Soil Mechanics II

2 – Basics of Mechanics

1. Definitions
2. Analysis of stress and strain in 2D – Mohr's circle
3. Basic mechanical behaviour
4. Testing of soils - apparatuses

Continuum

Continuous mathematical functions describing the material properties

Homogeneity

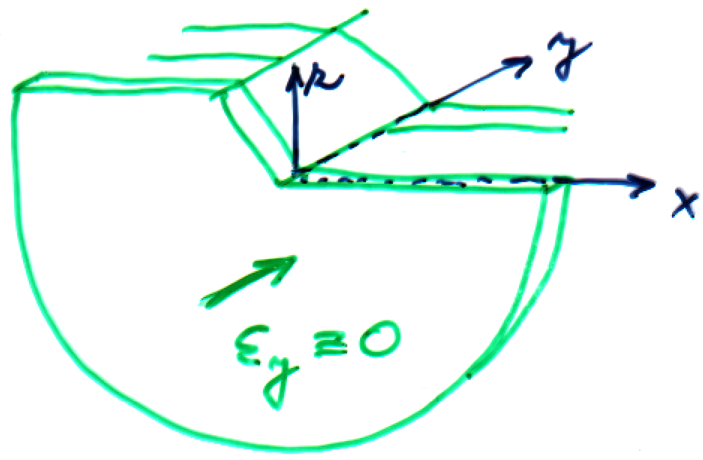
Smallest ($V \rightarrow 0$) volumes occupied by physically and chemically identical material / matter

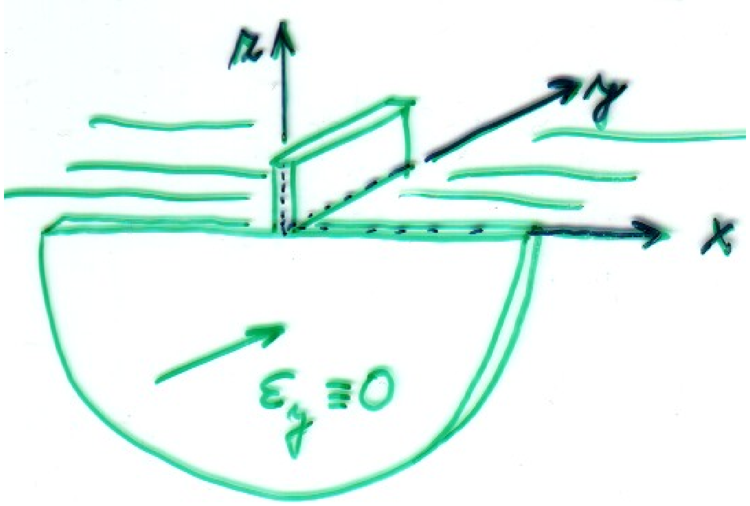
Isotropy

Physical – mechanical properties identical in all directions from the given (studied) point

3D → 2D simplifying the problem whenever possible

Plane strain – in EG/GT frequently applicable

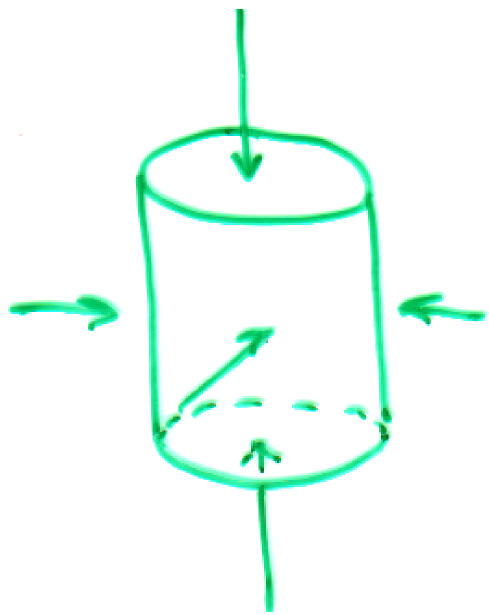




cf Plane stress – without practical use in EG/GT

(see above: $\sigma_y \neq 0$ since $\varepsilon_y = 0$)

..... another simplification of algebra: axial symmetry ... (not 2D though!)

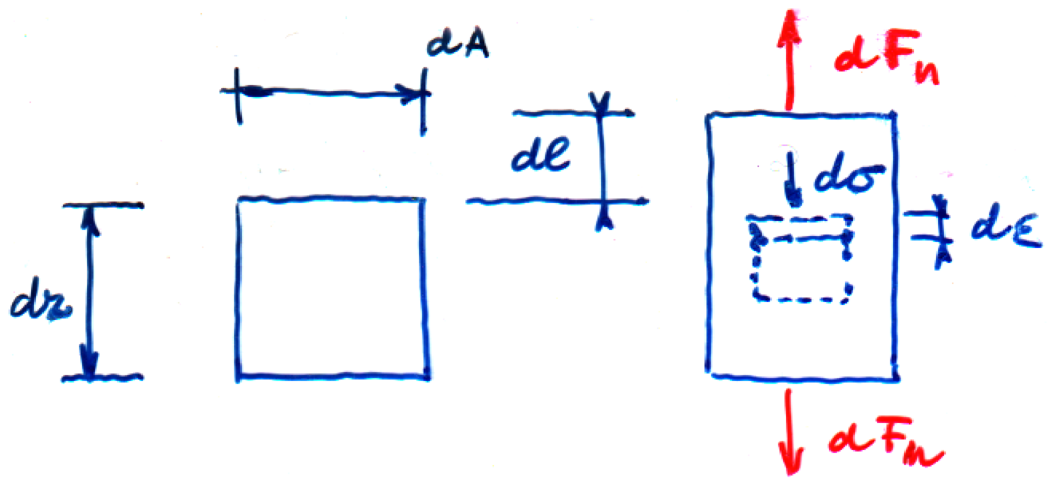


$$\sigma_x = \sigma_y = \sigma_r$$

$$\varepsilon_x = \varepsilon_y = \varepsilon_r$$

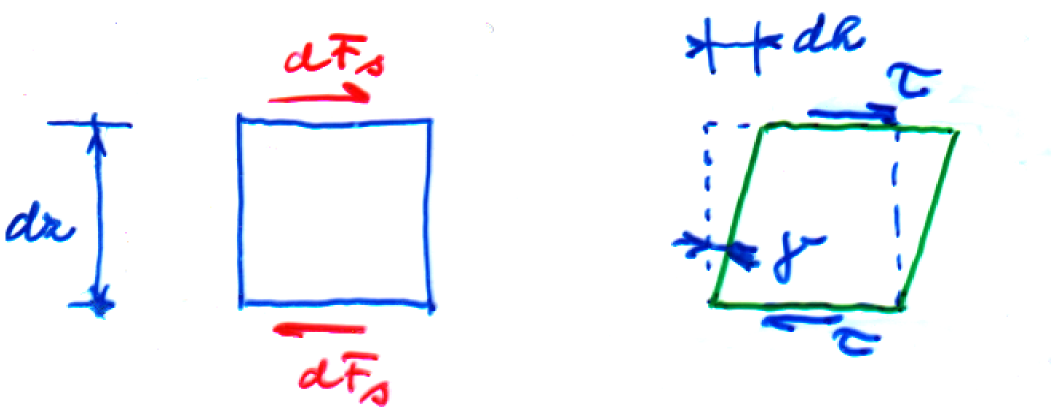
Stress = Force / Area

Strain = Change in dimension / Original dimension (or change of right angles)



$d\sigma = dF_n / dA \quad \rightarrow \sigma = F_n / A$

$d\varepsilon = dl / dz \quad (\rightarrow \varepsilon = \delta l / \delta z)$



$d\tau = dF_s / dA \quad \rightarrow \tau = F_s / A$

$d\gamma = dh / dz \quad (\rightarrow \gamma = \delta h / \delta z)$

Definitions

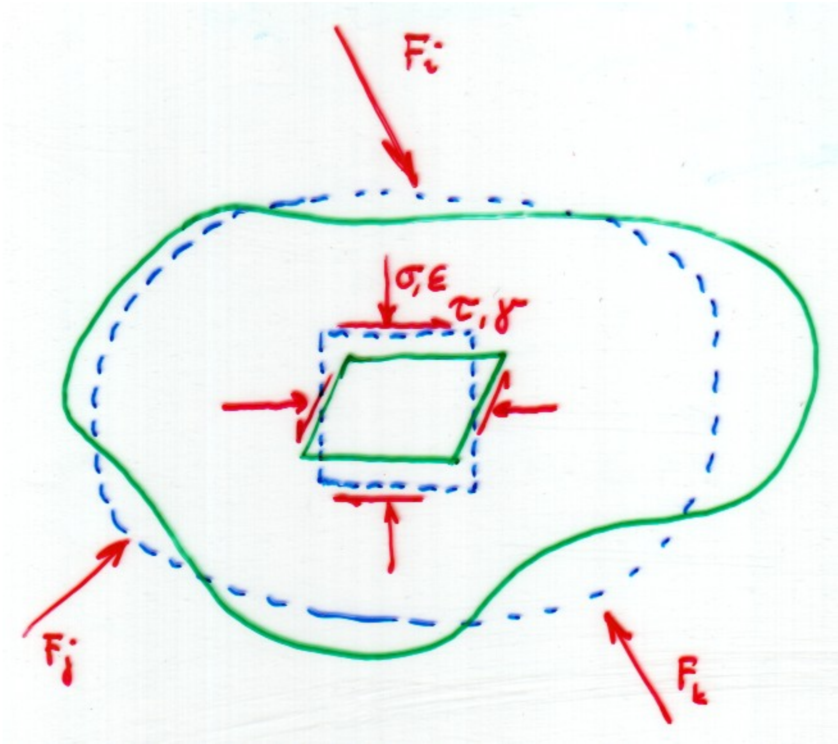
Deformation (≈ result of loading)
- change in shape and/or size of a continuum body
- depends on the size of the body, i.e. structure / model / specimen

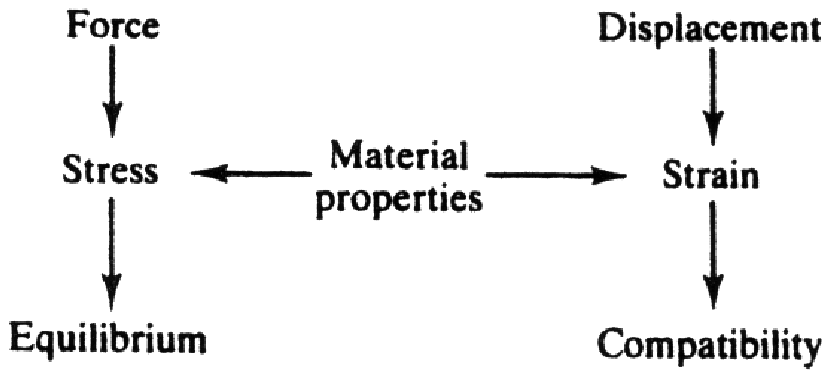
Strain
- the geometrical measure of deformation - the relative displacement between particles of the body (contrary to the rigid-body displacement).
  - normal strain
    - the amount of stretch or compression along a material line elements or fibers
  - shear strain
    - the amount of distortion associated with the sliding of plane layers over each other

Sign convention in Geotechnics

Compression is positive
Extension negative

Constitutive equation (= physical, material eq.)

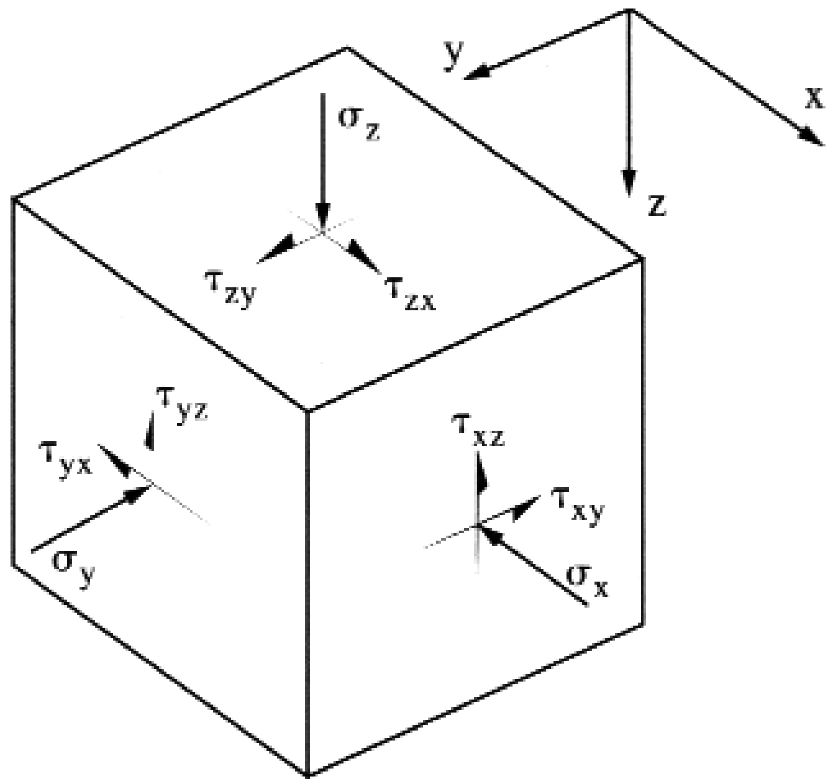




[1]

Normal stress σ_x, σ_y, σ_z,

Shear stress τ_{xy}, τ_{yz}, τ_{zx}, τ_{yx}, τ_{zy}, τ_{xz} ($\tau_{zy} = \tau_{yz}$ etc)



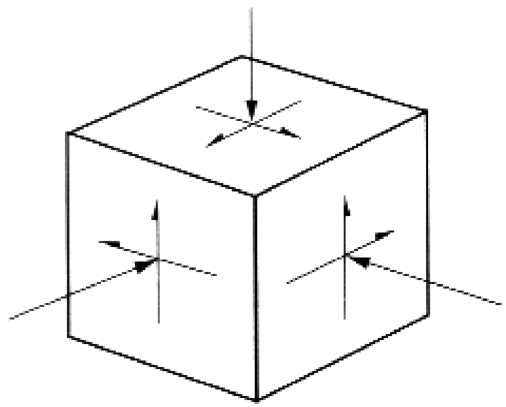
[3]

