^{10}$$
$$100 + 200(.75941) + 300(.75941)^2$$
$$= 100 + 151.882 + 173.011 = 424.893$$
$$600v^{10} = 424.893, \ \ v^{10} = .708155, \ \ i = 3.5\%$$
ANS. **(A)**

2. AV in 10 years:

$$\text{Joe: } 10[1 + (10)(.11)] + 30[1 + (5)(.11)] = 21 + 46.5 = 67.5$$
$$\text{Tina: } 10(1.0915)^{10-n} + 30(1.0915)^{10-2n} = 24v^n + 72v^{2n}$$
$$72v^{2n} + 24v^n - 67.5 = 0$$

For simplicity, let $x = v^n$.

$$72x^2 + 24x - 67.5 = 0, \ \text{ or } 3x^2 + x - 2.8125 = 0$$
$$x = \frac{-1 \pm \sqrt{1 - (4)(3)(-2.8125)}}{6} = .815819 \text{(positive root)}$$
$$x = v^n \text{ at } 9.15\% = .815819, \ \ n = 2.325$$
ANS. **(B)**

3.
$$e^{20\delta} = 2, \ \ 20\delta = \ln 2, \ \ \delta = \frac{\ln 2}{20} = .03466$$
$$\left(1 + \frac{.03466}{3}\right)^{3x} = 3$$
$$3x \ln 1.01155 = \ln 3$$
$$x = \frac{1.09861}{(3)(.01148)} = 31.90$$
ANS. **(B)**

4. Let n = time of first payment.

$$2{,}500\left(v^n + v^{n+5}\right) = 2{,}607$$
$$v^n\left(1 + v^5\right) = \frac{2{,}607}{2{,}500}, \ v^n = \frac{2{,}600}{(1 + v^5)(2{,}500)} = .584684$$
$$n = 11, \ \ \ n + 5 = 16$$

John's age at which he receives second payment = $30 + 16 = 46$. ANS. **(C)**

5.
$$5{,}000 + 3{,}000v^{12} + 2{,}000v^{60} = 10{,}000v^{12t} \text{ at } 1\%$$
$$v^{12t} = .876325, \ \ t = 1.1056$$
ANS. **(D)**

6.
$$X\left(1 + v_{.06}^2\right) = 3{,}000v_{.05}^8 + 4{,}000v_{.04}^{10}$$
$$= 3{,}000(.6768) + 4{,}000(.6756) = 4{,}732.80$$
$$X = \frac{4{,}732.80}{1 + .8900} = 2{,}504$$
ANS. **(C)**

7. With the penalties, the withdrawals are $1.05K$, $1.05K$, K and K at times 4, 5, 6 and 7, respectively. The equation of value is (comparison date time 10):

$$10{,}000(1.04)^{10} = 1.05K\left(1.04^6 + 1.04^5\right) + K\left(1.04^4 + 1.04^3\right) + 10{,}000$$

$$K = \frac{10{,}000\left(1.04^{10} - 1\right)}{1.05\left(1.04^6 + 1.04^5\right) + 1.04^4 + 1.04^3}$$

$$= \frac{4{,}802.44}{4.900794} = 979.93 \qquad \text{ANS. } \textbf{(C)}$$

8. Method of equated time (weighted average time):

$$X = \frac{300(5) + 500(6) + 700(8)}{300 + 500 + 700} = 6.733$$

Exact time Y:

$$1{,}500v^Y = 300v^5 + 500v^6 + 700v^8$$

$$v^Y = .7688, \quad Y = 6.704$$

$$X + Y = 13.437 \qquad \text{ANS. } \textbf{(A)}$$

9.

$$240(1+i)^{12} + 200(1+i)^6 + 300 = X$$

$$360(1+i)^6 + 700 = X + 100$$

$$Y(1+i)^{12} + 600(1+i)^6 = X$$

For simplicity, let $x = (1+i)^6$. Subtracting the second equation from the first:

$$240x^2 - 160x - 300 = 0$$

$$12x^2 - 8x - 15 = 0$$

$$x = \frac{8 \pm \sqrt{64 - (4)(12)(-15)}}{24} = 1.5 \text{(positive root)}$$

In the second line of this solution, substitute $x = (1+i)^6 = 1.5$:

$$360(1.5) + 700 = X + 100, \quad X = 1{,}140$$

In the third line of this solution, substitute $(1+i)^6 = 1.5$ and $x = 1{,}140$:

$$Y(1.5)^2 + 600(1.5) = 1{,}140, \quad Y = 106.67 \qquad \text{ANS. } \textbf{(D)}$$

10. First determine the equivalent monthly effective rate j:

$$(1+j)^{12} = 1.05, \quad j = 1.05^{\frac{1}{12}} - 1 = .4074\%$$

$$1{,}004v^T = 314v + 271v^{18} + 419v^{24} \text{ at } j$$

$$= 312.73 + 251.88 + 380.05 = 944.66$$

$$v^T = \frac{944.66}{1{,}004} = .940897, \quad T = 14.98 \qquad \text{ANS. } \textbf{(B)}$$

11. $1{,}000 = 400v^4 + 800v^{20} + Rv^{40}$ at 3%

$$R = \frac{1{,}000 - 400(.8885) - 800(.5537)}{.3066} = 657.66 \qquad \text{ANS. } \textbf{(B)}$$

