SOLID MECHANICS

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INTRODUCTION

The application of the principles of mechanics to bulk matter is conventionally divided into the *mechanics of fluids* and the *mechanics of solids*. The entire subject is often called *continuum mechanics*, particularly when we adopt the useful model of matter as being continuously divisible, making no reference to its discrete structure at microscopic length scales well below those of the application or phenomenon of interest. *Solid mechanics* is concerned with the stressing, deformation and failure of solid materials and structures. What, then, is a solid? Any material, fluid or solid, can support *normal* forces. These are forces directed perpendicular, or normal, to a material plane across which they act. The force per unit of area of that plane is called the *normal stress*. Water at the base of a pond, air in an automobile tire, the stones of a Roman arch, rocks at base of a mountain, the skin of a pressurized airplane cabin, a stretched rubber band and the bones of a runner all support force in that way (some only when the force is compressive). We call a material *solid* rather than *fluid* if it can also support a substantial *shearing force* over the time scale of some natural process or technological application of interest. Shearing forces are directed parallel, rather than perpendicular, to the material surface on which they act; the force per unit of area is called *shear stress*. For example, consider a

vertical metal rod that is fixed to a support at its upper end and has a weight attached at its lower end. If we consider a horizontal surface through the material of the rod, it will be evident that the rod supports normal stress. But it also supports shear stress, and that becomes evident when we consider the forces carried across a plane through the rod that is neither horizontal nor vertical. Thus, while water and air provide no long term support of shear stress, normally granite, steel, and rubber do so, and are called solids. Materials with tightly bound atoms or molecules, like the crystals formed below melting temperature by most substances or simple compounds, or the amorphous structures formed in glass and many polymer substances at sufficiently low temperature, are usually considered solids.

The distinction between solids and fluids is not precise and in many cases will depend on the time scale. Consider the hot rocks of the Earth's mantle. When a large earthquake occurs, an associated deformation disturbance called a seismic wave propagates through the adjacent rock and the whole earth is set into vibrations which, following a sufficiently large earthquake, may remain detectable with precise instruments for several weeks. We would then describe the rocks of the mantle as solid. So would we on the time scale of, say, tens to thousands of years, over which stresses rebuild enough in the source region to cause one or a few repetitions of the earthquake. But on a significantly longer time scale, say of order of a million years, the hot rocks of the mantle are unable to support shearing stresses and flow as a fluid. Also, many children will be familiar with a substance called *silly putty*, a polymerized silicone gel. If a ball of it is left to sit on a table at room temperature, it flows and flattens on a time scale of a few minutes to an hour. But if picked up and tossed as a ball against a wall, so that large forces act only over the short time of the impact, it bounces back and retains its shape like a highly elastic solid.

In the simple but very common case when such a material is loaded at sufficiently low temperature and/or short time scale, and with sufficiently limited stress magnitude, its deformation is fully recovered upon unloading. We then say that the material is *elastic*. But substances can also deform permanently, so that not all deformation is recovered. For example, if you bend a metal coat hanger substantially and then release the loading, it springs back only partially towards its initial shape, but does not fully recover and remains bent. We say that the metal of the coat hanger has been permanently deformed and in this case, for which the permanent deformation is not so much a consequence of long time loading at sufficiently high temperature, but more a consequence of subjecting the material to large stresses (above the yield stress), we describe the permanent deformation as *plastic* deformation, and call the material

elastic-plastic. Permanent deformation of a sort that depends mainly on time of exposure to a stress, and that tends to increase significantly with time of exposure, is called *viscous* or *creep* deformation and materials which exhibit that, as well as tendencies for elastic response, are called *viscoelastic* solids (or sometimes *visco-plastic* solids when we focus more on the permanent strain than on the tendency for partial recovery of strain upon unloading).

Who uses solid mechanics? All those who seek to understand natural phenomena involving the stressing, deformation, flow and fracture of solids, and all those who would have knowledge of such phenomena to improve our living conditions and accomplish human objectives, have use for solid mechanics. The latter activities are, of course, the domain of engineering and many important modern sub fields of solid mechanics have been actively developed by engineering scientists concerned, for example, with mechanical, structural, materials, civil or aerospace engineering. Natural phenomena involving solid mechanics are studied in geology, seismology and tectonophysics, in materials science and the physics of condensed matter, and in parts of biology and physiology. Further, because solid mechanics poses challenging mathematical and computational problems, it (as well as fluid mechanics) has long been an important topic for applied mathematicians concerned, for example, with partial differential equations and with numerical techniques for digital computer formulations of physical problems.

Here is a sampling of some of the issues addressed using solid mechanics concepts: How do flows develop in the earth's mantle and cause continents to move and ocean floors to slowly subduct beneath them? How do mountains form? What processes take place along a fault during an earthquake, and how do the resulting disturbances propagate through the earth as seismic waves, and shake, and perhaps collapse, buildings and bridges? How do landslides occur? How does a structure on a clay soil settle with time, and what is the maximum bearing pressure which the footing of a building can exert on a soil or rock foundation without rupturing it? What materials do we choose, and how do we proportion and shape them and control their loading, to make safe, reliable, durable and economical structures, whether airframes, bridges, ships, buildings, chairs, artificial heart valves, or computer chips, and to make machinery such as jet engines, pumps, bicycles, and the like? How do vehicles (cars, planes, ships) respond by vibration to the irregularity of surfaces or media along which they move, and how are vibrations controlled for comfort, noise reduction and safety against fatigue failure? How rapidly does a crack grow in a cyclically loaded structure, whether a bridge, engine, or airplane wing or fuselage, and when will it propagate catastrophically? How do we control the deformability of structures during impact so as to design crash worthiness into vehicles? How do we form the

materials and products of our technological civilization, e.g., by extruding metals or polymers through dies, rolling material into sheets, punching out complex shapes, etc.? By what microscopic processes do plastic and creep strains occur in polycrystals? How can we fashion different materials together, like in fiber reinforced composites, to achieve combinations of stiffness and strength needed in applications? What is the combination of material properties and overall response needed in downhill skis or in a tennis racket? How does the human skull respond to impact in an accident? How do heart muscles control the pumping of blood in the human body, and what goes wrong when an aneurysm develops?

HISTORICAL SKETCH

Solid mechanics developed in the outpouring of mathematical and physical studies following the great achievement of Isaac Newton (1642-1727) in stating the laws of motion, although it has earlier roots. The need to understand and control the fracture of solids seems to have been a first motivation. Leonardo da Vinci (1452-1519) sketched in his notebooks a possible test of the tensile strength of a wire. The Italian experimental scientist Galileo Galilei (1564-1642), who died in the year of Newton's birth, had investigated the breaking loads of rods in tension and concluded that the load was independent of length and proportional to the cross section area, this being a first step towards a concept of stress. He also investigated how the breaking of heavy stone columns, laid horizontally in storage as beams, depended on the number and condition of their supports.

Concepts of stress, strain and elasticity. The English scientist Robert Hooke discovered in 1660, but published only in 1678, the observation that for many materials that displacement under a load was proportional to force, thus establishing the notion of (linear) elasticity but not yet in a way that was expressible in terms of stress and strain. E. Mariotte in France published similar discoveries in 1680 and, also, reached an understanding of how beams like those studied by Galileo resisted transverse loadings or, more precisely, resist the torques caused by those transverse loadings, by developing extensional and compressional deformations, respectively, in material fibers along their upper and lower portions. It was for Swiss mathematician and mechanician James Bernoulli (1654-1705) to observe, in the final paper of his life, in 1705, that the proper way of describing deformation was to give force per unit area, or stress, as a function of the elongation per unit length, or strain, of a material fiber under tension. Swiss mathematician and mechanician Leonhard Euler (1707-1783), who was taught mathematics by

James' brother John Bernoulli (1667-1748), proposed, among many contributions, a linear relation between stress σ and strain ε in 1727, of form $\sigma = E\,\varepsilon$ where the coefficient E is now generally called Young's modulus after English naturalist Thomas Young who developed a related idea in 1807.

The notion that there is an internal tension acting across surfaces in a deformed solid was expressed by German mathematician and physicist Gottfried Wilhelm Leibniz in 1684 and James Bernoulli in 1691. Also, Bernoulli and Euler (see below) introduced the idea that at a given section along the length of a beam there were internal tensions amounting to a net force and a net torque. Euler introduced the idea of compressive normal stress as the pressure in a fluid in 1752. The French engineer and physicist Charles-Augustine Coulomb (1736-1806) was apparently the first to relate the theory of a beam as a bent elastic line to stress and strain in an actual beam, in a way never quite achieved by Bernoulli and, although possibly recognized, never published by Euler. He developed the famous expression $\sigma = M\,y / I$ for the stress due to the pure bending of a homogeneous linear elastic beam; here M is the torque, or *bending moment*, y is the distance of a point from an axis that passes through the section centroid, parallel to the torque axis, and I is the integral of y^2 over the section area. The French mathematician Parent introduced the concept of shear stress in 1713, but Coulomb was the one who extensively developed the idea in connection with beams and with the stressing and failure of soil in 1773, and studies of frictional slip in 1779. It was the great French mathematician Augustin Louis Cauchy (1789-1857), originally educated as an engineer, who in 1822 formalized the stress concept in the context of a general three-dimensional theory, showed its properties as consisting of a 3 by 3 symmetric array of numbers that transform as a tensor, derived the equations of motion for a continuum in terms of the components of stress, and gave the specific development of the theory of linear elastic response for isotropic solids. As part of this work, Cauchy also introduced the equations which express the six components of strain, three extensional and three shear, in terms of derivatives of displacements for the case when all those derivatives are much smaller than unity; similar expressions had been given earlier by Euler in expressing rates of straining in terms of the derivatives of the velocity field in a fluid.

Beams, columns, plates, shells. The 1700's and early 1800's were a productive period in which the mechanics of simple elastic structural elements were developed well before the beginnings in the 1820's of the general three-dimensional theory. The development of beam theory by Euler, who generally modeled beams as elastic lines which resist bending, and by several members of the Bernoulli family and by Coulomb, remains among the most immediately

useful aspects of solid mechanics, in part for its simplicity and in part because of the pervasiveness of beams and columns in structural technology. James Bernoulli proposed in his final paper of 1705 that the curvature of a beam was proportional to bending moment. Euler in 1744 and John's son, Daniel Bernoulli (1700-1782) in 1751 used the theory to address the transverse vibrations of beams, and Euler gave in 1757 his famous analysis of the buckling of an initially straight beam subjected to a compressive loading; the beam is then commonly called a column. Following a suggestion of Daniel Bernoulli in 1742, Euler in 1744 introduced the strain energy per unit length for a beam, proportional to the square of its curvature, and regarded the total strain energy as the quantity analogous to the potential energy of a discrete mechanical system. By adopting procedures that were becoming familiar in analytical mechanics, and following from the principle of virtual work as introduced by John Bernoulli for discrete systems such as pin-connected rigid bodies in 1717, Euler rendered the energy stationary and in this way developed the *calculus of variations* as an approach to the equations of equilibrium and motion of elastic structures.

That same variational approach played a major role in the development by French mathematicians in the early 1800's of a theory of small transverse displacements and vibrations of elastic plates. This theory was developed in preliminary form by Sophie Germain and partly improved upon by Simeon Denis Poisson in the early 1810's; they considered a flat plate as an elastic plane which resists curvature. Navier gave a definitive development of the correct energy expression and governing differential equation a few years later. An uncertainty of some duration in the theory arose from the fact that the final partial differential equation for the transverse displacement is such that it is impossible to prescribe, simultaneously, along an unsupported edge of the plate, both the twisting moment per unit length of middle surface and the transverse shear force per unit length. This was finally resolved in 1850 by German physicist Gustav Robert Kirchhoff in an application of virtual work and variational calculus procedures, in the framework of simplifying kinematic assumptions that fibers initially perpendicular to the plate middle surface remain so after deformation of that surface. As first steps in the theory of thin shells, in the 1770's Euler addressed the deformation of an initially curved beam, as an elastic line, and provided a simplified analysis of vibration of an elastic bell as an array of annular beams. John's grandson, through a son and mathematician also named John, James Bernoulli "the younger" (1759-1789) further developed this model in the last year of his life as a two dimensional network of elastic lines, but could not develop an acceptable treatment. Shell theory was not to attract attention for a century after Euler's work, as the outcome of many researches following the first consideration of shells from a three-dimensional elastic viewpoint

by H. Aron in 1873. Acceptable thin-shell theories for general situations, appropriate for cases of small deformation, were developed by English mathematician, mechanician and geophysicist A. E. H. Love in 1888 and mathematician and physicist Horace Lamb in 1890 (there is no uniquely correct theory as the Dutch applied mechanician and engineer W. T. Koiter and Russian mechanician V. V. Novozhilov were to clarify in the 1950's; the difference between predictions of acceptable theories is small when the ratio of shell thickness to a typical length scale is small). Shell theory remained of immense interest well beyond the mid-1900's in part because so many problems lay beyond the linear theory (rather small transverse displacements often dramatically alter the way that a shell supports load by a combination of bending and membrane action), and in part because of the interest in such light-weight structural forms for aeronautical technology.

Elasticity, general theory. Linear elasticity as a general three-dimensional theory was in hand in the early 1820's based on Cauchy's work. Simultaneously, Navier had developed an elasticity theory based on a simple corpuscular, or particle, model of matter in which particles interacted with their neighbors by a central-force attractions between particle pairs. As was gradually realized following works by Navier, Cauchy and Poisson in the 1820's and 1830's, the particle model is too simple and makes predictions concerning relations among elastic moduli which are not met by experiment. In the isotropic case it predicts that there is only one elastic constant and that the Poisson ratio has the universal value of 1/4. Most subsequent development of the subject was in terms of the continuum theory. Controversies concerning the maximum possible number of independent elastic moduli in the most general anisotropic solid were settled by English mathematician George Green in 1837, through pointing out that the existence of an elastic strain energy required that of the 36 elastic constants, relating the six stress components to the six strains, at most 21 could be independent. Scottish physicist Lord Kelvin (William Thomson) put this consideration on sounder ground in 1855 as part of his development of macroscopic thermodynamics, in much the form as it is known today, showing that a strain energy function must exist for reversible isothermal or adiabatic (isentropic) response, and working out such results as the (very modest) temperature changes associated with isentropic elastic deformation.

The middle and late 1800's were a period in which many basic elastic solutions were derived and applied to technology and to the explanation of natural phenomena. French mathematician Barre de Saint-Venant derived in the 1850's solutions for the torsion of non-circular cylinders, which explained the necessity of warping displacement of the cross section in the direction parallel to the axis of twisting, and for flexure of beams due to transverse loadings; the latter

allowed understanding of approximations inherent in the simple beam theory of Bernoulli, Euler and Coulomb. The German physicist Heinrich Rudolph Hertz developed solutions for the deformation of elastic solids as they are brought into contact, and applied these to model details of impact collisions. Solutions for stress and displacement due to concentrated forces acting at an interior point of a full space were derived by Kelvin, and on the surface of a half space by mathematicians J. V. Bousinesq (French) and V. Cerruti (Italian). The Prussian mathematician L. Pochhammer analyzed the vibrations of an elastic cylinder and Lamb and the Prussian physicist P. Jaerisch derived the equations of general vibration of an elastic sphere in the 1880's, an effort that was continued by many seismologists in the 1900's to describe the vibrations of the Earth. Kelvin derived in 1863 the basic form of the solution of the static elasticity equations for a spherical solid, and these were applied in following years to such problems as deformation of the Earth due to rotation and to tidal forcing, and to effects of elastic deformability on the motions of the Earth's rotation axis.

The classical development of elasticity never fully confronted the problem of finite elastic straining, in which material fibers change their lengths by other than very small amounts. Possibly this was because the common materials of construction would remain elastic only for very small strains before exhibiting either plastic straining or brittle failure. However, natural polymeric materials show elasticity over a far wider range (usually also with enough time or rate effects that they would more accurately be characterized as viscoelastic), and the widespread use of natural rubber and like materials motivated the development of finite elasticity. While many roots of the subject were laid in the classical theory, especially in the work of Green, G. Piola and Kirchhoff in the mid-1800's, the development of a viable theory with forms of stress-strain relations for specific rubbery elastic materials, and an understanding of the physical effects of the nonlinearity in simple problems like torsion and bending, is mainly the achievement of British-American engineer and applied mathematician Ronald S. Rivlin in the 1940's and 1950's.

Waves. Poisson, Cauchy and George G. Stokes showed that the equations of the theory predicted the existence of two types of elastic deformation waves which could propagate through isotropic elastic solids. These are called *body waves*. In the faster type, called *longitudinal*, or *dilational*, or *irrotational* waves, the particle motion is in the same direction as that of wave propagation; in the slower, called *transverse*, or *shear*, or *rotational* waves, it is perpendicular to the propagation direction. No analog of the shear wave exists for propagation through a fluid medium, and that fact led seismologists in the early 1900's to understand that the Earth has a liquid core (at the center of which there is a solid inner core).

Lord Rayleigh (John Strutt) showed in 1887 that there is a wave type that could propagate along surfaces, such that the motion associated with the wave decayed exponentially with distance into the material from the surface. This type of *surface wave*, now called a Rayleigh wave, propagates typically at slightly more than 90% of the shear wave speed, and involves an elliptical path of particle motion that lies in planes parallel to that defined by the normal to the surface and the propagation direction. Another type of surface wave, with motion transverse to the propagation direction and parallel to the surface, was found by Love for solids in which a surface layer of material sits atop an elastically stiffer bulk solid; this defines the situation for the Earth's crust. The shaking in an earthquake is communicated first to distant places by body waves, but these spread out in three-dimensions and must diminish in their displacement amplitude as r^{-1}, where r is distance from the source, to conserve the energy propagated by the wave field. The surface waves spread out in only two dimensions and must, for the same reason, diminish only as fast as $r^{-1/2}$. Thus shaking in surface waves is normally the more sensed, and potentially damaging, effect at moderate to large distances from a crustal earthquake. Indeed, well before the theory of waves in solids was in hand, Thomas Young had suggested in his 1807 Lectures on Natural Philosophy that the shaking of an earthquake "is probably propagated through the earth in the same manner as noise is conveyed through air". (It had been suggested by American mathematician and astronomer Jonathan Winthrop, following his experience of the "Boston" earthquake of 1755, that the ground shaking was due to a disturbance propagated like sound through the air.)

With the development of ultrasonic transducers operated on piezoelectric principles, the measurement of the reflection and scattering of elastic waves has developed into an effective engineering technique for the non-destructive evaluation of materials for detection of potentially dangerous defects such as cracks. Also, very strong impacts, whether from meteorite collision, weaponry, or blasting and the like in technological endeavors, induce waves in which material response can be well outside the range of linear elasticity, involving any or all of finite elastic strain, plastic or viscoplastic response, and phase transformation. These are called *shock waves*; they can propagate much beyond the speed of linear elastic waves and are accompanied with significant heating.

Stress concentrations and fracture. In 1898 G. Kirsch derived the solution for the stress distribution around a circular hole in a much larger plate under remotely uniform tensile stress. The same solution can be adapted to the tunnel-like cylindrical cavity of circular section in a

bulk solid. His solution showed a significant concentration of stress at the boundary, by a factor of three when the remote stress was uniaxial tension. Then in 1907 the Russian mathematician G. Kolosov, and independently in 1914 the British engineer Charles Inglis, derived the analogous solution for stresses around an elliptical hole. Their solution showed that the concentration of stress could become far greater as the radius of curvature at an end of the hole becomes small compared to the overall length of the hole. These results provided the insight to sensitize engineers to the possibility of dangerous stress concentrations at, for example, sharp re-entrant corners, notches, cut-outs, keyways, screw threads, and the like in structures for which the nominal stresses were at otherwise safe levels. Such stress concentration sites are places from which a crack can nucleate.

The elliptical hole of Kolosov and Inglis defines a crack in the limit when one semi-axis goes to zero, and the Inglis solution was adopted in that way by British aeronautical engineer A. A. Griffith in 1921 to describe a crack in a brittle solid. In that work Griffith made his famous proposition that spontaneous crack growth would occur when the energy released from the elastic field just balanced the work required to separate surfaces in the solid. Following a hesitant beginning, in which Griffith's work was initially regarded as important only for very brittle solids such as glass, there developed, largely under the impetus of American engineer and physicist George R. Irwin, a major body of work on the mechanics of crack growth and fracture, including fracture by fatigue and stress corrosion cracking, starting in the late 1940's and continuing into the 1990's. This was driven initially by the cracking of American fleet of Liberty ships during the Second World War, by the failures of the British Comet airplane, and by a host of reliability and safety issues arising in aerospace and nuclear reactor technology. The new complexion of the subject extended beyond the Griffith energy theory and, in its simplest and most widely employed version in engineering practice, used Irwin's *stress intensity factor* as the basis for predicting crack growth response under service loadings in terms of laboratory data that is correlated in terms of that factor. That stress intensity factor is the coefficient of a characteristic singularity in the linear elastic solution for the stress field near a crack tip, and is recognized as providing a proper characterization of crack tip stressing in many cases, even though the linear elastic solution must be wrong in detail near the crack tip due to non-elastic material response, large strain, and discreteness of material microstructure.

Dislocations. The Italian elastician and mathematician V. Volterra introduced in 1905 the theory of the elastostatic stress and displacement fields created by dislocating solids. This involves making a cut in a solid, displacing its surfaces relative to one another by some fixed

amount, and joining the sides of the cut back together, filling in with material as necessary. The initial status of this work was simply as an interesting way of generating elastic fields but, in the early 1930's, Geoffrey Ingram Taylor, Egon Orowan and Michael Polanyi realized that just such a process could be going on in ductile crystals and could provide an explanation of the low plastic shear strength of typical ductile solids, much like Griffith's cracks explained low fracture strength under tension. In this case the displacement on the dislocated surface corresponds to one atomic lattice spacing in the crystal. It quickly became clear that this concept provided the correct microscopic description of metal plasticity and, starting with Taylor in the 1930's and continuing into the 1990's, the use of solid mechanics to explore dislocation interactions and the microscopic basis of plastic flow in crystalline materials has been a major topic, with many distinguished contributors.

The mathematical techniques advanced by Volterra are now in common use by Earth scientists in explaining ground displacement and deformation induced by tectonic faulting. Also, the first elastodynamic solutions for the rapid motion of a crystal dislocations by South African materials scientist F. R. N. Nabarro, in the early 1950's, were quickly adapted by seismologists to explain the radiation from propagating slip distributions on faults. Japanese seismologist H. Nakano had already shown in 1923 how to represent the distant waves radiated by an earthquake as the elastodynamic response to a pair of force dipoles amounting to zero net torque. (All of his manuscripts were destroyed in the fire in Tokyo associated with the great Kwanto earthquake in that same year, but some of his manuscripts had been sent to Western colleagues and the work survived.)

Continuum plasticity theory. The macroscopic theory of plastic flow has a history nearly as old as that of elasticity. While in the microscopic theory of materials, the word "plasticity" is usually interpreted as denoting deformation by dislocation processes, in macroscopic continuum mechanics it is taken to denote any type of permanent deformation of materials, especially those of a type for which time or rate of deformation effects are not the most dominant feature of the phenomenon (the terms *viscoplasticity* or *creep* or *viscoelasticity* are usually used in such cases). Coulomb's work of 1773 on the frictional yielding of soils under shear and normal stress has been mentioned; yielding denotes the occurrence of large shear deformations without significant increase in applied stress. This work found applications to explaining the pressure of soils against retaining walls and footings in work of the French mathematician and engineer J. V. Poncelot in 1840 and Scottish engineer and physicist W. J. M. Rankine in 1853. The inelastic deformation of soils and rocks often takes place in situations for which the deforming mass is

infiltrated by groundwater, and Austrian-American civil engineer Karl Terzaghi in the 1920's developed the concept of effective stress, whereby the stresses which enter a criterion of yielding or failure are not the total stresses applied to the saturated soil or rock mass, but rather the effective stresses, which are the difference between the total stresses and those of a purely hydrostatic stress state with pressure equal to that in the pore fluid. Terzaghi also introduced the concept of consolidation, in which the compression of a fluid-saturated soil can take place only as the fluid slowly flows through the pore space under pressure gradients, according to the law of Darcy; this effect accounts for the time-dependent settlement of constructions over clay soils.

Apart from the earlier observation of plastic flow at large stresses in the tensile testing of bars, the continuum plasticity of metallic materials begins with Henri Edouard Tresca in 1864. His experiments on the compression and indentation of metals led him to propose that this type of plasticity, in contrast to that in soils, was essentially independent of the average normal stress in the material and dependent only on shear stresses, a feature later rationalized by the dislocation mechanism. Tresca proposed a yield criterion for macroscopically isotropic metal polycrystals based on the maximum shear stress in the material, and that was used by Saint-Venant to solve a first elastic-plastic problem, that of the partly plastic cylinder in torsion, and also to solve for the stresses in a completely plastic tube under pressure. German applied mechanician Ludwig Prandtl developed the rudiments of the theory of plane plastic flow in 1920 and 1921, with an analysis of indentation of a ductile solid by a flat-ended rigid indenter, and the resulting theory of *plastic slip lines* was completed by H. Hencky in 1923 and Hilda Geiringer in 1930. Additional developments include the methods of plastic limit analysis, which allowed engineers to directly calculate upper and lower bounds to the plastic collapse loads of structures or to forces required in metal forming. Those methods developed gradually over the early 1900's on a largely intuitive basis, first for simple beam structures and later for plates, and were put on a rigorous basis within the rapidly developing mathematical theory of plasticity around 1950 by Daniel C. Drucker and William Prager in the United States and Rodney Hill in England.

German applied mathematician Richard von Mises proposed in 1913 that a mathematically simpler theory of plasticity than that based on the Tresca yield criterion could be based on the *second tensor invariant* of the *deviatoric stresses* (that is, of the total stresses minus those of a hydrostatic state with pressure equal to the average normal stress over all planes). An equivalent yield condition had been proposed independently by Polish engineer M.-T. Huber. The Mises theory incorporates a proposal by M. Levy in 1871 that components of the plastic strain increment tensor are in proportion to one another just as are the components of deviatoric stress.