Stress Tensor: 9 components, 6 independent

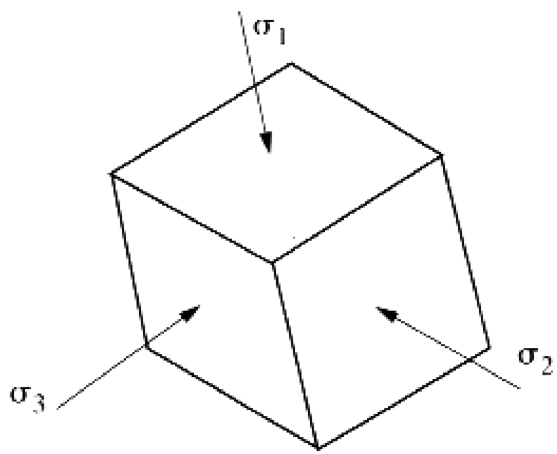
$$\begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

Tensor – numerical value, direction, orientation of coordinate system

Rotation of coordinate system



$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

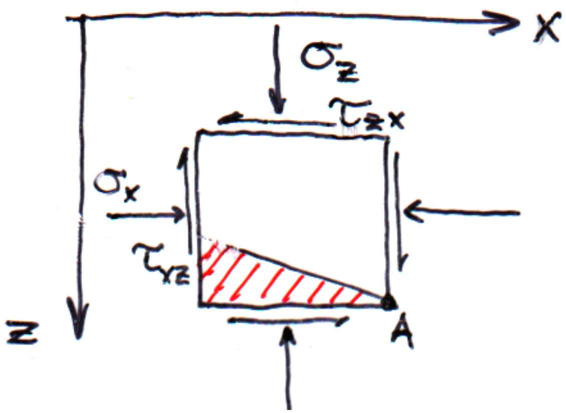


$$\begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

at every point three perpendicular planes exist (= a rotation exists) where shear stresses zero and normal stresses extreme values – principal stresses $\sigma_1 > \sigma_2 > \sigma_3$

$\sum \sigma_{ii}$ = konst. .... the first invariant of stress tensor

$p = 1/3(\sigma_{xx}+\sigma_{yy}+\sigma_{zz})$ = mean normal stres = const.
useful quantity for stress



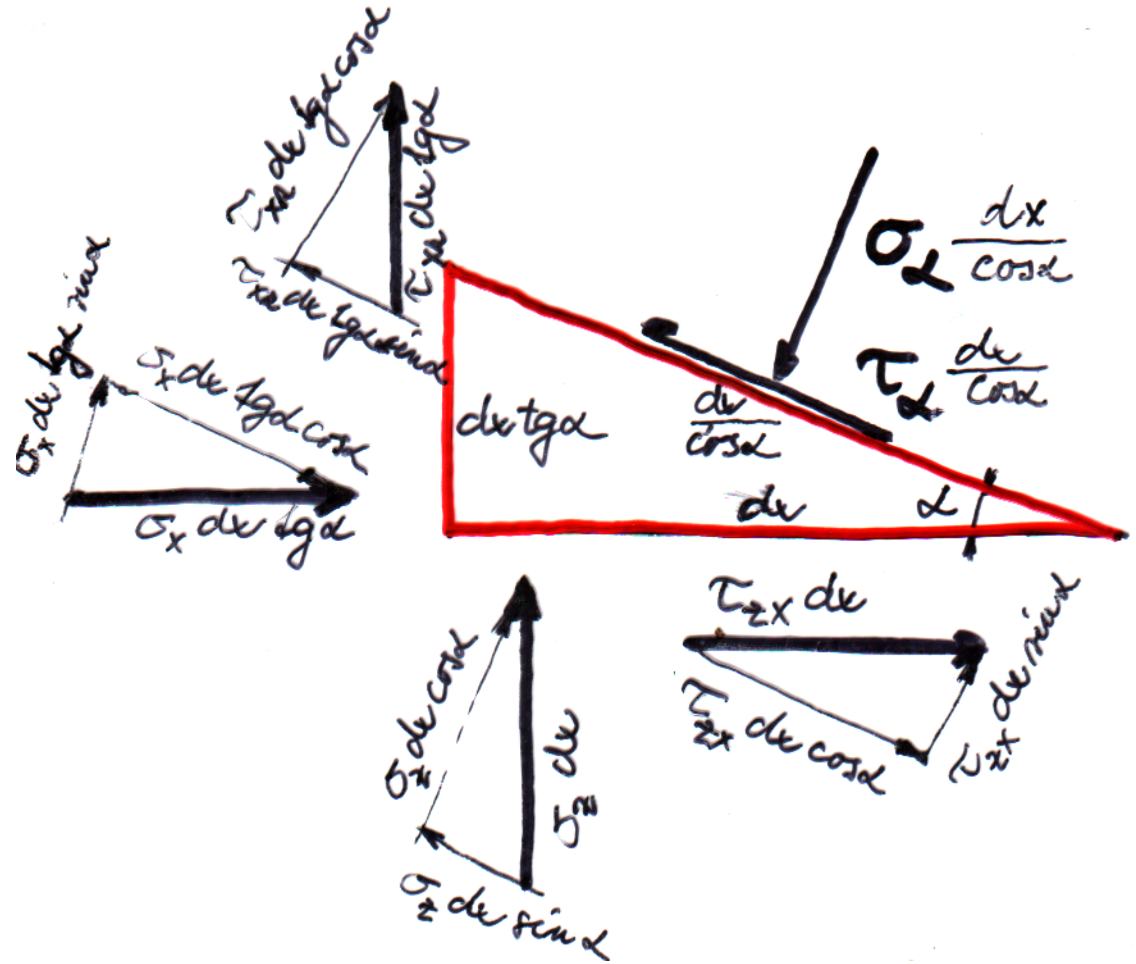
Equilibrium in a point in 2D: three equilibrium conditions: forces in two direction and moment

1. Moment = 0:

$\tau_{zx} \times dx \times dz = \tau_{xz} \times dz \times dx$

$\tau_{zx} = \tau_{xz}$

On two neighbouring planes shear stresses are equal and of opposite direction



2. Sum of forces in two perpendicular directions = 0

$$\sigma_\alpha dx / \cos\alpha = \sigma_z dx \cos\alpha + \tau_{zx} dx \sin\alpha + \tau_{xz} dx \sin\alpha + \sigma_x dx \sin^2\alpha / \cos\alpha$$

$$\tau_\alpha dx / \cos\alpha = - \sigma_z dx \sin\alpha + \tau_{zx} dx \cos\alpha - \tau_{xz} dx \sin^2\alpha / \cos\alpha + \sigma_x dx \sin\alpha$$

$$\sigma_\alpha \, dx / \cos\alpha = \sigma_z \, dx \cos\alpha + \tau_{zx} \, dx \sin\alpha + \tau_{xz} \, dx \sin\alpha + \sigma_x \, dx \sin^2\alpha / \cos\alpha$$

$$\sigma_\alpha = \sigma_z \cos^2\alpha + \sigma_x \sin^2\alpha + 2 \, \tau_{zx} \sin\alpha \cos\alpha$$

$$\cos^2\alpha = 1/2(1+\cos 2\alpha); \; \sin^2\alpha = 1/2(1-\cos 2\alpha)$$

$$\sigma_\alpha = \sigma_z /2 + \sigma_z /2 \cos 2\alpha + \sigma_x /2 - \sigma_x /2 \cos 2\alpha + \tau_{zx} \sin 2\alpha$$

$$\sigma_\alpha = (\sigma_z + \sigma_x)/2 + (\sigma_z - \sigma_x)/2 \cos 2\alpha + \tau_{zx} \sin 2\alpha \qquad (1)$$

$$\tau_\alpha \, dx / \cos\alpha = - \sigma_z \, dx \sin\alpha + \tau_{zx} \, dx \cos\alpha - \tau_{xz} \, dx \sin^2\alpha / \cos\alpha + \sigma_x \, dx \sin\alpha$$

$$\cos^2\alpha - \sin^2\alpha = \cos 2\alpha$$

$$\tau_\alpha = (\sigma_x - \sigma_z)/2 \sin 2\alpha + \tau_{zx} \cos 2\alpha \qquad (2)$$

Principal normal stress = extremes at $\alpha=\alpha_0$

(1): $\sigma_\alpha=(\sigma_z + \sigma_x)/2 + (\sigma_z - \sigma_x)/2 \cos 2\alpha + \tau_{zx} \sin 2\alpha$ ....derivation = 0...

direction of two perpendicular planes, so called principal planes, on which extreme normal stresses act:

$$tg2\alpha_0 = \tau_{zx} / ((\sigma_z - \sigma_x)/2) \quad (3)$$

(the same expression is obtained from (2) for $\tau_\alpha = 0$ (i.e., on principal planes there are zero shear stresses)

....manipulation using goniometric expressions:

$\cos 2\alpha = 1/(1+tg^2 2\alpha)^{1/2}$; $\sin 2\alpha = tg2\alpha/(1+tg^2 2\alpha)^{1/2}$

$\rightarrow \cos 2\alpha_0 = 1/(1+4\tau_{zx}^2 / (\sigma_z - \sigma_x)^2)^{1/2} = (\sigma_z - \sigma_x)/((\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2)^{1/2}$

$\rightarrow \sin 2\alpha_0 = (2\tau_{zx}/(\sigma_z - \sigma_x))/(1+4\tau_{zx}^2 / (\sigma_z - \sigma_x)^2)^{1/2} = 2\tau_{zx}/((\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2)^{1/2}$

and using $tg2\alpha_0$ due to (3)

values of principal (normal) stress:

$$\sigma_{1,2} = (\sigma_z + \sigma_x)/2 \pm (((\sigma_z - \sigma_x)/2)^2 + \tau_{zx}^2)^{1/2} \quad (4)$$

....the meaning of the previous page:

at all points of continuum at the given stress state such rotation of planes/axes ($\alpha=\alpha_0$) can be found at which the normal stresses are extremes and shear stress is zero

in 2D: minimum a maximum normal stress = 2 principle stresses acting on principal planes

convention: $\sigma_1 > \sigma_2$

in 3D: 3 principle stresses acting on principal planes

convention: $\sigma_1 > \sigma_2 > \sigma_3$

Similarly, a different rotation (angle α, i.e. different planes) can be found at every point, where the shear stresses reach extreme values:

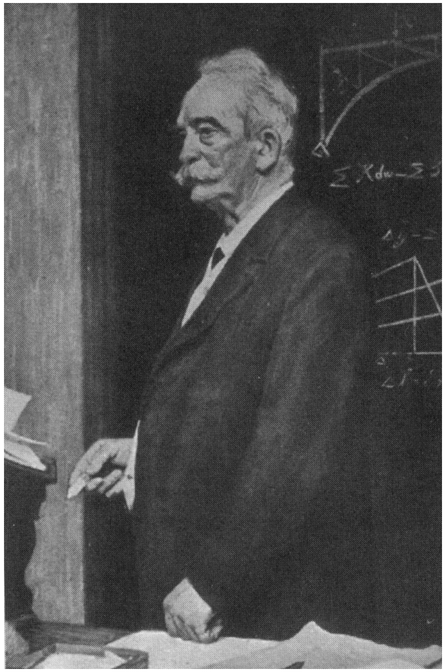
(2) $\tau_\alpha = (\sigma_x - \sigma_z)/2 \sin 2\alpha + \tau_{zx} \cos 2\alpha$

...derivation=0... → $tg2\alpha_{\tau max} = (\sigma_x - \sigma_z) / 2\tau_{zx}$ (5)

...putting into (2): $\tau_{max,min} = \pm\, (((\sigma_x - \sigma_z)/2)^2 + \tau_{zx}{}^2)^{1/2}$ (6)

Relations (3) to (6) are the results of the stress analysis in 2D (all the needed quantities / values are derived

K. Culmann (1866) and O. Mohr (1882) – graphic representation of the equations (3) až (6), i.e., equations (1) a (2), using a circle.



[2]

$$\sigma_\alpha - (\sigma_z + \sigma_x)/2 \quad = \quad (\sigma_z - \sigma_x)/2 \cos 2\alpha + \tau_{zx} \sin 2\alpha \qquad (1)$$

$$\tau_\alpha \quad = \quad (\sigma_x - \sigma_z)/2 \sin 2\alpha + \tau_{zx} \cos 2\alpha \qquad (2)$$

squaring and summing (1) a (2):

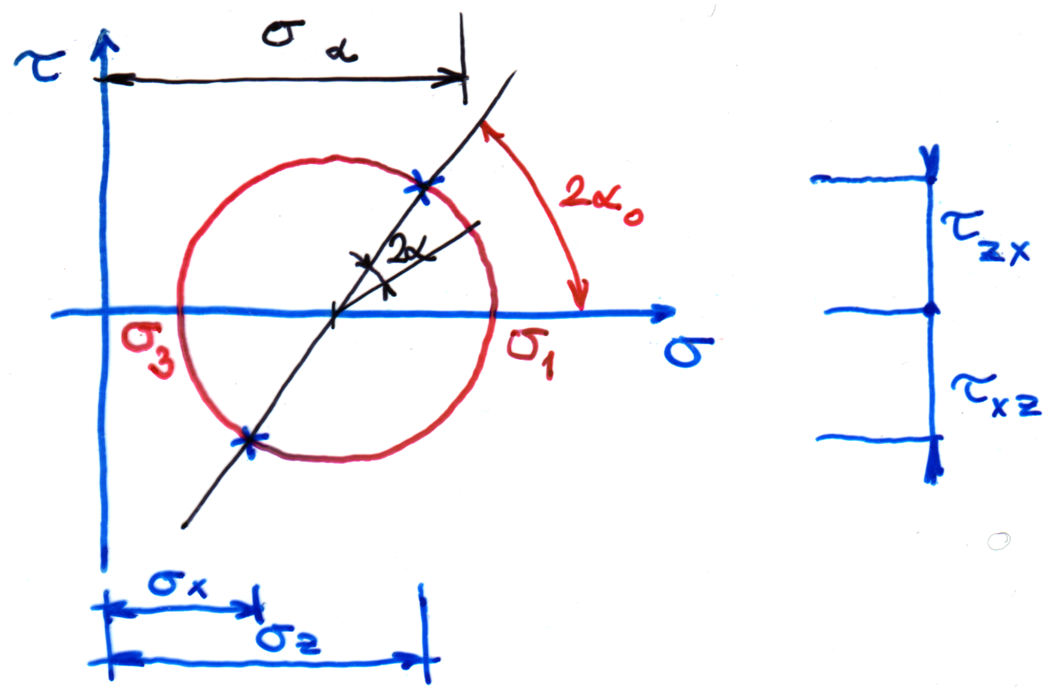
$$(\sigma_\alpha - (\sigma_z + \sigma_x)/2)^2 + \tau_\alpha^{\ 2} = (\sigma_z - \sigma_x)^2/4 \cos^2 2\alpha + 2\tau_{zx}(\sigma_z - \sigma_x)/2 \cos 2\alpha \sin 2\alpha + \tau_{zx}^{\ 2} \sin^2 2\alpha + (\sigma_x - \sigma_z)^2/4 \sin^2 2\alpha + 2\tau_{zx}(\sigma_x - \sigma_z)/2 \sin 2\alpha \cos 2\alpha + \tau_{zx}^{\ 2} \cos^2 2\alpha$$

$$(\sigma_\alpha - (\sigma_z + \sigma_x)/2)^2 + \tau_\alpha^{\ 2} = ((\sigma_z - \sigma_x)/2)^2 + \tau_{zx}^{\ 2}$$