12. To be more precise, the question should have said that i is the effective annual interest rate equivalent to δ_t *over the first year.*

$$1 + i = e^{\int_0^1 \frac{2t^3+8t}{t^4+8t^2+16}dt} = e^{\frac{1}{2}\ln\left(t^4+8t^2+16\right)\big]_0^1}$$

$$= \left(\frac{25}{16}\right)^{\frac{1}{2}} = 1.25, \ \ i = .25$$

Let $f(t) =$ Fund $X -$ Fund Y at time t

$$= (1 + .25t) - e^{\int_0^t \frac{2r^3+8r}{r^4+8r^2+16}dr}$$

$$= \frac{4+t}{4} - \left(\frac{t^4 + 8t^2 + 16}{16}\right)^{\frac{1}{2}}$$

Note that $\frac{t^4+8t^2+16}{16} = \left(\frac{t^2+4}{4}\right)^2$, so that the second term above is equal to $\frac{t^2+4}{4}$. To determine a relative maximum, take the first derivative and set it equal to 0:

$$f'(t) = \frac{1}{4} - \frac{t}{2} = 0$$

$$t = 0.5 \qquad \text{ANS. } \textbf{(C)}$$

(Check for maximum: 2nd derivative is negative)

13.
$$F(t) = e^{\int_0^t \frac{1}{1+r}dr} = e^{\ln(1+r)]_0^t} = 1 + t$$

$$G(t) = e^{\int_0^t \frac{4r}{1+2r^2}dr} = e^{\ln(1+2r^2)\big]_0^t} = 1 + 2t^2$$

(Note: We assumed $F(0) = G(0) = 1$, since the initial amount invested in each fund will cancel out anyway.)

$$H(t) = F(t) - G(t) = t - 2t^2$$

To determine a relative maximum, set $H'(t) = 0$:

$$H'(t) = 1 - 4t = 0, \ \ t = \frac{1}{4} \qquad \text{ANS. } \textbf{(A)}$$

(There is a relative maximum at $t = \frac{1}{4}$, since 2nd derivative is negative.)

14.
$$X\left(2v^3 + v^6\right) = 2{,}000v^4 + 3{,}000v^8$$

$$X = \frac{1{,}470.06 + 1{,}620.81}{1.587664 + .630170} = \frac{3{,}090.87}{2.217834} = 1{,}393.64 \qquad \text{ANS. } \textbf{(D)}$$

15. Using time 5 as the comparison date:

$$200 + 500v^5 = 400.94, \ \ v^5 = .40188$$

$$P = 100(1+i)^{10} + 120(1+i)^5$$

Note that $(1+i)^5 = \frac{1}{v^5} = \frac{1}{.40188}$ and $(1+i)^{10} = \frac{1}{.40188^2}$.

$$P = \frac{100}{.40188^2} + \frac{120}{.40188} = 918 \qquad \text{ANS. } \textbf{(B)}$$

16. PV of payments under Option (i) $= 100(v + v^2 + \cdots + v^{60})$

$$= 100v\left(\frac{1 - v^{60}}{1 - v}\right) \quad \text{at } 1\%$$

$$= 4{,}495.5038$$

PV of payment under Option (ii) $= 6{,}000v^K$

$$v^K = \frac{4{,}495.5036}{6{,}000} = .749251$$

$$K = 29.0 \qquad \text{ANS. } \textbf{(A)}$$

Note: in Section 3a, we will see that $v + v^2 + \cdots + v^{60}$ is the PV of an annuity with the symbol $a_{\overline{60|}}$.

17. The effective interest rate is 4% per half year. Set up an equation of value using the date the premium is due as the comparison date (i.e., 6 months after the effective date, or time 1 in terms of interest periods):

$$P = 10(1.04) + 10 + 10\% \times 1{,}000v^1 + 3\% \times 10{,}000v^5$$
$$= 20.40 + 100v^1 + 300v^5$$
$$= 20.40 + 96.15 + 246.58 = 363.13 \qquad \text{ANS. } \textbf{(B)}$$

Practice Exam 1

Note to Students: These practice exams follow the format of the actual exams in 2008 and subsequent: 35 questions in three hours. The actual exam will be in CBT format. Several of the questions (perhaps five) will be pilot questions that will not be graded, but you will have no way of knowing which ones they are.

When you take these exams, stick to the time limit and simulate exam conditions.

Questions for Practice Exam 1

1. Which of the following is *not* correct with respect to an annual effective interest rate of $i = 10\%$?

(A) $\delta = e^{0.10} - 1$
(B) $i^{(2)} = 2 \times [(1.10)^{0.50} - 1]$
(C) $\delta = \ln(1.10)$
(D) $d = \frac{0.10}{1.10}$
(E) $d^{(4)} = 4 \times [1 - (1.10^{-0.25})]$

2. You can receive one of the following two sets of cash flows. Under Option A, you will receive 10 annual payments of \$1,000, with the first payment to occur 4 years from now. Under Option B, you will receive X at the end of each year, forever, with the first payment to occur 1 year from now. The annual effective rate of interest is 8%. Which of the following equations should be solved to find the value of X such that you are indifferent between these two options?

(A) $80a_{\overline{10|}}v^4 = X$ (B) $80a_{\overline{13|}}v^3 = X$ (C) $80a_{\overline{10|}}v^3 = X$ (D) $80a_{\overline{10|}}v^3(0.08) = X$ (E) $80a_{\overline{13|}}v^2 = X$

3. An annuity will pay you \$500 two years from now, and another \$1,000 four years from now. The present value of these two payments is \$1,200. Find the implied effective annual interest rate, i.