This criterion was found to generally provide slightly better agreement with experiment than did that of Tresca, and most work on the application of plasticity theory uses this form. Following a suggestion of Prandtl, E. Reuss completed the theory in 1930 by adding an elastic component of strain increments, related to stress increments in the same way as for linear elastic response. This formulation was soon generalized to include strain hardening, whereby the value of the second invariant for continued yielding increases with ongoing plastic deformation, and was extended to high-temperature creep response in metals or other hot solids by assuming that the second invariant of the plastic (now generally called "creep") strain rate is a function of that same invariant of the deviatoric stress, typically a power law type with Arrhenius temperature dependence. This formulation of plastic and viscoplastic or creep response has been applied to all manner of problems in materials and structural technology and in flow of geological masses. Representative problems addressed include the large growth to coalescence of microscopic voids in the ductile fracture of metals, the theory of the indentation hardness test, the extrusion of metal rods and rolling of metal sheets, the auto-frettage of gun tubes, design against collapse of ductile steel structures, estimation the thickness of the Greenland ice sheet, and modeling the geologic evolution of the Tibetan plateau. Other types of elastic-plastic theories intended for analysis of ductile single crystals originate from the work of G. I. Taylor and Hill, and bases the criterion for yielding on E. Schmid's concept from the 1920's of a critical resolved shear stress along a crystal slip plane, in the direction of an allowed slip on that plane; this sort of yield condition has approximate support from the dislocation theory of plasticity.

Viscoelasticity. The German physicist Wilhelm Weber noticed in 1835 that a load applied to a silk thread produced not only an immediate extension but also a continuing elongation of the thread with time. This type of *viscoelastic* response is especially notable in polymeric solids but is present to some extent in all types of solids and often does not have a clear separation from what could be called viscoplastic or creep response. In general, if all the strain is ultimately recovered when a load is removed from a body, the response is termed viscoelastic, but the term is also used in cases for which sustained loading leads to strains which are not fully recovered. The Austrian physicist Ludwig Boltzmann developed in 1874 the theory of linear viscoelastic stress-strain relations. In their most general form these involve the notion that a step loading (suddenly imposed stress, subsequently maintained constant) causes an immediate strain followed by a time-dependent strain which, for different materials, may either have a finite long time limit or may increase indefinitely with time. Within the assumption of linearity, the strain at time t in response to a general time dependent stress history $\sigma(t)$ can then be written as the sum (or integral) of terms that involve the step-loading strain response at time t-t' due to a step

loading dt' $d\sigma(t')/dt'$ at time t'. The theory of viscoelasticity is important for consideration of the attenuation of stress waves and the damping of vibrations.

A new class of problems arose with the mechanics of very long molecule polymers, without significant cross-linking, existing either in solution or as a melt. These are fluids in the sense that they cannot long support shear stress but have, at the same time, remarkable properties like those of finitely deformed elastic solids. A famous demonstration is to pour one of these fluids slowly from a bottle and to suddenly cut the flowing stream with scissors; if the cut is not too far below the place of exit from the bottle, the stream of falling fluid immediately contracts elastically and returns to the bottle. The molecules are elongated during flow but tend to return to their thermodynamically preferred coiled configuration when forces are removed. The theory of such materials came under intense development in the 1950's after British applied mathematician James Gardner Oldroyd showed in 1950 how viscoelastic stress-strain relations of a memory type could be generalized to a flowing fluid. This involves subtle issues on assuring that the *constitutive relation*, or *rheological relation*, between the stress history and the deformation history at a material "point" is properly invariant to a superposed history of rigid rotation , which should not affect the local physics determining that relation (the resulting Coriolis and centrifugal effects are quite negligible at the scale of molecular interactions). Important contributions on this issue were made by S. Zaremba and G. Jaumann in the first decade of the 1900's; they showed how to make tensorial definitions of stress rate that were invariant to superposed spin and thus were suitable for use in constitutive relations. But it was only during the 1950's that these concepts found their way into the theory of constitutive relations for general viscoelastic materials and, independently and a few years later, properly invariant stress rates were adopted in continuum formulations of elastic-plastic response.

Computational mechanics. The digital computer revolutionized the practice of many areas of engineering and science, and solid mechanics was among the first fields to benefit from its impact. Many computational techniques have been used in that field, but the one which emerged by the end of the 1970's as, by far, the most widely adopted is the *finite element method*. This method was outlined by the mathematician Richard Courant in 1943 and was developed independently, and put to practical use on computers, in the mid-1950's by aeronautical structures engineers M. J. Turner, R. W. Clough, H. C. Martin and L. J. Topp in the United States and by J. H. Argyris and S. Kelsey in Britain. Their work grew out of earlier attempts at systematic structural analysis for complex frameworks of beam elements. The method was soon recast in a variational framework and related to earlier efforts at approximate solution of

problems described by variational principles, by substituting trial functions of unknown amplitude into the variational functional which is then rendered stationary as an algebraic function of the amplitude coefficients. In the most common version of the finite element method, the domain to be analyzed is divided into cells, or *elements*, and the displacement field within each element is interpolated in terms of displacements at a few points around the element boundary, and sometimes within it, called *nodes*. The interpolation is done so that the displacement field is continuous across element boundaries for any choice of the nodal displacements. The strain at every point can thus be expressed in terms of nodal displacements, and it is then required that the stresses associated with these strains, through the stress-strain relations of the material, satisfy the principle of virtual work for arbitrary variation of the nodal displacements. This generates as many simultaneous equations as there are degrees of freedom in the finite element model, and numerical techniques for solving such systems of equations are programmed for computer solution.

The finite element method and other computational techniques (finite differences, spectral expansions, boundary integral equations) have made a major change in the practice of, and education for, engineering in the various areas that draw on solid mechanics. Previously, many educators saw little point in teaching engineers much of the subject beyond the techniques of elementary beam theory developed in the 1700's by Bernoulli, Euler and Coulomb. More advanced analyses involved sufficiently difficult mathematics as to be beyond the reach of the typical practitioner, and were regarded as the domain of advanced specialists who would, themselves, find all but the simpler cases intractable. The availability of software incorporating the finite element method, and other procedures of computational mechanics and design analysis, has placed the advanced concepts of solid mechanics into the hands of a far broader community of engineers. At the same time, it has created a necessity for them and other users to have a much deeper education in the subject, so that the computational tools are used properly and at full effectiveness.

BASIC PRINCIPLES

In addressing any problem in continuum or solid mechanics, we need to bring together the following considerations: (1) The Newtonian equations of motion, in the more general form recognized in the subsequent century by Euler, expressing conservation of linear and angular momentum for finite bodies (rather than just for point particles), and the related concept of stress

as formalized by Cauchy; (2) Consideration of the geometry of deformation and thus expression of strains in terms of gradients in the displacement field; and (3) Use of relations between stress and strain that are characteristic of the material in question, and of the stress level, temperature and time scale of the problem considered.

These three considerations suffice for most problems in solid and structural mechanics for simple materials. They must be supplemented for solids undergoing diffusion processes in which one material constituent moves relative to another (as of interest sometimes for a fluid-infiltrated soils or petroleum reservoir rocks), and in cases for which the induction of a temperature field by deformation processes and the related heat transfer cannot be neglected. The latter cases require that we supplement the above three considerations with the following: (4) Equations for conservation of mass of diffusing constituents; (5) The first law of thermodynamics, which introduces the concept of heat flux and relates changes in energy to work and heat supply, and (6) Relations which express the diffusive fluxes and heat flow in terms of spatial gradients of appropriate chemical potentials and of temperature. Also, in many important technological devices, electric and magnetic fields affect the stressing, deformation and motion of matter. Examples are provided by piezoelectric crystals and other ceramics for electric or magnetic actuators, and the coils and supporting structures of powerful electromagnets. In these cases, we must add the following: (7) Scottish physicist James Clerk Maxwell's set of equations which interrelate electric and magnetic fields to polarization and magnetization of material media, and to the density and motion of electric charge; and (8) Augmented relations between stress and strain which now, for example, express all of stress, polarization and magnetization in terms of strain, electric field and magnetic intensity, and of temperature. Also, the second law of thermodynamics, combined with the principles mentioned above, serves to constrain physically allowed relations between stress, strain and temperature in (3). Such considerations were first brought to bear in a purely mechanical context by Green in 1837, as based on the existence of a strain energy which generalized, for a continuum, the potential energy of the discrete dynamical systems of analytical mechanics; they were later rooted in the development of macroscopic thermodynamics by Kelvin. The second law also constrains the other types of relations described in (6) and (8) above. Such relations are commonly referred to as *constitutive relations*.

In general, the stress-strain relations are to be determined by experiment. A variety of mechanical testing machines and geometrical configurations of material specimens have been devised to measure them. These allow, in different cases, simple tensile, compressive, or shear

stressing, and sometimes combined stressing with several different components of stress, and the determination of material response over a range of temperature, strain rate and loading history. The testing of round bars under tensile stress, with precise measurement of their extension to obtain the strain, is common for metals and for technological ceramics and polymers. For rocks and soils, which generally carry load in compression, the most common test involves a round cylinder that is compressed along its axis, often while being subjected to confining pressure on its curved face. Often, a measurement interpreted by solid mechanics theory is used to determine some of the properties entering stress-strain relations. For example, measuring the speed of deformation waves or the natural frequencies of vibration of structures can be used to extract the elastic moduli of materials of known mass density, and measurement of indentation hardness of a metal can be used to estimate its plastic shear strength.

In some favorable cases, stress strain relations can be calculated approximately by applying appropriate principles of mechanics at the microscale of the material considered. In a composite material, the microscale could be regarded as the scale of the separate materials making up the reinforcing fibers and matrix. When their individual stress-strain relations are known from experiment, continuum mechanics principles applied at the scale of the individual constituents can be used to predict the overall stress-strain relations for the composite. In the case of a polycrystalline metal undergoing elastic or plastic deformation, the overall stress-strain relations are sometimes estimated by applying continuum mechanics principles to the heterogeneous aggregate of joined crystals, assuming that we know the stress-strain relations of the single crystals constituting the individual grains. For rubbery polymer materials, made up of long chain molecules which would randomly configure themselves into coil-like shapes, some aspects of the elastic stress-strain response can be obtained by applying principles of statistical thermodynamics to the partial uncoiling of the array of molecules by imposed strain. In the case of a single crystallite of an element like silicon or aluminum, or simple compound like silicon carbide, the relevant microscale is that of the atomic spacing in the crystals, and principles governing atomic force laws at that scale can be used to estimate elastic constants. For example, quantum mechanical principles, implemented on digital computers in the framework of *density functional theory*, in which one solves for the density distribution of electrons amidst an array of fixed atomic nuclei, are the basis for such calculations. For consideration of plastic flow processes in metals and in sufficiently hot ceramics, the relevant microscale involves the network of *dislocation* lines that move within crystals. These lines shift atom positions relative to one another by one atomic spacing as they move along slip planes. Important features of elastic-plastic and viscoplastic stress-strain relations can be understood by modeling the stress

dependence of dislocation generation and motion, and the resulting dislocation entanglement and immobilization processes which account for strain hardening. For rubbery polymeric solids showing viscoelastic response, like gradual relaxation of stress with time after a strain is imposed and subsequently held constant, the microscale processes involve the gradual sliding of long molecules relative to the network of like molecules with which they have entangled. In such cases prediction of viscoelastic stress-strain relations involves, very roughly, the modeling of slow pulling of molecules along and out of the "tubes" formed by the other molecules with which they are entangled.

To examine the mathematical structure of the theory, considerations (1) to (3) above are now further developed. For this purpose, we adopt a continuum model of matter, making no detailed reference to its discrete structure at molecular, or possibly other larger microscopic, scales that are far below those of the intended application.

Linear and Angular Momentum Principles: Stress, and Equations of Motion

Let $\mathbf{x}$ denote the *position vector* of a point in space as measured relative to the origin of a Newtonian reference frame; $\mathbf{x}$ has the components (x_1, x_2, x_3) relative to a Cartesian set of axes, fixed in the reference frame, which we denote as the 1, 2 and 3 axes, Figure 1. (This form of notation proves more convenient for the subject than a convention which may be more familiar to many readers, in which positions are denoted as (x, y, z) and the reference axes as the X, Y and Z axes.) Suppose that a material occupies the part of space considered and let $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ be the velocity vector of the material point which occupies position $\mathbf{x}$ at time t; that same material point will be at position $\mathbf{x} + \mathbf{v}\, dt$ an infinitesimal time interval dt later. Let $\rho = \rho(\mathbf{x}, t)$ be the mass density of the material. Here $\mathbf{v}$ and ρ are *macroscopic* variables. What we idealize in the continuum model as a material point, moving as a smooth function of time, will correspond on molecular (or larger but still "microscopic") length scales to a region with strong fluctuations of density and velocity. In terms of phenomena at such scales, ρ corresponds to an average of mass per unit of volume, and $\rho\, \mathbf{v}$ to an average of linear momentum per unit volume, as taken over spatial and temporal scales that are large compared to those of the microscale processes but still small compared to those of the intended application or phenomenon under study. Thus $\mathbf{v}$ of the continuum theory is a mass-weighted average velocity, from the microscopic viewpoint. (There do not generally exist tractable formulations of macroscopic mechanics when such a separation of scales does not apply. This is an important area of research since many important phenomena involving the fracture of solids and fine scale inelastic deformation processes do not have a clear

separation of length scales. Large scale digital computer simulations of systems of discrete particles allow some of the simpler of such cases to be addressed; these include atomistic modeling of fracture and plastic flow processes in small regions of crystals, and flows of granular solids with highly idealized particle interactions.)

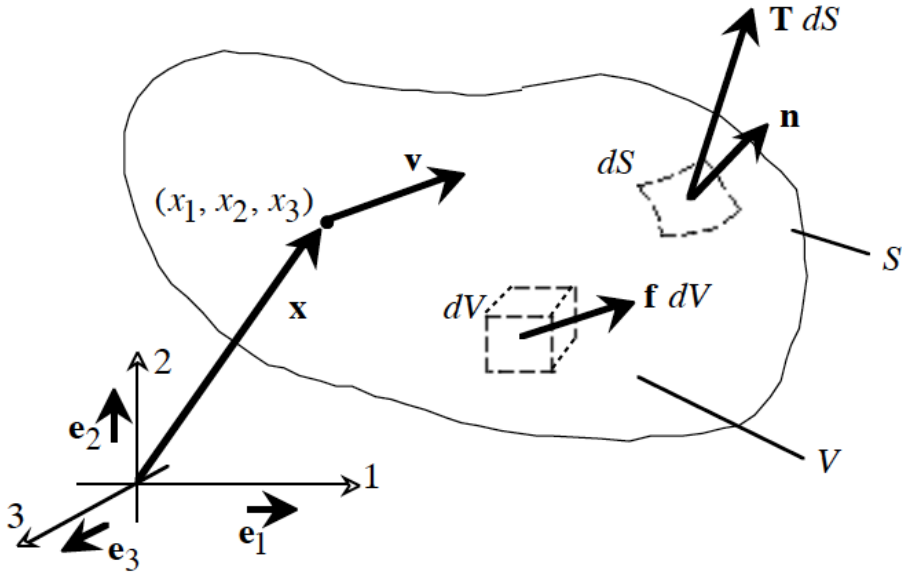

Figure 1. Coordinate system; position (**x**) and velocity (**v**) vectors; body force **f** *dV* acting on element *dV* of volume, and surface force **T** *dS* acting on element *dS* of surface.

We observe that an infinitesimal element of material occupying volume dV at **x** moves and distorts in such a way that ρ dV, which corresponds to the (conserved) mass of the element, remains constant. The linear momentum of the element is ρ **v** dV and its angular momentum relative to the coordinate origin is given as the *vector*, or *cross*, product $\mathbf{x} \times (\rho\, \mathbf{v}\, dV)$. Thus the linear momentum **P**, and angular momentum **H** relative to the coordinate origin, of the matter instantaneously occupying any volume V of space are then given by summing up the linear and angular momentum vectors of each element of material. Such summation over infinitesimal elements is represented mathematically by the integrals

$$\mathbf{P} = \int_V \rho \mathbf{v}\, dV, \quad \mathbf{H} = \int_V \rho \mathbf{x} \times \mathbf{v}\, dV$$

We limit attention to situations in which relativistic effects can be ignored.

Let **F** denote the total *force* and **M** the total *torque* or *moment* (relative to the coordinate origin) acting instantaneously on the material occupying any arbitrary volume V. The basic laws

of Newtonian mechanics are the linear and angular momentum principles that

$$\mathbf{F} = d\mathbf{P}/dt \ , \qquad \mathbf{M} = d\mathbf{H}/dt \ ,$$

where time derivatives of **P** and **H** are calculated following the motion of the matter which occupies V at time t. Newton's focus was on matter in situations for which the particle point of view is valid, so that only $\mathbf{F} = d\mathbf{P}/dt$ is required. It was Euler who recognized the need for the two vectorial laws of motion for general finite bodies; he explicitly stated the pair of laws $\mathbf{F} = d\mathbf{P}/dt$ and $\mathbf{M} = d\mathbf{H}/dt$ in 1776, then for a rigid body, but had implicitly recognized them as early as 1752. When either **F** or **M** vanish, these equations of motion correspond to conservation of linear or angular momentum. An important, very common, and non-trivial class of problems in solid mechanics involves determining the deformed and stressed configuration of solids or structures that are in *static equilibrium*; in that case the relevant basic equations are $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$. The understanding of such conditions for equilibrium, at least in a rudimentary form, long predates Newton. Indeed, Archimedes of Syracuse (3rd Century BC), the great Greek mathematician and arguably the first theoretically and experimentally minded physical scientist, understood these equations at least in a nonvectorial form appropriate for systems of parallel forces. That is shown by his treatment of the hydrostatic equilibrium of a partially submerged body and his establishment of the principle of the lever (torques about the fulcrum sum to zero) and the concept of center of gravity. Archimedes' approach to natural philosophy is now the standard model of science but was overshadowed for about 2000 years by Aristotle's (4th Century BC) style of ex-cathedra, if sometimes insightful, speculation. The Dutch mathematician and engineer, Simon Stevin recognized the vectorial nature of the equations $\mathbf{F} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$ for static equilibrium, developing the parallelogram law of vectorial force addition in 1586 and correctly analyzing the principle of the lever for systems of nonparallel forces.

Stress vector and equations of motion in integral form. We now assume that **F** and **M** derive from two type of forces, namely *body forces* **f**, like gravitational attractions, defined such that force **f** dV acts on volume element dV (see Figure 1), and *surface forces* which represent the mechanical effect of matter immediately adjoining that along the surface, *S*, of the volume *V* that we consider. Cauchy formalized in 1822 a basic assumption of continuum mechanics that such surface forces could be represented as a vector distribution $\mathbf{T}^{force}$, defined so that $\mathbf{T}^{force}\, dS$ is an element of force acting over the area *dS* of the surface, Figure 1. (Shortly, we will want to define a *stress vector* **T**, of which this $\mathbf{T}^{force}$ will be a part; it is typically the dominant part for solid materials.) Thus, for any arbitrarily chosen region, like in Figure 1, we assume that total

force and torque acting can be written, respectively, as

$$\mathbf{F} = \int_S \mathbf{T}^{force}\, dS + \int_V \mathbf{f}\, dV \ , \quad \text{and} \quad \mathbf{M} = \int_S \mathbf{x} \times \mathbf{T}^{force}\, dS + \int_V \mathbf{x} \times \mathbf{f}\, dV \ .$$

These should now be equated, respectively, to the rates of change of linear and angular momentum, $d\mathbf{P}/dt$ and $d\mathbf{H}/dt$.

To calculate $d\mathbf{P}/dt$, note that the integrand for $\mathbf{P}$ contains the product ρdV times $\mathbf{v}$. Since the mass element ρdV is invariant in the motion, it has zero time derivative, and we need only calculate the derivative of its velocity $\mathbf{v}$, which is acceleration $\mathbf{a}$. However, the correct expression for $d\mathbf{P}/dt$ contains the term which has just been motivated *plus* a second term:

$$d\mathbf{P} \,/\, dt = \int_V \rho \mathbf{a}\, dV + \int_S \mathbf{T}^{mom\,flux} dS \ .$$

That second term arises because there is a microscopic motion, in general, relative to the mass-averaged macroscopic motion, and that relative motion causes some *momentum flux* $\mathbf{T}^{mom.flux}$ per unit area, across the surfaces S. In the continuum model, the surface S moves through space such that the velocity of the surface in a direction normal to itself is $\mathbf{n} \cdot \mathbf{v}$, where $\mathbf{n}$ is the unit outer normal to S at the point considered and $\mathbf{v}$ is the velocity at that point. Since $\mathbf{v}$ is a mass-weighted average of fluctuating velocities on a molecular (or larger microscopic) scale, this assures that there is no mass transferred across S, but *not* that there is no momentum transferred; $\mathbf{T}^{mom.flux}$ accounts for that latter transfer. In a similar way, the rate of change of angular momentum can be calculated and we obtain the expression

$$d\mathbf{H} \,/\, dt = \int_V \rho \mathbf{x} \times \mathbf{a}\, dV + \int_S \mathbf{x} \times \mathbf{T}^{mom\,flux} dS \ .$$

The acceleration $\mathbf{a} = \mathbf{a}(\mathbf{x},t) = d\mathbf{v}/dt$ is calculated such that the time derivative of $\mathbf{v}$ is taken following the motion of a material point. Thus $\mathbf{a}(\mathbf{x},t)dt$ corresponds to the difference between $\mathbf{v}(\mathbf{x} + \mathbf{v}\, dt,\, t + dt)$ and $\mathbf{v}(\mathbf{x},\, t)$. Also, in deriving the expression for $d\mathbf{H}/dt$, $\mathbf{v} = d\mathbf{x}/dt$ has been used, with the derivative again following the motion, and it has been noted that $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

We now define the *stress*, or *traction, vector* $\mathbf{T}$ by

$$\mathbf{T} = \mathbf{T}^{force} - \mathbf{T}^{mom.flux} \ .$$

In gases and, at least for viscous shearing effects, in liquids the microscale momentum transports $\mathbf{T}^{mom.flux}$, resulting from fast moving molecules randomly moving into regions of slower motion and *vice-versa*, are the main contribution to **T**. In solids they generally are a small contribution, especially at low temperatures compared to that for melting. It will generally be the case that microscopic mass elements moving at a velocity different from **v**, and hence contributing to $\mathbf{T}^{mom.flux}$, will undergo collisional interactions with other mass elements within a short distance of the surface element considered, reducing their speed, on average, to **v**. This delivers impulsive forces $-\mathbf{T}^{mom.flux}$ per unit area in the near vicinity of the surface so, including their effect, it is legitimate to refer to the total stress vector **T** as a *force* per unit area, as is often done in the literature of the subject.

Then ,using the definition of **T**, when we equate the expressions for **F** and **M** above to those for $d\mathbf{P}/dt$ and $d\mathbf{H}/dt$, we obtain the *equations of motion, in integral form*, for a continuum,

$$\int_S \mathbf{T}\,dS + \int_V \mathbf{f}\,dV = \int_V \rho\mathbf{a}\,dV, \quad \int_S \mathbf{x}\times\mathbf{T}\,dS + \int_V \mathbf{x}\times\mathbf{f}\,dV = \int_V \rho\mathbf{x}\times\mathbf{a}\,dV$$

We now assume these to hold good for every conceivable choice of region *V*.

Stress components. Nine quantities σ_{ij} ($i, j = 1, 2, 3$) called *stress components* may be defined at each point of the medium; these will vary with position and time, $\sigma_{ij} = \sigma_{ij}(\mathbf{x}, t)$, and have the following interpretation. Suppose that we consider an element of surface dS through a point **x** with dS oriented so that its outer normal (pointing away from the region V, bounded by S) points in the positive x_i direction, where *i* is any of 1, 2 or 3. Then σ_{i1}, σ_{i2} and σ_{i3} at **x** are defined as the Cartesian components of the stress vector **T** (call it $\mathbf{T}^{(i)}$) acting on this dS. Figure 2 shows the components of such stress vectors for faces in each of the three coordinate directions. To use a vector notation with $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ denoting unit vectors along the coordinate axes (Figure 1),

$$\mathbf{T}^{(i)} = \sigma_{i1}\,\mathbf{e}_1 + \sigma_{i2}\,\mathbf{e}_2 + \sigma_{i3}\,\mathbf{e}_3 .$$

Thus the stress σ_{ij} at **x** is the stress in the *j* direction associated with an *i*-oriented face through point **x**; the physical dimension of the σ_{ij} is [Force]/[Length]^2. The components σ_{11}, σ_{22} and σ_{33} are stresses directed perpendicular, or *normal*, to the faces on which they act and are *normal stresses*; the σ_{ij} with $i \neq j$ are directed parallel to the plane on which they act and are *shear*

stresses.

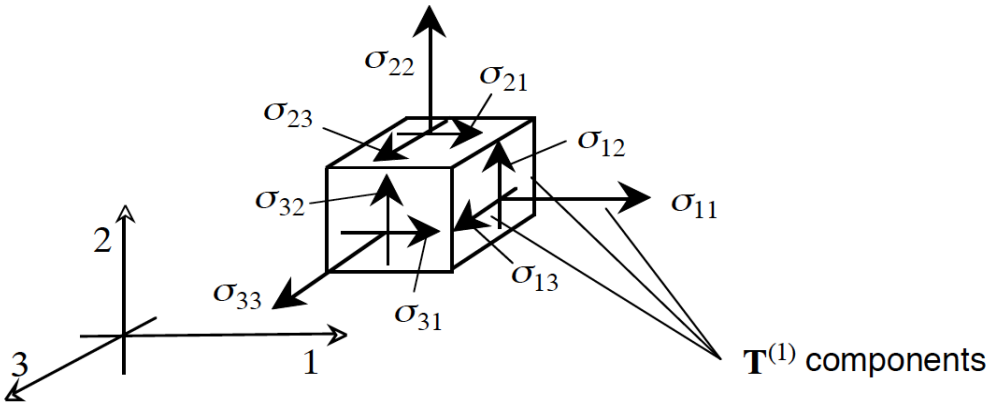

Figure 2. Stress components; first index denotes plane, second denotes direction.

By hypothesis, the linear momentum principle applies for any volume V. If we first apply it to a small region including a general position **x**, and consider the limit of the resulting equation as both the volume V and bounding surface area S of the region approach zero, so that the region shrinks onto point **x**. We can observe that the volume integrals, when divided by area S, approach zero in that limit. Thus, for such choices of region, the linear momentum principle requires that we

set to zero the limit of $(1 / S)\int_S \mathbf{T}\, dS$ as S approaches zero. Consider a thin sliver of material at **x**, (Figure 3) with thin direction along the x_1 axis, let that thickness approach zero, and then let the diameter of the region approach zero so that it shrinks onto **x**. We thus derive that $\mathbf{T}^{(-1)} + \mathbf{T}^{(1)} = \mathbf{0}$, which is a special case of the *action-reaction* principle. (It, like other variants of the action-reaction principle, can be regarded as a derivable consequence of the hypothesis that the linear momentum principle applies for every choice of region, including two subregions which act upon one another and exert the forces referred to as the action and reaction.) The result tells us that, for any direction *i*,

$$\mathbf{T}^{(-i)} = -\mathbf{T}^{(i)} = -\sigma_{i1}\,\mathbf{e}_1 - \sigma_{i2}\,\mathbf{e}_2 - \sigma_{i3}\,\mathbf{e}_3$$

and hence that $-\sigma_{i1}$, $-\sigma_{i2}$ and $-\sigma_{i3}$ are the Cartesian components of the stress vector acting on a surface element dS through **x** whose outer normal points in the *negative i* direction.