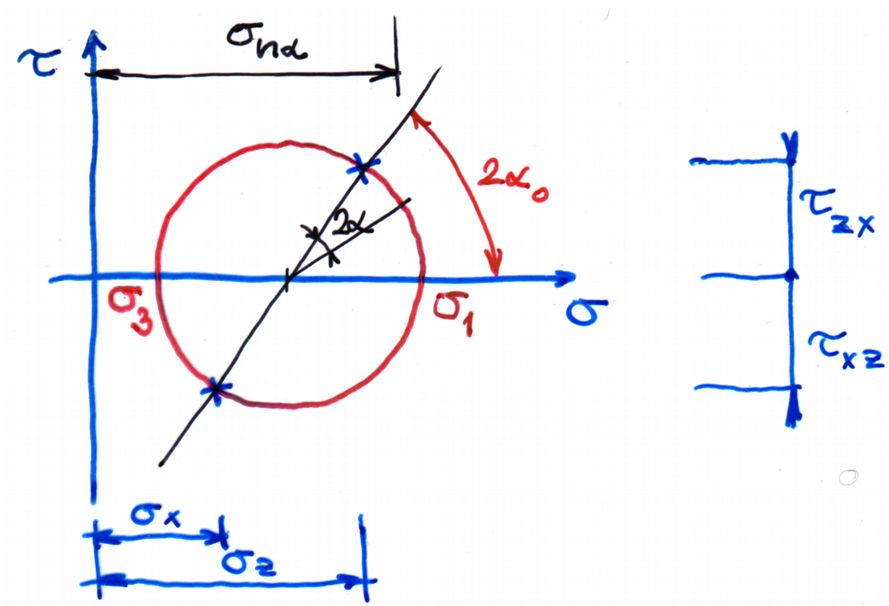
$$(\sigma - m)^2 + \tau^2 = r^2$$

i.e., equation of a circle for variables

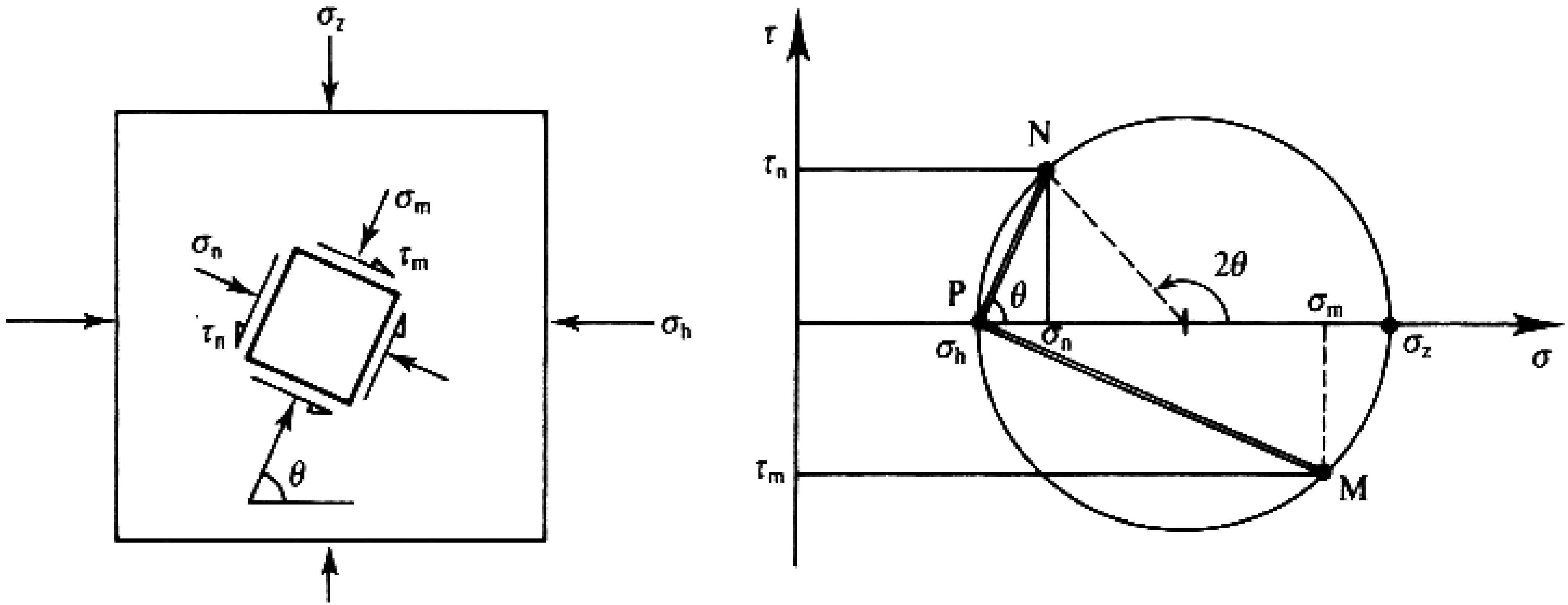
$\sigma_\alpha; \tau_\alpha$ $(\sigma ; \tau)$



Knowing σ_z, σ_x, τ_{zx}, τ_{xz}, it is straightforward to



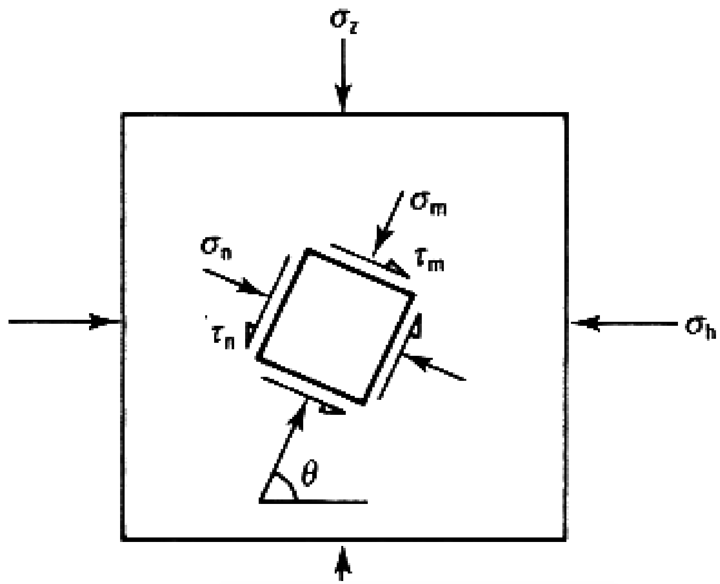
- draw Mohr's circle of stresses
- determine principal stresses
- determine the directions of principal planes (α_0)

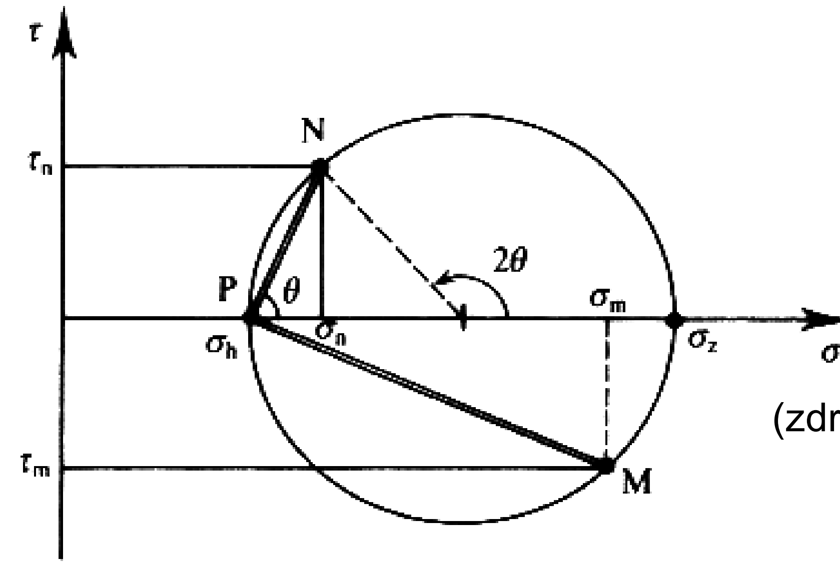


Pole of planes: a point on the M.C. A parallel line with any arbitrary direction (plane) intersects the M.C. at the stress point defining the stresses acting on the particular plane.

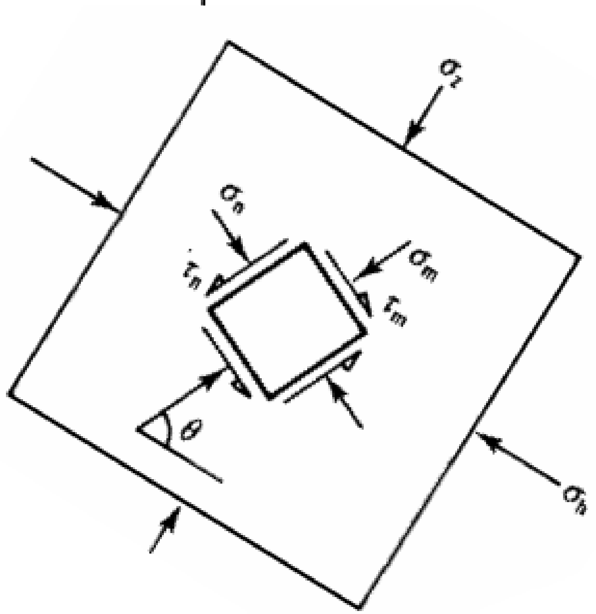
Usage: 1 Find pole; 2 Draw parallel line with the direction; 3 Read the stress.

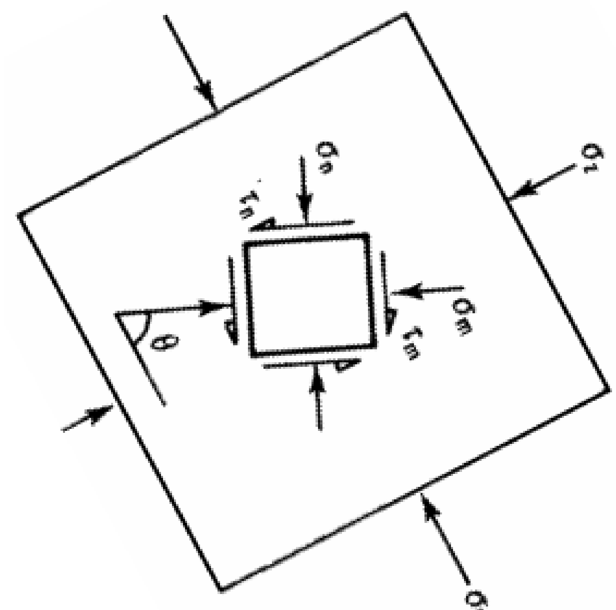
Pole of stress directions also may be used





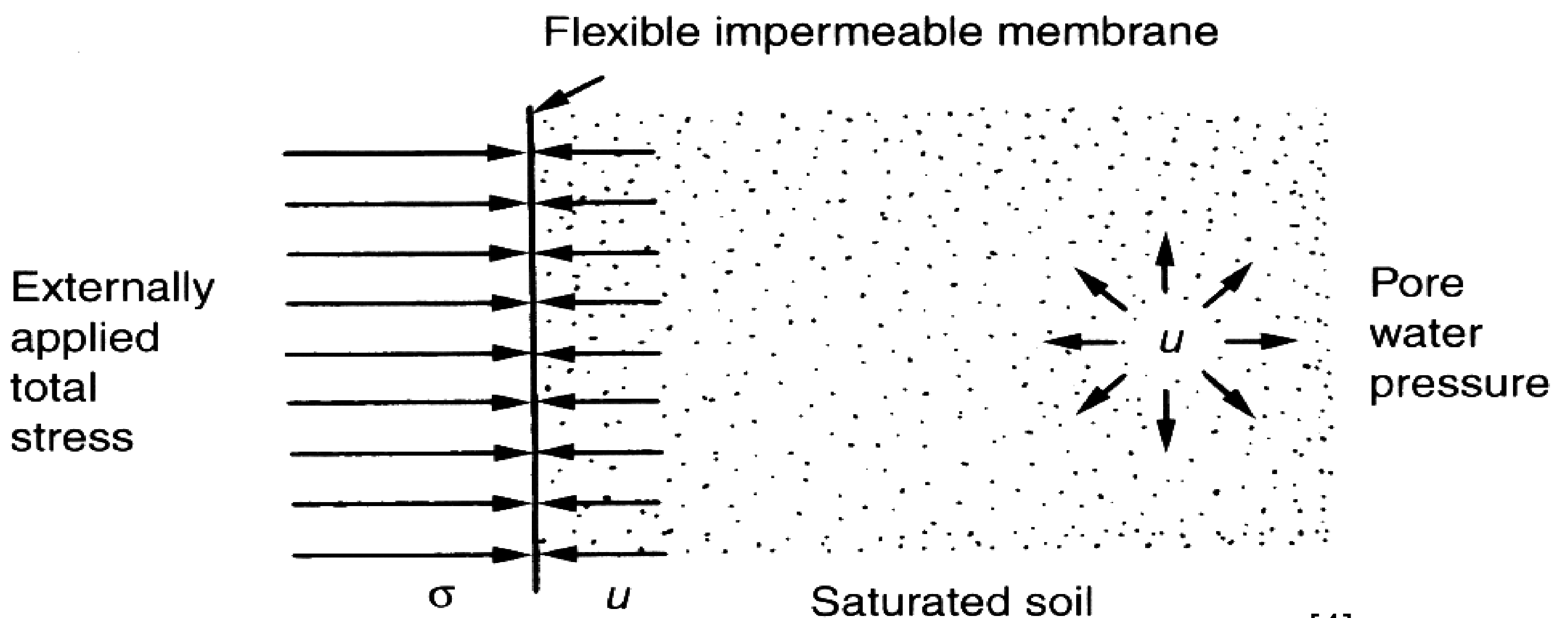
(zdroj: [1])





NB: on rotating the drawing the poles shift – change their positions;
NB: the angle θ remains at its position.