(A) $i \le 4.5\%$ (B) $4.5\% < i \le 5.5\%$ (C) $5.5\% < i \le 6.5\%$ (D) $6.5\% < i \le 7.5\%$ (E) $7.5\% < i$

4. An investor took out a 30-year loan which he repays with annual payments of 1,500 at an annual effective interest rate of 4%. The payments are made at the end of the year. At the time of the 12$^{\text{th}}$ payment, the investor pays an additional payment of 4,000 and wants to repay the remaining balance over 10 years. Calculate the revised annual payment.

(A) 1,682 (B) 1,729 (C) 1,783 (D) 1,825 (E) 1,848

5. A 25-year loan is being paid off via level amortization payments made at the end of each quarter. The nominal annual interest rate is 12% convertible monthly. The amount of principal in the 29th payment is 1,860. Find the amount of principal in the 61st payment

(A) 4,535 (B) 4,635 (C) 4,735 (D) 4,835 (E) 4,935

6. Suppose you are the actuary for an insurance company. Your company, in response to a policyholder claim involving physical injury, is responsible for making annual medical payments. The first payment will occur on January 1, 2008, and the final payment will occur on January 1, 2031. The first payment will be \$100,000; after that, the payments will increase annually for inflation, at a rate of 5% per year. The real interest rate is 3% per year. Find the present value of these future payments as of December 31, 2005.

(A) 1,491,000 (B) 1,501,000 (C) 1,511,000 (D) 1,521,000 (E) 1,531,000

7. A company must pay the following liabilities at the end of the years shown:

End of Year	Liability
2	\$1,000
4	X
6	1,000

The company achieves Redington immunization by purchasing assets that have two cash inflows: \$733 at the end of one year and Y at the end of 5 years. The effective annual rate of interest is 10%. Determine Y.

(A) 1,789 (B) 1,934 (C) 2,152 (D) 2,201 (E) 2,376

8. A share of stock is expected to pay a dividend of 2 one year from now, and a dividend of 3 two years from now. Thereafter, dividends will be paid annually, with each dividend being g% greater than the previous dividend. The effective annual interest rate is 8.5%, and the price of the share of stock is 112.50. Find g.

(A) 5.6 (B) 5.7 (C) 5.8 (D) 5.9 (E) 6.0

9. At any moment t, a continuously-varying continuous 5-year annuity makes payments at the rate of t^2 per year at moment t. The force of interest is 6%. Which of the following represents a correct expression of the present value of this annuity?

(A) $\int_0^5 t^2 e^{0.06t}\,dt$
(B) $\int_0^5 t^2 e^{-0.06t}\,dt$
(C) $\int_0^5 te^{-0.12t}\,dt$
(D) $\int_0^5 t^2(1.06)^{-t}\,dt$
(E) None of (A), (B), (C), or (D) is a correct expression of the present value of the annuity.

10. A loan of 45,000 is being repaid with level annual payments of 3,200 for as long as necessary plus a final drop payment. All payments are made at the end of the year. The principal portion of the 9th payment is 1.5 times the principal portion of the 2nd payment. Calculate the drop payment.

(A) 1,495 (B) 1,521 (C) 1,546 (D) 1,584 (E) 1,597

11. A project requires an investment of 50,000 now (time 0), and will provide returns of X at the end of each of years 3 through 10. The effective annual rate of interest is 10%. The net present value of this project is 2,500. Find X.

(A) 11,300 (B) 11,500 (C) 11,700 (D) 11,900 (E) 12,100

12. Two growing perpetuities have the same yield rate. The first perpetuity—a perpetuity-immediate—has an initial payment of 500 one year from now, and each subsequent annual payment increases by 4%. This first perpetuity has a present value of 9,500. The second perpetuity—also a perpetuity-immediate—has an initial payment of 400 one year from now, and each subsequent annual payment increases by 20. Find the present value, P, of this second perpetuity.

(A) $P \le 6{,}500$
(B) $6{,}500 < P \le 6{,}600$

(C) $6{,}600 < P \leq 6{,}700$
(D) $6{,}700 < P \leq 6{,}800$
(E) $6{,}800 < P$

13. You invest \$2,000 in a fund on January 1, 2007. On August 20, 2007, your fund is worth \$1,800, and you deposit another X into the fund. On December 31, 2007, your fund is worth \$3,400. The time-weighted rate of return on your investment during 2007 was 20%. Find X.

(A) 650 (B) 750 (C) 850 (D) 950 (E) 1,050

14. A 10-year 200,000 loan is being paid off with level amortization payments at the end of each month. The effective annual interest rate is 15%. Find the amount of interest in the 56th monthly payment.

(A) 1,576 (B) 1,607 (C) 1,652 (D) 1,714 (E) 1,789

15. A 30-year \$300,000 loan involves level amortization payments at the end of each year. The effective annual interest rate is 9%. Let P be the ratio of total dollars of interest paid by the borrower divided by total aggregate payment dollars made by the borrower over the life of the loan. Find P.

(A) $P \leq 0.525$ (B) $0.525 < P \leq 0.575$ (C) $0.575 < P \leq 0.625$ (D) $0.625 < P \leq 0.675$ (E) $0.675 < P$

16. At the end of each year, for the next 19 years, you make deposits into an account, as follows:

Deposit at end of year $t = 100t$ for $t = 1, 2, 3, \ldots, 10$.

Deposit at end of year $t = 1{,}000 - \{100(t - 10)\}$ for $t = 11, 12, 13, \ldots, 19$

The effective annual interest rate is 10%. Find the present value, at time $t = 0$, of this annuity.