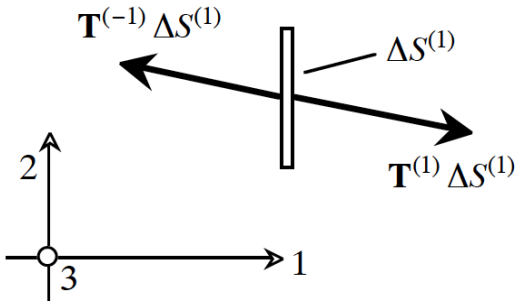

Figure 3. Linear momentum principle leads to action-reaction, $\mathbf{T}^{(-1)} + \mathbf{T}^{(1)} = \mathbf{0}$.

Next we consider a small tetrahedron (Figure 4) at $\mathbf{x}$ with inclined face having outward unit normal vector $\mathbf{n}$, and other three faces oriented perpendicular to the three coordinate axes. Let S denote the area of the inclined face and $S^{(1)}$, $S^{(2)}$ and $S^{(3)}$ the areas of the faces with outer normals respectively in the negative 1, 2 and 3 directions; we note from geometry that $S^{(i)}/S = n_i$ (this also leads to the correct result when the normal to face $S^{(i)}$ points in the positive i direction, in which case $n_i < 0$). Letting the size of the tetrahedron approach zero, so that it shrinks onto $\mathbf{x}$, the linear momentum principle requires that $\mathbf{T} + n_1\,\mathbf{T}^{(-1)} + n_2\,\mathbf{T}^{(-2)} + n_3\,\mathbf{T}^{(-3)} = \mathbf{0}$. Thus, using the expression above for the $\mathbf{T}^{(-i)}$ we derive the result that the stress vector $\mathbf{T}$ on a surface element through $\mathbf{x}$ with outward normal $\mathbf{n}$ can be expressed as a linear function of the σ_{ij} at $\mathbf{x}$. The relation is such that the j component of the stress vector $\mathbf{T}$ is

$$T_j \equiv \mathbf{e}_j \cdot \mathbf{T} = n_1 \sigma_{1j} + n_2 \sigma_{2j} + n_3 \sigma_{3j} = \sum_{i=1}^{3} n_i \sigma_{ij} \quad (j = 1,\ 2,\ 3)$$

Summation convention. It turns out that almost always, when we have a sum over an index like i in the last equation, the index on which we sum is repeated precisely twice but other indices (j there) appear only once. Also, in equations with multiple summations, as will be found subsequently, the various indices on which we sum are each repeated twice. Thus, many authors prefer to drop the summation signs and adopt the *summation convention* that one always understands a repeated index to denote a sum. In that convention, the last equation would be written as $T_j = n_i \sigma_{ij}$. Similarly, the earlier equation defining stress components would be written as $\mathbf{T}^{(i)} = \sigma_{ij}\mathbf{e}_j$. Occasionally we encounter an equation with a repeated index that it not intended to be summed, or perhaps that index appears more than two times. In such (rare) cases, we simly say that the summation convention is suspended.

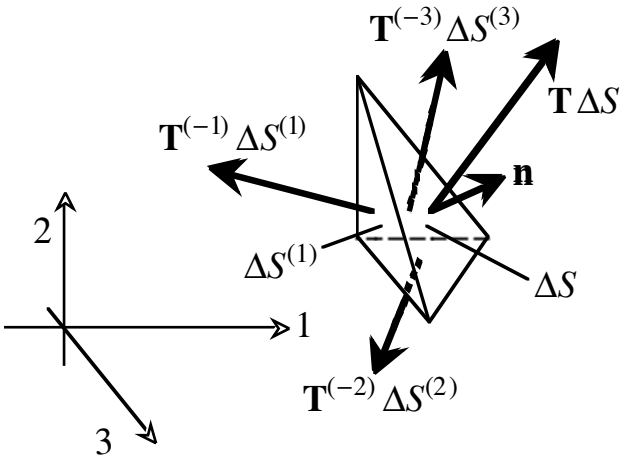

Figure 4. Cauchy tetrahedron with inclined face having an arbitrary orientation **n**; constructed about some material point, and to be shrunk onto that point in the limit to be taken. Linear momentum principle relates **T** for such an inclined face to the σ_{ij}.

Tensors; stress transformations. This relation for **T** (or T_j) also tells us that the σ_{ij} have the mathematical property of being the components of a *second rank tensor*. To show that, suppose that a different set of Cartesian reference axes $1'$, $2'$ and $3'$ have been chosen. Let x_1', x_2', x_3' denote the components of the position vector of point **x** and let $\sigma_{k\,l'}$ $(k,\, l' = 1,\, 2,\, 3)$ denote the 9 stress components relative to that coordinate system. Choose **n** in the above equation to coincide with the direction of the unit vector $\mathbf{e}'_k$ along the k' axis. Then, by the definition of stress components we can write

$$\mathbf{T} = \mathbf{T}^{(k')} = \sigma'_{k1}\,\mathbf{e}'_1 + \sigma'_{k2}\,\mathbf{e}'_2 + \sigma'_{k3}\,\mathbf{e}'_3,$$

whereas from the result derived just above, we can also write

$$\mathbf{T} = \sum_{j=1}^{3} T_j \mathbf{e}_j = \sum_{j=1}^{3} \Big(\sum_{i=1}^{3} n_i \sigma_{ij}\Big)\mathbf{e}_j$$

Using the first form for **T**, we can form the scalar product $\mathbf{e}'_l \cdot \mathbf{T} = \mathbf{e}'_l \cdot \mathbf{T}^{(k')} = \sigma'_{kl}$, which must also hold when we use the second form. Noting also that since $\mathbf{n} = \mathbf{e}'_k$, $n_i = \mathbf{e}'_k \cdot \mathbf{e}_i$, we thus obtain

$$\sigma'_{kl} = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ki}\alpha_{lj}\sigma_{ij} \quad (\text{where } \alpha_{pq} = \mathbf{e}'_p \cdot \mathbf{e}_q \text{ for } p, q = 1,2,3)$$

which is the defining equation of a *second rank tensor*. Here α_{pq} gives the cosine of the angle between the p' and q axes, and defines a component of the orthogonal transformation matrix $[\alpha]$, satisfying $[\alpha]^T[\alpha] = [\alpha][\alpha]^T = [I]$. (The first index of a quantity like α_{pq} denotes row number and the second the column number in the corresponding matrix; superscript T denotes transpose, i.e., interchange rows and columns; [I] denotes the unit matrix, a 3 by 3 matrix with unity for every diagonal element and zero elements elsewhere; and if [A] = [B][C], then $A_{ij} = B_{i1}C_{1j} + B_{i2}C_{2j} + B_{i3}C_{3j}$ for 3 by 3 matrices like here.). The coordinates themselves are related by

$$x'_k = \sum_{i=1}^{3} \alpha_{ki} x_i ,$$

and the same relation which applies to components of vectors such as **v**, **a**, and **f** introduced above, which may be called *first rank tensors*.

Note that with the *summation convention*, some of the above transformations would be written as $\sigma'_{kl} = \alpha_{ki}\alpha_{lj}\sigma_{ij}$ (this involves two repeated indices, *i* and *j*, and hence implies a summation over both), and $x'_k = \alpha_{ki}x_i$. Also, the properties of $[\alpha]$ are expressed by

$$\alpha^T_{ik}\alpha_{kj} \equiv \alpha_{ki}\alpha_{kj} = \alpha_{ik}\alpha^T_{kj} \equiv \alpha_{ik}\alpha_{jk} = \delta_{ij} ,$$

where δ_{ij} is called the *Kronecker delta*, and describes the components of the unit matrix [I]. Hence δ_{ij} is unity when *i* and *j* coincide, and is zero otherwise. Also, we may note that, for example, $\delta_{ij}v_j = v_i$, where we sum on the repeated *j*. As a little practice in this notation, let us multiply both sides of $x'_k = \alpha_{ki}x_i$ by α_{kl}, so that *k*, now repeated, becomes an index over which there is summation too. We thus get $\alpha_{kl}x'_k = \alpha_{kl}\alpha_{ki}x_i = \delta_{li}x_i = x_l$, and hence have inverted the coordinate transformation. Similarly, let us multiply both sides of $\sigma'_{kl} = \alpha_{ki}\alpha_{lj}\sigma_{ij}$ by $\alpha_{kr}\alpha_{ls}$. Now *k* and *l* are repeated and hence we are summing over them, as well as over *i* and *j* on the right side of the equation, in getting $\alpha_{kr}\alpha_{ls}\sigma'_{kl} = \alpha_{kr}\alpha_{ki}\alpha_{ls}\alpha_{lj}\sigma_{ij} = \delta_{ri}\delta_{sj}\sigma_{ij} = \sigma_{rs}$, thus inverting the second-rank tensor transformation.

Equations of motion, local form. Now let us apply the linear momentum principle to an arbitrary finite body. The *divergence theorem* of multivariable calculus shows that integrals over the area of a closed surface S, with integrand $n_i f(\mathbf{x})$, may be rewritten as integrals over the

volume V enclosed by S, with integrand $\partial f(\mathbf{x})/\partial x_i$, when $f(\mathbf{x})$ is a differentiable function. Thus, using the expression for T_j above, we may see that

$$\int_S T_j \, dS = \int_S (n_1 \sigma_{1j} + n_2 \sigma_{2j} + n_3 \sigma_{3j}) \, dS = \int_V \left(\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} \right) dV$$

at least when the σ_{ij} are continuous and differentiable, which is the typical case. If we now insert this expression for the surface integral in the linear momentum principle, that principle reduces to an equality in terms of volume integrals. It must hold no matter how we choose the volume and this can be so only if the same equation holds in terms of the integrands and thus, if the linear momentum principle is to apply for every conceivable choice of region V, we must satisfy the three equations

$$\frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3} + f_j \equiv \sum_{i=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_i} + f_j = \rho a_j \quad (j = 1,\ 2,\ 3)$$

These are the *equations of motion, in local form,* for a continuum. In the *summation convention*, they would be written as $\partial \sigma_{ij} / \partial x_i + f_j = \rho a_j$, or more concisely as $\sigma_{ij,i} + f_j = \rho a_j$ when we use also the *comma notation* for partial derivatives, in which $\partial F / \partial x_i$ is written as $F_{,i}$.

Consequence of angular momentum: stress symmetry. Once the above consequences of the linear momentum principle are accepted, the only further result which can be derived from requiring that the angular momentum principle apply for every conceivable choice of region *V* is that

$$\sigma_{ij} = \sigma_{ji} \quad (i,\ j = 1,\ 2,\ 3).$$

Thus the stress tensor is *symmetric*. To see the origin of this, let us use the *summation convention* for conciseness and note that the angular momentum principle can be written equivalently as

$$\int_S (x_i T_j - x_j T_i) dS + \int_V (x_i f_j - x_j f_i) dV = \int_V \rho (x_i a_j - x_j a_i) dV \ .$$

Now, using $T_j = n_k \sigma_{kj}$, then the divergence theorem, and then $\partial \sigma_{kj} / \partial x_k + f_j = \rho a_j$, the first term can be rearranged to

$$\int_S (x_i T_j - x_j T_i) dS = \int_S (x_i n_k \sigma_{kj} - x_j n_k \sigma_{ki}) dS = \int_V \frac{\partial}{\partial x_k} (x_i \sigma_{kj} - x_j \sigma_{ki}) dV$$

$$= \int_V [(\sigma_{ij} - \sigma_{ji}) + (x_i \frac{\partial \sigma_{kj}}{\partial x_k} - x_j \frac{\partial \sigma_{ki}}{\partial x_k})] dV = \int_V [(\sigma_{ij} - \sigma_{ji}) - (x_i f_j - x_j f_i) + \rho(x_i a_j - x_j a_i)] dV .$$

When we substitute that into the above form of the angular momentum principle, we are left with

$$\int_V (\sigma_{ij} - \sigma_{ji}) dV = 0$$

and, since that must hold for every choice of region V, $\sigma_{ij} = \sigma_{ji}$.

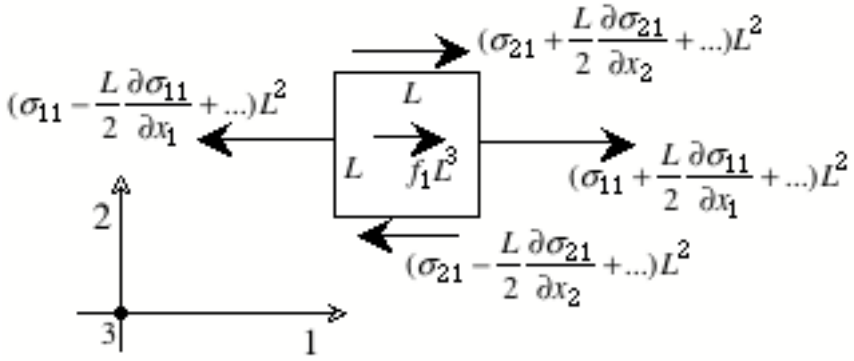

Figure 5. Forces acting in the 1 direction on a cube of side length L centered on some point **x** of interest; we will let $L \to 0$, and thus need the forces accurately only to order L^3; stresses around the cube faces are developed in a Taylor series expansion about their values at **x**.

Alternative derivation, local equations of motion. For another way of thinking about the origin of the local equations of motion, consider a small cube of material of side length L centered on some arbitrarily chosen point **x**. Shortly, we will let $L \to 0$, which turns out to mean that we will only need to know forces acting on the cube accurately to terms of order L^3. Consider all forces in the 1 direction acting on the cube; some are shown in Figure 5. The total body force is $f_1 L^3$. We write the stress components around the surface of the cube by doing a Taylor series expansion about point **x** at the cube center. In the limit $L \to 0$, the same stress σ_{11} acts on the face of the cube oriented in the +1 direction as on the face oriented in the –1 direction, so that the main forces $\sigma_{11} L^2$ due to the stress σ_{11} on those faces balance each other. However, to get the net force from σ_{11} correct to order L^3, we must recognize that the average stress will be $\sigma_{11} + (L/2)\partial \sigma_{11} / \partial x_1 + ...$ on the cube face oriented in the +1 direction, and

$\sigma_{11} - (L/2)\partial\sigma_{11}/\partial x_1 + ...$ on that in the –1 direction. Here "+ ... " stands for terms from the Taylor expansion which are of higher order in L than those explicitly shown, and which make no contribution in the limit $L \to 0$. Recognizing that these average stresses on the ± 1 faces act over an area L^2, the net force (stress times area) due to variations of stress σ_{11} is therefore $L^3\partial\sigma_{11}/\partial x_1 + ...$. (For simplicity, we identify stress as force per unit area in this discussion, although the stresses also contain the momentum flux contributions discussed above.) Similarly, an average stress $\sigma_{21} + (L/2)\partial\sigma_{21}/\partial x_2 + ...$ will act on the cube face oriented in the +2 direction, and $\sigma_{21} - (L/2)\partial\sigma_{21}/\partial x_2 + ...$ on that in the –2 direction, contributing a net force $L^3\partial\sigma_{21}/\partial x_2 + ...$, and stresses of type σ_{31} (not illustrated in Figure 5) contribute net force $L^3\partial\sigma_{31}/\partial x_3 + ...$.

When we sum all these forces and set them equal to the mass of the cube, ρL^3, times its acceleration a_1 in the 1 direction, there results

$$(\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3})L^3 + ... + f_1 L^3 = \rho L^3 a_1 \ ,$$

where now +... represents terms of order L^4 and higher. Thus, when we divide by L^3 and let $L \to 0$, we obtain

$$\frac{\partial\sigma_{11}}{\partial x_1} + \frac{\partial\sigma_{21}}{\partial x_2} + \frac{\partial\sigma_{31}}{\partial x_3} + f_1 = \rho a_1$$

which reproduces the first (corresponding to $j = 1$) of the three local equations of motion obtained above. The other equations for $j = 2$ and 3 may be derived similarly.

As a simple route to understanding the symmetry of the stress tensor, refer to Figure 2 and let us similarly consider the element shown there as a cube of side lengths L centered on some point **x** of interest. The stress component σ_{12} generates force $+\sigma_{12}L^2 + ...$ on the face oriented in the +1 direction, and $-\sigma_{12}L^2 + ...$ on that in the –1 direction, of which the leading terms constitutes a force couple separated by distance L, and hence their effect is to generate a torque $\sigma_{12}L^3 + ...$ about an axis in the 3 direction passing through the mass center of the element. The stress component σ_{21} on the cube faces oriented in the ± 2 directions generates torque $-\sigma_{21}L^3 + ...$ about the axis in the 3 direction, so that the net torque is $(\sigma_{12} - \sigma_{21})L^3 + ...$. That net torque should be equated to the 3-component of the rate of change of angular momentum of the element about its mass center. However, that angular momentum and its rate of change are of higher

order than L^3, and therefore this leading term in the torque, of apparent order L^3, must actually vanish. Thus we must have $\sigma_{12} = \sigma_{21}$. By extending the same argument to torques about other directions, we conclude that $\sigma_{ij} = \sigma_{ji}$ in general.

Principal stresses. Symmetry of the stress tensor, together with the tensor transformation property, has the important consequence that, at each point **x**, there exist three mutually perpendicular directions along which there are no shear stresses. These directions are called the *principal directions* and the corresponding normal stresses are called the *principal stresses*. If we order the principal stresses algebraically as σ_{I}, σ_{II}, σ_{III} (Figure 6), then the normal stress on any face (given as $\sigma_n = \mathbf{n} \cdot \mathbf{T}$) satisfies

$$\sigma_{\mathrm{I}} \le \sigma_{\mathrm{n}} \le \sigma_{\mathrm{III}} .$$

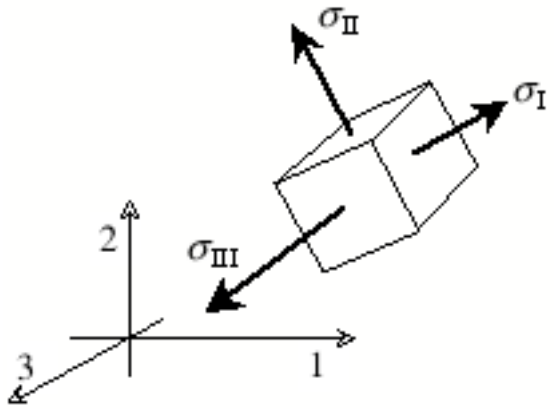

Figure 6. Principal stresses.

In fact, the principal stresses σ and principal directions **n** are the solutions of the *eigenvalue* (or *characteristic value*) problem

$$\sum_{j=1}^{3} \sigma_{ij} n_j - \sigma n_i = 0 \quad (i = 1,\ 2,\ 3)$$

which follows from asking the equivalent questions: For what directions is σ_{n} stationary relative to infinitesimal variations of the direction **n**? For what directions is the stress vector **T** aligned with **n**? The determinant of the coefficients of the components of **n** must vanish, so that

$$\det\begin{bmatrix}\sigma_{11}-\sigma & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma\end{bmatrix}=-\sigma^3+I_1\sigma^2+I_2\sigma+I_3=0$$

where

$$I_1=\sum_{i=1}^{3}\sigma_{ii},\quad I_2=-\frac{1}{2}I_1^2+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\sigma_{ji}\ ,\quad I_3=\det[\sigma].$$

Here $[\sigma]$ denotes the 3 by 3 matrix whose elements are σ_{ij}. Since the principal stresses are determined by I_1, I_2, I_3 and can have no dependence on how we chose the coordinate system with respect to which we refer the components of stress, I_1, I_2 and I_3 must be independent of that choice and are therefore called *stress invariants*. One may readily verify that they have the same values when evaluated in terms of σ'_{ij} above as in terms of σ_{ij} by using the tensor transformation law and properties noted for the orthogonal transformation matrix $[\alpha]$.

Stress transformation in a plane, Mohr circle. Stress transformation in a plane is often of interest. Referring to Figure 7, suppose that the in-plane stress components acting in the x_1, x_2 plane are given, as in the upper left of the figure. We wish to find the in-plane stress components acting across a surface which is tilted about the x_3 axis at angle θ, measured positive anti-clockwise relative to the x_1 axis as shown. That surface has unit normal **n** and we let **s** be an orthogonal unit vector along the surface in the x_1, x_2 plane, also as shown. The normal stress to be determined is called σ_n and the shear stress τ_{ns}; see the upper center diagram in Figure 7. Letting **T** be the stress vector on the inclined plane, we obtain the stress components as $\sigma_n=\mathbf{n}\cdot\mathbf{T}$ and $\tau_{ns}=\mathbf{s}\cdot\mathbf{T}$. Thus, recalling the previous expression for **T**,

$$\sigma_n=\mathbf{n}\cdot[\sum_{j=1}^{3}(\sum_{i=1}^{3}n_i\sigma_{ij})\mathbf{e}_j]=\sum_{j=1}^{3}\sum_{i=1}^{3}n_in_j\sigma_{ij}\ ,\text{ and }\tau_{ns}=\mathbf{s}\cdot[\sum_{j=1}^{3}(\sum_{i=1}^{3}n_i\sigma_{ij})\mathbf{e}_j]=\sum_{j=1}^{3}\sum_{i=1}^{3}n_is_j\sigma_{ij}\ .$$

Since $\mathbf{n}=(-\sin\theta,\ \cos\theta,\ 0)$ and $\mathbf{s}=(\cos\theta,\ \sin\theta,\ 0)$, these reduce (recalling that $\sigma_{12}=\sigma_{21}$) to

$$\sigma_n=\sigma_{11}\sin^2\theta-2\sigma_{12}\sin\theta\cos\theta+\sigma_{22}\cos^2\theta=\frac{\sigma_{11}+\sigma_{22}}{2}+\frac{\sigma_{22}-\sigma_{11}}{2}\cos2\theta-\sigma_{12}\sin2\theta\ ,$$
$$\tau_{ns}=-\sigma_{11}\sin\theta\cos\theta+\sigma_{12}(\cos^2\theta-\sin^2\theta)+\sigma_{22}\sin\theta\cos\theta=\frac{\sigma_{22}-\sigma_{11}}{2}\sin2\theta+\sigma_{12}\cos2\theta\ .$$

A little analysis shows that the latter expressions are the parametric equations (with parameter θ) of a circle in a *Mohr plane* whose axes are σ_n and τ_{ns}; the circle is called the *Mohr circle*. It has center at $(\sigma_{11}+\sigma_{22})/2$ along the σ_n axis, has radius $\sqrt{(\sigma_{22}-\sigma_{11})^2/4+\sigma_{12}^2}$, and rotation of the inclined interface anti-clockwise by θ in the physical (x_1, x_2) plane corresponds to anti-clockwise rotation by 2θ in the Mohr plane.

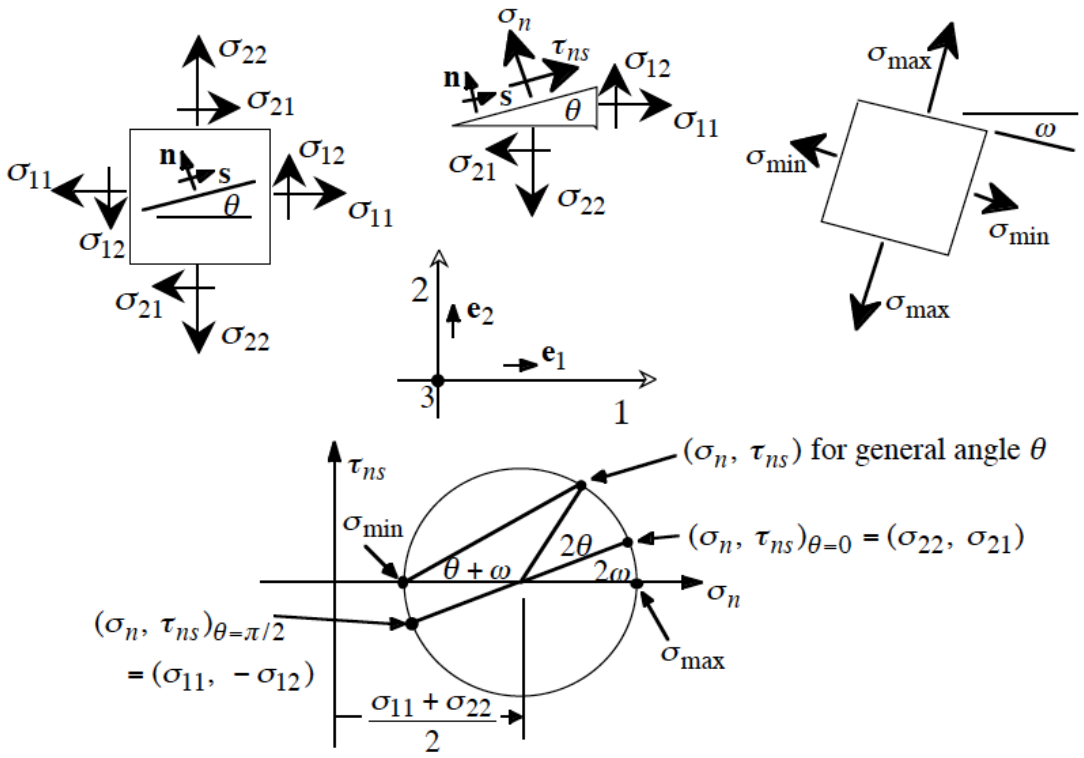

Figure 7. Mohr circle representation of stress transformation in a plane. A general stress state is shown at the upper left. The circle is used to determine the normal stress σ_n and shear stress τ_{ns}, upper center, acting on a plane inclined at angle θ.