PRINCIPLE OF EFFECTIVE STRESSES



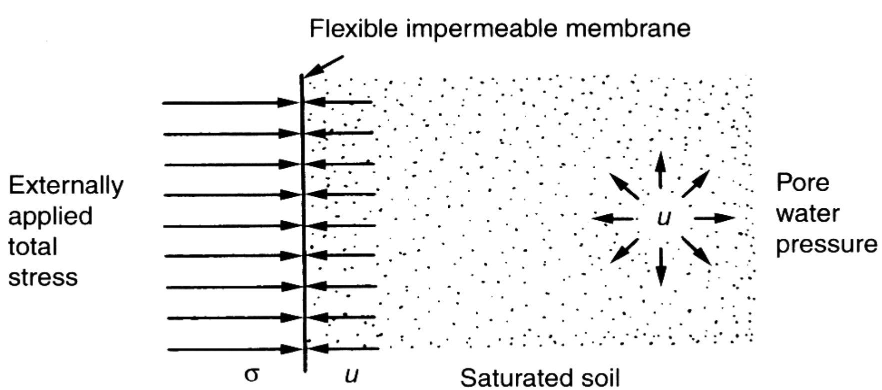
[4]

Terzaghi (1936):

The stresses in any point of a section through a mass of earth can be computed from the total principal stresses n_I', n_{II}' and n_{III}' which act in this point. If the voids of the earth are filled with water under a stress n_w, the total principal stresses consist of two parts. One part, n_w, acts in the water and in the solid in every direction with equal intensity. It is called the neutral stress. The balance, $n_I = n_I' - n_w$, $n_{II} = n_{II}' - n_w$ and $n_{III} = n_{III}' - n_w$, represents an excess over the neutral stress n_w and it has its seat exclusively in the solid phase of the earth.

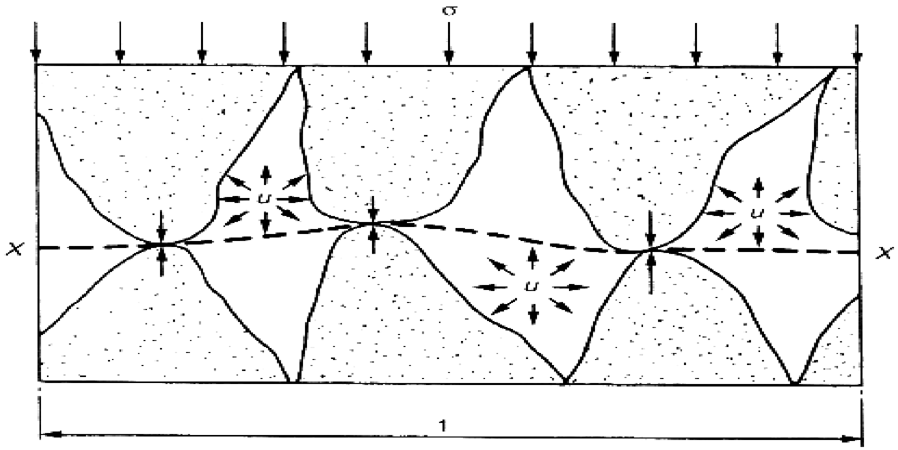
This fraction of the total principal stresses will be called the effective principal stresses. For equal values of the total principal stresses, the effective stresses depend on the value of n_w. In order to determine the effect of a change of n_w at a constant value of the effective stresses, numerous tests were made on sand, clay and concrete, in which n_w was varied between zero and several hundred atmospheres. All these tests led to the following conclusions, valid for the materials mentioned:

A change of the neutral stress n_w produces practically no volume change and has practically no influence on the stress conditions for failure. Each of the porous materials mentioned was found to react on a change of n_w as if it were incompressible and as if its internal friction were equal to zero. All the measurable effects of a change of the stress, such as compression, distortion and a change of the shearing resistance are exclusively due to changes in the effective stresses, n_I, n_{II} and n_{III}. Hence every investigation of the stability of a saturated body of earth requires the knowledge of both the total and the neutral stresses.



$$\sigma' = \sigma - u$$

What is NOT effective stress:

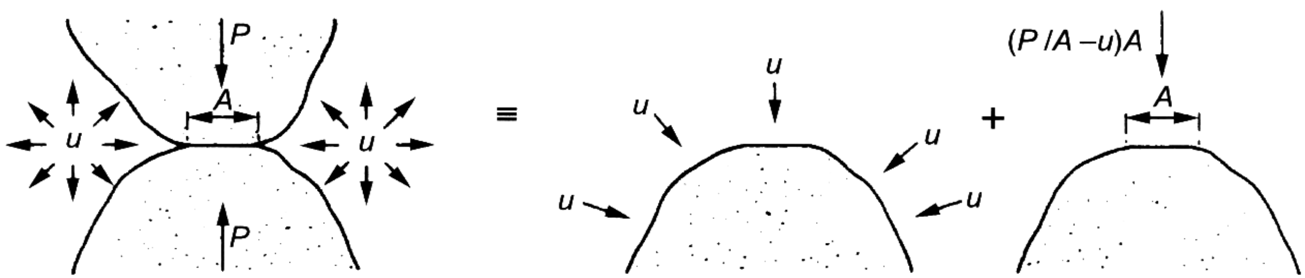


[4]

P average contact force

n number of contacts in X-X

σ_i = nP intergranular force per unit area (intergranular stress)



Incompressible grains; only the stress fraction over pore pressure can cause deformation:

Summing over all *n* (average) contacts:

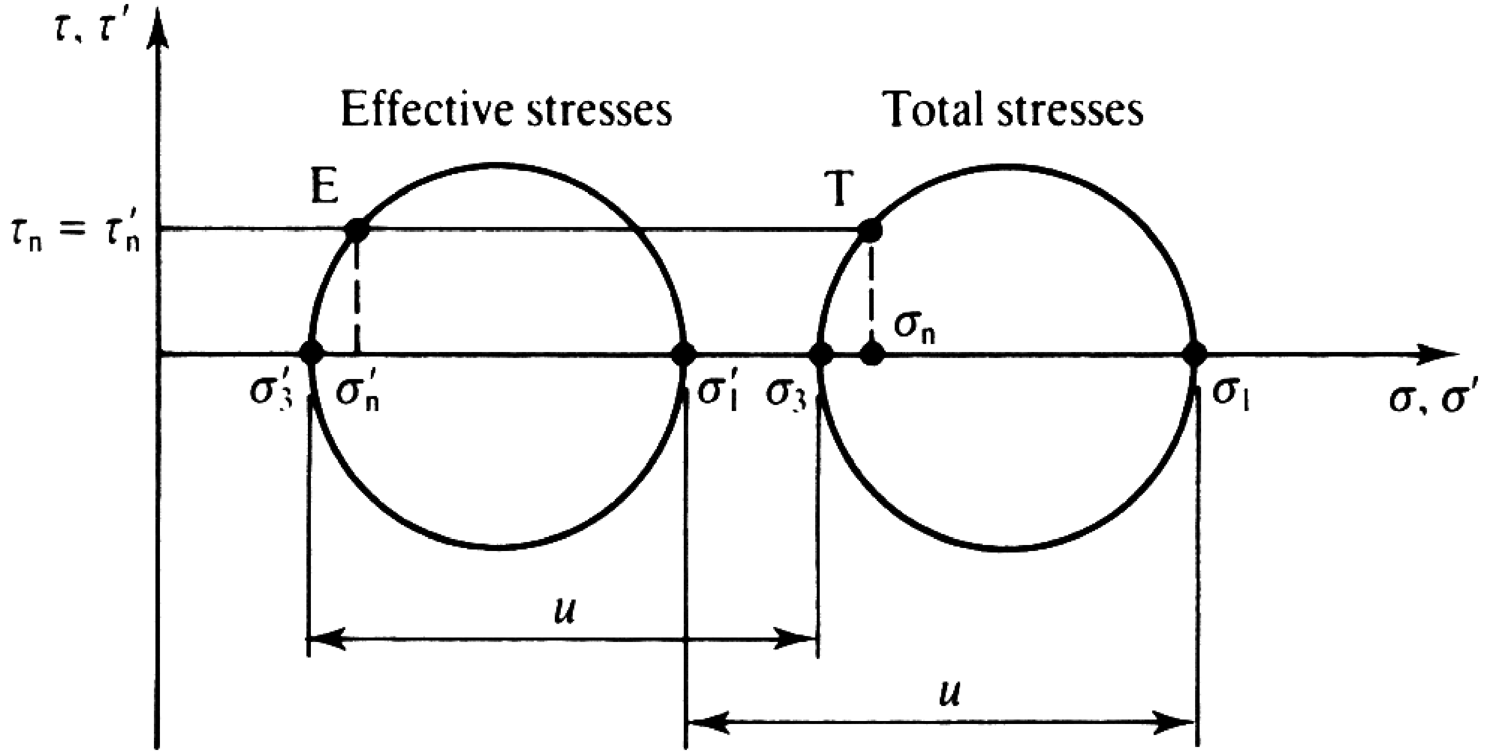
$$\sigma' = n\,((P / A) - u)\,A = n\,P - u\,n\,A = \sigma_i - u\,n\,A$$

$$\sigma' \neq \sigma_i$$

Effective stress IS NOT intergranular stress

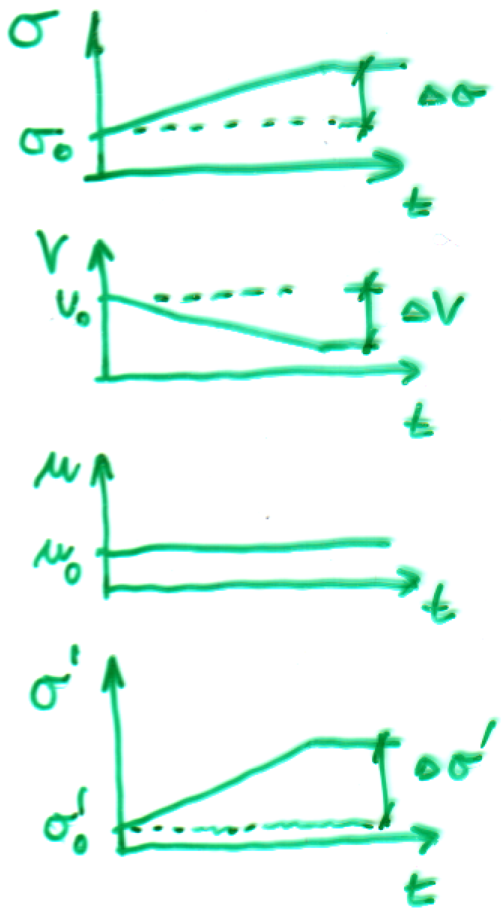
(Effective stress is less than the average stress between grains.)

→ MOHR CIRCLES FOR TOTAL AND EFFECTIVE STRESSES

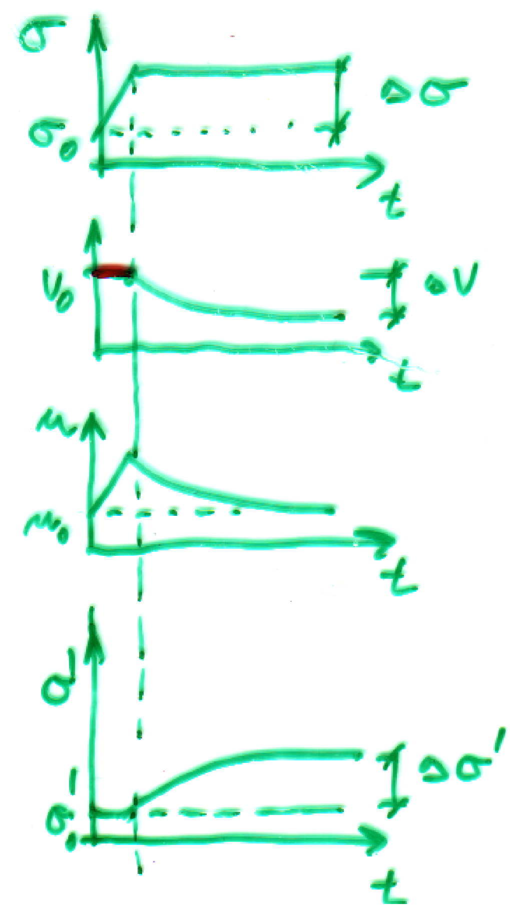


[1]

DRAINED LOADING



UNDRAINED LOADING + CONSOLIDATION



1. Relation between volumetric and normal strain:

initial state / dimensions: index 0

final state: index f

volumetric strain: $\varepsilon_V = -\Delta dV/dV_0 = -(dV_f - dV_0)/dV_0$

normal strain: $\varepsilon_x = -\Delta dx/dx_0 = -(dx_f - dx_0)/dx_0 \rightarrow dx_f = (1-\varepsilon_x)dx_0$

$$\varepsilon_V = -((1-\varepsilon_x)dx_0(1-\varepsilon_y)dy_0(1-\varepsilon_z)dz_0 - dx_0dy_0dz_0)/(dx_0dy_0dz_0)$$
$$= -(1-\varepsilon_x)(1-\varepsilon_y)(1-\varepsilon_z) + 1$$
$$= -1+\varepsilon_x+\varepsilon_y+\varepsilon_z+1 + \text{multiples of a higher order....}$$