(A) 4,053 (B) 4,103 (C) 4,153 (D) 4,203 (E) 4,253

17. An investment opportunity has the following characteristics: payments of \$10,000 will be made to you and invested into a fund at the beginning of each year, for the next 20 years. These payments will earn a 7% effective annual rate, and the interest payments (paid at the end of each year) will immediately be reinvested into a second account earning a 4% effective annual rate. Find the purchase price of this investment opportunity, given that it has an annual yield of 6% over the 20-year life of the investment.

(A) 92,000 (B) 102,000 (C) 112,000 (D) 122,000 (E) 132,000

18. A 30-year bond with par value 1,000 has annual coupons and sells for 1,300. The write down in the first year is 4.60. What is the yield-to-maturity for this bond?

(A) 4.73% (B) 4.89% (C) 4.98% (D) 5.15% (E) 5.27%

19. On January 1, 2007, you initiate an investment account, with the following value and deposit/withdrawal activity during the year:

Date (2007)	Account Value	Activity
January 1	—	10,000 deposit
June 30	12,000	X
December 31	10,000	—

(The "account values" represent the amount in the account immediately before the deposit or withdrawal activity on that date.) The time-weighted and dollar-weighted rates of return on the account during 2007 are equal. Find the non-zero value of X—both its magnitude, and whether it's a deposit or a withdrawal. (For the dollar-weighted rate of return, assume simple interest from the date of each deposit or withdrawal.)

(A) 4,000 deposit

(B) 4,000 withdrawal
(C) 2,000 deposit
(D) 2,000 withdrawal
(E) Cannot be determined from the given information.

20. A 20-year 100 par value bond with 8% semiannual coupons is purchased for 108.50. What is the book value of the bond just after the 13^{th} coupon is paid?

(A) 102.24 (B) 103.32 (C) 104.89 (D) 105.73 (E) 106.91

21. Yield rates to maturity for zero coupon bonds are currently quoted at 6% for one-year maturity, 7% for two-year maturity, and 7.5% for three-year maturity. Find the present value, two years from now, of a one-year 1000-par-value zero-coupon bond.

(A) 902 (B) 922 (C) 942 (D) 962 (E) 982

22. Determine the modified duration (or "volatility") of a growing perpetuity. The perpetuity will make annual payments, with the first payment being \$1 one year from now, and thereafter each subsequent payment will be \$1 greater than the preceding payment. Assume an annual effective interest rate of 8%.

(A) 12 (B) 16 (C) 20 (D) 24 (E) 28

23. You purchase a 7.5% annual coupon bond with a face value of 1,000 to yield a minimum interest rate of 8% effective. The bond is a callable corporate bond, with a call price of 1,050, and can be called by the issuing corporation after five years. The bond matures at par in 30 years. Immediately after the 12th coupon payment, the issuing corporation redeems the bond. Determine the effective annual yield you achieved on this twelve-year investment.

(A) 6.5% (B) 7.0% (C) 7.5% (D) 8.0% (E) 8.5%

24. A one-year zero-coupon bond has an annual yield of 6.25%. A two-year zero-coupon bond has an annual yield of 7.00%. A three-year zero-coupon bond has an annual yield of 7.50%. A three-year 12% annual coupon bond has a face value of \$1,000. Find the yield to maturity on this three-year 12% annual coupon bond.

(A) 6.6% (B) 7.0% (C) 7.4% (D) 7.8% (E) 8.2%

25. Bond A is an n-year 100 par value bond with 8% annual coupons and sells for 140.25. Bond B is an n-year 100 par value bond with 3% annual coupons and sells for 80.17. Both bonds have the same yield rate i. Determine i.

(A) 3.82% (B) 4.65% (C) 4.85% (D) 5.15% (E) 5.52%

26. A 30-year 1,000 par value bond pays 10% annual coupons. Using an interest rate of 12%, find the Macaulay duration of this bond.

(A) 9.2 (B) 10.2 (C) 11.2 (D) 12.2 (E) 13.2

27. The price of a 1-year zero-coupon bond with a maturity value of 1.00 is .943. The price of a 2-year zero-coupon bond with a maturity value of 1.00 is X. A 2-year interest rate swap contract with annual interest payments and a constant notional value has a swap rate of 6.5826%. Determine X.

(A) .865 (B) .870 (C) .875 (D) .880 (E) .885

28. A 2-year interest rate swap contract with a constant notional amount of 200,000 is established on January 1, 2021. Under the contract, Corp. A pays fixed interest to Corp. B and Corp. B pays variable interest to Corp. A. You are given the following spot rates:

Term (Years)	Spot Rates	
	As of 1/1/2021	As of 1/1/2022
1	4.50%	5.15%
2	4.75%	5.40%

The underlying loan has a variable interest rate equal to the 1-year spot interest rate in effect at the beginning of the year. Determine the net swap payment at the end of the 2nd year and determine whether this payment is made by Corp. A to Corp. B, or by Corp. B to Corp. A.

(A) 12 by Corp. A to Corp. B
(B) 812 by Corp. A to Corp. B
(C) 1,312 by Corp. A to Corp. B
(D) 12 by Corp. B to Corp. A
(E) 812 by Corp. B to Corp. A

29. An insurer must pay 3,000 and 4,000 at the ends of years 1 and 2, respectively. The only investments available to the company are a one-year zero-coupon bond (with a par value of 1,000 and an effective annual yield of 5%), and a two-year 8% annual coupon bond (with a par value of 1,000 and an effective annual yield of 6%). Which of the following is closest to the cost to the company today to match its liabilities exactly?