The quickest way to construct the Mohr circle is usually to identify two points which lie on it, such uniquely locating the circle given that its center lies on the σ_n axis. Thus, we first observe that $(\sigma_n, \tau_{ns}) = (\sigma_{22}, \sigma_{21})$ must be the point on the circle corresponding to $\theta = 0$, and then that $(\sigma_n, \tau_{ns}) = (\sigma_{11}, -\sigma_{12})$ must be another point on it, corresponding to $\theta = \pi/2$. Both of those points are labeled in Figure 7. Once the circle is drawn, the stress state for a general orientation at angle θ is given by rotating in the same sense around the circle, by angle 2θ, from the point $(\sigma_{22}, \sigma_{21})$ on it corresponding to $\theta = 0$. As seen in the figure, an angle 2ω may be defined as that marked, and then $\theta = -\omega$ is the face orientation for which the maximum

in-plane normal stress $\sigma_{\max}$ acts. That orientation is shown in the upper right of the figure, where $\sigma_{\max}$ and the least in-plane normal stress $\sigma_{\min}$ are the extremity points of the Mohr circle along the σ_n axis. Finally, by an elementary geometric relation, the angle $\theta + \omega$ between the general orientation considered and that of the maximum in-plane normal stress can be identified as marked.

Further remarks. Occasionally there is need for, or use of, continuum theories in which distributed surface couples (torques without net force) are assumed to act over each element dS of surface, thus defining a *couple stress* vector, and also there may be appeal to the notion of body couples in addition to body forces acting on elements dV of volume. We do not consider such cases here but, in such theories, the stress tensor is not symmetric and its anti-symmetric part balances the body couple as well as the gradient of a third rank couple stress tensor which may then be defined. Such notions appear to have originated with W. Voigt in 1887; the formal theory was developed by the Italian elasticians and mathematicians, the brothers E. and F. Cosserat, in 1909, and was revived and greatly extended by Mindlin in the 1960's.

While the angular momentum principle must be accepted as an independent law of physics, supplementing the linear momentum principle, there are situations for which the angular principle can be derived by combining the linear principle with some special hypothesis concerning interactions of material elements with one another. For example, a view of matter as particles which exert equal and opposite forces pairwise on one another, directed along the line joining a particle pair, leads to the angular momentum principle as a derived consequence. (However, that particle model is known to be too simple a model for atoms in a crystal; it leads to a "Cauchy relation" between elastic constants, for which the elastic moduli C_{ijkl} to be introduced later are unaffected by interchange of any two indices and, for an isotropic material, to a Poisson ratio of 1/4. Such relations were widely discussed following the studies of that particle model by Navier, Cauchy, and Poisson in the 1820's and 1830's, but they are not generally observed to hold experimentally.) For elastic solids, the simple assumption that the strain energy of deformed material is unaffected by a superposed rigid rotation is enough to derive $\sigma_{ij} = \sigma_{ji}$, and hence to derive the angular principle as a consequence of the linear principle.

This exposition has sought consequences of the basic laws of mechanics in the most generally useful context for three dimensional solids. Very often, both in nature and technology, there is interest in structural elements in forms that might be identified as *strings, wires, rods,*

bars, beams, or *columns*, or as *membranes, plates,* or *shells*. These are usually idealized as, respectively, one- or two-dimensional continua. One possible approach is to develop the consequences of the linear and angular momentum principles entirely within that idealization, working in terms of net axial and shear forces, and bending and twisting torques, at each point along a one-dimensional continuum, or in terms of forces and torques per unit length of surface in a two-dimensional continuum. In fact such notions for the bending of beams as modeled as one-dimensional elastic lines predate Cauchy's formalization of the stress principle and, as has been noted, were introduced by James Bernoulli and Euler in the flowering of mechanics in the 1700's following Newton's work. Formulations for such structures can also be based directly on the three dimensional theory, as generally simplified by making approximate assumptions concerning the variation of certain stress or strain components along the thin direction(s), typically these being assumptions which become exact in long wavelength limits. Such approaches to beams began also long before Cauchy and Navier, with Coulomb's 1776 analysis of stresses induced by the bending of a beam of finite cross section.

Geometry of Deformation: Strain, Strain-Displacement Relations, Compatibility

The shape of a solid or structure changes with time during a deformation process. To characterize deformation, we adopt a certain *reference* configuration which we agree to call *undeformed*. Often, that reference configuration is chosen as an unstressed state, but such is neither necessary nor always convenient. Measuring time from zero at a moment when the body exists in that reference configuration, we may then use the upper case $\mathbf{X}$ to denote the position vectors of material points when t = 0. At some other time t, a material point which was at $\mathbf{X}$ will have moved to some spatial position $\mathbf{x}$. We thus describe the deformation as the mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, with $\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$. The *displacement vector* $\mathbf{u} = \mathbf{x}(\mathbf{X}, t) - \mathbf{X}$ and, also, $\mathbf{v} = \partial\mathbf{x}(\mathbf{X}, t)/\partial t$ and $\mathbf{a} = \partial^2\mathbf{x}(\mathbf{X}, t)/\partial t^2$.

It is simplest to write equations for strain in a form which, while approximate in general, is suitable for the case when any infinitesimal line element $d\mathbf{X}$ of the reference configuration undergoes extremely small rotations and fractional change in length, in deforming to the corresponding line element $d\mathbf{x}$. These conditions are met when $\left|\partial u_i / \partial X_j\right| << 1$. The solids with which we deal are very often sufficiently rigid, at least under the loadings typically applied to them, that these conditions are realized in practice. Linearized expressions for strain in terms of $[\partial u/\partial X]$, appropriate to this situation, are called *small strain* or *infinitesimal strain*. Expressions for strain will also be given that are valid for rotations and fractional length changes of arbitrary

magnitude; such expressions are called *finite strain*.

Two simple types of strain are *extensional* strain and *shear* strain. Consider a rectangular parallelpiped, a brick-like block of material with mutually perpendicular planar faces, and let the edges of the block be parallel to the 1, 2 and 3 axes. If we deform the block homogeneously, so that each planar face moves perpendicular to itself and such that the faces remain orthogonal (i.e., the parallelpiped is deformed into another rectangular parallelpiped), then we say that the block has undergone extensional strain relative to each of the 1, 2 and 3 axes, but no shear strain relative to these axes. Denote the edge lengths of the undeformed parallelpiped as ΔX_1, ΔX_2 and ΔX_3, and those of the deformed parallelpiped as Δx_1, Δx_2 and Δx_3; see Figure 8, where the dashed-line figure represents the reference configuration and solid-line the deformed configuration. Then the quantities $\lambda_1 = \Delta x_1/\Delta X_1$, $\lambda_2 = \Delta x_2/\Delta X_2$, and $\lambda_3 = \Delta x_3/\Delta X_3$ are called *stretch ratios*. There are various ways that extensional strain can be defined in terms of them. Note that the change in displacement in, say, the x_1 direction between points at one end of the block and those at the other is $\Delta u_1 = (\lambda_1 - 1)\,\Delta X_1$. For example, if E_{11} denotes the extensional strain along the x_1 direction, then the most commonly understood definition of strain is

$$E_{11} = (\text{change in length})/(\text{initial length}) = (\Delta x_1 - \Delta X_1)/\Delta X_1 = \Delta u_1/\Delta X_1 = \lambda_1 - 1\ .$$

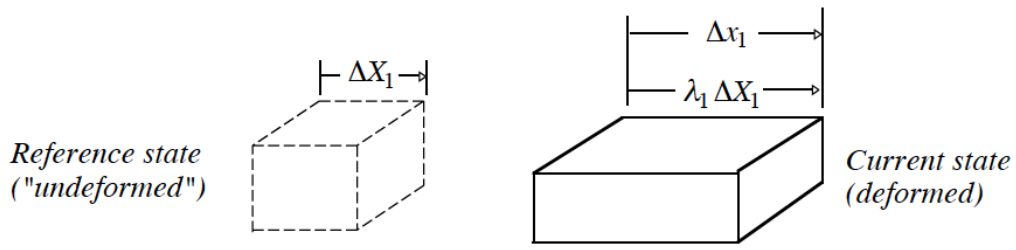

Figure 8. Extensional strain; element in reference configuration shown with dashed lines; λ_1 is the stretch ratio for a fiber initially oriented in the 1 direction.

A variety of other measures of extensional strain can be defined by $E_{11} = g(\lambda_1)$ where the function $g(\lambda)$ satisfies $g(1) = 0$ and $g'(1) = 1$, so as to agree with the above definition when λ_1 is very near 1. Two such in common use are the strain E_{11}^M based on the change of *metric* tensor, and the *logarithmic* strain E_{11}^L:

$$E_{11}^M = (\lambda_1^2 - 1)/2, \quad E_{11}^L = \ln(\lambda_1)\ .$$

We say that the parallelpiped considered has undergone homogeneous strain if the strain of each smaller parallelpiped that can be drawn within it has the same set of stretch ratios λ_1, λ_2, λ_3. Thus, for the homogeneous extensional strain considered $x_1 = \lambda_1 X_1 + C_1$, $x_2 = \lambda_2 X_2 + C_2$, $x_3 = \lambda_3 X_3 + C_3$, where C_1, C_2 and C_3 are constants.

To define a *simple shear* strain, consider the same rectangular parallelpiped but now deform it so that every point on a plane of type X_2 = constant moves only in the x_1 direction, and by an amount that increases linearly with X_2. Thus the deformation $x_1 = \gamma X_2 + X_1$, $x_2 = X_2$, $x_3 = X_3$ defines a homogeneous simple shear strain of amount γ, and is illustrated in Figure 9. Note that this strain causes no change of volume.

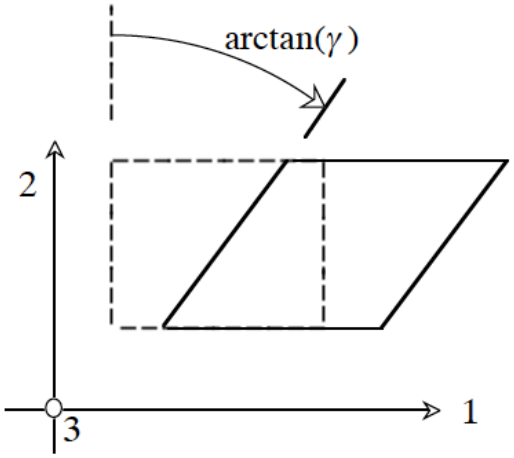

Figure 9. Simple shear strain; element in reference configuration shown with dashed lines.

For small strain we can identify the shear strain γ as the change in angle between two initially perpendicular lines. That is, choose a point and draw, in the undeformed configuration, two small line elements from it, one in the direction of increasing X_1 and the other of increasing X_2. The angle between the line elements as measured in radians is initially $\pi/2$ but, when $|\gamma| << 1$, reduces to $\pi/2 - \gamma$. (For finite simple shear strain as above, γ would be replaced by arctan γ in this interpretation.)

Small strain tensor. To define the *small strains*, or *infinitesimal strains*, appropriate for situations with $\left|\partial u_i / \partial X_j\right| << 1$ for all i and j, and for which we use the symbols ε_{ij}, we can proceed as follows: Two infinitesimal line segments, one initially in the 1 direction and the other in the 2 direction, are shown in Figure 10 as dashed lines in the reference configuration and as solid lines in the deformed configuration. Displacements are labeled so as to recognize that if u_i

denotes a displacement of the point initially at (X_1, X_2, X_3) then the corresponding displacement should, for example, be denoted as $u_i + dX_2\ \partial u_i/\partial X_2$ for the point which is initially an infinitesimal distance away at (X_1, $X_2 + dX_2$, X_3). If the material fibers shown in the 1 and 2 directions did not rotate, then the strains defined from their fractional changes in length would be exactly $\varepsilon_{11} = \partial u_1/\partial X_1$ and $\varepsilon_{22} = \partial u_2/\partial X_2$. These expressions remain correct, to the first order accuracy in components of [$\partial u/\partial X$] sought at present, in the general case when the fibers do rotate. The angle reduction between two fibers initially in the directions of increasing X_1 and X_2 is $\partial u_2/\partial X_1 + \partial u_1/\partial X_2$ to first order accuracy in [$\partial u/\partial X$], and this defines a shear strain which we can call γ_{12}. For the shear strain denoted ε_{12}, however, we use half of γ_{12}. Thus, considering all extensional and shear strains associated with infinitesimal fibers in the 1, 2 and 3 direction at a point of the material, the set of strains is given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j}\right) \qquad (i, j = 1,\ 2,\ 3).$$

Note that this definition implies that $\varepsilon_{ij} = \varepsilon_{ji}$.

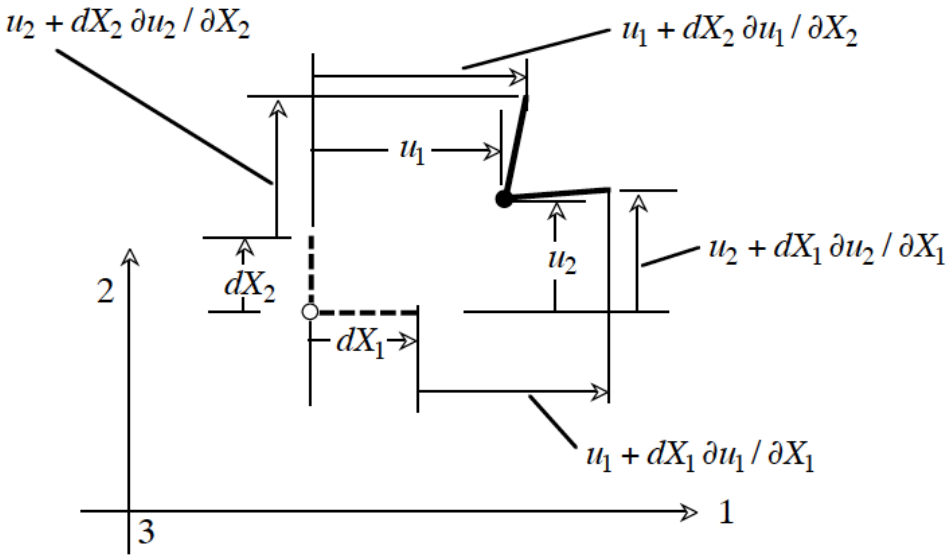

Figure 10. Deformation of line elements dX_1 and dX_2; origin of line elements has displaced by u_1 and u_2, and extremities of the elements have displaced by slightly different amounts, due to the displacement gradients $\partial u_i / \partial X_j$ (presumed small for purposes of this diagram, which is used to introduce infinitesimal strain ε_{ij}).

Strain tensor transformation. The mode of definition of the ε_{ij} above confers on them the property of being a second rank tensor. That is, if we chose Cartesian reference axes 1´, 2´, 3´ instead, and formed ε'_{kl}, then ε'_{kl} is related to the ε_{ij} by the same equations which relate the stresses σ'_{kl} to the σ_{ij}. That is

$$\varepsilon'_{kl} = \sum_{i=1}^{3} \sum_{j=1}^{3} \alpha_{ki} \alpha_{lj} \varepsilon_{ij} \ ,$$

which is the defining property of a second-rank tensor, where $\varepsilon'_{kl} = (1/2)(\partial u'_k / \partial x'_l + \partial u'_l / \partial x'_k)$. In proving that the ε'_{kl} do indeed transform according to such equation, the occasion will be taken to get a little more familiarity with the *summation convention*, adopted in the rest of this paragraph. Consider $\partial u'_k / \partial x'_l$ and use the chain rule to write

$$\partial u'_k / \partial x'_l = (\partial u'_k / \partial x_j)(\partial x_j / \partial x'_l).$$

Since **u** and **x** are vectors, we know that $u'_k = \alpha_{ki} u_i$ and $x'_l = \alpha_{lm} x_m$, where the latter has the inversion $x_j = \alpha_{nj} x'_n$ so that $\partial x_j / \partial x'_l = \alpha_{lj}$. Thus the above expression becomes

$$\partial u'_k / \partial x'_l = (\partial u'_k / \partial x_j)(\partial x_j / \partial x'_l) = (\alpha_{ki} \partial u_i / \partial x_j)(\alpha_{lj}) = \alpha_{ki} \alpha_{lj} \partial u_i / \partial x_j$$

which shows, incidentally, that the $\partial u'_k / \partial x'_l$ form a second-rank tensor. Now, the repeated *i* and *j* are just summation (or "dummy") indices in the above expression, and we could replace them with any other indices, say, *p* and *q*, to write equivalently $\partial u'_k / \partial x'_l = \alpha_{kp} \alpha_{lq} \partial u_p / \partial x_q$. We can use this to form the other term, $\partial u'_l / \partial x'_k$, needed in the strain expression, as

$$\partial u'_l / \partial x'_k = \alpha_{lp} \alpha_{kq} \partial u_p / \partial x_q = \alpha_{lj} \alpha_{ki} \partial u_j / \partial x_i = \alpha_{ki} \alpha_{lj} \partial u_j / \partial x_i$$

where, for the second equality, summation indices *p* and *q* have been traded, respectively, for *j* and *i*. Thus, adding together the two last transformation expressions

$$(1/2)(\partial u'_k / \partial x'_l + \partial u'_l / \partial x'_k) = \alpha_{lj} \alpha_{ki} [(1/2)(\partial u_i / \partial x_j + \partial u_j / \partial x_i)]$$

which is the same as $\varepsilon'_{kl} = \alpha_{lj} \alpha_{ki} \varepsilon_{ij}$.

Principal strains. We have thus proven that ε_{ij} is a second-rank tensor. It is also

symmetric, $\varepsilon_{ij} = \varepsilon_{ji}$. Paralleling the case for the stress tensor σ_{ij}, these mathematical features require that there exist *principal strain directions*. That is, at every point of the strained continuum it is possible to identify three mutually perpendicular directions along which there is purely extensional strain, with no shear strain between these special directions. The directions are the principal directions and we denote the corresponding strains, ordered algebraically, as ε_{I}, $\varepsilon_{\mathrm{II}}$, and $\varepsilon_{\mathrm{III}}$; these include the least and greatest extensional strains experienced by fibers through the material point considered. While developed here within small (infinitesimal) strain analysis, the result on the existence of mutually perpendicular principal directions, relative to which there is extensional strain but no shear strain, applies at finite strain too. In general, for a fiber with orientation along the direction of the unit vector $\mathbf{n}^o$ in the reference configuration, the exensional strain, according to the small strain measure ε_{ij} now considered, is

$$\varepsilon = \sum_{i=1}^{3} \sum_{j=1}^{3} n_i^o n_j^o \varepsilon_{ij}$$

Invariants of the strain tensor ε_{ij} may be defined in a way paralleling those for the stress tensor.

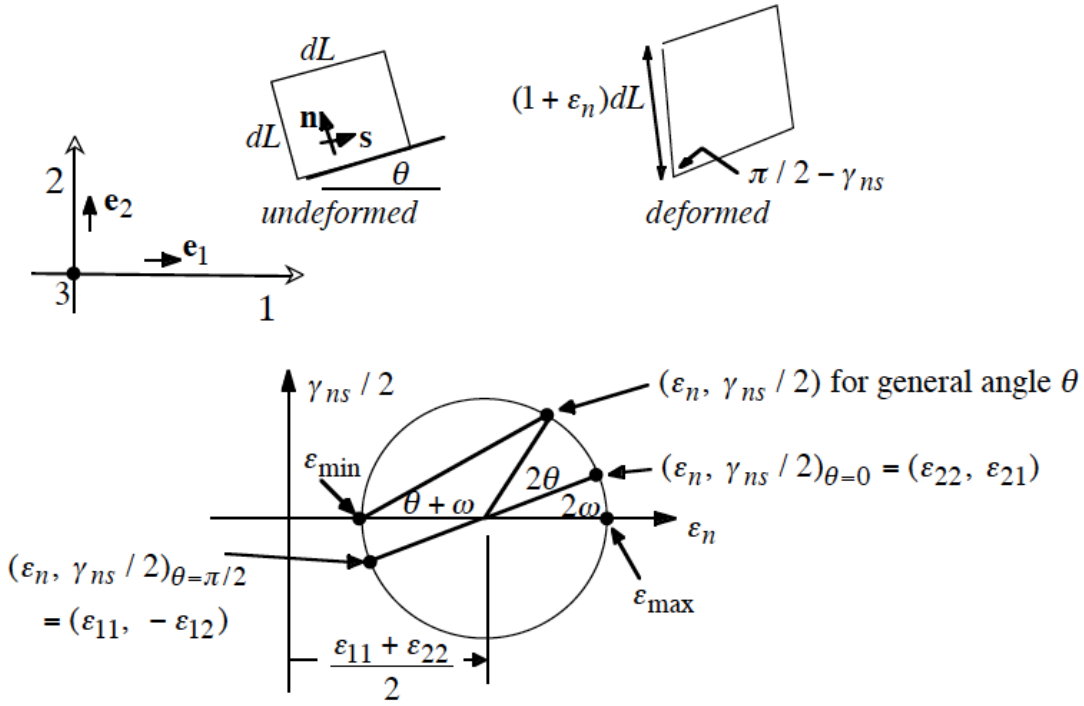

Figure 11. Mohr circle for strain transformation in a plane.

Mohr circle for strain transformation in a plane. Since the same transformation equations

apply as for stress, there must be a Mohr circle for strain transformation in a plane. This is illustrated in Figure 11. An infinitesimal element in the form of a cube of side dL is shown in the undeformed reference configuration. One of its faces is perpendicular to $\mathbf{e}_3$ but the others have been rotated by θ anticlockwise about that vector, so that the orthogonal unit vectors **s** and **n** shown, rather than $\mathbf{e}_1$ and $\mathbf{e}_2$, are perpendicular to the other cube faces. Extensional strain ε_n parallel to **n** and shear strain γ_{ns}, defined as an angle reduction between the **s** and **n** directions, are marked on the deformed sketch of the element. We recall that the tensor component of shear strain is $\gamma_{ns}/2$, so that the Mohr circle will apply in terms of axes labelled ε_n and $\gamma_{ns}/2$. The simplest route to construction of the circle is to recognize that we know the Mohr coordinate locations $(\varepsilon_n,\ \gamma_{ns}/2)$ along the circle, in terms of the given ε_{ij}, when θ=0 and when $\theta=\pi/2$. The extremities of the Mohr circle mark the maximum and minimum in-plane strains, obtained respectively in the **n** and **s** directions when those are oriented such that $\theta = -\omega$, where 2ω is defined in Figure 11.

Strain compatibility. An important fact to note is that the strains cannot vary in an arbitrary manner from point to point in the body. That is because the six strain components are all derivable from three displacement components. Restrictions on strain resulting from such considerations are called *compatibility relations*; the body would not fit together after deformation unless they were satisfied. Consider, for example, a state of *plane strain* in the 1, 2 plane (so that $\varepsilon_{33} = \varepsilon_{23} = \varepsilon_{31} = 0$). The three non-zero strains, $\varepsilon_{11} = \partial u_1/\partial X_1$, $\varepsilon_{12} = (1/2)(\partial u_2/\partial X_1 + \partial u_1/\partial X_2)$ and $\varepsilon_{22} = \partial u_2/\partial X_2$, are in this case derivable from the two displacements u_1 and u_2. So those strains cannot vary arbitrarily from point to point but must satisfy some single equation of constraint. In this case (plane strain) that equation of constraint , or *compatibility equation*, is

$$\partial^2\varepsilon_{11}/\partial X_2^2 + \partial^2\varepsilon_{22}/\partial X_1^2 = 2\partial^2\varepsilon_{12}/\partial X_1\partial X_2$$

as may be verified by directly inserting the relations for strains in terms of displacements.

In the general 3D case, the strain compatibility equations are the set

$$\partial^2\varepsilon_{ij}/\partial X_k\partial X_l + \partial^2\varepsilon_{kl}/\partial X_i\partial X_j = \partial^2\varepsilon_{ik}/\partial X_j\partial X_l + \partial^2\varepsilon_{jl}/\partial X_i\partial X_k\ .$$

These comprise 81 equations, but most are identities (all cases when 3 or more of i, j, k, l are identical) or replications of one another. In the end we have 6 equations. Three of them,

resulting when $i = j$, $k = l$, but $i \neq k$, have the same form of that written for plane strain above (which corresponds to $i = j = 1$ and $k = l = 2$). The other three, resulting when $i = j$ but $i \neq k$, $i \neq l$ and $k \neq l$ have the form (e.g., corresponding to $i = j = 1$, $k = 2$, $l = 3$)

$$\partial^2 \varepsilon_{11} / \partial X_2 \partial X_3 = \partial(\partial \varepsilon_{12} / \partial X_3 - \partial \varepsilon_{23} / \partial X_1 + \partial \varepsilon_{31} / \partial X_2) / \partial X_1 \ .$$

These 6 equations are not completely independent of one another, in that we have, fundamentally, need for only three constraints (six strains derivable from three displacements).

Small strain analysis. When the smallness of stretch and rotation of line elements allows use of the infinitesimal strain tensor, a derivative $\partial/\partial X_i$ will be very nearly identical to $\partial/\partial x_i$. Frequently, but not always, it will then be acceptable to ignore the distinction between the deformed and undeformed configurations in writing the governing equations of solid mechanics. For example, the differential equations of motion in terms of stress are rigorously correct only with derivatives relative to the deformed configuration but, in the circumstances considered, the equations of motion can be written relative to the undeformed configuration. This is what is done in the most widely used variant of solid mechanics, in the form of the theory of *linear elasticity*. The procedure can go badly wrong in some important cases, like for columns under compressive loadings so that buckling occurs, or for elastic-plastic materials when the slope of the stress versus strain relation is of the same order as existing stresses; these cases are best approached through finite deformation theory.

Finite deformation and strain tensors. The theory of finite deformations, with extension and rotations of line elements being unrestricted as to size, contains the small strain formulation as a limiting case. Consider an infinitesimal fiber through the point considered which is, initially, given as the vector d**X**. If that fiber deforms at the time considered to the vector d**x** then, defining $F_{ij} = \partial x_i(\mathbf{X}, t) / \partial X_j$, we can write

$$dx_i = \sum_{j=1}^{3} F_{ij} dX_j$$

The 3 by 3 matrix [F], with components F_{ij}, is called the *deformation gradient*; the F_{ij} transform as the components of a second rank tensor when we change from one set of Cartesian reference axes to another. Two special types of [F] are of interest, and the most general [F] can be composed of the two of them. The first type is a *rigid rotation*, [F] = [R] where [R] has the same

properties as an orthogonal transformation matrix [α]. When [F] = [R], each d**x** has the same length as the corresponding d**X**, and the angle between two line elements $d\mathbf{X}'$ and $d\mathbf{X}''$ is the same as between $d\mathbf{x}'$ and $d\mathbf{x}''$. The other special type of [F] is [F] = [U] where the components U_{ij} of [U] are symmetric, $U_{ij} = U_{ji}$, and det[U] > 0; [U] corresponds to a *pure deformation*. It is possible to show that when [F] = [U], there exist three special, mutually orthogonal directions, called principal directions, with the property that if d**X** lies along one of these directions, then so does d**x**. Thus fibers in these three special directions do not rotate during the deformation; and there is no shearing deformation between them. Further these fibers include the fibers with the least and greatest stretch ratios, denoted λ_I and λ_{III}, respectively, among all fibers through the point considered.