....with small ε, the multiples can be neglected:

$$\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

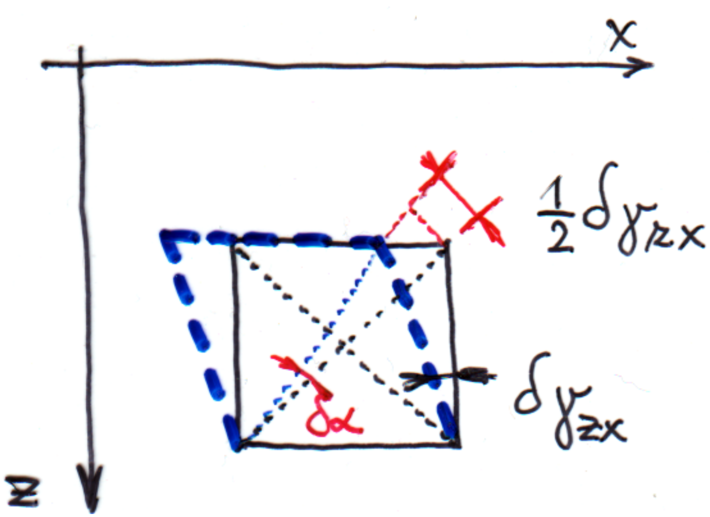
For small strains volumetric strain is a sum of normal strains

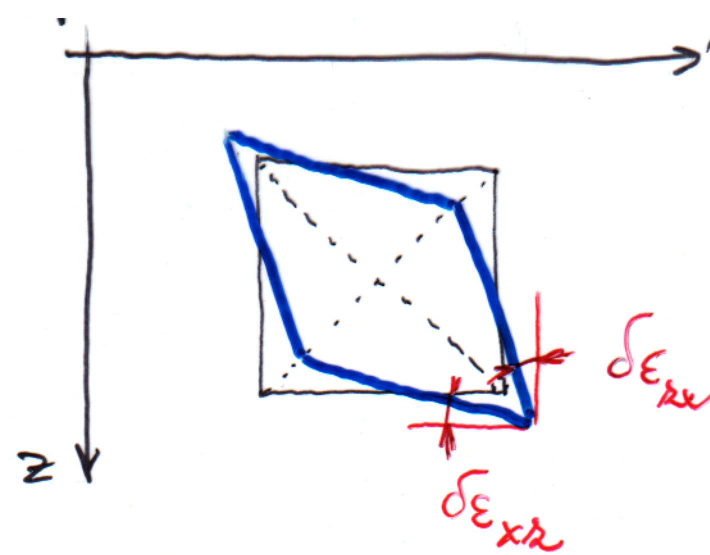
Analysis of strain in 2D

→ Mohr's circle of strain

In comparison with stress:

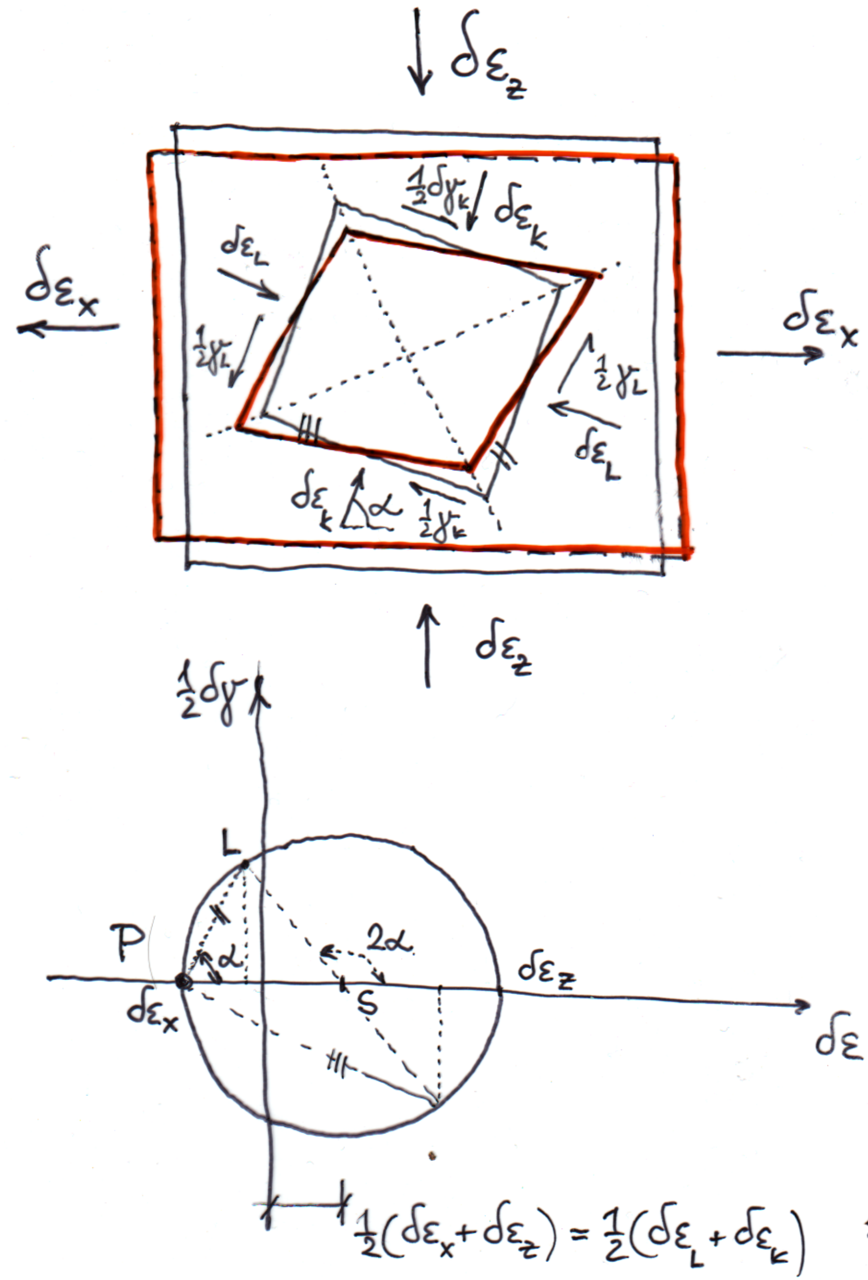
1. an initial value of strain - zero - does not exist → increments must be considered
2. normal strain typically exhibit both positive and negative values (opposite signs) during the loading event
3. for mathematical expressions engineering definition of shear strain (change of right angles) is not sufficient (as it consists of both change in shape and movement of the body) $\delta\varepsilon_{xz} = \delta\varepsilon_{zx} = \frac{1}{2}\gamma_{zx}$





Analysis of strain in 2D

$\delta\varepsilon_{xz} = \delta\varepsilon_{zx} = ½\, \gamma_{zx}$



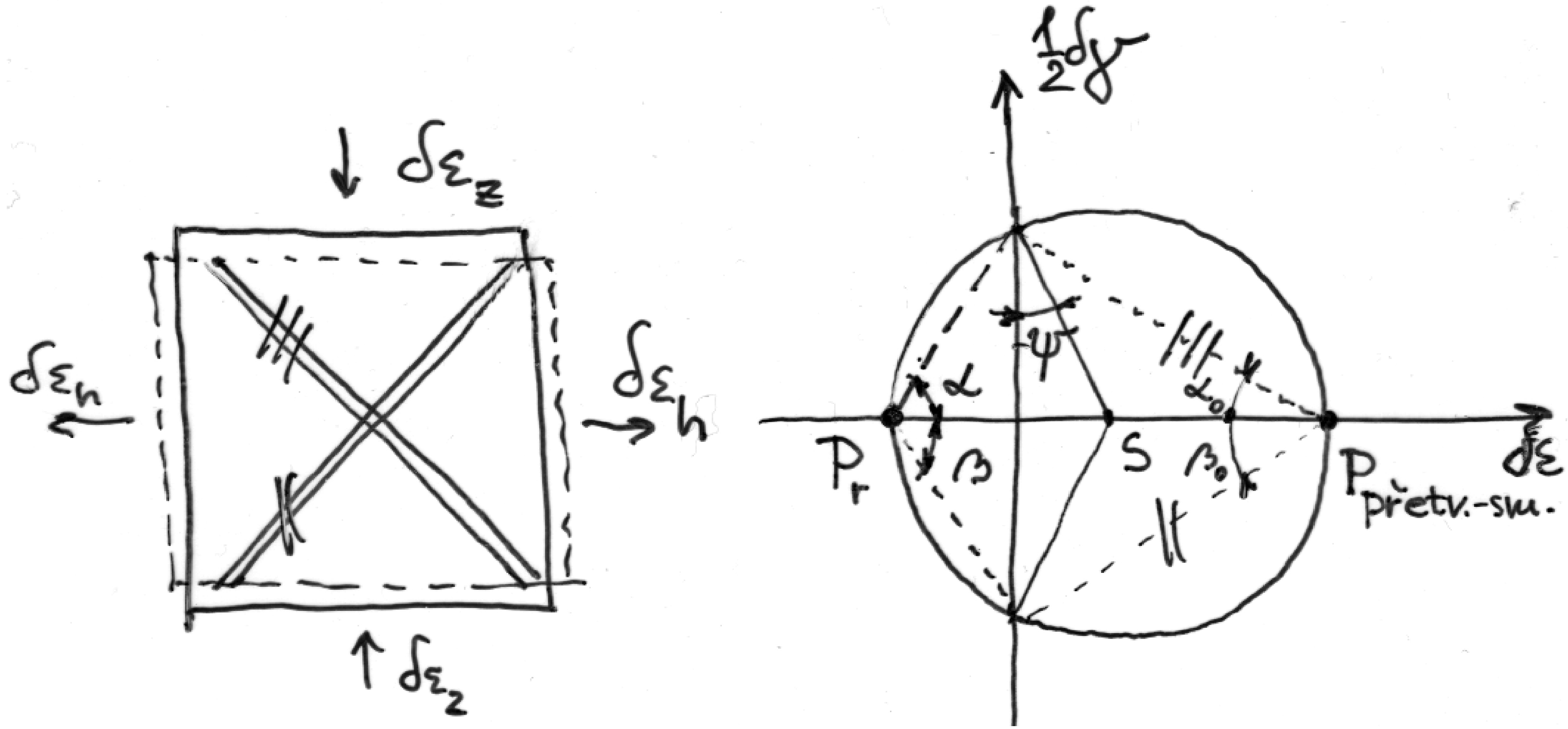
From M.C. od strain follows:

1. $\delta\varepsilon_V = 2 \times OS$

2. two planes exist with δε = 0, only shear strains act ≡ shear surfaces „planes of zero extension“

Analysis of strain in 2D

planes of zero extension, slip planes, angle of dilation

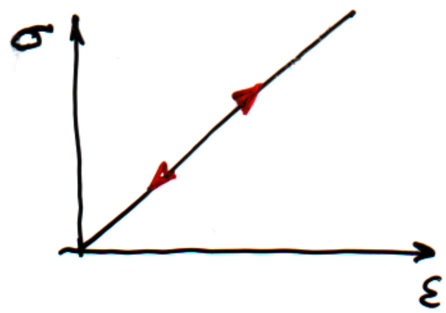


$\sin \psi = -(\delta\varepsilon_z + \delta\varepsilon_h) / (\delta\varepsilon_z - \delta\varepsilon_h)$

$\tan \psi = -\delta\varepsilon_V / \delta\gamma$

direction of zero extension: $-\psi + 2\alpha_0 = 90^o \rightarrow \alpha_0 = \beta_0 = 45^o + ½\psi$

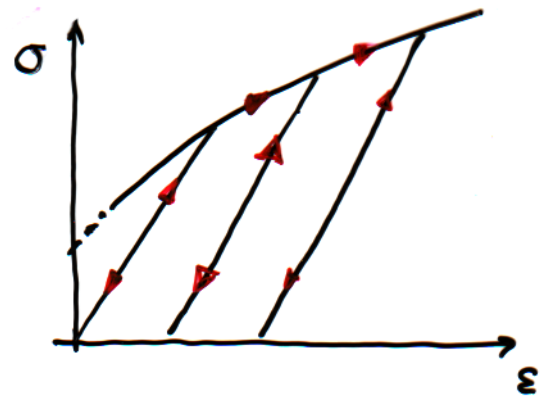
ELASTICITY



reversible strains
non / linear elasticity

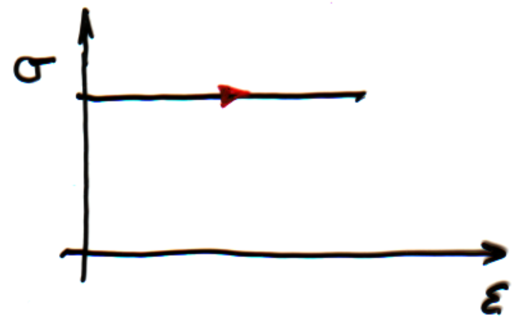
PLASTICITY

ELASTOPLASTICITY

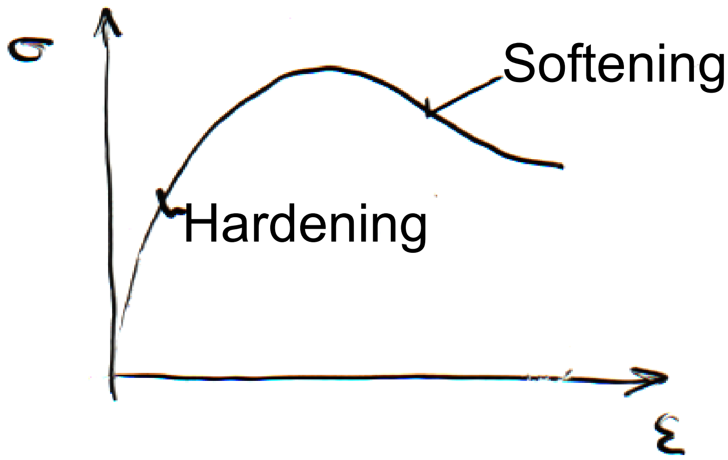


yielding
irrecoverable strains
(plastic)