(A) 6,014 (B) 6,114 (C) 6,214 (D) 6,314 (E) 6,414

30. Patricia buys a 180-day \$10,000 U.S. Treasury Bill at a quoted rate of 2%. Two months later, she sells the bill for \$9,950. What is Patricia's yield rate on this transaction, expressed as an annual effective rate of interest?

(A) 2.00% (B) 2.35% (C) 2.79% (D) 2.98% (E) 3.07%

31. Christine deposits \$100 into an account which earns interest at an effective annual rate of discount of d. At the same time, Douglas deposits \$100 into a separate account earning interest at a force of interest of $\delta_t = 0.001t^2$. After 10 years, both accounts have the same value. Find d.

(A) 3.3% (B) 3.6% (C) 3.9% (D) 4.2% (E) 4.5%

32. You are given the following information about two annual-coupon bonds, each with a face and redemption value of \$ 1,000, and each 3 years in length:

- Bond A: A 3-year 6% annual coupon bond with a price of \$955.57.
- Bond B: A 3-year 8% annual coupon bond with a price of \$1,008.38.

Using this data, find the annual yield on a 3-year zero-coupon bond.

(A) 6.40% (B) 6.95% (C) 7.30% (D) 7.85% (E) 8.40%

33. Suppose the FMOC significantly increases the target level of the federal funds rate over a period of months. Which of the following statements is (are) true?

(I) Banks will have an incentive to make more loans.
(II) Interest rates in the economy will tend to increase.
(III) Employment in the economy will tend to increase.

(A) I only (B) II only (C) III only (D) I and II only
(E) The correct answer is not given by (A), (B), (C), or (D).

34. You have the following data regarding an investment account:

Date	Account Value (prior to deposit or withdrawal)	Deposit or Withdrawal
January 1, 2008	—	1,000 deposit
April 1, 2008	800	X
October 1, 2008	900	500 deposit
January 1, 2009	1,200	—

The time-weighted rate of return for 2008 was 23.4%. Find X.

(A) 500 withdrawal (B) 300 withdrawal (C) 0 (D) 300 deposit (E) 500 deposit

35. An investment opportunity has the following characteristics: payments of \$500 will be made to you and invested into an account at the end of each year, for the next 20 years. These payments will earn an effective annual interest rate of 8%, and the interest from this account (paid at the end of each year) can be reinvested at an effective annual rate of 5%. Find the purchase price of this investment opportunity assuming an effective annual yield of 7% over the 20-year life of the investment.

(A) 4,885 (B) 4,985 (C) 5,085 (D) 5,185 (E) 5,285

Solutions to Practice Exam 1

1. All of the formulas except the first (answer (A)) are valid equivalencies when the effective rate of interest is 10%. The correct relationship between the effective rate and the force of interest is $e^{\delta} = 1 + i$ or $i = e^{\delta} - 1$ or $\delta = \ln(1 + i)$. ANS. **(A)**

2. "Indifference" between two alternatives means that a person considers the present values of the two options to be equal. Setting up this equivalency relationship:

$$1{,}000 \cdot a_{\overline{10}|.08} \cdot v^3 = \frac{X}{0.08}$$

which is equivalent to answer (C). The three-year present value factor on the left-hand side is necessary because the first payment is four years away, and the annuity-immediate formula provides a PV one year prior to the first payment (leaving three more years of discounting to invoke). ANS. **(C)**

3. Set up the present value formula. The key is to recognize this as a quadratic in v^2:

$$\begin{aligned}
1{,}200 &= 500v^2 + 1{,}000v^4 \\
10(v^2)^2 + 5v^2 - 12 &= 0 \\
v^2 &= \frac{-5 \pm \sqrt{25 + 480}}{20} = 0.873610 \\
v &= 0.934671 \\
i &= 0.0699 \qquad \text{ANS. } \textbf{(D)}
\end{aligned}$$

4. The outstanding balance at time 12 prior to the additional payment is:

$$B_{12} = 1{,}500a_{\overline{18}|.04} = 18{,}988.95.$$

After the additional payment, the outstanding balance is 14,988.95.

To pay this remaining balance in 10 years, the revised annual payment is such that:

$$Pa_{\overline{10}|.04} = 14{,}988.95 \quad \text{that gives } P = 1{,}848.00 \qquad \text{ANS. } \textbf{(E)}$$

5. The key in this problem is to use the $(1 + i)$ multiplicative factor relationship between the principal components of sequential amortization payments. This is a consequence of the formula $P_t = R \cdot v^{n-t+1}$. Thus, once the appropriate interest rate is determined, the answer can be found quickly:

$$\begin{aligned}
j &= (1.01)^3 - 1 = 0.030301/qtr \\
P_{61} &= P_{29} \cdot (1 + j)^{32} = 4{,}834.65 \qquad \text{ANS. } \textbf{(D)}
\end{aligned}$$

6. This is an application of a geometrically-growing annuity present value function. It can be done using either real payments and interest rates, or nominal payments and rates. Using the latter approach:

$$i_{\text{NOM}} = (1.05 \times 1.03) - 1 = 0.0815$$

$$PV = v_i \cdot 100{,}000 \cdot \left(\frac{1 - \left(\frac{1.05}{1.0815}\right)^{24}}{0.0815 - 0.05} \right) = 1{,}491{,}363 \qquad \text{ANS. } \textbf{(A)}$$