It may be shown that a general deformation gradient [F] may be represented as a pure deformation followed by a rigid rotation; this result is called the *polar decomposition theorem*, and takes the form, in matrix notation [F] = [R][U]. Thus, for an arbitrary deformation, there exist three mutually orthogonal principal stretch directions at each point of the material; call these directions in the reference configuration $\mathbf{N}^{(I)}$, $\mathbf{N}^{(II)}$, $\mathbf{N}^{(III)}$ and let the stretch ratios be λ_I, λ_{II}, λ_{III}. Fibers in these three principal directions undergo extensional strain but have no shearing between them. The three fibers in the deformed configuration remain orthogonal but are rotated by the operation [R] relative to their orientation in the reference configuration.

Recall that an extensional strain may be defined by E = g(λ) where g(λ) is any function satisfying g(1) = 0 and g´(1) = 1, with examples for g(λ) given above. We may then define a *finite strain* E_{ij} based on any particular function g(λ) by

$$E_{ij} = g(\lambda_I)N_i^{(I)}N_j^{(I)} + g(\lambda_{II})N_i^{(II)}N_j^{(II)} + g(\lambda_{III})N_i^{(III)}N_j^{(III)},$$

and any E_{ij} so defined may readily be shown to transform according to the law defining a second-rank tensor.

Usually, it is rather difficult to actually solve for the λ's and **N**'s associated with any general [F] (= [∂x/∂X]), so it is not easy to use this strain definition. However, for the special choice identified as $g^M(\lambda) = (\lambda^2 - 1)/2$ above, it may be shown that

$$E_{ij}^M = \frac{1}{2}\left(\sum_{k=1}^{3} F_{ki}F_{kj} - \delta_{ij}\right)$$

Remembering that $x_i = X_i + u_i$, so that $F_{ij} = \partial x_i / \partial X_j = \delta_{ij} + \partial u_i / \partial X_j$, we may show that

$$E_{ij}^M = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} \right)$$

which, like the finite strain generated by any other g(λ), reduces to ε_{ij} when $\left|\partial u_k / \partial X_l\right| << 1$ so that we need retain only the linear terms.

To illustrate the *summation convention* and *comma notation*, the last two expressions for E_{ij}^M could instead be written as $E_{ij}^M = (1/2)(F_{ki}F_{kj} - \delta_{ij}) = (1/2)(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j})$, but where here we must explain that $G_{,i}$ corresponds to $\partial G / \partial X_i$, not to $\partial G / \partial x_i$.

It is sometimes convenient to think in terms of lines in the material, parallels to the 1, 2 and 3 axes, being carried along, or *convected*, with the deformation, so that the labels X_1, X_2, X_3 correspond to the labeling of a system of curvilinear coordinates, convected with the deformation. This viewpoint was introduced by Hencky in 1925. When the square of the length of an infinitesimal line element is written as a quadratic form in the corresponding increments of curvilinear reference coordinates, the coefficients in the quadratic form are called the *metric tensor* g_{ij}. Thus, observing that the E_{ij}^M appear in the equation

$$d\mathbf{x} \cdot d\mathbf{x} = \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\delta_{ij} + 2E_{ij}^M \right) dX_i dX_j ,$$

we can identify g_{ij} with $\delta_{ij} + 2E_{ij}^M$, so that $2E_{ij}^M$ is the change in metric tensor of the convected coordinates in going from the reference to deformed configuration. The strain E_{ij}^M is also known as the *Green* strain or the *Lagrangian* strain.

Some Comments on Work and Energy

Virtual work. Consider a solid in its deformed configuration at some time t and let $\delta\mathbf{u} = \delta\mathbf{u}(\mathbf{x})$ be some virtual, or imagined, infinitesimal displacement field. Using the various equations derived from the linear and angular momentum principles above we may derive the *Principle of Virtual Work*, namely that

$$\int_S \mathbf{T}\cdot\delta\mathbf{u}\,dS+\int_V(\mathbf{f}-\rho\mathbf{a})\cdot\delta\mathbf{u}\,dV=\int_V\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,\delta\varepsilon_{ij}\,dV$$

where

$$\delta\varepsilon_{ij}=\frac{1}{2}\left[\frac{\partial(\delta u_i)}{\partial x_j}+\frac{\partial(\delta u_j)}{\partial x_i}\right]$$

is the infinitesimal strain, as measured from the deformed configuration of the solid at time *t*, associated with the virtual displacement field.

If we assume that $\sigma_{ij}=\sigma_{ji}$ and that the Principle of Virtual Work holds for all possible continuous fields $\delta\mathbf{u}$ of virtual displacement and compatible strains [$\delta\varepsilon$], then it may be shown that all of the equations derived from the linear momentum principle are implied. The form of the principle here is an extension of what John Bernoulli introduced in 1717 for systems such as interconnected rigid bodies with frictionless joints, but reduces to his form when we consider only virtual displacements which are consistent with rigidity, so that the stress working terms of type $\sigma_{ij}\,\delta\varepsilon_{ij}$ vanish. The virtual work principle is usually taken as the preferred formulation in developing the *finite-element method* of computational solid mechanics analysis discussed earlier.

If we let $\delta\mathbf{u}$ be the actual displacement field throughout the solid over some infinitesimal time increment δt at *t*. Then, writing $\delta\mathbf{u}/\delta t = \mathbf{v}$ and $D_{ij}=(1/2)(\partial v_i/\partial x_j+\partial v_j/\partial x_i)$ (D_{ij} is called the *deformation rate* tensor), there follows the work-energy relation

$$\int_S \mathbf{T}\cdot\mathbf{v}\,dS+\int_V\mathbf{f}\cdot\mathbf{v}\,dV=\frac{d}{dt}\int_V\frac{1}{2}\rho\,\mathbf{v}\cdot\mathbf{v}\,dV+\int_V\sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}\,D_{ij}\,dV\,.$$

This shows that the rate of work of volume and surface forces equals the rate of change of kinetic energy when the solid moves as a rigid body (i.e., when each $D_{ij}=0$), but not when the solid deforms.

First law of thermodynamics. The first law of thermodynamics has a similar form,

$$\int_S (\mathbf{T}\cdot\mathbf{v} - q_n)\,dS + \int_V (\mathbf{f}\cdot\mathbf{v} + r)\,dV = \frac{d}{dt}\int_V \rho(e + \frac{1}{2}\mathbf{v}\cdot\mathbf{v})\,dV$$

where e is the *internal energy* per unit mass, q_n dS is the rate at which heat flows out through area dS the surface of the region considered, and r dV is the rate of heat supply by radiation to the volume dV. It then follows that

$$-\int_S q_n dS + \int_V (r + \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}D_{ij})dV = \int_V \rho\ de/dt\ \ dV$$

where de/dt is a derivative following the motion. From this equation we can show (consider a tetrahedron, size approaching zero) that the heat outflow at a place on the surface with normal **n** has the form $q_n = \mathbf{n}\cdot\mathbf{q}$, where we call **q** the *heat flux vector*. Also, by requiring the expression to hold for an arbitrary region, one concludes that

$$-\sum_{i=1}^{3}\partial q_i / \partial x_i + r + \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_{ij}D_{ij} = \rho\ de/dt$$

which is the local form of the first law of thermodynamics. This justifies an interpretation of the summation involving $\sigma_{ij}D_{ij}$ as the rate of stress working per unit volume.

Stress-Strain Relations

Linear elastic isotropic solid. The simplest case is that of the *linear elastic solid*, considered in circumstances for which $\left|\partial u_i / \partial X_j\right| << 1$ and distinctions between the deformed and undeformed configurations are neglected in writing equations of motion or of equilibrium. Consider first the case of *isotropic* materials, whose mechanical response is independent of the direction of stressing. If a material point sustains a stress state $\sigma_{11} = \sigma$, with all other $\sigma_{ij} = 0$, it is subjected to *uniaxial tensile stress*. This can be realized in a homogeneous bar loaded by an axial force. We then will expect extensional strain ε_{11} to develop in the direction of the stress, but that there will be no shearing relative to the 1, 2, 3 axes, and that negative extensional strains will develop in directions transverse to the tension. Hence we may write the strain in response to uniaxial tension $\sigma_{11} = \sigma$ as

$$\varepsilon_{11} = \sigma/E\ ,\quad \varepsilon_{22} = \varepsilon_{33} = -\nu\varepsilon_{11} = -\nu\sigma/E\ ,\quad \varepsilon_{12} = \varepsilon_{23} = \varepsilon_{31} = 0$$

Two new parameters have been introduced here, E and ν. The first, E is called *Young's modulus* and it has dimensions of [Force]/[Length]^2 and is measured in units such as Pa (= Pascal = 1 N/m^2), or dyne/cm^2, or psi (pounds per square inch). The second, ν, is dimensionless and is called the *Poisson ratio*. More generally, if stresses of type σ_{11}, σ_{22} and σ_{33} are applied, no shear strains are produced and we can write the extensional strains as the sum of responses to each stress individually (because the material is linear):

$$\varepsilon_{11} = \sigma_{11}/E - \nu\,(\sigma_{22} + \sigma_{33})/E\,, \quad \varepsilon_{22} = \sigma_{22}/E - \nu\,(\sigma_{33} + \sigma_{11})/E\,,$$
$$\varepsilon_{33} = \sigma_{33}/E - \nu\,(\sigma_{11} + \sigma_{22})/E\,.$$

If the isotropic solid is subjected only to shear stress, $\sigma_{12} = \sigma_{21} = \tau$, with all other $\sigma_{ij} = 0$, then the response is shearing strain of the same type,

$$\varepsilon_{12} = \tau/2G\,, \quad \varepsilon_{23} = \varepsilon_{31} = \varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 0\,.$$

Notice that because $2\varepsilon_{12} = \gamma_{12}$, this is equivalent to $\gamma_{12} = \tau/G$. The constant G introduced is called the *shear modulus*. Frequently, the symbol μ is used instead for it. If shear stresses of all types are applied the shear strains are

$$\varepsilon_{12} = \sigma_{12}/2G,\ \varepsilon_{23} = \sigma_{23}/2G,\ \varepsilon_{31} = \sigma_{31}/2G$$

but none of the shear stresses cause extensional strains.

Relation of G to E and ν. It is now clear that the strain response to a completely general stress state is given by this last set of equations and those for normal stress application above. However, the shear modulus G is not independent of E and ν, but is related to them by

$$G = E\,/\,[2(1+\nu)]\,.$$

This can be seen from the following arguments, presented in two ways because both are instructive. First consider the solid under stress $\sigma_{12} = \sigma_{21} = \tau$, producing strain $\varepsilon_{12} = \varepsilon_{21} = \tau/2G$. Now suppose we look at the same loading relative to a new set of axes 1´,2´,3´ produced by rotating the original set by $\pi/4$ (45 °) about the 3 axis so that 3 and 3´ coincide but 1´ is rotated away from 1, halfway towards 2, and 2´ away from 2, halfway towards negative 1. By the laws

of transformation discussed earlier, we calculate that our assumed stress and strain state is equivalent to

$$\sigma_{11}' = \tau, \quad \sigma_{22}' = -\tau, \quad \text{all other } \sigma_{kl}' = 0$$
$$\varepsilon_{11}' = \tau/2G, \quad \varepsilon_{22}' = -\tau/2G, \quad \text{all other } \varepsilon_{kl}' = 0$$

Since the material is isotropic, the response to normal stresses described above must also apply relative to the 1′, 2′, 3′ axes, and thus it is necessary that

$$\varepsilon_{11}' = \sigma_{11}'/E - \nu\sigma_{22}'/E, \quad \text{and hence that} \quad \tau/2G = \tau/E + \nu\tau/E$$

from which we see that $G = E/[2\,(1+\nu)]$.

For the second mode of derivation, suppose that we had chosen axes 1′, 2′, 3′ to coincide with the principal directions of stress at the point considered, so that the stresses σ_{kl}' contain no shear components. The principal axes of strain will align with those of stress when the material is isotropic, so that the strains ε_{kl}' likewise contain no shear component. Thus stress-strain relations will involve only E and ν. For example,

$$\varepsilon_{11}' = \sigma_{11}'/E - \nu(\sigma_{22}' + \sigma_{33}')/E = (1+\nu)\sigma_{11}'/E - \nu(\sigma_{11}' + \sigma_{22}' + \sigma_{33}')/E$$

Noting the last rearrangement, we see that the proper relation between all components of stress and strain along principal axes, for which shears vanish, is

$$\varepsilon_{kl}' = (1+\nu)\sigma_{kl}'/E - \nu\delta_{kl}'(\sigma_{11}' + \sigma_{22}' + \sigma_{33}')/E$$

where δ_{kl}' is the analog, on the 1′, 2′, 3′ axis system, to the δ_{ij} on the 1, 2, 3 system; both are defined as 1 when their indices agree and 0 otherwise. Using a property of the orthogonal transformation matrix $[\alpha]$, it may be observed that

$$\delta_{kl}' = \sum_{i=1}^{3} \alpha_{ki}\alpha_{li} = \sum_{i=1}^{3}\sum_{j=1}^{3} \alpha_{ki}\alpha_{lj}\delta_{ij},$$

so that the δ_{kl}' relate to the δ_{ij} by the same transformation rule of a second rank tensor which relates the σ_{kl}' to the σ_{ij} and the ε_{kl}' to the ε_{ij}. If we apply the inverse of that transformation to both sides of the above stress-strain relations on the 1′, 2′, 3′ system, and remember that the sum

of normal stresses is invariant to choice of axes, we conclude that for every choice of coordinate axes the stress-strain relations are

$$\varepsilon_{ij} = (1 + \nu)\sigma_{ij}/E - \nu\,\delta_{ij}(\sigma_{11} + \sigma_{22} + \sigma_{33})/E\,.$$

This confirms that $\varepsilon_{12} = (1 + \nu)\sigma_{12}/E$ and hence also demonstrates that $G = E/[2(1 + \nu)]$. Along the way, we have seen the working of an important concept: The laws of physics can be formulated as relations between tensors (not necessarily second order tensors, as they are in this case) and have the same form in all coordinate systems.

Other constants. The linear elastic stress-strain relations can be inverted to read

$$\sigma_{ij} = \lambda\delta_{ij}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu\varepsilon_{ij}$$

where here we have used μ rather than G as the notation for the shear modulus, following convention, and where $\lambda = 2\nu\mu/(1 - 2\nu)$. The elastic constants λ and μ are sometimes called the *Lamé constants*. Since ν is typically in the range 1/4 to 1/3 for hard polycrystalline solids, λ falls often in the range between μ and 2μ. (Navier's particle model with central forces leads to $\lambda = \mu$ for an isotropic solid.)

Another elastic modulus often cited is the *bulk modulus* K, defined for a linear solid under pressure p ($\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$) such that the fractional decrease in volume is p/K. If we consider a small cube of side length L in the reference state, observe that shearing strain does not change volume, and that the length along, say, the 1 direction changes to $(1 + \varepsilon_{11})L$, we see that the fractional change of volume is

$$(1 + \varepsilon_{11})(1 + \varepsilon_{22})(1 + \varepsilon_{33}) - 1 = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\,,$$

neglecting quadratic and cubic order terms in the ε_{ij} compared to linear, as appropriate when using linear elasticity. Thus $K = E/[3(1 - 2\nu)] = \lambda + 2\mu/3$.

Thermal strains. Temperature change can also cause strain. In an isotropic material the thermally induced extensional strains are equal in all directions, and there are no shear strains. In the simplest cases, we can treat these *thermal* strains as being linear in the temperature change, $\theta - \theta_0$ (where θ_0 is the temperature of the reference state) writing

$$\varepsilon_{ij}^{\text{thermal}} = \delta_{ij}\,\alpha\,(\theta - \theta_0)$$

for the strain produced by temperature change in the absence of stress. Here α is called the *coefficient of thermal expansion*. We now regard the strain expressed above, in terms of stress, as the *mechanical* part of the strain, $\varepsilon_{ij}^{\text{mech}}$, with $\varepsilon_{ij} = \varepsilon_{ij}^{\text{mech}} + \varepsilon_{ij}^{\text{thermal}}$. Thus, in cases of temperature change, we replace ε_{ij} in the stress-strain relations above with $\varepsilon_{ij} - \varepsilon_{ij}^{\text{thermal}}$, with the thermal part given as a function of temperature. Typically, when temperature changes are modest, we can neglect the small dependence of E and ν on temperature.

Anisotropy. Anisotropic solids are also common in nature and technology. Examples are: single crystals; polycrystals in which the grains are not completely random in their crystallographic orientation but have a "texture", typically due to some plastic or creep flow process which has left a preferred grain orientation; fibrous biological materials like wood or bone; and composite materials which, on a microscale, have the structure of reinforcing fibers in a matrix, with fibers oriented in a single direction or in multiple directions (e.g., to ensure strength along more than a single direction), or may have the structure of a lamination of thin layers of separate materials. In the most general case the application of any of the six components of stress induces all six components of strain, and there is no shortage of elastic constants. There would seem to be $6 \times 6 = 36$ in the most general case but, as will be seen, a consequence of the laws of thermodynamics is that the maximum number of independent elastic constants is 21 (compared to 2 for isotropic solids). In many cases of practical interest, *symmetry considerations* reduce the number to far below 21. Crystals of cubic symmetry, like rocksalt (NaCl), or face-centered-cubic metals such as aluminum, copper, or gold, or body-centered-cubic metals like iron at low enough temperature or tungsten, or non-metals such as diamond, germanium or silicon, have only 3 independent elastic constants. Also solids with a special direction, and with identical properties along any direction perpendicular to that direction, are called *transversely isotropic*, and have 5 independent elastic constants. Examples are provided by fiber-reinforced composite materials, with fibers that are randomly emplaced but aligned in a single direction in an isotropic, or transversely isotropic, matrix, and by single crystals of hexagonal-close-packing such as zinc.

General linear elastic stress strain relations have the form

$$\sigma_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl}\, \varepsilon_{kl}$$

where, because the ε_{kl} are symmetric, we can write $C_{ijkl} = C_{ijlk}$, and because the σ_{ij} are symmetric, $C_{ijkl} = C_{jikl}$. Hence the $3 \times 3 \times 3 \times 3 = 81$ components of C_{ijkl} reduce to the $6 \times 6 = 36$ mentioned, which thermodynamics further reduces. In cases of temperature change, we replace ε_{ij} with $\varepsilon_{ij} - \varepsilon_{ij}^{\text{thermal}}$ where $\varepsilon_{ij}^{\text{thermal}} = \alpha_{ij}(\theta - \theta_o)$ and α_{ij} is the set of thermal strain coefficients, with $\alpha_{ij} = \alpha_{ji}$.

An alternative matrix notation is sometimes employed, especially in the literature on single crystals. That approach introduces 6-element columns of stress and strain $\{\sigma\}$ and $\{\varepsilon\}$, defined so that the columns, when transposed (super-script T) or laid out as rows, are

$$\{\sigma\}^T = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})$$
$$\{\varepsilon\}^T = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31}).$$

These forms assure that the scalar $\{\sigma\}^T\{d\varepsilon\}$ is an increment of stress working per unit volume. The stress-strain relations are then written $\{\sigma\} = [c]\{\varepsilon\}$ where [c] is the 6 by 6 matrix of elastic moduli. Thus, $c_{13} = C_{1133}$, $c_{15} = C_{1123}$, $c_{44} = C_{1212}$, $c_{42} = C_{1222}$, etc. The thermodynamic requirement as explained below is that [c] be a symmetric matrix.

For isotropic elastic materials the independent constants may be chosen as any two of c_{11} ($= \lambda + 2\mu$), c_{12} ($= \lambda$), and c_{44} ($= \mu$). In solids of cubic symmetry they are chosen and tabulated in many materials data sources as c_{11}, c_{12} and c_{44}. For solids that are transversely isotropic about the 1 axis, the independent constants are c_{11}, c_{12}, c_{22}, c_{23}, c_{44} and, for example, others are given by $c_{55} = c_{22} - c_{23}$, $c_{21} = c_{12}$, etc. Sometimes elastic compliances, corresponding to elements of $[c]^{-1}$, the matrix inverse of [c], appearing as $\{\varepsilon\} = [c]^{-1}\{\sigma\}$ are given.

Thermodynamic considerations; second law. Thermodynamic principles constrain stress-strain relations for elastic materials as follows. The most primitive notion of elasticity is that stress is a function of strain and of temperature . (For rigorous theory applicable to arbitrarily large rotations and to however large stretches the solid can sustain while still remaining elastic, we modify this to say that the six stresses σ_{ij} are functions of the nine deformation gradients F_{ij}, and of temperature.) We need only consider states of spatially homogeneous deformation and temperature, since we have tacitly assumed that the local relation between stress, strain and

temperature is the same as for such homogeneous states. In thermodynamic terminology, a state of purely elastic material response corresponds to an *equilibrium* state, and a process during which there is purely elastic response corresponds to a sequence of equilibrium states and hence to a *reversible process*. Then, the *second law of thermodynamics* assures us that the heat absorbed per unit mass can be written $\theta\, ds$ where θ is *thermodynamic (absolute) temperature* and s is the *entropy* per unit mass. Hence, writing the work per unit volume of reference configuration in a manner appropriate to cases when infinitesimal strain can be used, and letting ρ_o be the density in that configuration, we have from the first law of thermodynamics that

$$\rho_o \theta\, ds + \sum_{k=1}^{3} \sum_{l=1}^{3} \sigma_{ij}\, d\varepsilon_{ij} = \rho_o de$$

This relation shows that if we express internal energy e as a function of entropy s and strains $[\varepsilon]$, and if we write e so as to depend identically on ε_{ij} and ε_{ji}, then

$$\sigma_{ij} = \rho_o\, \partial e([\varepsilon], s)/\partial \varepsilon_{ij}$$

Alternatively, we may introduce the *Helmholtz free energy f* per unit mass, $f = e - \theta s = f([\varepsilon], \theta)$, and show that

$$\sigma_{ij} = \rho_o\, \partial f([\varepsilon], \theta)/\partial \varepsilon_{ij}$$

The later form corresponds to the variables with which the stress-strain relations were written above. Sometimes $\rho_o f$ is called the *strain energy* for states of isothermal (constant θ) elastic deformation; $\rho_o e$ has the same interpretation for *isentropic* (s = constant) elastic deformation, achieved when the time sale is too short to allow heat transfer to or from a deforming element. Since the mixed partial derivatives must be independent of order, a consequence of the last equation is that

$$\partial \sigma_{ij}([\varepsilon], \theta)/\partial \varepsilon_{kl} = \partial \sigma_{kl}([\varepsilon], \theta)/\partial \varepsilon_{ij},$$

which requires that $C_{ijkl} = C_{klij}$, or equivalently that the matrix [c] be symmetric, $[c] = [c]^T$, reducing the maximum possible number of independent elastic constraints from 36 to 21. At constant temperature θ_o, the strain energy $W([\varepsilon]) = f([\varepsilon], \theta_o) = (1/2)\, \{\varepsilon\}^T [c] \{\varepsilon\}$.

Isothermal versus isentropic response. The $C_{ijkl} = \rho_0 \partial^2 f([\varepsilon], \theta)/\partial\varepsilon_{ij}\,\partial\varepsilon_{kl}$ are elastic moduli for isothermal response. Let $\overline{C}_{ijkl} = \rho_0 \partial^2 e([\varepsilon], s)/\partial\varepsilon_{ij}\,\partial\varepsilon_{kl}$ be the corresponding moduli for isentropic response. Reasoning from the first and second laws as embodied in eq. (xx), it follows that

$$\overline{C}_{ijkl} = C_{ijkl} + (\theta_o / \rho_o c_\varepsilon)\,\beta_{ij}\,\beta_{kl}, \quad \text{where } \beta_{pq} = \sum_{r=1}^{3}\sum_{s=1}^{3} C_{pqrs}\alpha_{rs};$$

here $c_\varepsilon = \theta_o \partial s([\varepsilon], \theta)/\partial\theta$, evaluated at $\theta = \theta_o$ and $[\varepsilon] = [0]$, and gives the specific heat at constant strain in the reference configuration. In the case of the elastically isotropic material, it is most convenient to give results in terms of G and K, the isothermal shear and bulk moduli. We find that the isentropic moduli are

$$\overline{G} = G, \quad \overline{K} = K\ (1 + 9\,\theta_o\,K\,\alpha^2 / \rho_o\,c_\varepsilon).$$

The shear modulus is unchanged and the fractional change in bulk modulus, given by the second term in the parenthesis, is very small, typically of the order of 1% or less even for metals and ceramics of relatively high α, of order 10^{-5}/degree Kelvin. The fractional change in absolute temperature during an isentropic deformation is found to involve the same small parameter:

$$[(\theta - \theta_o)/\theta_o]_{s\,=\,\text{const}} = -(9\,\theta_o\,K\,\alpha^2 / \rho_o\,c_\varepsilon)\,[(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) / 3\alpha\theta_o]$$

Values of α for most solid elements and inorganic compounds are in the range 10^{-6} to 4×10^{-5} / degree Kelvin, and room temperature is around 300 Kelvin, so $3\alpha\theta_o$ is typically in the range 10^{-3} to 4×10^{-2}. Thus, even if the fractional change in volume is of the order of 1%, which is quite large for a metal or ceramic deforming in its elastic range, the fractional change in absolute temperature is only of order 1%.

For those reasons, it is usually appropriate to neglect the alteration of the temperature field due to elastic deformation, and hence to use purely mechanical formulations of elasticity in which distinctions between isentropic and isothermal response are neglected. Temperature changes, inducing thermal strains, are then considered only as specified quantities, perhaps calculated from considerations of heat transfer, but in a manner that is de coupled from the elastic deformation of the solid. The same approach will not always be adequate when significant non-elastic deformation occurs (a metal wire that is flexed back and forth in the

plastic range gets noticeably warm), or in the presence of strong shock waves, or when we wish to consider what it is that ultimately damps free vibrations of a solid under conditions for which there is negligible inelastic deformation, or resistance from surrounding fluids, or frictional slip at support points. The latter point refers to *thermoelastic dissipation* due to small fluctuating heat conduction currents that act to level-out small temperature fluctuations due to tendencies towards locally isentropic response in a vibrating solid. In order to deal with coupled thermal and mechanical phenomena, we must add to our set of governing equations the local form of the first law of thermodynamics, as on page xx above, expressing the heat flux within it by

$q_i = -\sum_{j=1}^{3} k_{ij} \partial\theta / \partial x_j$ where the conductivities [k]

must, according to the second law of thermodynamics, form a positive definite matrix.