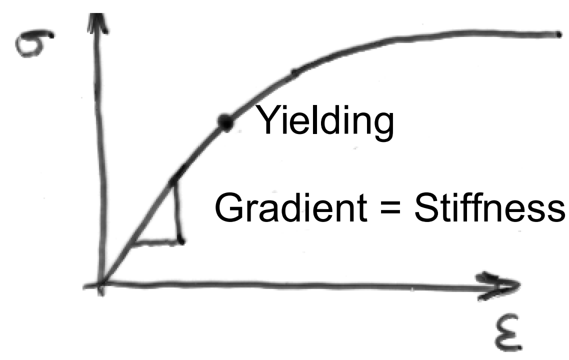
IDEAL PLASTICITY

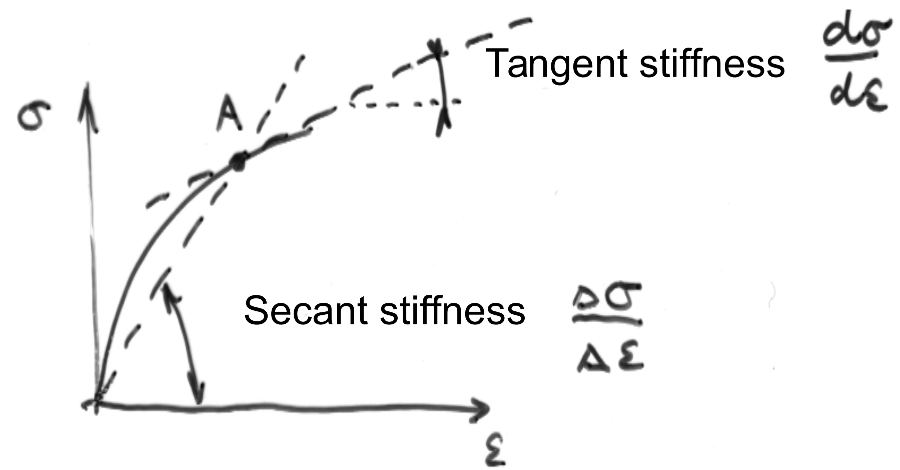


HARDENING - SOFTENING



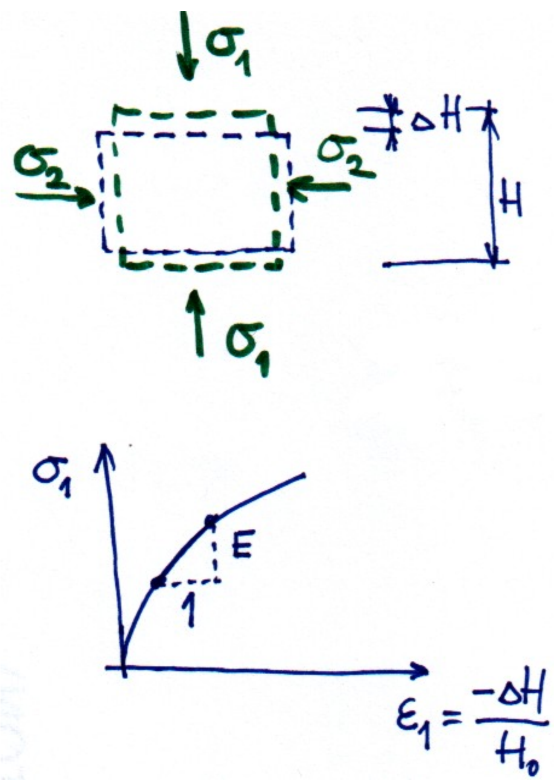
STIFFNESS (Moduli)



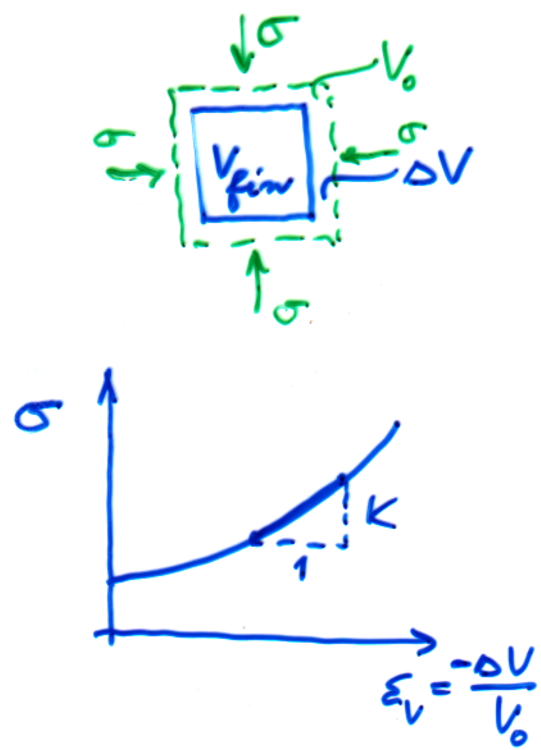


STIFFNESS

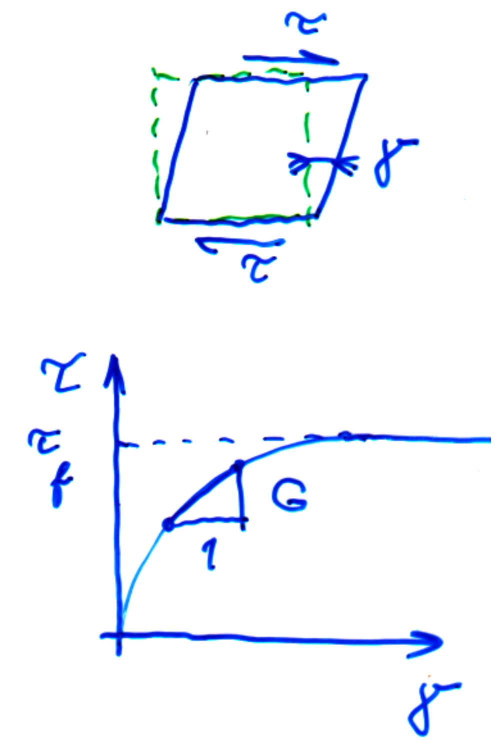
Young modulus
$\sigma_2 = \sigma_3 = \text{const}$



bulk modulus
$\sigma_1 = \sigma_2 = \sigma_3 \; (= \sigma = p)$



shear modulus



Poisson's ratio

Strains at one-dimensional increase of stress:

Poisson's ratio: $-\nu = \varepsilon_h / \varepsilon_v$ $(\equiv -\mu)$

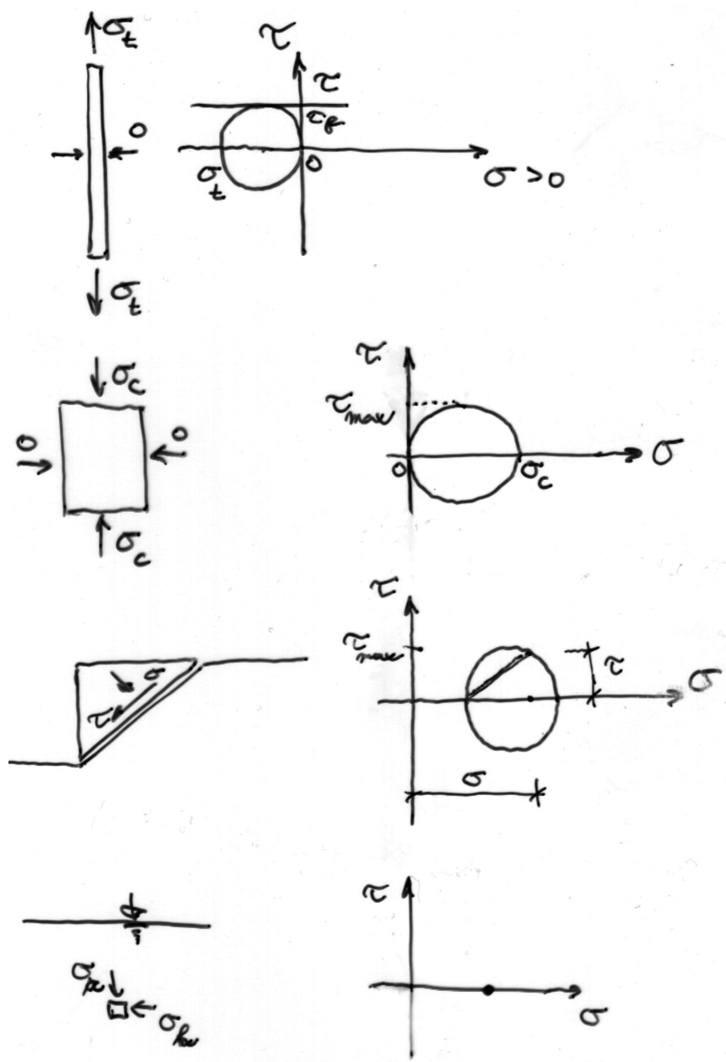
Poisson's constant: $m = 1 / \nu$

Incompressible material, e.g. $\Delta\sigma_x \neq 0$:

$$\varepsilon_V = 0$$
$$\varepsilon_V = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_x (1 - 2\nu) = 0$$
$$\nu = 0{,}5$$

→ saturated soil at undrained loading: $\nu = 0{,}5$

Strength



„in tension“

„compressive“

„in shear“

strength of water......?

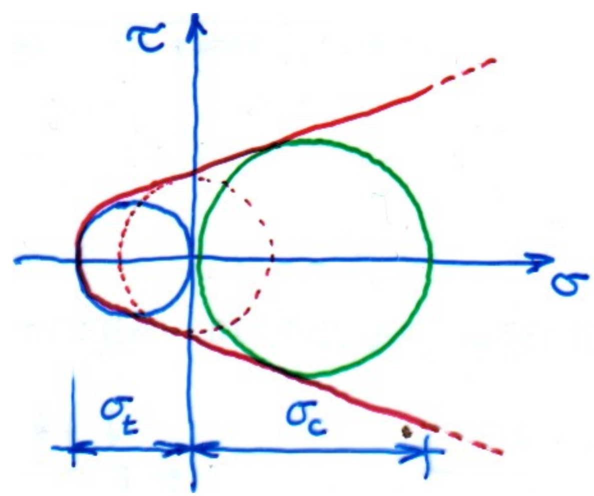
...strength is the largest Mohr Circle

STRENGTH

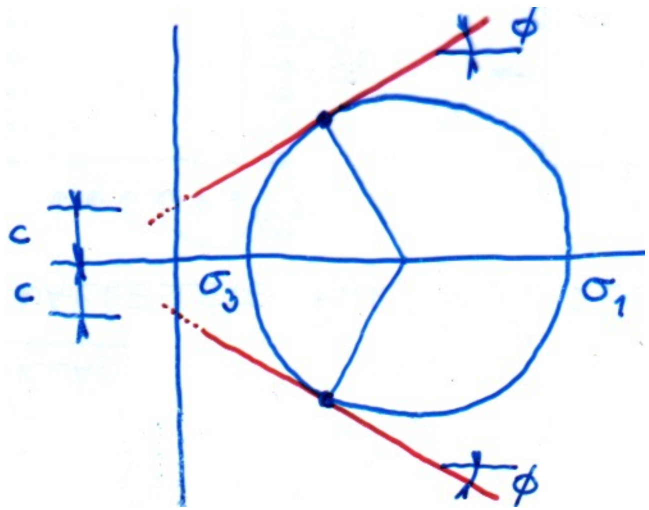
Coulomb (1776): S = c A + 1/n N (S = shear force at failure); c = cohesion; A area; N= normal force; 1/n = friction coefficient);
i.e. failure due to reaching limiting shearing stress
Present formulation: $\tau_{max} = c + \sigma \, tg\varphi$

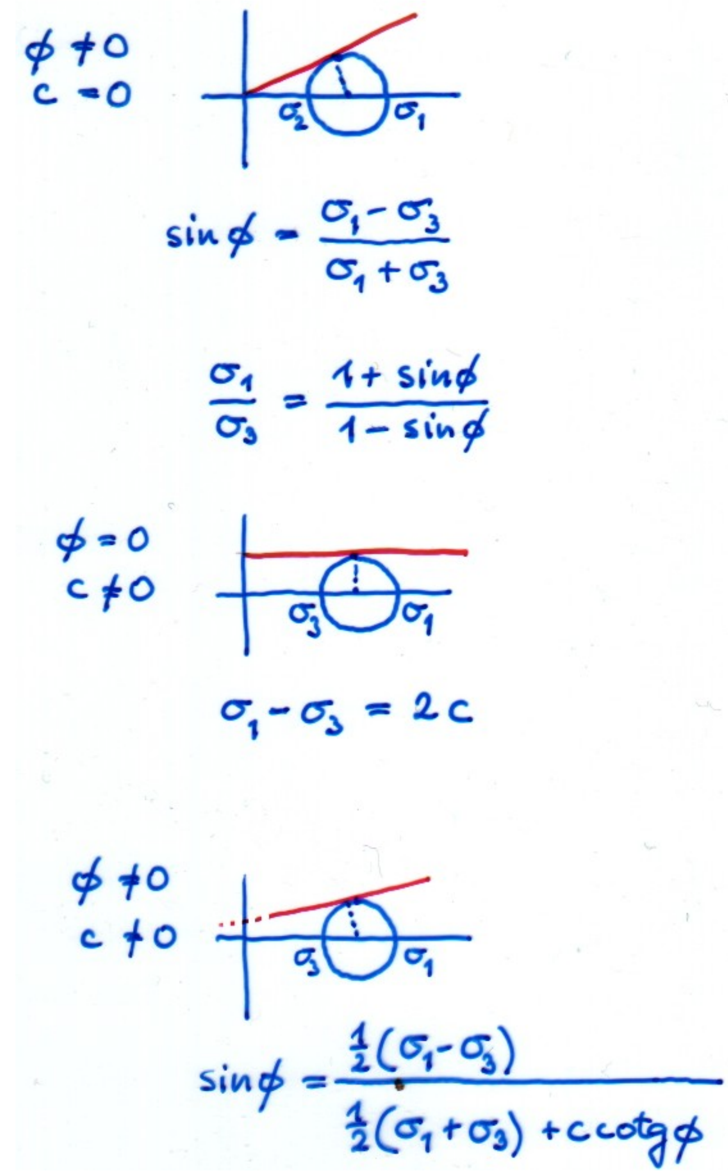
(Saint Vénant's failure criterion: failure at $\varepsilon \geq \varepsilon_{max}$)