7. The first condition of Redington immunization is $P_A = P_L$, where P_A is the PV of the assets and P_L is the PV of the liabilities:

(1) $733v + Yv^5 = 1000v^2 + Xv^4 + 1000v^6$

Dividing (1) by v:

(2) $733 + Yv^4 = 1000v + Xv^3 + 1000v^5$

The second condition is $P'_A = P'_L$:

(3) $-733v^2 - 5Yv^6 = -2000v^3 - 4Xv^5 - 6000v^7$

Dividing (3) by $-v^2$:

(4) $733 + 5Yv^4 = 2000v + 4Xv^3 + 6000v^5$

Multiplying (2) by 4:

(5) $2932 + 4Yv^4 = 4000v + 4Xv^3 + 4000v^5$

Subtracting (5) from (4):

(6) $-2199 + Yv^4 = -2000v + 2000v^5$

Solving for Y:

(7) $Y = (2000v^5 - 2000v + 2199)/v^4 = 2375.74$ ANS. **(E)**

Note: We wouldn't test for the third condition ($P''_A > P''_L$) unless we had the time.

8. When individual expected future dividends do not fall into the geometric growth pattern, they can be treated separately:

$$112.50 = \frac{2}{1.085} + v_{0.085}\left(\frac{3}{0.085 - (g/100)}\right)$$

$\therefore\ g = 6.00$ ANS. **(E)**

9. Consider a "slice" of time, and the payment made during that slice of time. Conceptually, at time t, the payment would be the rate at which payments are being made (t^2), multiplied by the slice of time (dt). That payment slice would need to be discounted back to time zero from time t using a 6% force of interest. Thus:

$$PV = \int_0^5 t^2 e^{-0.06t} dt$$ ANS. **(B)**

10. The principal portions of the 9th and the 2nd payments are such that: $P_9 = P_2(1+i)^7$. This is a consequence of the formula $P_t = Rv^{n-t+1}$.

Given the principal portion of the 9th payment is 1.5 times the principal portion of the 2nd payment, then:

$$(1+i)^7 = 1.5 \quad \text{and} \quad i = 1.5^{1/7} - 1 = .059634.$$

Compute the number of full payments by solving for n the equation: $45{,}000 = 3{,}200a_{\overline{n}|}$. Get $n = 31.49$.

Thus, there are 31 full payments of 3,200 and an additional drop payment of P at time 32.

$$45{,}000 = 3{,}200a_{\overline{31}|i} + Pv^{32} = 44{,}751.76 + Pv^{32}$$

Solving for P: $P = 248.24(1.059634)^{32} = 1{,}584.37$. ANS. **(D)**

Note: The drop payment could easily be found using a calculator: Enter 5.9634 into $\boxed{\text{I/Y}}$, 45,000 $\boxed{\text{+/-}}$ into $\boxed{\text{PV}}$, 3,200 into $\boxed{\text{PMT}}$, $\boxed{\text{CPT}}$, $\boxed{\text{N}}$.

Enter 31 into $\boxed{\text{N}}$, $\boxed{\text{CPT}}$ $\boxed{\text{FV}}$, then multiply by $1 + i$. (Answers may differ because of rounding.)

11. Set up the net present value equation, and solve for X:

$$NPV = 2{,}500 = -50{,}000 + v_{.10}^2 \cdot X \cdot a_{\overline{8}|.10}$$

$$\therefore\ X = 11{,}907.38 \qquad \text{ANS. } \mathbf{(D)}$$

12. Use the information regarding the first perpetuity to find the interest rate. Then, use that rate to price the second perpetuity:

$$9{,}500 = \frac{500}{i - 0.04}$$

$$i = 0.092632$$

$$P = \frac{400}{i} + \frac{20}{i^2} = 6{,}649.02 \qquad \text{ANS. } \mathbf{(C)}$$

13. The time-weighted rate of return is a function of the product of ratios of ending to beginning balances within subperiods:

$$1.20 = \left(\frac{1800}{2000}\right)\left(\frac{3400}{1800 + X}\right)$$

$$\therefore\ X = 750.00 \qquad \text{ANS. } \mathbf{(B)}$$

14. Find the size of each monthly payment, and the effective monthly interest rate. Then determine the amount of interest in the 56th payment:

$$j = 1.15^{1/12} - 1 = 0.011715$$

$$200{,}000 = R \cdot a_{\overline{120}|j}$$

$$R = 3{,}112.295$$

$$I_{56} = R \cdot (1 - v^{120-56+1}) = 1{,}652.47 \qquad \text{ANS. } \mathbf{(C)}$$

15. Calculate the total amount of payments during the life of the loan. The difference between this figure and the original principal that needs to be paid off is the total amount of interest paid during the life of the loan:

$$300{,}000 = R \cdot a_{\overline{30}|.09}$$

$$R = 29{,}200.905$$

$$P = \frac{30R - 300{,}000}{30R} = 0.658 \qquad \text{ANS. } \mathbf{(D)}$$

16. One can either treat the first 9 years or the first 10 years as the increasing annuity, and then the remaining years as the decreasing annuity. Using the latter approach:

$$PV = 100(Ia)_{\overline{10}|.10} + v_{.10}^{10} \cdot 100(Da)_{\overline{9}|.10}$$

$$= 2{,}903.59 + 1{,}249.54 = 4{,}153.13$$

An alternative, and somewhat quicker approach (if you happen to remember Section 4l of the manual, is to express the PV of this palindromic annuity as $100a_{\overline{10}|} \cdot \ddot{a}_{\overline{10}|}$ or $100 \cdot (1.10) \cdot (a_{\overline{10}|})^2 = 4{,}153.13$. ANS. **(C)**