Finite elastic deformations. When we deal with elastic response under arbitrary deformation gradients, because rotations, if not strains, are large or, in a material such as rubber, because the strains are large too, it is necessary to dispense with the infinitesimal strain theory. Instead, the first and second laws of thermodynamics, combined as on page 36, have the form

$$\rho_o \theta ds + \det[F] \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} F_{ij}^{-1} \sigma_{jk} dF_{ki} = \rho_o de \, .$$

Here the F_{ij}^{-1} are components of $[F]^{-1}$, which is the matrix inverse of [F]. Also, det[F] – 1 is the fractional change in volume, from the reference to the deformed configuration, so that mass conservation requires that $\rho = \rho_o / \det[F]$. It may be noted that the form written for the stress working here incorporates no assumption that $\sigma_{ij} = \sigma_{ji}$, and hence may be considered an equation that is conceptually independent of the angular momentum principle.

For elastic response in this context, the free energy $f = e - \theta s$ is a function of [F] and θ, $f = f([F], \theta)$, and there results

$$\sigma_{ij} = \rho \sum_{k=1}^{3} F_{ik} \partial f([F], \theta) / \partial F_{jk}$$

for the general form of elastic stress-deformation gradient relations. If we have deformed a parcel of material by [F] and then give it some additional rigid rotation, we would insist that the free energy be unchanged in that rotation. In terms of the polar decomposition [F] = [R][U], this

is equivalent to saying that f is independent of the rotation part [R] or [F]. Thus f depends only on the pure deformation [U], which is equivalent to saying that f depends on the F_{ij} only as they appear in components of $[F]^T[F]$ (=[U][U]). This is equivalent to saying that f is a function of the finite strain measure $[E^M]$ based on change of metric or, for that matter, on any member of the family of material strain tensor discussed. Let us, then assume that f is a function of $[E^M]$ and θ, $f = f([E^M], \theta)$. In that case we may show that

$$\sigma_{ij} = (\rho / \rho_o) \sum_{k=1}^{3} \sum_{l=1}^{3} F_{ik} F_{jl} S_{kl}([E^M], \theta) \ ,$$

where S_{kl} (= S_{lk}) is sometimes called the *second Piola-Kirchhoff* stress, and is given by

$$S_{kl} = \rho_o \partial f([E^M], \theta) / \partial E^M_{kl} \ ,$$

it being assumed that f has been written so as to have identical dependence on E^M_{kl} and E^M_{lk}. This shows that $\sigma_{ij} = \sigma_{ji}$, so that the assumption on invariance of the free energy to rigid rotation makes the angular momentum principle redundant in this case.

Inelastic response. The above mode of expressing $[\sigma]$ in terms of [S] is valid for solids showing *viscoelastic* or *plastic* response as well, except that [S] is then to be regarded not only as a function of the present $[E^M]$ and θ, but to depend on the prior history of both. Assuming that such materials show elastic response to sudden stress changes, or to small unloading from a plastically deforming state, we may still express [S] as a derivative of f, as above, but the derivative is understood as being with respect to an elastic variation of strain and is to be taken at fixed θ and with fixed prior inelastic deformation and temperature history. Such dependence on history is sometimes represented as a dependence of f on *internal state variables* whose laws of evolution are part of the inelastic constitutive description. There are also simpler models of plastic or viscoelastic response and the most commonly employed forms for isotropic solids are presented next.

Plasticity and creep. To a good approximation, plastic deformation of crystalline solids causes no change in volume, and *hydrostatic* changes in stress, amounting to equal change of all normal stresses, have no effect on plastic flow, at least for changes that are of the same order or magnitude as the strength of the solid in shear. Thus plastic response can be formulated in terms of *deviatoric stress* , defined by $\tau_{ij} = \sigma_{ij} - \delta_{ij} (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$. This stress has a zero first

invariant and thus, in isotropic materials, plastic flow must be controlled by its second and third invariants. Following von Mises, in a procedure which is found to agree moderately well with experiment, the plastic flow relation is formulated in terms of the second invariant of deviatoric stress. That invariant is commonly rewritten as $\bar{\sigma}$, called the *equivalent tensile stress*, where

$$\bar{\sigma}^2 = (3/2)\sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}\tau_{ji}.$$

The definition is arranged so that for a state of uniaxial tension, $\bar{\sigma}$ equals the tensile stress, and the stress-strain relation for general stress states is formulated in terms of data from the tensile test. In particular, a plastic strain $\bar{\varepsilon}^p$ in a uniaxial tension test is defined from $\bar{\varepsilon}^p = \bar{\varepsilon} - \bar{\sigma}/E$, where here $\bar{\varepsilon}$ is interpreted as the strain in the tensile test according to the logarithmic definition, $\bar{\varepsilon} = \ln\lambda$, and the elastic modulus E is assumed to remain unchanged with deformation; also, in the situations considered $\bar{\sigma}/E << 1$.

Thus in the *rate-independent plasticity* version of the theory, tensile data (or compressive, with appropriate sign reversals) from a monotonic load test is assumed to define a function $\bar{\varepsilon}^p(\bar{\sigma})$. In the *viscoplastic* or *high-temperature creep* versions of the theory, tensile data is interpreted to define $d\bar{\varepsilon}^p/dt$ as a function of $\bar{\sigma}$ in the simplest case representing, for example, *secondary creep*, and as a function of $\bar{\sigma}$ and $\bar{\varepsilon}^p$ in theories intended to represent transient creep effects or rate-sensitive response at lower temperatures.

Rigid-plastic model. Consider first the *rigid-plastic* material model in which elastic deformability is ignored altogether, as sometimes appropriate for problems of large plastic flow as in metal forming or long term creep in the Earth's mantle, or for analysis of plastic collapse loads on structures. Then the rate of deformation [D] can be equated to what we will shortly consider to be its *plastic* part [D^p]. Following Levy, in an idea that would be said, since the formalization of the mathematical theory of plasticity starting in the early 1950's, to embody an *associated flow rule* with *normality* of [D^p] to surfaces of form $\bar{\sigma}$ = constant in a stress hyper-space, one assumes that [D^p] is distributed among its components just as is [τ], so that

$$D_{ij}^p = (3\tau_{ij}/2\bar{\sigma})d\bar{\varepsilon}^p/dt \qquad (i,j = 1,2,3).$$

The numerical factors secure agreement between D_{11}^p and $d\bar{\varepsilon}^p/dt$ for uniaxial tension in the 1

direction. Also, the equation implies that

$$D_{11}^p + D_{22}^p + D_{33}^p = 0\,,$$

which implies incompressibility of plastic flow, and that

$$(d\bar{\varepsilon}^p \,/\, dt)^2 = (2/3)\sum_{i=1}^{3}\sum_{j=1}^{3} D_{ij}^p D_{ji}^p\,,$$

which must be integrated over previous history to get $\bar{\varepsilon}^p$ as required for viscoplastic models in which $d\bar{\varepsilon}^p \,/\, dt$ is a function of $\bar{\sigma}$ and $\bar{\varepsilon}^p$.

Since $\bar{\sigma}$ is defined as a function of $[\tau]$, this formulation expresses $[D^p]$ as a function of $[\tau]$,, and possibly of the previous $[\tau]$, history, in its *viscoplastic* version. It expresses $[D^p]$ as an explicit function of $[\tau]$, and d$[\tau]$,/dt in its *rate-independent plasticity* version, since then one has $d\bar{\varepsilon}^p \,/\, dt = [d\bar{\varepsilon}^p(\bar{\sigma}) / d\bar{\sigma}]d\bar{\sigma} / dt$. In the rate-independent version, we define $[D^p]$ as *zero* whenever $\bar{\sigma}$ is less than the highest value that it has attained in the previous history, or when the current value of $\bar{\sigma}$ is the highest value but $d\bar{\sigma}/dt < 0$. (In the *elastic-plastic* context, this means that "unloading" involves only elastic response.) In the case of the *ideally-plastic solid*, which is idealized to be able to flow without increase of stress when $\bar{\sigma}$ equals the yield strength level, we regard $d\bar{\varepsilon}^p \,/\, dt$ as an undetermined but necessarily non-negative parameter, which can be determined (sometimes not uniquely) only through the complete solution of a solid mechanics boundary value problem.

Elastic-plastic model, co-rotational stress rate. The *elastic-plastic* material model is then formulated by writing

$$D_{ij} = D_{ij}^e + D_{ij}^p$$

where $[D^p]$ is given in terms of stress and possibly stress rate just as explained above, and where the elastic deformation rates $[D^e]$ are related to stresses by the usual linear elastic expression,

$$D_{ij}^e = \frac{1+\nu}{E}\dot{\sigma}_{ij}^* - \frac{\nu}{E}\delta_{ij}(\dot{\sigma}_{11}^* + \dot{\sigma}_{22}^* + \dot{\sigma}_{33}^*) \qquad (i,j = 1,2,3).$$

Here the stress rates are expressed as the Jaumann *co-rotational* rates

$$\dot{\sigma}^*_{ij} = \dot{\sigma}_{ij} + \sum_{k=1}^{3} (\sigma_{ik}\Omega_{kj} - \Omega_{ik}\sigma_{kj}) \ ;$$

where $\dot{\sigma}_{ij} = d\sigma_{ij}/dt$ is a derivative following the motion of a material point, and the *spin* Ω_{ij} is defined by

$$\Omega_{ij} = (1/2)(\partial v_i / \partial x_j - \partial v_j / \partial x_i) \ .$$

The co-rotational stress rates $\dot{\sigma}^*_{ij}$ are those calculated by an observer who spins with the average angular velocity of a material element. The elastic part of the stress strain relation should be consistent with the existence of a free energy f as discussed above. This is not strictly satisfied by the form just given, but the differences between it and one which is consistent in that way involves additional terms which are of order $\overline{\sigma} / E$ times the $\dot{\sigma}^*_{ij}$. These are quite negligible in typical cases in which the theory is used, since $\overline{\sigma} / E$ is usually an extremely small fraction of unity, say, 10^{-4} to 10^{-2}. A *small-strain* version of the theory is in common use for purposes of elastic-plastic stress analysis. In that one replaces $[D]$ with $\partial[\varepsilon(\mathbf{X}, t)] / \partial t$, where $[\varepsilon]$ is the small strain tensor, $\partial / \partial x$ with $\partial / \partial X$ in all equations, and $[\dot{\sigma}^*]$ with $\partial[\sigma(\mathbf{X}, t)] / \partial t$. The last two steps cannot always be justified even in cases of very small strain if, for example, in the case of a rate-independent material, $d\overline{\sigma} / d\overline{\varepsilon}^p$ is not large compared to $\overline{\sigma}$, or, just as is a concern for buckling problems in purely elastic solids, if rates of rotation of material fibers are much larger than rates of stretching.

Linear viscoelasticity. Here the focus is on situations for which the small strain tensor may be used. This excludes viscoelastic liquids, for which a treatment more in the spirit of that above for large plastic flow is necessary. Consider a bar under uniaxial stress σ and let ε be its strain. In the *creep test* the stress σ is applied and subsequently maintained constant. The strain is then denoted $\varepsilon = \sigma\, C(t)$ where $C(t)$ is called the *creep compliance* and corresponds to $1/E$ for a purely elastic solid; $C(0^+) = 1/E_u$ and, if the limit exists, $C(\infty) = 1/E_r$, where E_u and E_r ($< E_u$) are, respectively, the *unrelaxed* and *relaxed* elastic moduli. Also, $dC(t)/dt > 0$ for all finite t. The complementary test, in which a strain ε is applied and subsequently maintained constant, is called the *relaxation test*, and the resulting stress can be written as $\sigma = \varepsilon\, R(t)$. Here $R(t)$ is the *relaxation function*; $R(0^+) = E_u$ and $R(\infty) = E_r$; also, $dR(t)/dt < 0$ for all finite t. The Boltzmann superposition then represents response in a general loading history, starting at $t = 0$,

by
either of the equivalent forms

$$\varepsilon(t) = \int_{0-}^{t} C(t-s)\, d\sigma(s) / ds\, ds \quad \text{or} \quad \sigma(t) = \int_{0-}^{t} R(t-s)\, d\varepsilon(s) / ds\, ds$$

Such *convolutions* will subsequently be denoted as $\varepsilon = C*\sigma$ and $\sigma = R*\varepsilon$

The C(t) and R(t) are often approximated by solutions of the first order differential equation introduced by Jeffreys in 1929 and defining a *standard linear solid*,

$$Td\sigma / dt + \sigma = TE_u d\varepsilon / dt + E_r \varepsilon$$

where, in different circumstances, either T or $T E_u / E_r$ is called a *relaxation time*. The *Maxwell model*, first introduced in the hope of describing viscoelastic phenomena in gases, is given by setting $E_r = 0$ and writing $E_u = E$, so that $C(t) = (1 + t / T) / E$ and $R(t) = E \exp(-t / T)$. It is a special case of the *elastic-plastic* law of *secondary creep* type, discussed above, in which $d\bar{\varepsilon}^p / dt = \bar{\sigma} / TE$, and has the mechanical analog of a spring and viscous dashpot element in series. The *Kelvin model* is obtained by letting $E_u \to \infty$ and $T \to 0$ in such a way that $T E_u / E_r$ remains bounded. It has the analog of a spring and dashpot in parallel, and allows no long-term permanent deformation. A weakness is that it precludes instantaneous elastic response, but it can nevertheless be useful in modeling the damping of vibrations.

For the tensorial generalization in an isotropic material, it is simplest to consider the separate response to pure shear stress and to pure hydrostatic stress, introducing creep compliance functions S(t) and B(t) that correspond, respectively, to 1/G and 1/K. Thus if shear strain $\gamma = \tau S(t)$ is the response to a step loading by shear stress τ, and if $-p B(t)$ is the fractional change in volume in response to a step loading by hydrostatic pressure p, then the general stress-strain relations may be written as $\varepsilon_{ij} = (1/2) S*\tau_{ij} + (1/9) \delta_{ij} B*(\sigma_{11}+\sigma_{22}+\sigma_{33})$ where τ_{ij} is the deviatoric stress. In many cases the time dependence of B(t) is negligible compared to that of S(t) and one can just replace $B*(\sigma_{11}+\sigma_{22}+\sigma_{33})$ by $(\sigma_{11}+\sigma_{22}+\sigma_{33})/K$.

SOME PROBLEMS INVOLVING ELASTIC RESPONSE

Equations of linear elasticity, mechanical theory. The final equations of the purely

mechanical theory of linear elasticity (i.e., when we neglect coupling with the temperature field) and assume either isothermal or adiabatic response, with elastic moduli identified appropriately, are obtained as follows. We use the stress strain relations, given in the general case as on page xx write the strains in terms of displacement gradients as on page xx, and insert the final expressions for stress into the equations of motion given on page xx, replacing $\partial/\partial x$ with $\partial/\partial X$ in those equations, to obtain

$$\sum_{i=1}^{3} \frac{\partial}{\partial X_i} \left(\sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \frac{\partial u_k}{\partial X_l} \right) + f_j = \rho \frac{\partial^2 u_j}{\partial t^2} \quad (j = 1, 2, 3) .$$

Throughout this section we now use x_i and X_i interchangeably. In the case of an isotropic and homogeneous solid, the equations of motion reduce to the *Navier* equations

$$(\lambda + \mu) \frac{\partial}{\partial X_j} \left(\sum_{k=1}^{3} \frac{\partial u_k}{\partial X_k} \right) + \mu \sum_{k=1}^{3} \frac{\partial^2 u_j}{\partial X_k^2} + f_j = \rho \frac{\partial^2 u_j}{\partial t^2} \quad (j = 1, 2, 3),$$

which can be restated in the concise vector notation

$$(\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{f} = \rho \, \partial^2 \mathbf{u} / \partial t^2 ,$$

where $\nabla = \sum_{i=1}^{3} \mathbf{e}_i \frac{\partial}{\partial X_i}$ and ∇^2 is the Laplacian operator defined by $\nabla \cdot \nabla$. Such equations hold in the region V occupied by the solid; on the surface S one prescribes each component of **u**, or each component of **T** (expressed in terms of $[\partial u/\partial X]$), or sometimes mixtures of components or relations between them. For example, along a freely slipping planar interface with a rigid solid, the normal component of **u** and the two tangential components of **T** would be prescribed, all as zero.

Body wave solutions. By looking for *body wave* solutions in the form

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{p} \, f(\mathbf{n} \cdot \mathbf{X} - ct) ,$$

where unit vector **n** is the propagation direction, **p** is the *polarization*, or direction of particle motion, and c is the wave speed, one may show for the isotropic material that solutions exist for arbitrary functions $f(..)$ if either

$$c = c_d \equiv \sqrt{(\lambda + 2\mu)/\rho} \quad \text{and} \quad \mathbf{p} = \mathbf{n}\,, \quad \text{or} \quad c = c_s \equiv \sqrt{\mu/\rho} \quad \text{and} \quad \mathbf{p}\cdot\mathbf{n} = 0\,.$$

The first case, with particle displacements in the propagation direction, describes *longitudinal* or *dilatational* waves and the latter case, which corresponds to two linearly independent displacement directions, both transverse to the propagation direction, describes *transverse* or *shear* waves. For any given propagation direction **n** in a general anisotropic material, the wave speeds (squared) are the eigenvalues, and the polarizations the eigenvectors, solving

$$\sum_{k=1}^{3}\left(\sum_{i=1}^{3}\sum_{l=1}^{3} n_i\ C_{ijkl} n_l\right)\ p_k = \rho c^2 p_j \quad (j = 1,\ 2,\ 3)$$

Here the positive definiteness of the strain energy suffices to assure that there are three real positive solutions for c^2 with associated polarizations **p** that are mutually orthogonal (or can be chosen to be such when two c^2 values coincide like in the isotropic case). Further, as any such body wave propagates, the stress vector **T** it creates, on a surface with normal in the propagation direction, and the particle velocity **v** of the wave field satisfy $\mathbf{T} = -\rho\, c\, \mathbf{v}$. Thus, for example, if a uniform pressure p is suddenly applied to a certain area of a planar boundary of an isotropic elastic solid, points within that area will suddenly acquire the velocity $p/\rho c_d$, and will maintain that velocity until dilational waves arrive from the edges of the pressurized area or from other boundaries after reflection of the outgoing wave.

Linear elastic beam. The case of a beam treated as a linear elastic one-dimensional line may also be considered. Let the line lie along the 1 axis, Figure 12, have properties that are uniform along its length, and have sufficient symmetry that bending it by applying a torque about the 3 direction causes the line to deform into an arc lying in the 1,2 plane. Make an imaginary cut through the line and let the forces and torque acting at that section on the part lying in the direction of decreasing X_1 be denoted as a shear force V in the positive 2 direction, an axial force P in the positive 1 direction, and a torque M, commonly called a *bending moment*, about the positive 3 direction. The linear and angular momentum principles then require that the actions at that section on the part of the line lying in the direction of increasing X_1 are of equal magnitude but opposite sign. Now let the line be loaded by transverse force F per unit length, directed in the 2 direction, and make assumptions on the smallness of deformation consistent with those of linear elasticity. Let ρA be the mass per unit length (so that A could be interpreted as the cross section area of a homogeneous beam of density ρ) and let u the transverse displacement in the 2

direction.

Then, writing X for X_1, the linear and angular momentum principles require that

$$\partial V/\partial X + F = \rho A\, \partial^2 u/\partial t^2, \quad \partial M/\partial X + V = 0$$

where rotatory inertia has been neglected in the second equation, as is appropriate for long-wavelength disturbances compared to cross section dimensions. The curvature κ of the elastic line can be approximated by $\kappa = \partial^2 u / \partial X^2$ for the small deformation situation considered, and the equivalent of the stress-strain relation is to assume that κ is a function of M at each point along the line. The function can be derived by the analysis of stress and strain in pure bending and is $M = EI\kappa$ with $I = \int_A (X_2)^2 dA$ for uniform elastic properties over all the cross section, and with the 1 axis passing through the section centroid. Hence the equation relating transverse load and displacement of a linear elastic beam is

$$-\partial^2 (EI\ \partial^2 u/\partial X^2) / \partial X^2 + F = \rho A\, \partial^2 u/\partial t^2$$

and this is to be solved subject to two boundary conditions at each end of the elastic line. Examples are $u = \partial u/\partial X = 0$ at a completely restrained ("built in") end, $u = M = 0$ at an end that is restrained against displacement but not rotation, and $V = M = 0$ at a completely unrestrained (free) end. The beam will be reconsidered later in an analysis of response with initial stress present.

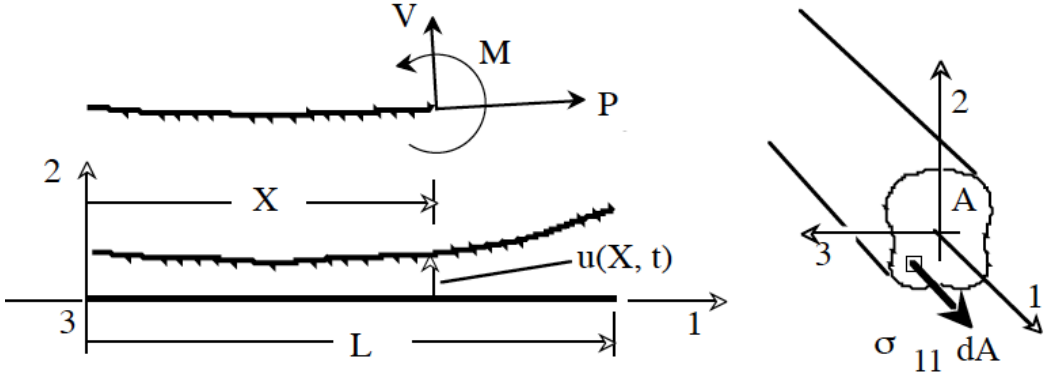

Figure 12. Transverse motion of an initially straight beam, shown at left as a one-dimensional elastic line, and at right as a three-dimensional solid of finite section.

The preceding derivation was presented in the spirit of the model of a beam as the *elastic line*

of Euler. We may obtain the same equations of motion by the following five steps:

(1) Integrate the three-dimensional equations of motion over a section, writing $V = \int_A \sigma_{12} dA$.

(2) Integrate X_2 times those equations over a section, writing $M = -\int_A X_2 \sigma_{11} dA$.

(3) Assume that planes initially perpendicular to fibers lying along the 1 axis remain perpendicular during deformation, so that $\varepsilon_{11} = \varepsilon_o(X, t) - X_2 \kappa(X, t)$ where $X \equiv X_1$, $\varepsilon_o(X, t)$ is the strain of the fiber along the 1 axis and $\kappa(X, t) = \partial^2 u/\partial X^2$ where $u(X, t)$ is u_2 for the fiber initially along the 1 axis.

(4) Assume that the stress σ_{11} relates to strain as if each point was under uniaxial tension, so that $\sigma_{11} = E \varepsilon_{11}$.

(5) Neglect terms of order h^2/L^2 compared to unity, where h is a typical cross section dimension and L is a scale length for variations along the 1 direction.

In step (1) the average of u_2 over area A enters but may be interpreted as the displacement u of step (3) to the order retained in (5). The kinematic assumption (3) together with (5), if implemented under conditions that there are no loadings to generate a net axial force P, requires that $\varepsilon_o(X, t) = 0$ and that $\kappa(X, t) = M(X, t)/EI$ when the 1 axis has been chosen to pass through the centroid of the cross section. Hence $\sigma_{11} = -X_2 M(X, t) / I = -X_2 E \partial^2 u(X, t)/\partial X^2$ according to these approximations. The expression for σ_{11} is exact for static equilibrium under *pure bending*, since assumptions (3) and (4) are exact and (5) is irrelevant then. That is, of course, what motivates the use of assumptions (3) and (4) in a situation which does not correspond to pure bending. The beam is discussed again in the final subsection on elasticity, when we consider the effects of significant initial stresses on vibrations and buckling.

Finite-element stiffness procedure. To illustrate the finite-element procedure (which is in no sense restricted to our present context of linear elasticity, or to the displacement-based formulation about to be outlined) let us rewrite the Principle of Virtual Work making use of the 6-member columns $\{\sigma\}$ and $\{\varepsilon\}$ to describe stress and strain, and introducing 3-member columns $\{u\}$, $\{f\}$ and $\{T\}$ to describe the vectors **u**, **f** and **T** of displacement, body force and surface traction. Then the principle requires that

$$\int_S \delta\{u\}^T\{T\}\, dS + \int_V \delta\{u\}^T\{f\}\, dV = \int_V \rho\, \delta\{u\}^T \partial^2\{u\}/\partial t^2\, dV + \int_V \delta\{\varepsilon\}^T\{\sigma\}\, dV$$

for every compatible set of variations $\delta\{u\}$ and $\delta\{\varepsilon\}$. The approximation now consists of

dividing the domain into cells, called elements, with {u} within each interpolated from the values of {u} at nodes. The set of such nodal values of displacements defines the column {Q}, which has 3n members if n is the number of nodes, and the interpolation is done so that {u} is continuous across element boundaries for any set of {Q} whatsoever. There are many ways of accomplishing this. The simplest is to divide the domain into tetrahedra (a tetrahedron is a solid with four triangular faces, for example, like in Figure 4) with nodes located at vertices, and to interpolate {u} as a linear function of the three spatial coordinates within each such element. Then {u} within is uniquely determined by the values of {u} at the four nodes of each element (these are a subset of the members of the column {Q}). The value of {u} along any face is readily seen to be determined uniquely by the three nodes at the vertices of that face this interpolation assures that {u} is continuous from one to the other element which meet at that face. In practice, far more elaborate elements and interpolation procedures are often used. The interpolation approximates the displacement field in the form $\{u(\mathbf{X}, t)\} = [N(\mathbf{X})]\{Q(t)\}$ where [N] is the 3 by 3n matrix describing the interpolation over elements, and from the strain-displacement relations there will be an associated representation of the strain in the form $\{\varepsilon(X, t)\} = [B(\mathbf{X})]\{Q(t)\}$ where [B] is a 6 by 3n matrix, although within any given element all entries of [N] and [B] will be zero except those associated with the nodes bordering that element.