Mohr suggested the criterion of τ_{max} - maximum stress envelope combined with Coulomb's criterion



STRENGTH - MOHR-COULOMB failure criterion





$\tau_{max} = c + \sigma\ tg\varphi$

effective stress: $\tau_{max} = c' + \sigma'\ tg\varphi'$

Soil description, state, classification ..... the procedures have been explained

For mechanical parameters → Field and laboratory tests

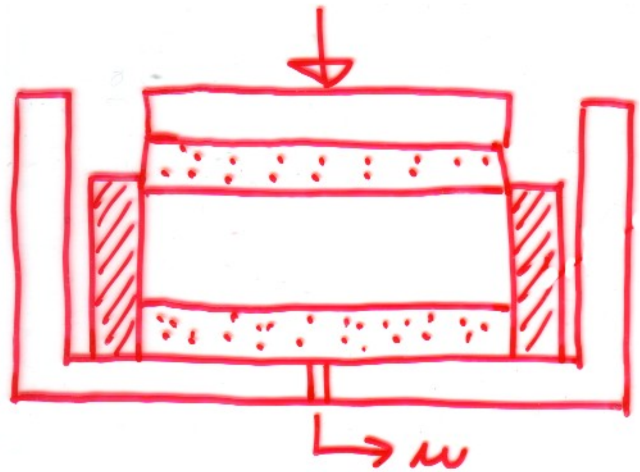
Requirements:

- measurement and controlling of total and pore pressures (→ σ')
- control of drainage (drained vs. undrained event)
- range of values - accuracy: strength – large strains vs. stiffness – small strains
- determination of Mohr circle (stress known) for interpretation

Field tests – σ' and interpretation is a problem

Lab - specimen is a problem

One-dimensional compressibility – oedometer

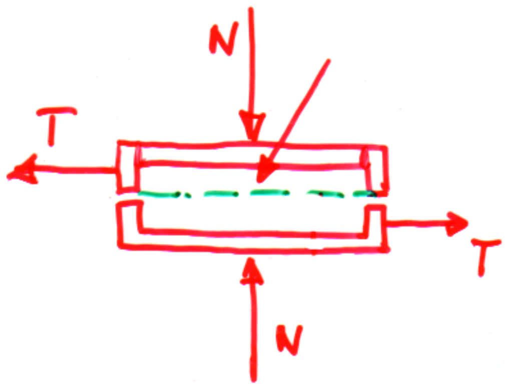


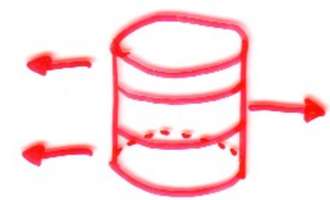
Standard procedure:

- undrained loading in steps
- waiting for pore pressure dissipation → effective stress known → one point of the compressibility curve

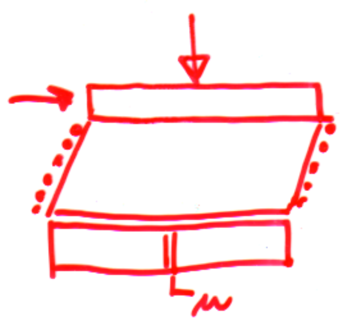
Determining mechanical parameters in SM

Strength – shear box – different modifications – always direct measurement of shear force





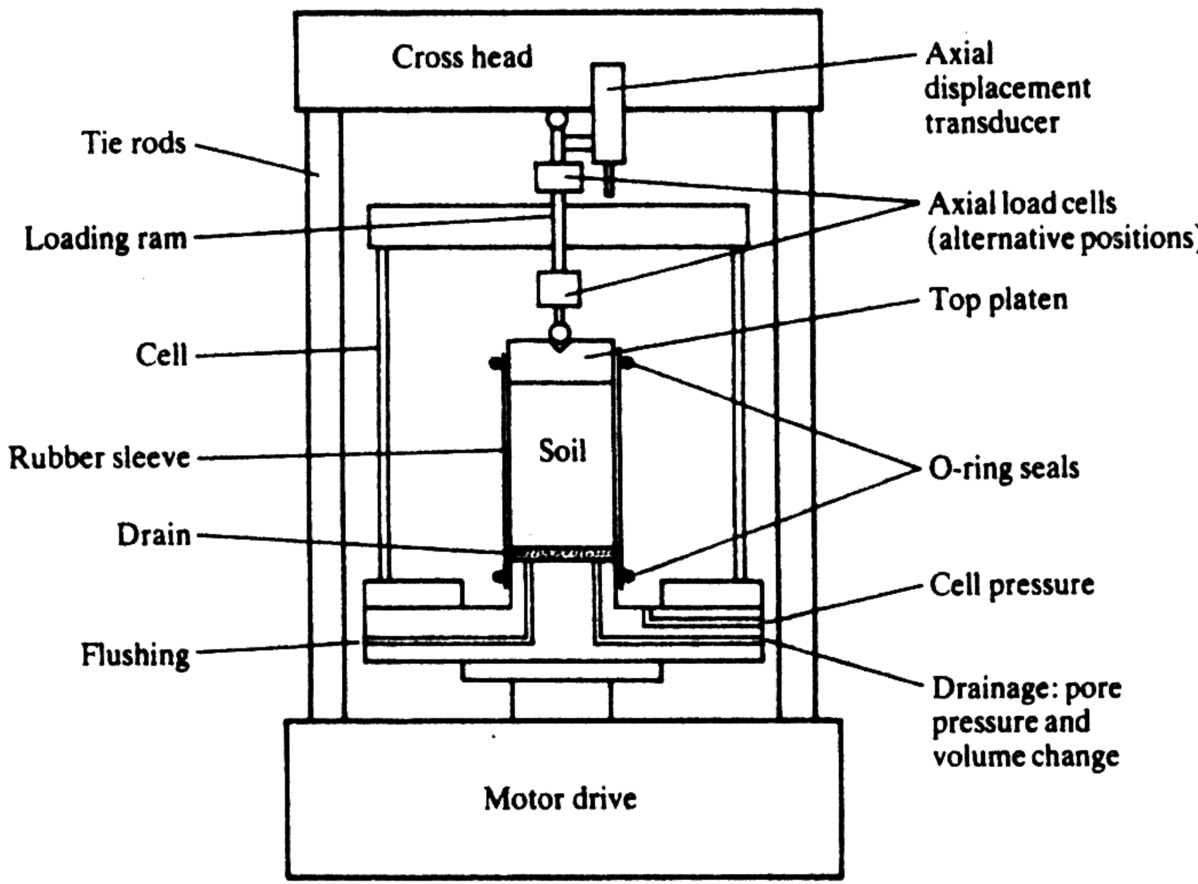
translation



simple shear

ring shear (rotation, torsion)

Strength and stiffness – triaxial apparatus



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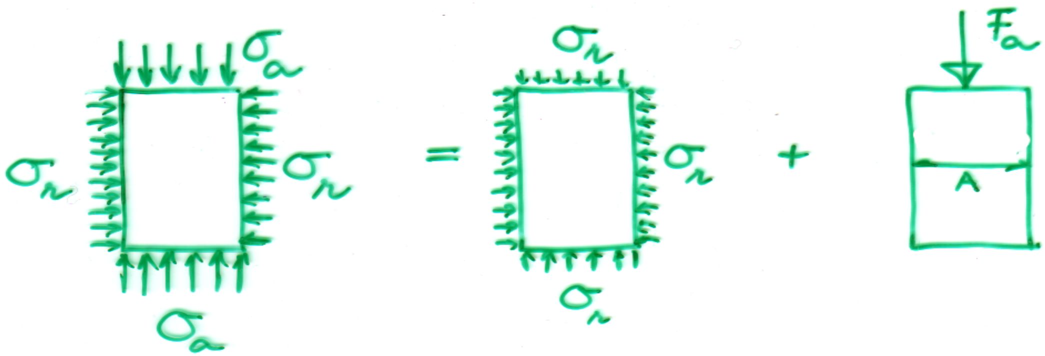
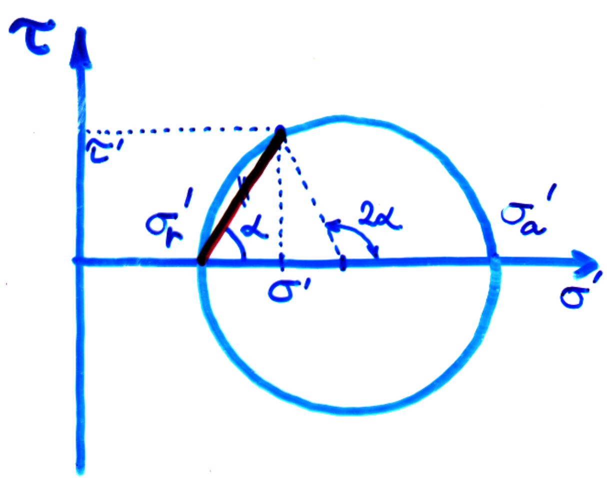
Strength and stiffness – triaxial apparatus

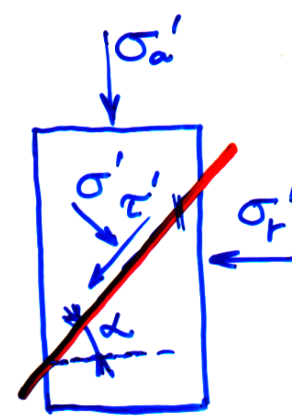
Standard „compression“ triaxial test:



$\sigma_a = \sigma_r + F_a / A$

$F_a / A = \sigma_a - \sigma_r = \sigma_a' – \sigma_r' = q$
(deviatoric stress)





Invariants for stress and strain in soil mechanics

$p = 1/3(\sigma_a + 2\sigma_r)$ $\quad$ $p' = 1/3(\sigma_a' + 2\sigma_r') = p - u$

$q = \sigma_a - \sigma_r$ $\quad$ $q' \equiv q$

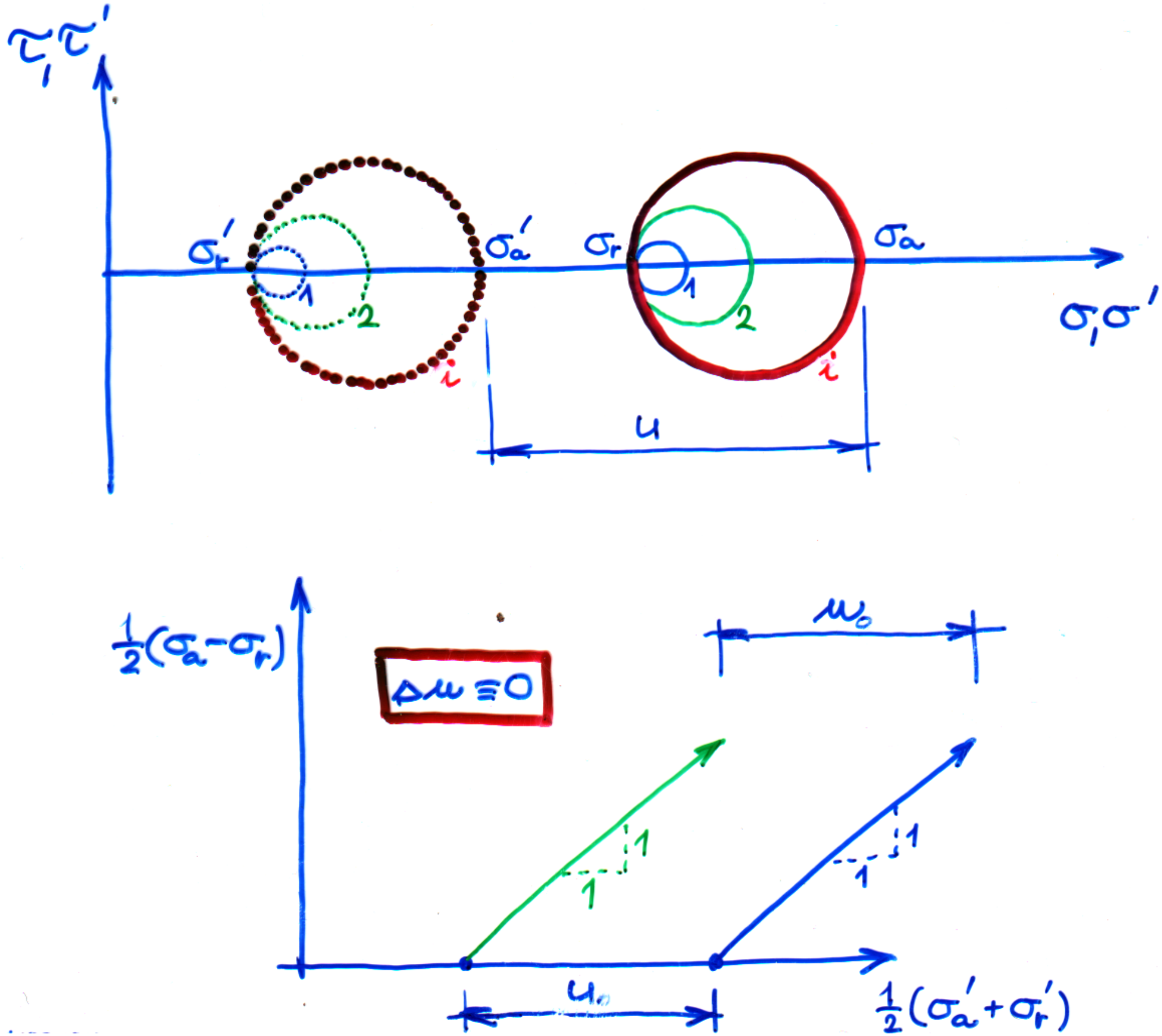
$\varepsilon_v = \varepsilon_a + 2\varepsilon_r$