17. First, calculate the accumulated value of the combined original and reinvestment accounts, and then determine the present value of that accumulated amount at the specified yield rate:

$$\begin{aligned} A(20) &= 20(10{,}000) + 700(Is)_{\overline{20}|.04} \\ &= 200{,}000 + 191{,}961.03 = 391{,}961.03 \\ P &= \frac{A(20)}{(1.06)^{20}} = 122{,}215.3 \qquad \text{ANS. } \mathbf{(D)} \end{aligned}$$

18. The quickest way is to recall that the write downs are in geometric progression, with common ratio $(1+i)$. The sum of the write downs is equal to the premium paid for the bond, i.e., is equal to 300. $(300 = 1{,}300 - 1{,}000)$. We have:

$$\begin{aligned} (4.60)[1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{29}] &= 300 \\ 4.60 s_{\overline{30}|} &= 300 \end{aligned}$$

Using the calculator, we get $i = 4.89\%$. ANS. **(B)**

19. Assume, for example, that X is a deposit. Once we solve for X, its sign, properly interpreted, will indicate whether it is indeed a deposit (a positive value for X) or a withdrawal (a negative value for X). Doing the calculations in thousands:

$$\begin{aligned} 1 + r_T &= \left(\frac{12}{10}\right)\left(\frac{10}{12+X}\right) = \frac{12}{12+X} \\ r_T &= \frac{-X}{12+X} \\ r_D &= \frac{10 - 10 - X}{10(1) + X(0.5)} = \frac{-X}{10 + 0.5X} \\ \frac{-X}{12+X} &= \frac{-X}{10+0.5X} \\ \therefore\ X &= 0 \text{ or } -4 \end{aligned}$$

Thus, the non-trivial (non-zero) solution is a 4,000 withdrawal. ANS. **(B)**

20. First, using the calculator, find the yield rate. You should get 3.5960% as the effective semiannual yield rate.

Prospectively, the outstanding balance at any time is the present value of the future bond payments (coupons and maturity value). This PV is the same as the price of the bond at that point in time, using the original yield rate.

$$B_{13} = 4a_{\overline{27}|} + 100v^{27} \text{ at } 3.596\% = 106.91 \qquad \text{ANS. } \mathbf{(E)}$$

21. Determine the two-year forward rate and use it to compute the present value of the 1,000 par value zero-coupon bond:

$$\begin{aligned} j &= \frac{1.075^3}{1.070^2} - 1 = 0.085070 \\ PV &= 1{,}000 v_j^1 = 921.60 \qquad \text{ANS. } \mathbf{(B)} \end{aligned}$$

22. One way of calculating a modified duration (volatility) is by determining the Macaulay duration and dividing it by (one plus the interest rate). But volatility is also a rate-of-change concept, and thus can be calculated via the derivative of the underlying price-yield equation. Using this approach:

$$P = \frac{1}{i} + \frac{1}{i^2}$$

$$P' = -i^{-2} - 2i^{-3}$$

$$\mathrm{D}_{\text{mod}} = -\frac{P'}{P} = 24.07 \qquad \text{ANS. } \mathbf{(D)}$$

23. First, determine the price for which the bond was originally purchased to yield a minimum of 8% effective. To do this, we must find the lowest price for all possible redemption dates.

The price assuming maturity at par in 30 years is:

$$P = 75a_{\overline{30}|} + 1000v^{30} \text{ at } 8\% = \$943.71$$

If we assume that the bond is called at \$1,050 at the end of 6 to 29 years, the price would be (premimum/discount formula):

$$P = 1050 + (75 - 84)a_{\overline{n}|}$$

Thus, the bond would be purchased at a discount and the lowest price is for redemption in 29 years:

$$P = 1050 - 9a_{\overline{29}|} = \$949.57$$

Thus, to get a minimum yield of 8%, the bond was purchased at the lower of \$943.71 or \$949.57, i.e., \$943.71.

The bond was actually called at the end of the 12th year. To determine the yield rate, we set up the following equation of value and solve for i using the calculator:

$$943.71 = 75 \cdot a_{\overline{12}|i} + 1050 \cdot v_i^{12}$$

$$i = 8.52\% \qquad \text{ANS. } \mathbf{(E)}$$

24. Calculate the bond price as the present value of its cash flows at the zero-coupon rates, and then use a calculator to determine the bond yield which will produce that same price:

$$P_0 = \frac{120}{1.0625} + \frac{120}{1.07^2} + \frac{1120}{1.075^3} = 1{,}119.310$$

$$1{,}119.310 = 120 \cdot a_{\overline{3}|i} + 1000 \cdot v_i^3$$

$$\therefore\ i = 7.42\% \qquad \text{ANS. } \mathbf{(C)}$$

25. The best way to do this is to use the premium/discount formula for the price of a bond:

$$P = C + (Fr - Ci)a_{\overline{n}|}$$

$$\text{Bond A:} 140.25 = 100 + (8 - 100i)a_{\overline{n}|} \text{ or } 40.25 = (8 - 100i)a_{\overline{n}|}$$

$$\text{Bond B:} 80.17 = 100 + (3 - 100i)a_{\overline{n}|} \text{ or } -19.83 = (3 - 100i)a_{\overline{n}|}$$