We now insist that the Principal of Virtual Work hold for all variations $\delta\{u\} = [N(\mathbf{X})]\delta\{Q\}$ and $\delta\{\varepsilon\} = [B(\mathbf{X})]\delta\{Q\}$ that can be generated by arbitrary variations of {Q}, and thus obtain the 3n discretized equations of motion

$$\{F\} = [M]\frac{d^2}{dt^2}\{Q\} + \int_V [B]^T\{\sigma\}dV \ , \quad \text{where}$$

$$\{F(t)\} = \int_S [N]^T\{T\}dV + \int_V [N]^T\{f\}dV \ \text{ and } \ [M] = \int_V \rho[N]^T[N]dV .$$

By expressing {σ} in terms of {ε} (= [B]{Q}) through a stress-strain relation, this becomes a differential equation in {Q}. Thus, in the present case of linear elasticity with {σ} = [D]{ε} = [D][B]{Q}, we obtain

$$\{F\} = [M]\frac{d^2}{dt^2}\{Q\} + [K]\{Q\} \ ,$$

where $[K] = \int_V [B]^T[D][B]dV$ is called the *stiffness matrix*. Also, {F} is called the

force column and [M] the *(consistent) mass matrix*. In the general problem not all of the members of {F} are known because, for example, some may be associated with forces at restrained portions of the boundary. But in such cases the corresponding members of {Q} will be known, as zero at those nodes corresponding to a restrained boundary. Thus one solves, on the computer, the subset of these equations that involves the known F's on the left and unknown Q's on the right, and then back-substitutes the solution to extract the unknown F's. Typically, in implementations, it is acceptable replace [M] by a diagonal matrix, which corresponds to lumping mass at the nodes.

Some elementary two-dimensional solutions. A sense of the sort of problems addressed by elasticity theory can be seen from the sample which now follows of some simple two-dimensional problems.

Plane strain and plane stress. *Plane strain* describes the situation in which all displacements occur parallel to a given plane and are a function of position in that plane. For example, $u_1 = u_1(X_1, X_2)$, $u_2 = u_2(X_1, X_2)$, $u_3 = 0$. Thus $\varepsilon_{33} = 0$, which requires that $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$, and the other non-zero stresses are σ_{11}, σ_{12} and σ_{22}. States of plane strain are exact solutions of the elasticity equations. *Plane stress*, by contrast, is an approximate theory for thin plates of, in the simplest case, uniform thickness that are loaded in their plane (say, the 1,2 plane) in such a way that the middle surface does not bend. The equations of plane stress incorporate the assumption that $\sigma_{33}=0$ in writing stress-strain relations. The condition $\sigma_{33}=0$ is exact at the unloaded faces of the plate but is, in general, only satisfied approximately within the interior of the plate (nevertheless, σ_{33} is very small compared to in-plane stresses when the scale length over which stress varies within the plate is large compared to plate thickness). The equilibrium equations in the plane, which now refer to averages of stress over the thickness, are the same as for plane strain, and the governing equations for the two cases can be considered together. It is often convenient for static problems to express the governing differential equations in terms of stresses. In that case one uses the equations of motion with $\mathbf{a} = \mathbf{0}$, which are then the *equilibrium equations*, together with the compatibility equation which assures that the in-plane strains ε_{11}, ε_{12} and ε_{22} are derivable from the two displacement components u_1 and u_2. When there are negligible body forces ($f_1 = f_2 = 0$) the governing equations are therefore

$$\partial\sigma_{11}/\partial X_1 + \partial\sigma_{12}/\partial X_2 = 0 \ , \quad \partial\sigma_{12}/\partial X_1 + \partial\sigma_{22}/\partial X_2 = 0 \ ,$$

and $$\partial^2\varepsilon_{22} / \partial X_1^2 + \partial^2\varepsilon_{11} / \partial X_2^2 = 2\partial^2\varepsilon_{12} / \partial X_1 \partial X_2 .$$

By using the stress-strain relations the last of these can be expressed in terms of the stresses. After some help from the equilibrium equations to simplify, one finds that

$$\nabla^2(\sigma_{11} + \sigma_{22}) = 0$$

for both cases when we assume an isotropic linear elastic material. Here $\nabla^2 = \partial^2 / \partial X_1^2 + \partial^2 / \partial X_2^2$. Thus the governing equations when expressed in terms of the stresses σ_{11}, σ_{12} and σ_{22} are the same for both plane stress and plane strain, and hence when the boundary conditions on a problem are expressed entirely in terms of stress, the same in-plane stress distribution will result in both cases. The problems differ in that $\varepsilon_{33} = 0$ for plane strain, which leads to $\sigma_{33}=\nu(\sigma_{11}+\sigma_{22})$, whereas $\sigma_{33} = 0$ for plane stress, which leads to the thickness-direction strain $\varepsilon_{33}=\nu(\sigma_{11}+\sigma_{22})/E$.

It is often convenient to introduce the *Airy stress function*, $U = U(X_1, X_2)$ which exactly satisfies the above equations of plane stress or plane strain equilibrium, in absence of body forces, by representing stresses as

$$\sigma_{11} = \partial^2 U / \partial X_2^2 , \quad \sigma_{12} = -\partial^2 U / \partial X_1 \partial X_2 \quad \text{and} \quad \sigma_{22} = \partial^2 U / \partial X_1^2 .$$

Then, if we insist on satisfying strain compatibility and express the strains in terms of stresses by the linear isotropic relations, so that $\nabla^2(\sigma_{11} + \sigma_{22}) = 0$ as above, the governing equation which determines U is $\nabla^2(\nabla^2 U) = 0$, which is called the *biharmonic* equation. The next sections present a few solutions.

Equations in polar coordinates. In developing such solutions, the use of polar coordinates r, θ in the X_1, X_2 plane will be helpful, with $X_1 = r\cos\theta$ and $X_2 = r\sin\theta$. Stress components $\sigma_{rr}, \sigma_{r\theta} = \sigma_{\theta r}, \sigma_{\theta\theta}$ relative to the polar coordinate system are shown in Figure 13. These may be expressed in term of the Cartesian stress components at the same point by an application of the stress transformation equations developed earlier, and in particular by application of the Mohr circle transformation of Figure 7. They are given by

$$\sigma_{rr} + \sigma_{\theta\theta} = \sigma_{11} + \sigma_{22} , \quad \text{and} \quad \sigma_{\theta\theta} - \sigma_{rr} + 2i\sigma_{r\theta} = \exp(2i\theta)(\sigma_{22} - \sigma_{11} + 2i\sigma_{12}) .$$

Here $i = \sqrt{-1}$, the unit imaginary number. Also, the equations of equilibrium in polar

coordinates for plane stress or plane strain states are

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+f_r=0\ ,\ \text{and}\quad \frac{\partial \sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+f_\theta=0\ ,$$

and in the absence of body force ($f_r = f_\theta = 0$) all solutions may likewise be expressed in terms of an Airy stress function U by

$$\sigma_{rr}=\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\ ,\quad \sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial U}{\partial \theta}\right),\ \text{and}\quad \sigma_{\theta\theta}=\frac{\partial^2 U}{\partial r^2}.$$

Further, for the linear isotropic elastic case, $\nabla^2(\nabla^2 U)=0$ as above, where in this coordinate system

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\ .$$

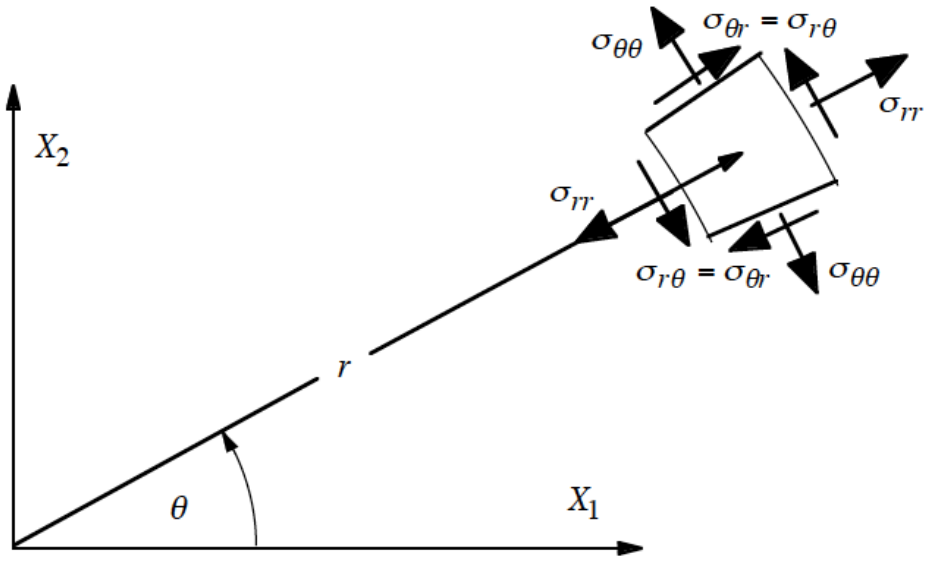

Figure 13. Stress components in polar coordinates.

Holes and stress concentrations. Consider a circular hole of radius a in a plate whose dimensions are much larger than a and can, for present purposes be taken as infinite (Figure 14). The plate is under remotely uniform stress σ^∞ in the 2 direction and the boundary of the hole is free of loading. The same conditions, interpreted as a plane strain problem, describe a circular

tunnel in a large solid. Thus we wish to solve $\nabla^2(\nabla^2 U) = 0$ subject to the requirements that the stresses associated with U satisfy $\sigma_{22} \rightarrow \sigma^{\infty}$, $\sigma_{11} \rightarrow 0$ and $\sigma_{12} \rightarrow 0$ as $r \rightarrow \infty$, and that $\sigma_{rr} = \sigma_{r\theta} = 0$ on $r = a$. To make the solution to this type of problem unique, we must also specify the value of the integral of $\partial \mathbf{u}/\partial s$ with respect to arc length s around the hole, which is zero in the present case for which we require a *single-valued* displacement field, but which would be non-zero if the hole was to represent the core of a dislocation.

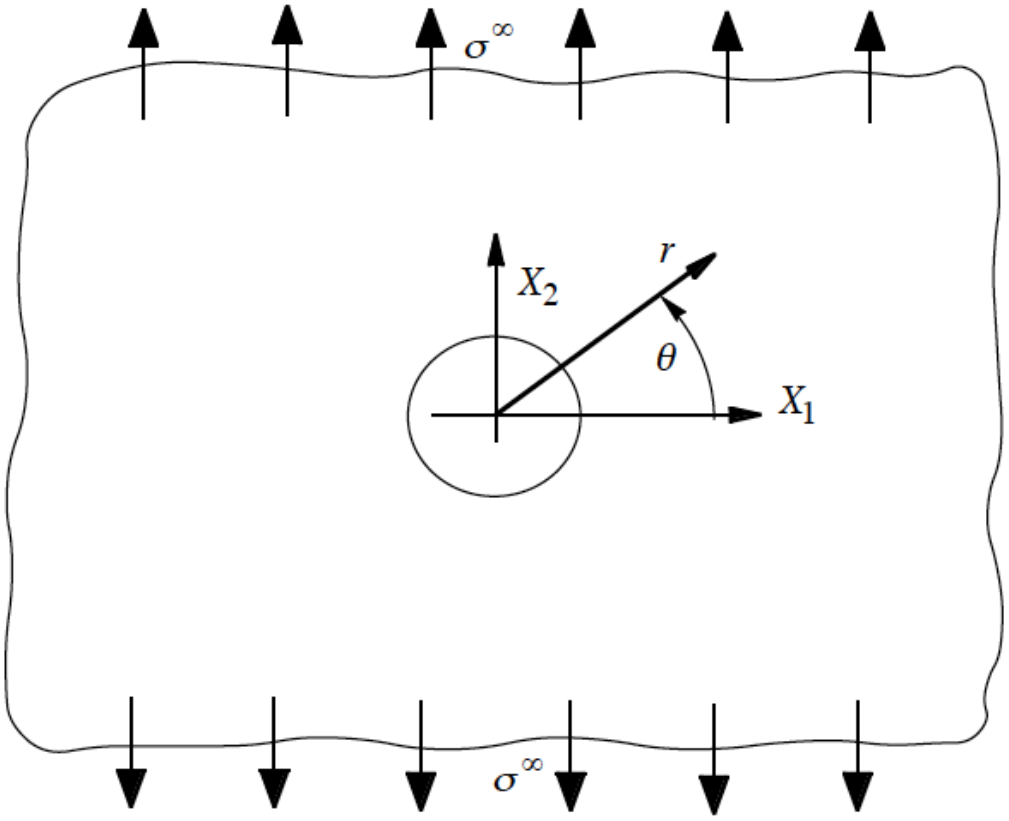

Figure 14. Circular hole on a large plate (or circular tunnel in a solid) under remote tensile stress σ^{∞}.

The proper stress state as $r \rightarrow \infty$ is given by writing $U = \sigma^{\infty} X_1^2 / 2 = \sigma^{\infty} r^2 (1 + \cos 2\theta) / 4$ and, while this does not meet the conditions on the boundary of the hole, it does encourage one to seek solutions to $\nabla^2(\nabla^2 U) = 0$, which is a *partial* differential equation, in the form $U = g(r) + h(r)\cos 2\theta$. The functions g(r) and h(r) must satisfy *ordinary* differential equations, which are easier to solve. After solving for the most general forms of g(r) and h(r) which, for example, for h is $h(r) = A\, r^4 + B\, r^2 + C + D\, r^{-2}$ where A, B, C and D are constants, and choosing all constants to meet the given conditions on stress at $r = a$ and as $r \rightarrow \infty$, and to give single-valued

displacements, one solves for U and from it finds the stresses:

$$\sigma_{rr} = (\sigma^{\infty}/2)[(1-a^2/r^2)-(1-4a^2/r^2+3a^4/r^4)\cos 2\theta]$$
$$\sigma_{r\theta} = -(\sigma^{\infty}/2)(1+2a^2/r^2-3a^4/r^4)\sin 2\theta \quad .$$
$$\sigma_{\theta\theta} = (\sigma^{\infty}/2)[(1+a^2/r^2)+(1+3a^4/r^4)\cos 2\theta]$$

Thus, setting $r = a$, $\sigma_{\theta\theta} = \sigma^{\infty}(1+2\cos 2\theta)$ is the stress created around the boundary of the hole. This amounts at $\theta=0$ and π, that is, at boundary points intersected by the X_1 axis, to a concentration of stress $\sigma_{\theta\theta} = \sigma_{22} = 3\,\sigma^{\infty}$. At $\theta=\pi/2$ and $-\pi/2$, points intersected by the X_2 axis, there is an oppositely signed stress $\sigma_{\theta\theta} = \sigma_{11} = -\sigma^{\infty}$. Thus, when we consider a circular hole in a brittle material which can support very little tensile stress, we expect failure to begin at $\theta=0$ or π under remote tensile loading, but at $\theta=\pi/2$ or $-\pi/2$, and at three times the load level, under remote compressive loading.

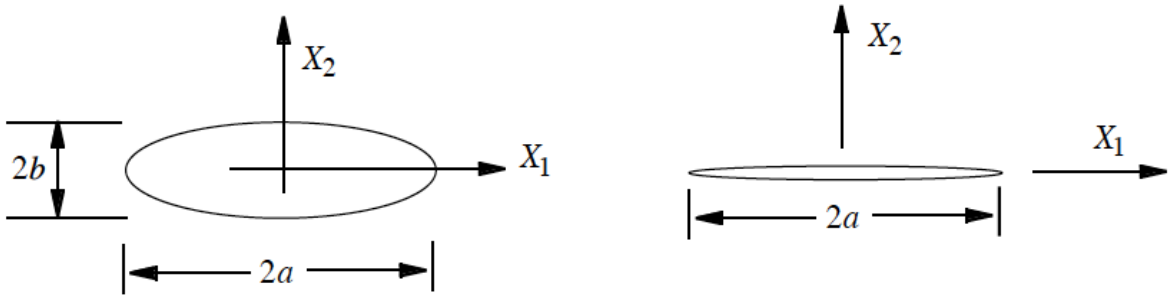

Figure 15. Elliptical tunnel hole, and limit as a flat crack.

As another problem showing an important aspect of stress concentration, consider the Kolosov-Inglis problem of an elliptical hole (Figure 15, at left) in large plate under remotely uniform stress σ^{∞} in the 2 direction as above. This also describes the tunnel cavity of elliptical cross section under plane strain. Let a denote the semi-axis of the ellipse along the 1 direction and b denote that along the 2 direction; the equation of the ellipse is $X_1^2/a^2 + X_2^2/b^2 = 1$. It is then found that the concentration of stress at points of the hole boundary intersected by the X_1 axis is $\sigma_{22} = (1+2b/a)\,\sigma^{\infty}$, which can be rewritten as $\sigma_{22} = (1+2\sqrt{a/\rho_{tip}})\sigma^{\infty}$. In the latter form, $\rho_{tip} = b^2/a$ is the radius of curvature of the hole boundary, at the tip of the hole at $X_1 = \pm a$. This illustrates a result of general validity for notches with relatively small root radii compared to length: The elevation of stress over the value (σ^{∞}) in absence of the notch is, very approximately, given by $2\sqrt{a/\rho_{tip}}\sigma^{\infty}$ in all cases, where a is the half length of an internal notch like the elliptical hole just discussed, and is the length of a notch that has been cut in from

the free surface of a solid. Thus good engineering design is always sensitive to the stress concentrating effect of holes and, especially, notches or other cut-outs of small root radius, avoiding them where possible. This is a lesson reinforced by the bitter experience of many structural failures beginning at unrecognized locations of stress concentration. In the particular case of the elliptical hole, the stress induced along the hole boundary where it is intersected by the X_2 axis is $\sigma_{11} = -\sigma^{\infty}$, independent of the *b*/*a* ratio.

Inclusions. Points around the boundary of the elliptical hole just discussed are found to displace according to the equations

$$u_1 = -(\sigma^{\infty} / E')X_1, \quad \text{and} \quad u_2 = (1 + 2a / b)(\sigma^{\infty} / E')X_2$$

where X_1 and X_2 are coordinates of points on the boundary, and satisfy $X_1^2 / a^2 + X_2^2 / b^2 = 1$. Also, $E' = E$ for the plane stress model but $E' = E / (1 - \nu^2)$ for plane strain. These results show that if we had considered not a solid with an elliptical hole but rather a solid with a uniform elliptical *inclusion* of another material, in this case a material with vanishing small elastic modulus compared to that of the surrounding solid (so that the situation in the surrounding solid is indistinguishable from that for the case of a hole), then that inclusion would undergo uniform strain, the strains $\varepsilon_{ij}^{\text{incl}}$ within it being given by rewriting the above equations as $u_1 = \varepsilon_{11}^{\text{incl}} X_1$ and $u_2 = \varepsilon_{22}^{\text{incl}} X_2$ and noting that $\varepsilon_{12}^{\text{incl}} = 0$. A little reflection on this result will convince one that if an inclusion of arbitrary but uniform and isotropic material properties (the stress-strain relation for the inclusion material need not even be linear) were placed in the hole, then the inclusion would undergo a uniform stress and strain that could be calculated from its material properties and the information given so far here.

This discussion generalizes to an important three-dimensional result discovered by J. D. Eshelby and which is this: Let a uniform, possibly anisotropic, linear elastic solid of infinite extent be loaded by a remotely uniform stress tensor and let it contain an ellipsoidal inclusion of a material of uniform but different mechanical properties. The inclusion material can even be such that, in its stress-free state, it takes an ellipsoidal shape which differs finitely from the stress-free shape of the ellipsoidal hole into which it is to be inserted. Eshelby's result is that, regardless of all these factors, the inclusion undergoes a spatially uniform stress and strain state. To develop that result, Eshelby first solved the *transformation* problem of a misfitting linear elastic inclusion of identical properties as those of the surrounding material. That involves taking an ellipsoidal region of a uniform solid and, at least conceptually, transforming its stress-

free state by a homogeneous infinitesimal strain, without changing its elastic properties. A uniform stress field within the transformed zone then suffices to deform it back to its original shape, and that stress field can be maintained *in-situ*, without disturbance of the region outside the ellipsoid, by application of a suitable layer of surface force. Since the actual transformation problem to be solved has no agent to supply that layer of force, the strains everywhere can be calculated as those due to removing such a layer (i.e., applying a force layer of opposite sign) within an elastically uniform full space. One calculates then that the stress and strain state induced within the inclusion is uniform, and the rest of what is needed for arbitrary inclusions can be developed from there.

Crack as limit of elliptical hole. Consider again the two-dimensional problem of the elliptical hole under remotely uniform tension, and let the semi-axis b go to zero (Figure 15, at right) so as to define a flat Griffith crack lying along the X_1 axis on $-a < X_1 < +a$. In this case the stress concentration at the hole becomes unbounded so that it can fairly be objected that the discussion lies outside the proper realm of linear elasticity. However, as will be seen, it proves quite useful to continue with the linear elastic model and to learn about its stress singularities. Letting $\Delta u_2 = u_2^+ - u_2^-$ denote the crack opening gap (superscripts + and – denote the upper and lower crack surfaces) we see that

$$\Delta u_2 = (4\sigma^\infty / E')\sqrt{a^2 - X_1^2}\ , \quad -a < X_1 < +a\ .$$

Also, the tensile stress transmitted across the X_1 axis outside the crack (i.e., on $X_2 = 0$) is

$$\sigma_{22} = \sigma^\infty |X_1| / \sqrt{X_1^2 - a^2}\ , \ -\infty < X_1 < -a \ \text{ and } \ +a < X_1 < +\infty\ .$$

Thus, very near the crack tip, say, at $X_1 = a$, the displacement of the crack walls varies in proportion to $\sqrt{a - X_1}$, whereas the stress a small distance ahead of the crack varies as $1/\sqrt{X_1 - a}$. These are, in fact, universal features of all linear elastic solutions to crack problems, whether two- or three-dimensional and whether in isotropic or anisotropic materials, when the crack is idealized as a mathematical cut across which no loading is transferred.

Crack tip fields. To understand the origin of this universality consider the stress state near a crack edge in a three-dimensional solid, where now the origin of coordinates is placed at a point of interest along the crack tip, with the 2 axis normal to the local plane of the crack, the 3 axis

tangent to the crack tip, and the 1 axis pointing in the direction of crack growth. For simplicity, assume that the material is isotropic and that the symmetry of the geometry and loading is such as to induce a tensile opening Δu_2 but no relative shear displacements Δu_1 or Δu_3 of the crack walls. One easily sees that unless the point chosen along the crack front happens to coincide with the intersection of the crack with a boundary of the solid, or to a kink along the front, all terms involving $\partial / \partial X_3$ in the governing set of equations must become negligible compared to those involving $\partial / \partial X_1$ and $\partial / \partial X_2$ as the crack tip singularity is approached. So also must terms involving $\partial / \partial t$ in the case of a non-propagating crack. Hence the singular field for a non-propagating crack satisfies the same equations as apply for the singular field at the tip of a crack under *static plane-strain conditions*, a situation which is readily analyzed.

Some generalizations are as follows: For the propagating crack, the singularity is likewise the same as for the plane strain case. The operator $\partial / \partial t$ in the governing equations can then be replaced by $- V_{cr} \partial / \partial X_1$ for purposes of establishing the form of the singular field; here V_{cr} is the instantaneous speed of crack propagation. For general loadings that also, or perhaps only, induce relative shear displacement discontinuities Δu_1 and Δu_3 along the crack surfaces, and for arbitrary elastic anisotropy, the same discussion holds good except that then the singular field is given by some combination of two-dimensional *plane strain* and *anti-plane strain* equations (i.e., all three displacements are functions of just X_1 and X_2). Cracks with shear displacement discontinuities are of interest as models for earthquake faulting, although then it is generally thought necessary to consider the crack walls as being subjected to surface forces of frictional type, possibly dependent on the normal stress, instantaneous slip rate, and slip history.

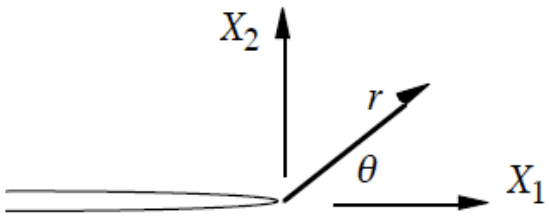

Figure 16. Coordinates for analysis of crack tip singular field.

Thus, returning to the case of the non-propagating, tensile loaded, crack tip, we consider the plane strain formulation and choose polar coordinates such that $\theta = 0$ is the direction of prospective growth, and that $\theta = \pm \pi$ denotes the crack walls (Figure 16). The problem then is to find the form of an Airy function U generating stresses consistent with the symmetry of the problem and such that $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$ on the crack walls. By following M. L. Williams in

assuming that U has the form $r^{\lambda+2}$ f(θ), one finds that $\nabla^2(\nabla^2 U) = 0$ will be satisfied in a way consistent with the symmetry of the problem if f(θ) = A *cos*λθ + B *cos*(λ+2)θ where A, B and λ are constants. This form of the Airy function produces stresses proportional to r^{λ}, so the values of λ of interest in describing a *singular* solution are $\lambda < 0$.

The stresses which enter the boundary conditions are then calculated as

$\sigma_{\theta\theta} = \partial^2 U / \partial r^2 = (\lambda+2)(\lambda+1)\, r^{\lambda}$ [A *cos*λθ + B *cos*(λ+2)θ], and

$\sigma_{r\theta} = -\partial(r^{-1}\partial U / \partial\theta) / \partial r = (\lambda+1)\, r^{\lambda}$ [λA *sin*λθ + (λ+2) B *sin*(λ+2)θ].

Setting both to zero at $\theta = \pm\pi$ then requires

(A + B) *cos*λπ = [λA + (λ+2) B] *sin*λπ = 0,

which has solutions

λ=..., −7/2, −5/2, −3/2, −1/2, 1/2, 3/2, 5/2, ... with B = – A λ/(λ+2),

and λ=..., −3, −2, −1, 0, 1, 2, 3, ... with B = – A.

When it is realized that one must reject any λ values which are so negative as to lead to unbounded total strain energy of some finite region, one concludes that only values $\lambda > -1$ are admissible. There is one and only one value of λ in the range $-1 < \lambda < 0$ which is admissible and allows a singular field meeting the crack surface boundary conditions. It is λ = –1/2 and, for it, B must be related in a specific way to A. Thus the singular field at the crack tip has one free parameter (A, which we now redefine as $1/\sqrt{2\pi}$ times a parameter designated K_I). The singular distribution of stress at a tensile crack tip is then found to be:

$$\sigma_{\theta\theta} = \frac{K_I}{\sqrt{2\pi r}}\cos^3\left(\frac{\theta}{2}\right),\ \sigma_{\theta r} = \frac{K_I}{\sqrt{2\pi r}}\cos^2\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right),\ \sigma_{rr} = \frac{K_I}{\sqrt{2\pi r}}\cos\left(\frac{\theta}{2}\right)\left[2-\cos^2\left(\frac{\theta}{2}\right)\right].$$

Also, $\sigma_{33} = \nu(\sigma_{rr}+\sigma_{\theta\theta})$. The same singular field applies within the simple plane stress model for a crack in a plate when we set $\sigma_{33} = 0$.