$\varepsilon_s = 2/3(\varepsilon_a - \varepsilon_r)$

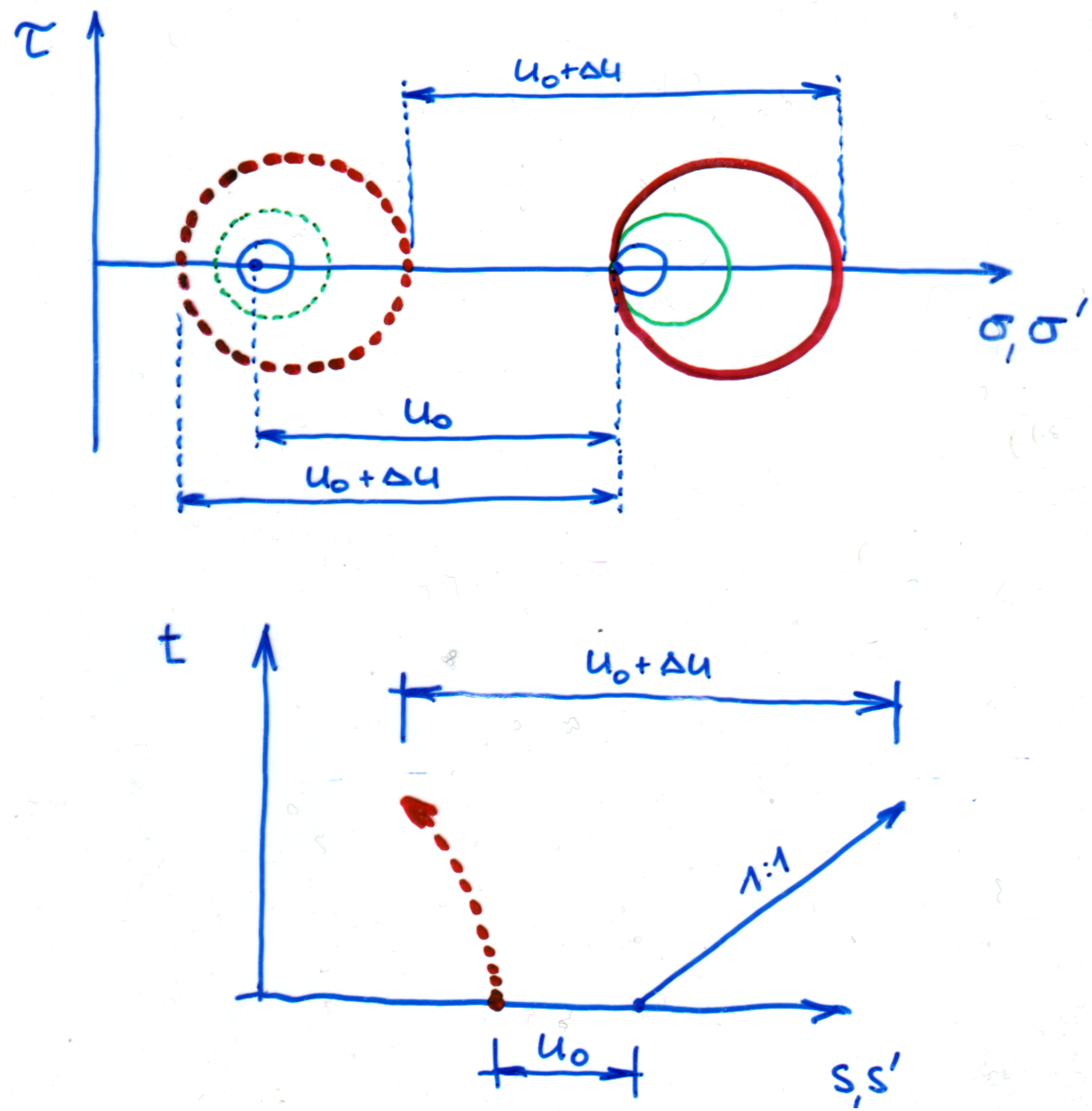
$s = 1/2(\sigma_a + \sigma_r)$ $\quad$ $s' = 1/2(\sigma_a' + \sigma_r') = s - u$

$t = 1/2(\sigma_a - \sigma_r)$ $\quad$ $t' \equiv t$

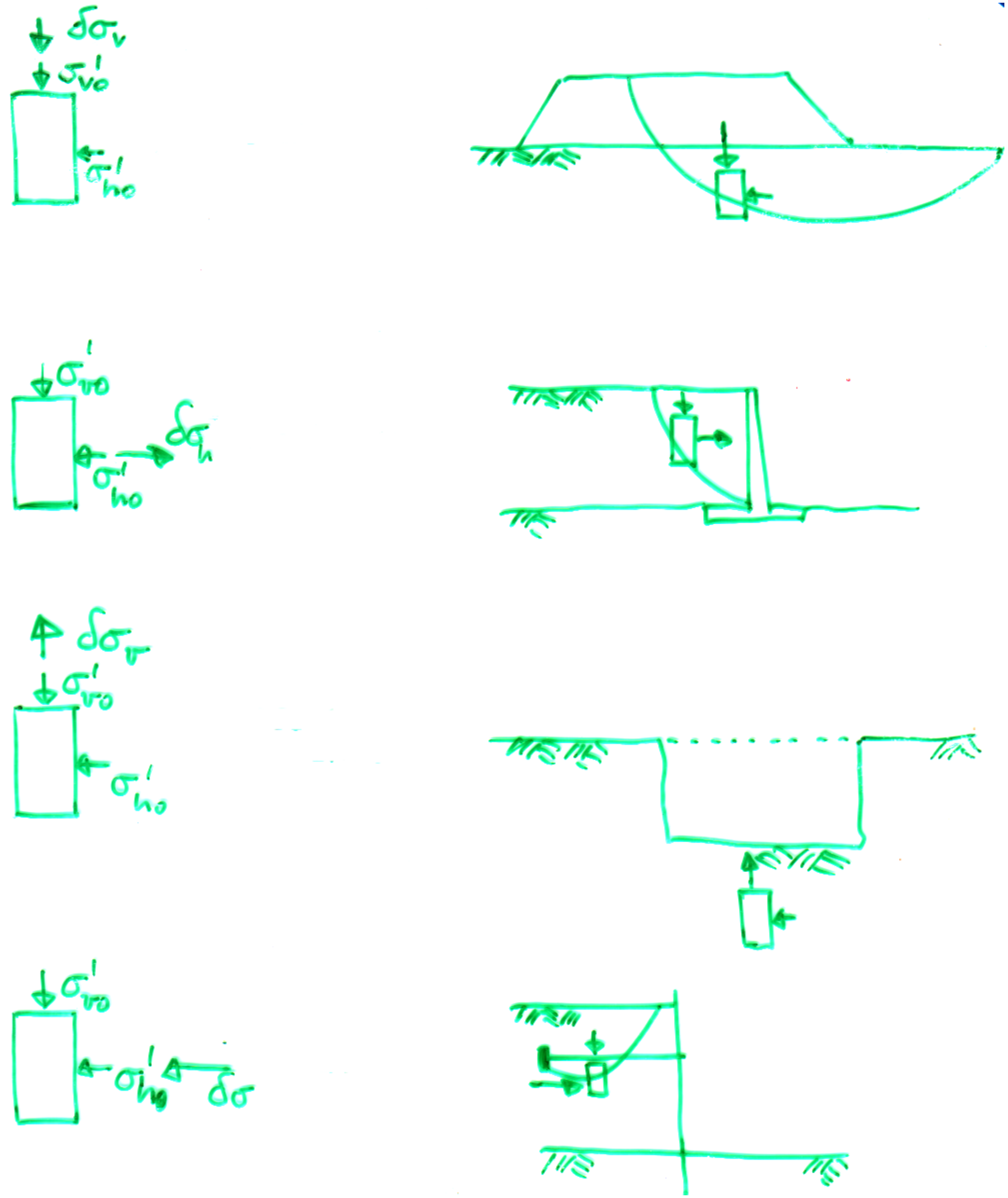
Drained standard triaxial test: Mohr circle + stress path



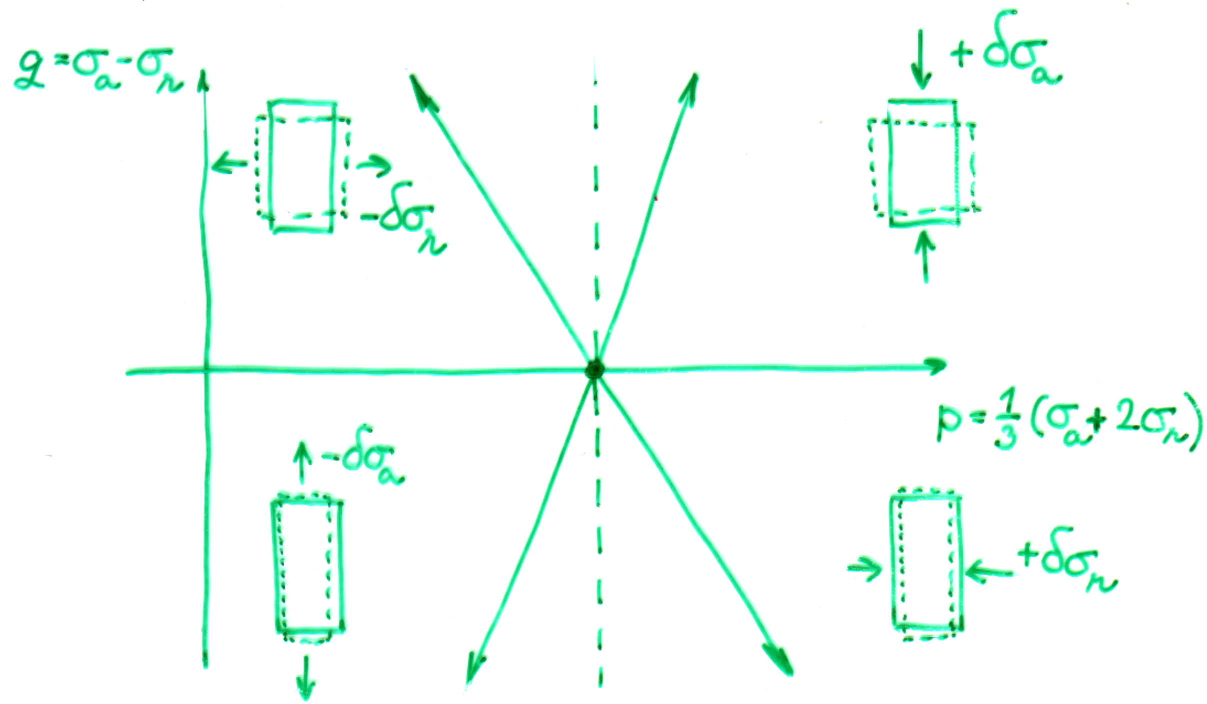
Undrained standard triaxial test: Mohr circle + stress path

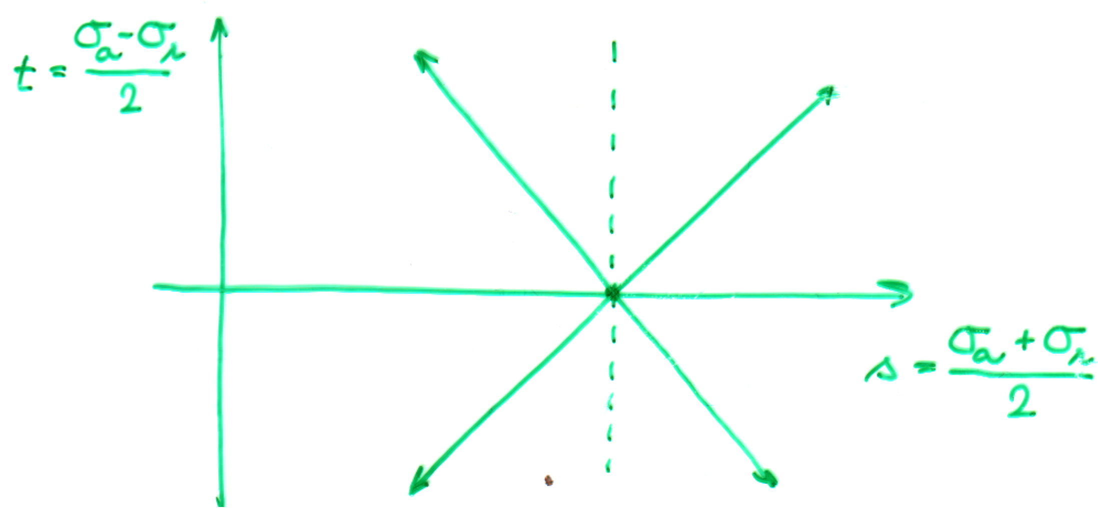


Stress paths in situ



Stress paths in situ





Literature for the course in Soil Mechanics

http://labmz1.natur.cuni.cz/~bhc/s/sm1/

Atkinson, J.H. (2007) The mechanics of soils and foundations. 2nd ed. Taylor & Francis.

Further reading:

Wood, D.M. (1990) Soil behaviour and critical state soil mechanics. Cambridge Univ.Press.

Mitchell, J.K. and Soga, K (2005) Fundamentals of soil behaviour. J Wiley.

Atkinson, J.H: and Bransby, P.L. (1978) The mechanics of soils. McGraw-Hill, ISBN 0-07-084077-2.

Bolton, M. (1979) A guide to soil mechanics. Macmillan Press, ISBN 0-33318931-0.

Craig, R.F. (2004) Soil mechanics. Spon Press.

Holtz, R.D. and Kovacs, E.D. (1981) An introduction to geotechnical engineering, Prentice-Hall, ISBN 0-13-484394-0

Feda, J. (1982) Mechanics of particulate materials, Academia-Elsevier.)

[1] Atkinson, J.H. (2007) The mechanics of soils and foundations. 2nd ed. Taylor & Francis.

[2] Parry, R.H.G. (1995) Mohr circles, stress paths and geotechnics. Spon, ISBN 0419192905.

[3] Hudson, J.A. and Harrison, J.P. (1997) Engineering rock mechanics. An introduction to the principles, Pergamon.

[4] Simons, N. et al. (2001) Soil and rock slope engineering. Thomas Telford, ISBN 0727728717.