Divide the left-hand and right-hand sides of the two equations:

$$-\frac{40.25}{19.83} = \frac{8 - 100i}{3 - 100i}$$

Cross multiply and solve for i:

$$6.008i = 279.39$$
$$i = 4.65\% \qquad \text{ANS. } \textbf{(B)}$$

26. Use a standard Macaulay duration formula:

$$\mathrm{D_{mac}} = \frac{100(Ia)_{\overline{30}|.12} + 30{,}000v_{.12}^{30}}{100a_{\overline{30}|.12} + 1{,}000v_{.12}^{30}} = 9.16 \qquad \text{ANS. } \textbf{(A)}$$

27. The easiest way to do this is to use the "special formula" for the swap rate, i.e., the formula in terms of prices of zero-coupon bonds:

$$R = \frac{1 - P_n}{P_1 + P_2 + \cdots + P_n}$$

Substituting, with $n = 2$:

$$.065826 = \frac{1 - P_2}{.943 + P_2}$$

Solving, we get $P_2 = .880$. ANS. **(D)**

28. First, determine the swap rate. The easiest way to do this is to use the "special formula" for R with a constant Q:

$$R = \frac{1 - P_2}{P_1 + P_2} = \frac{1 - 1.0475^{-2}}{1.045^{-1} + 1.0475^{-2}} = 4.7442\%$$

The fixed interest payment from Corp. A to Corp. B at the end of each of the two years is $(.047442)(200{,}000) = 9{,}488$.

The variable interest that Corp. B pays to Corp. A at the end of the 2nd year is based on the spot rate for a 1-year term at the beginning of that year, i.e., at the beginning of 2022. This spot rate is 5.15%, so Corp. B must pay Corp. A $(.0515)(200{,}000) = 10{,}300$. Since B's payment is larger than A's, B pays the net amount of 812 ($= 10{,}300 - 9{,}488$) to A. ANS. **(E)**

29. First, determine the price of each of the two bonds at our disposal:

$$P_A = 1000v_{.05}^{1} = 952.38$$

$$P_B = \frac{80}{1.06} + \frac{1080}{(1.06)^2} = 1036.67$$

Next, determine the numbers of the two bonds that are needed to match the liabilities, by starting with second liability payment (only Bond B has a cash flow at the time of that second payment):

$$n_B = \frac{4000}{1080} = 3.704$$

$$3000 = n_B(80) + n_A(1000) \quad \therefore\ n_A = 2.704$$

Finally, the total cost of the matching bond portfolio can be determined:

$$Cost = n_A P_A + n_B P_B = 6{,}414.47 \qquad \text{ANS. } \textbf{(E)}$$

30. First, determine the price that Patricia paid for the bill. The quoted rate on a U.S. T-Bill is a simple discount rate, so the price is:

$$P = \left(1 - \frac{n}{360}d\right) C$$

where n is the number of days in the bill, d is the quoted rate, and C is the maturity value. (A conventional 360-day year is used in this formula.)

$$P = \left[1 - \frac{180}{360}(.02)\right] 10{,}000 = \$9{,}900$$

If i is Patricia's effective annual yield rate, we have:

$$(9{,}900)(1+i)^{2/12} = 9{,}950$$
$$i = 3.07\% \qquad \text{ANS. } \textbf{(E)}$$

31. Set up and equate the accumulation formulas:

$$100 \cdot (1-d)^{-10} = 100 \cdot \exp\left(\int_0^{10} 0.001t^2 dt\right) = 139.5612$$
$$d = 0.0328 \qquad \text{ANS. } \textbf{(A)}$$

32. If the cash flows of an asset or liability are linear combinations of other assets or liabilities, then their prices must also have the same linear relationship. In this problem, which basically asks you to determine the three-year spot rate, you can combine Bonds A and B such that you are only left with a single cash flow at time 3. This can be done by taking $4/3$ units of Bond A, and subtracting from it Bond B. The cash flows at times 1 and 2 will zero-out, and the net cash flow at time three will be

$$CF_3 = \left(\frac{4}{3}\right) 1{,}060 - 1{,}080 = 333.333$$

Similarly, the price at time zero of the linear combination will be

$$P = \left(\frac{4}{3}\right) 955.57 - 1{,}008.38 = 265.713$$

Thus, if we buy $4/3$ of Bond A, and sell one Bond B, we will be left with a zero-coupon bond that costs 265.713 now, and pays off 333.333 three years from now. The implied yield is

$$\left(\frac{333.333}{265.713}\right)^{1/3} - 1 = 0.0785 \qquad \text{ANS. } \textbf{(D)}$$

33. Only II is true. See the discussion in Section 6.3 of the Study Note on Determinants of Interest Rates. ANS. **(B)**

34. Set up the time-weighted rate of return formula, leaving using X so that the sign tells you whether the quantity is a deposit (positive) or a withdrawal (negative).

$$1.234 = \left(\frac{800}{1{,}000}\right)\left(\frac{900}{800+X}\right)\left(\frac{1{,}200}{1{,}400}\right)$$
$$X = -299.88 \qquad \text{ANS. } \textbf{(B)}$$

35. Add together the two accumulated values, stemming from the original deposits, and the reinvested interest. Then, discount the total back to time zero at 7% per year.

$$A(20) = 500(20) + 40 \cdot (Is)_{\overline{19}|0.05} = 20{,}452.763$$

$$A(20) \cdot v_{0.07}^{20} = 5{,}285.38 \qquad \text{ANS. } \mathbf{(E)}$$