The quantity K_I is called the *Mode I stress intensity factor*. It will be proportional to the loading in a given problem and will depend on crack geometry, and gives the stress acting across the plane $\theta = 0$ ahead of the crack, very near its tip, according to $\sigma_{\theta\theta} = \sigma_{22} = K_I / \sqrt{2\pi r}$. There are analogous singular stress fields for cracks loaded so as to sustain in-plane (*Mode II*) relative shear displacements Δu_1, and anti-plane (*Mode III*) relative shear displacements Δu_3, of their surfaces near the tip, and stress intensity factors K_{II} and K_{III} characterize the strength of the singularity for those modes.

All the other allowed values of λ given above correspond to zero stress at $r = 0$, except for the $\lambda = 0$ value; it corresponds to a uniform stress state of type $\sigma_{11} = T$ (a constant) acting parallel to the crack. It is found that positive values of T promote the growth of a tensile loaded crack out of its initial plane, i. e., promote an instability of the planar crack path.

The displacement field associated with the singular stress state for the Mode I crack is

$$u_1 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}(\kappa - \cos\theta)\cos\left(\frac{\theta}{2}\right), \quad u_2 = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}}(\kappa - \cos\theta)\sin\left(\frac{\theta}{2}\right)$$

where $\kappa = 3 - 4\nu$ for plane strain, and the expression is valid also for plane stress if we set $\kappa = (3-\nu)/(1+\nu)$. Thus the displacement across the crack face, very near the tip, in the plane strain case is $\Delta u_2 = [4(1-\nu)K_I / \mu]\sqrt{r / 2\pi}$.

In the case of the tunnel crack of length $2a$ in a solid under remote tension σ^∞, $K_I = \sigma^\infty \sqrt{\pi a}$. This may be obtained either by comparing the complete expression for Δu_2 given above for the tunnel crack to the small r result, or by comparing the complete expression for σ_{22} above to the analogous small r result, $\sigma_{22} = K_I / \sqrt{2\pi r}$. For an edge crack penetrating by depth a into a half-space under the same remote tension, K_I is 1.12 times that value, and for a circular crack of radius a in a large solid it is $2/\pi$ times that value. Many problems have been solved, and computational mechanics techniques developed for the solution of more, such that expressions of K_I are readily available in the technical literature on fracture mechanics for many cases of interest.

The above representation of the field near the crack tip can be valid only at values of r that are large compared to the size of some inner cut-off zone, that being the zone where any of inelastic deformation, large geometry changes, or microscale discreteness of material invalidate

the homogeneous linear elastic continuum model. Nevertheless, there are many cases for which the cut-off zone size is very much smaller than overall overall sizes like crack length or specimen size and, since the continuum stress field is dominated by the $1/\sqrt{r}$ terms in that small size range, in such cases we can assert that K_I characterizes the intensity of loading sensed at the crack tip. This is so even though the linear elastic solution, and in many cases, perhaps any continuum solution, is wrong in detail immediately at the crack tip. Thus, for all such cases, a viable procedure, defining what is sometimes called *linear elastic fracture mechanics*, consists of the following: We measure crack growth data in the laboratory, whether under monotonically increasing load, sustained load in the presence of an active chemical environment that causes stress corrosion cracking, or fluctuating load as for fatigue crack growth, and correlate the measured cracking response as a function of K_I. Then, to make a prediction of crack growth in a structure of the same material, one uses the data correlated in terms of K_I, together with a linear elastic stress analysis determining the expression for K_I in terms of the loading on, and crack geometry in, the structure, as a means of predicting response in service. There is also a substantial body of work on crack growth in plastic or creeping solids which attempts to extract, from continuum solutions for such materials, parameters in terms of which growth can be correlated. There is, of course, an important component of research in materials science directed to understanding the mechanics of fracture processes at scales extending to those of microscale separation processes, whether by cleavage, or plastic or diffusional cavitation.

Griffith introduced the idea of an energy flow to a fracture. Based on the linear elastic (quasi)static crack tip field, the energy G which is lost per unit area of new crack surface is given as $G = K_I^2 / E'$; $G\,\delta A$ is the difference between the work done by applied forces and the change in strain energy, in an increment of crack growth over the infinitesimal area δA. For ideally brittle solids with no sources of dissipation by plastic or creep deformation, and with negligible "lattice trapping", the crack growth condition should be $G = 2\,\gamma_s$, as proposed by Griffith, where γ_s is the *surface free energy* and hence $2\,\gamma_s$ is the work of reversibly separating atomic bonds over a unit area in the solid. However, most solids, even rather brittle ones, show energy adsorption in fracture that is greatly in excess of $2\,\gamma_s$, and hence the *fracture energy* of a solid, which is the value of G for crack growth, is regarded as an empirical parameter to be measured directly in a cracking test or inferred from the relation of G to the critical K_I from such a test.

Torsion. Consider a cylindrical bar, not necessarily of circular cross section, with its axis along the 3 direction, and which supports torsional loading by a torque M_3 about that axis. A

simplified formulation was devised by Saint-Venant in 1852 which involves the same stress and strain distribution at each section along the length of the bar; the resulting solution may not be accurate near the ends of the bar where loadings are applied. Each cross section is assumed to displace in its own plane by a rotation which increases with distance along the bar and which creates no strain of fibers lying in a cross section plane; that is, $\varepsilon_{11} = \varepsilon_{12} = \varepsilon_{22} = 0$. The rotation is $\phi\, X_3$ where ϕ is the angle of twist per unit length and the displacements in such planes are $u_1 = -\phi X_3 X_2$, $u_2 = \phi X_3 X_1$. The earliest attempts to solve the problem made, at this point, the assumption that $u_3 = 0$ but, for other than a bar of circular section, that assumption cannot solve the problem; the strains associated with such a **u** field lead to stresses which do not make **T** = **0** on the lateral surfaces of the bar. However, $u_3 = 0$ is correct for the case of a bar with section in the form of a circle or circular annulus. The resulting non-zero stresses, $\sigma_{31} = -\mu\phi X_2$ and $\sigma_{32} = \mu\phi X_1$, satisfy the equilibrium equations and make **T** = **0** on the bar surface, provided that the coordinate origin is at the bar center. Thus if A denotes the cross section area of a bar with section in form of a circle or concentric circular annulus, the torque which must be applied to generate the angle of twist ϕ per unit length is calculated by

$$M_3 = \int_A (X_1\sigma_{32} - X_2\sigma_{31})dA = \mu\phi \int_A (X_1^2 + X_2^2)dA$$
$$= \mu\phi \int_A r^2(2\pi r)dr = (\pi / 2)(r_o^4 - r_i^4)$$

where r_i and r_o are the inner and outer radii of a circular annulus, and $r_i = 0$ for a full circular section of radius r_o. The stress distribution is more readily described in this case in terms of polar coordinates r and θ in the section; then the shear stress distribution is given by $\sigma_{3r} = 0$ and $\sigma_{3\theta} = M_3\, r / J$ where $J = (\pi / 2)(r_o^4 - r_i^4)$.

For other than circular section shapes a non-zero *warping displacement* $u_3 = u_3(X_1, X_2)$ must also develop. Then the shear stresses are given by

$$\sigma_{31} = \mu(\partial u_3 / \partial X_1 - \phi X_2), \quad \text{and} \quad \sigma_{32} = \mu(\partial u_3 / \partial X_2 + \phi X_1).$$

These will satisfy the equations of equilibrium if u_3 satisfies $\nabla^2 u_3 = 0$, and will produce zero tractions, $T_3 = 0$, on the lateral surface if $n_1 \partial u_3/\partial X_1 + n_2 \partial u_3/\partial X_2 = (n_1 X_2 - n_2 X_1)\,\phi$ at points along the boundary of the section, where (n_1, n_2) are the components of the outward unit normal to the surface.

It is generally somewhat easier to instead solve torsion problems by formulating the set of

governing differential equations in terms of the two shear stresses rather than in terms of the displacement. In that case we have to solve the following equations:

$$\partial\sigma_{31}/\partial X_1 + \partial\sigma_{32}/\partial X_2 = 0 \ , \quad \partial\sigma_{31}/\partial X_2 + \mu\phi = \partial\sigma_{32}/\partial X_1 - \mu\phi \ ,$$

where the first is the only equation of equilibrium not trivially satisfied by the assumed state of shear stress and the second is a *compatibility* condition written in terms of stress, and can be obtained by observing that $\partial(\partial u_3 / \partial X_2) / \partial X_1$ must equal $\partial(\partial u_3 / \partial X_1) / \partial X_2$. These equations are to be solved subject to the boundary condition $T_3 = n_1\sigma_{31} + n_2\sigma_{32} = 0$. Solutions are easiest to develop if we introduce a stress function $\Psi(X_1, X_2)$ which automatically satisfies the equilibrium (first) equation above. This is done by relating Ψ to the stresses by $\sigma_{31} = \mu\phi\,\partial\Psi/X_2$ and $\sigma_{32} = -\mu\phi\,\partial\Psi/X_1$. The compatibility (second) equation above then requires that $\nabla^2\Psi = -2$ within the section, and the boundary condition may be shown to be equivalent to requiring that $\partial\Psi/\partial s = 0$ on the outer surface of the bar (s = arc length around the bounding curve of a cross section), so that Ψ = constant on the outer surface. That constant is irrelevant to determining the stress state and, to simplify a later expression, we take $\Psi = 0$ on the outer surface.

If the bar is multiply connected, that is, if it has one or more internal holes in the section, then Ψ must be assigned a constant value on the boundary of each such hole (it is then convenient to also define Ψ within each such hole, making it equal to that constant). The constant associated with each hole is not known *a priori* but, rather, follows from requiring that the integral of $dX_1\ \partial u_3/\partial X_1 + dX_2\ \partial u_3/\partial X_2$ around the boundary of each such hole is zero. When expressed in terms of the stresses, and hence Ψ, this requires that the integral of $n_1\partial\Psi / \partial X_1 + n_2\partial\Psi / \partial X_2$ with respect to arc length around the boundary of every such hole be equal to twice the cross section area of the hole; here (n_1, n_2) are components of the unit normal to the wall of the hole, pointing away from the material and into the hole. This requirement provides a supplemental set of conditions that lets one determine the constants.

The torque necessary to create the angle of twist ϕ per unit length is then given by

$$M_3 = 2\mu\phi\int_A \Psi dA$$

where the integral extends over the entire area within the outer boundary of the section, the area of material and that of any holes too.

Membrane analogy for torsion. Consider first a bar of solid cross section. Let the lateral boundary of the bar be represented by the bounding contour Γ in the X_1,X_2 plane of the cross section A. Then we wish to solve $\nabla^2\Psi = -2$ in A subject to boundary condition $\Psi = 0$ on Γ. This is exactly the equation which solves the following problem: A thin soap film membrane is laid over a hole A, with bounding contour Γ, in a plate lying in the in the X_1,X_2 plane. Then the under side of the membrane is lightly pressurized by p relative to the upper side to create an upward displacement w. The equation governing w is $T_o\nabla^2 w = -p$ in A subject to boundary condition $w = 0$ on Γ. This is the same equation as for the torsion problem, and if we make the identification $\Psi = 2T_o w / p$, then the solution of the membrane problem (which is easy to visualize, and can be produced without calculations in the laboratory) provides a solution to the torsion problem. Indeed, the slope $-\partial w / \partial n$ of the membrane at its boundary, in the direction perpendicular the supporting curve Γ, gives the shear stress σ_{3s} (where the s direction is along Γ) at the boundary of the bar; precisely, $\sigma_{3s} / \mu\phi = -(2T_o / p)\partial w / \partial n$. The component σ_{3n} is zero there, which is a boundary condition already imposed in developing the formulation. Also, the volume displaced by the membrane gives the torsional stiffness factor $\int_A \Psi dA$.

The same membrane analogy applies for torsion of a bar with one or more holes along the X_3 axis. This is a much harder case to produce in the laboratory, but not much harder to visualize, and so is useful. Now we imagine that wherever there is a hole in the section, the corresponding part of the membrane is removed and replaced with a thin rigid plate, of equally negligible weight, to which the membrane is attached. The entire arrangement, membrane and one or more rigid plate inserts within it, is pressurized as previously, but now the inserts are constrained so that they can only displace without tilting, which means that w is uniform within each such insert. We then have the governing equations that $T_o\nabla^2 w = -p$ within the region of membrane, that $w = 0$ on the external boundary Γ, and that the integral of $T_o(n_1\partial w / \partial X_1 + n_2\partial w / \partial X_2)$ with respect to arc length around the boundary of every such insert be equal to p times the area of the insert (for equilibrium of each insert, where the convention for n_1,n_2 is as above). Under the interpretation $\Psi = 2T_o w / p$, these are the same governing equations as for the torsion problem and, for example, the torsional stiffness factor $\int_A \Psi dA$ is then given in terms of the volume displaced by the membrane and the inserts within it.

Equations of finite deformation. The solutions thus far discussed are in the framework of linear elasticity. Now we turn briefly to finite deformation and the associated topic of small deformations from an equilibrium state of significant initial stress. Piola identified the stresses which precisely satisfy the equations of motion when these are written in terms of coordinates of

the reference configuration. These stresses are modified from the σ_{ij} because they also include the effect of changing from ∂/∂x to ∂/∂X in the equations of motion, and are

$$N_{ij} = \sum_{q=1}^{3} S_{iq} F_{jq}$$

which define the components of what may be called the *nominal*, or *first Piola-Kirchhoff*, stress. Here it is to be recalled that [S] is the second Piola-Kirchhoff stress, introduced earlier, and [F] denotes [$\partial x/\partial X$]. The N_{ij} are not symmetric, in general, are given by

$$N_{ij} = \rho_o \, \partial f([F], \theta)/\partial F_{ji}$$

for an elastic solid, have the property that

$$\sum_{i=1}^{3} n_i^o N_{ij} = \overline{T}_j$$

gives the j component of the nominal surface stress vector $\overline{\mathbf{T}}$ as reckoned per unit area of surface of the reference configuration at a place where its normal is $\mathbf{n}^o$, and satisfy

$$\sum_{i=1}^{3} \frac{\partial N_{ij}}{\partial X_i} + \bar{f}_j = \rho_o \frac{\partial^2 x_j(\mathbf{X},t)}{\partial t^2} \qquad (j = 1,2,3)$$

where $\bar{\mathbf{f}} = \mathbf{f} \det[F]$ are body forces as reckoned per unit volume of the reference configuration. The latter provides the system of differential equations for the $x_j(\mathbf{X},t)$ since the N_{ij} are expressed by the above considerations in terms of the components of $F_{kl} = \partial x_k(\mathbf{X}, t)/\partial X_l$.

There is often interest in solids which undergo extremely small strain but, possibly, quite large rotations. An example is provided by a rod which is straight in its unstressed state but which is sufficiently long that it can be bent elastically into a circular hoop. This will involve maximum extensional and compressional strains of approximately πD/L, where D is the diameter and L the length, so by making L sufficiently large compared to D it will always be possible to accomplish this elastically, with arbitrarily small strains, even though the rotation is as large as 2π (= 360°). In such cases it is appropriate to write [S] in terms of [E^M] by the same linear elastic stress-strain relations as written earlier and giving [σ] in terms of [ε]. In the case of

a beam modeled as an elastic line, this is equivalent to maintaining the linear elastic relation between moment and curvature but writing the equations of motion, and the expression for curvature in terms of displacements, in a manner that is correct for arbitrary displacement gradients.

Small deformation from an initially stressed state. Sometimes we wish to deal with solids which are already under stress in what we choose as the reference configuration, from which we measure strain. Call those stresses σ^o_{kl} which are, then, also the values of S_{kl} and N_{kl} in the reference configuration, and suppose that those stresses are in equilibrium with the body force distribution $\mathbf{f}^o$, based on unit volume of the reference state. Let us now deal with very small departures from that stressed reference configuration such that $\left|\partial u_i / \partial X_j\right| << 1$. In that case we can equate the E^M_{kl} with the infinitesimal strain tensor ε_{kl}, and hence write

$$S_{ij} = \sigma^o_{ij} + \sum_{k=1}^{3} \sum_{l=1}^{3} C_{ijkl} \varepsilon_{kl}$$

where C_{ijkl} has all the symmetries in its indices as elastic constants discussed earlier. In this case the expression for the N_{ij}, truncated at linear order in the (very small) displacement gradients is

$$N_{ij} = \sigma^o_{ij} + \sum_{l=1}^{3} (\sum_{k=1}^{3} C_{ijkl} \varepsilon_{kl} + \sigma^o_{il} \frac{\partial u_j}{\partial X_l}).$$

The term involving σ^o_{il} times a displacement gradient would usually be negligible in typical cases for which the stresses are extremely small fractions of moduli. An exception occurs, however, when the parts of displacement gradients corresponding to rotation can be very much larger than those corresponding to strain. This is exactly what happens at the onset of buckling in a slender strut under compression, and hence provides an example for which the conventional simplifications of linear elasticity cannot be used even in a case for which stresses are extremely small compared to elastic moduli. Also, for calculations involving large self-gravitating bodies such as the Earth, the initial stresses at depth can be quite non-negligible compared to moduli. Those stresses cannot be known precisely, but it is regarded as a good approximation to assume that they amount to a purely hydrostatic pressure (large shear stresses should relax over geological time), which is then uniquely determined by the radial distribution of density.

The linearized equations of motion for small departures from an initially stressed equilibrium state are obtained by substituting for above in the equations of motion, remembering that the $[\sigma^o]$ equilibrate the body force $\mathbf{f}^o$:

$$\sum_{i=1}^{3} \frac{\partial}{\partial X_i}[\sum_{l=1}^{3} (\sum_{k=1}^{3} C_{ijkl}\frac{\partial u_k}{\partial X_l} + \sigma_{il}^o \frac{\partial u_j}{\partial X_l})] + \bar{f}_j - f_j^o = \rho_o \frac{\partial^2 u_j(\mathbf{X},t)}{\partial t^2} \qquad (j = 1,2,3) .$$

For a solid in which the body forces are gravitational, which is the usual case, the two body forces $\bar{\mathbf{f}}$ and $\mathbf{f}^o$ will be identical if there is no change in the intensity of gravitational attraction per unit mass associated with the disturbance leading to the displacement field **u**. Generally, any such changes in the gravity field could be neglected for phenomena at the human scale, like for the buckling of a compressed column or plate. However, for large scale phenomena in the Earth, such as for calculation of its elastic distortion due to rotation, or of its response to tidal loading, or for the calculation of its lower frequency (hence greater length scale) vibration modes, the alteration of the gravity field is an important contributor to the resistance to deformation, of the same order of magnitude of that due to elastic stiffness, and in all such cases we must regard $\bar{\mathbf{f}} - \mathbf{f}^o$ as a linear functional of the distribution of displacement **u** throughout the Earth, calculated according to the usual equations of Newtonian gravitation.

Initially stressed beam. Consider the beam discussed earlier, and suppose that it is under an initial uniform tensile stress $\sigma_{11} = \sigma^o$; that is, the axial force $P = \sigma^o A$. If σ^o is negative and of significant magnitude, one generally refers to the beam as a column; if it is large and positive, the beam might respond more like a taut string. Proceeding to model the beam by integrating the equations of motion over a section, following the same five steps outlined earlier, we now recognize that the initial stress σ^o contributes a term to the equations of small transverse motion so that these now become

$$\partial V/\partial X + \sigma^o A\, \partial^2 u/\partial X^2 + F = \rho A\, \partial^2 u/\partial t^2, \quad \partial M/\partial X + V = 0 ,$$

where here V is defined by $V = \int_A N_{21} dA$ and we recognize that $N_{12} - N_{21} = \sigma^o\, \partial^2 u_2/\partial X^2$. Again one obtains $M = EI\, \partial^2 u/\partial X^2$, so that the equation of motion with initial stress is

$$-\partial^2 (EI\, \partial^2 u/\partial X^2)/\partial X^2 + \sigma^o A\, \partial^2 u/\partial X^2 + F = \rho A\, \partial^2 u/\partial t^2 .$$

Free vibrations. As a particular case of some interest, suppose that the beam is of length L, is of uniform properties, and is hinge-supported at its ends at $X = 0$ and $X = L$ so that $u = M = 0$ there. Then free transverse motions of the beam, solving the above equation with F = 0, are described by any linear combination of the real part of solutions of the form

$$u(X,t) = C_n \exp(i\omega_n t)\sin(n\pi X / L)$$

where n is any positive integer, C_n is an arbitrary complex constant, and where

$$\rho\, A\, \omega_n^2 = (n\,\pi / L)^4\, E\, I\, [1 + (\sigma^o / E)(A\, L^2 / n^2\, \pi^2\, I)]$$

This expression is arranged so that the bracket shows the correction, from unity, of what would be the expression giving the frequencies of free vibration for a beam when there is no σ^o.

The correction from unity can be quite significant, even though σ^o/E is always much smaller than unity (for interesting cases, a few times 10^{-6} to, say, 10^{-3} would be a representative range; few materials in bulk form would remain elastic or resist fracture at higher σ^o/E, although good piano wire could reach about 10^{-2}). That is because σ^o/E is multiplied by a term which can become enormous for a beam which is long compared to its thickness. For a square section of side length h, that term (at its largest, when n = 1) is $AL^2 / \pi^2 I \approx 1.2L^2 / h^2$, which can combine with a small σ^o/E to produce a correction term within the brackets which is quite non-negligible compared to unity. When $\sigma^o > 0$ and L is large enough to make the bracketed expression much larger than unity, the EI term cancels out and the beam simply responds like a stretched string (here *string* denotes an object which is unable to support a bending moment), although at large enough vibration mode number n, the string-like effects become negligible and beam like response takes over. At sufficiently high n that L/n is reduced to the same order as h, the simple beam theory becomes inaccurate and should be replaced by three-dimensional elasticity or, at least, an improved beam theory which takes account rotary inertia and shear deformability. (While the option of using three-dimensional elasticity for such a problem posed an insurmountable obstacle over most of the history of the subject, by 1990 the availability of computing power and easily used software reduced it to a routine problem which could be studied by an undergraduate engineer or physicist using the finite element method or some other computational mechanics technique.)

Buckling. An important case of that of compressive loading, $\sigma^o < 0$, which can lead to

buckling. Indeed, we see that if $\sigma^o A < -\pi^2 EI/L^2$, then the ω_n^2 for at least for n = 1 is negative, which means that the corresponding ω_n is of the form $\pm ib$, where b is a positive real number, so that the $\exp(i\omega_n t)$ term has a time dependence of type which no longer involves oscillation but, rather, exponential growth, $\exp(bt)$. The critical compressive force, $\pi^2 EI/L^2$, which causes this type of behavior is called the Euler buckling load. Different numerical factors are obtained for different end conditions than the pinned ends assumed here. We may see that the acceleration associated with the $n = 1$ mode becomes small in the vicinity of the critical load, and vanishes at that load. Thus solutions are possible, at the buckling load, for which the column takes a deformed shape without acceleration; for that reason, an approach to buckling problems which is equivalent for what, in dynamical terminology, are called conservative systems is to seek the first load at which an alternate equilibrium solution $u = u(X)$, other than $u = 0$, may exist.

Instability by *divergence*, that is, with growth of displacement in the form $\exp(bt)$, is representative of conservative systems. Columns under non-conservative loadings, for example, by a follower force which has the property that its line of action rotates so as to always be tangent to the beam center line at its place of application, can exhibit a *flutter* instability in which the dynamic response is proportional to the real or imaginary part of a term like $\exp(iat)\exp(bt)$, that is, an oscillation with exponentially growing amplitude. Such instabilities also arise in the coupling between fluid flow and elastic structural response, as in the subfield called *aeroelasticity*, and the prototype is the flutter of an airplane wing. That is a torsional oscillation of the wing, of growing amplitude, which is driven by the coupling between rotation of the wing and the development of aerodynamic forces related to the angle of attack; the coupling feeds more energy into the structure with each cycle. Of course, instability models based on linearized theories, and predicting exponential growth in time, really tell us no more than that the system is deforming out of the range for which our mathematical model applies. Proper nonlinear theories, that take account of the finiteness of rotation and, sometimes, the large and possibly non-elastic strain of material fibers are necessary to really understand the phenomena. An important subclass of such nonlinear analyses for conservative systems involves the static post-buckling response of a perfect structure, such as a perfectly straight column or perfectly spherical shell. That post-buckling analysis allows one to determine if increasing force is required for very large displacement to develop during the buckle, or whether the buckling is of a more highly unstable type for which the load must diminish with buckling amplitude in order to still satisfy the equilibrium equations. The latter type of behavior describes a structure which may show strong sensitivity of the maximum load that it can support to very small imperfections of material or geometry, as do many shell structures.

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An introduction to plate and shell structures is given in the book by Den Hartog cited above, and the topic treated at an advanced level in those cited by Love and by Green and Zerna; books specializing on the topic are

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- Bazant, Z. P. and L. Cedolin, *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press, New York, 1991,

whereas a treatment of plasticity in a way tht updates the book cited by Hill is given by

- Lubliner, J., *Plasticity Theory*, Macmillan, New York, 1990.

Appendix: ***Comments on notation used here, versus that often used in elementary texts and other sources***

Coordinate axes: Denoted $1, 2, 3$ or x_1, x_2, x_3 here, versus x, y, z

Stress: $\boldsymbol{\sigma} = [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ here, versus $\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$

Displacement: $\mathbf{u} = \{u\} = [u_1, u_2, u_3]^{\mathrm{T}}$ here, versus $[u, v, w]^{\mathrm{T}}$

Strain: $\boldsymbol{\varepsilon} = [\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \gamma_{12}/2 & \gamma_{13}/2 \\ \gamma_{21}/2 & \varepsilon_{22} & \gamma_{23}/2 \\ \gamma_{31}/2 & \gamma_{32}/2 & \varepsilon_{33} \end{bmatrix}$ here,

versus $\begin{bmatrix} \varepsilon_x & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_y & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_z \end{bmatrix}$

Note on relation to other works by the author

The *Introduction* and *Historical Sketch* here closely follow parts of the article *Mechanics of Solids* published by the author as a section of the article on *Mechanics* in the 1993 printing of the 15th edition of *Encyclopaedia Britannica* (volume 23, pages 734 - 747 and 773). Some subsequent portions are expanded versions of that *Encyclopaedia Britannica* text. Also, parts have been presented in a slightly modified form, sometimes at a more advanced level, as Chapter 3, *Foundations of Solid Mechanics,* of the book *Mechanics and Materials: Fundamentals and Linkages* (eds. M. A. Meyers, R. W. Armstrong, and H. Kirchner), Wiley, 1999.

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