Solution of ODEs using Laplace Transforms

Process Dynamics and Control

Linear ODEs

- For linear ODEs, we can solve without integrating by using Laplace transforms

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_{0^+}^{\infty} f(t)e^{-st}dt$$

- Integrate out time and transform to *Laplace domain*

$$\frac{dy}{dt} = ay(t) + bu(t),\ \ y(0) = 0$$

$$\xrightarrow[\textit{Multiplication}]{\textit{Integration}}$$

$$Y(s) = G(s)U(s)$$

Common Transforms

Useful Laplace Transforms

1. Exponential $f(t) = e^{-bt}$

$$\begin{aligned}\mathcal{L}\left[e^{-bt}\right] &= \int_0^\infty e^{-bt}e^{-st}dt = \int_0^\infty e^{-(s+b)t}dt \\ &= -\left.\frac{e^{-(s+b)t}}{s+b}\right]_0^\infty = \frac{1}{s+b}\end{aligned}$$

2. Cosine $f(t) = \cos\omega t = \dfrac{e^{j\omega t} + e^{-j\omega t}}{2}$

$$\begin{aligned}\mathcal{L}\left[\cos\omega t\right] &= \frac{1}{2}\left\{\int_0^\infty e^{-(s-j\omega)t}dt + \int_0^\infty e^{-(s+j\omega)t}dt\right\} \\ &= \frac{1}{2}\left\{\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right\} = \frac{s}{s^2+\omega^2}\end{aligned}$$

Useful Laplace Transforms

3. Sine

$$f(t) = \sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$\begin{aligned}
\mathcal{L}\left[\sin \omega t\right] &= \frac{1}{2j}\left\{\int_0^\infty e^{-(s-j\omega)t}dt - \int_0^\infty e^{-(s+j\omega)t}dt\right\} \\
&= \frac{1}{2j}\left\{\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right\} = \frac{\omega}{s^2+\omega^2}
\end{aligned}$$

Common Transforms

Operators

1. Derivative of a function, $f(t)$, $\frac{df(t)}{dt}$

$$\begin{aligned}\mathcal{L}\left[\frac{df(t)}{dt}\right] &= f(t)e^{-st}\Big]_0^\infty - \int_0^\infty (-sf(t)e^{-st})dt \\ &= s\int_0^\infty f(t)e^{-st}dt - f(0) = sF(s) - f(0)\end{aligned}$$

2. Integral of a function $f(t)$

$$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt = \frac{F(s)}{s}$$

Operators

3. Delayed function $f(t-\tau)$

$$g(t)=\begin{cases} 0 & t<\tau \\ f(t-\tau) & t\geq\tau \end{cases}$$

$$\begin{aligned} \mathcal{L}\left[g(t)\right] &= \int_0^{\tau} e^{-st}(0)dt+\int_{\tau}^{\infty} e^{-st}f(t-\tau)d\tau \\ &= e^{-s\tau}F(s) \end{aligned}$$

Input Signals

1. Constant

$$f(t) = a$$

$$\mathcal{L}[a] = \int_0^\infty a e^{-st} dt = -\frac{a e^{-st}}{s}\bigg]_0^\infty = \frac{a}{s}$$

2. Step

$$f(t) = \begin{cases} 0 & t < 0 \\ a & t \geq 0 \end{cases}$$

$$\mathcal{L}[a] = \int_0^\infty a e^{-st} dt = -\frac{a e^{-st}}{s}\bigg]_0^\infty = \frac{a}{s}$$

3. Ramp function

$$f(t) = \begin{cases} 0 & t < 0 \\ at & t \geq 0 \end{cases}$$

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty a t e^{-st} dt = -\frac{a t e^{-st}}{s}\bigg]_0^\infty \frac{a e^{-st}}{s} dt = \frac{a}{s^2}$$

Common Transforms

Input Signals

4. Rectangular Pulse

$$f(t) = \begin{cases} 0 & t < 0 \\ a & t \le t_w \\ 0 & t \ge t_w \end{cases}$$

$$\mathcal{L}\left[f(t)\right] = \int_0^{t_w} ae^{-st}dt = \frac{a}{s}\left(1 - e^{-t_w s}\right)$$

5. Unit impulse $a = \frac{1}{t_w}$

$$\mathcal{L}\left[\delta(t)\right] = \lim_{t_w \to 0} \frac{1}{t_w s}\left(1 - e^{-t_w s}\right)$$

$$\mathcal{L}\left[\delta(t)\right] = \lim_{t_w \to 0} \frac{se^{-st}}{s} = 1$$

Final Value Theorem

$$\boxed{\lim_{t\to\infty}[y(t)] = \lim_{s\to 0}[sY(s)]}$$

Limitations:

$$y(t) \in C^1$$

$$\lim_{s\to 0}[sY(s)] \text{ exists } \forall s,\ \mathrm{Re}(s) \geq 0.$$

Initial Value Theorem

$$\boxed{y(0) = \lim_{s\to\infty}[sY(s)]}$$

Solution of ODEs

We can continue taking Laplace transforms and generate a catalogue of Laplace domain functions.

The final aim is the solution of ordinary differential equations.

Example

Using Laplace Transform, solve

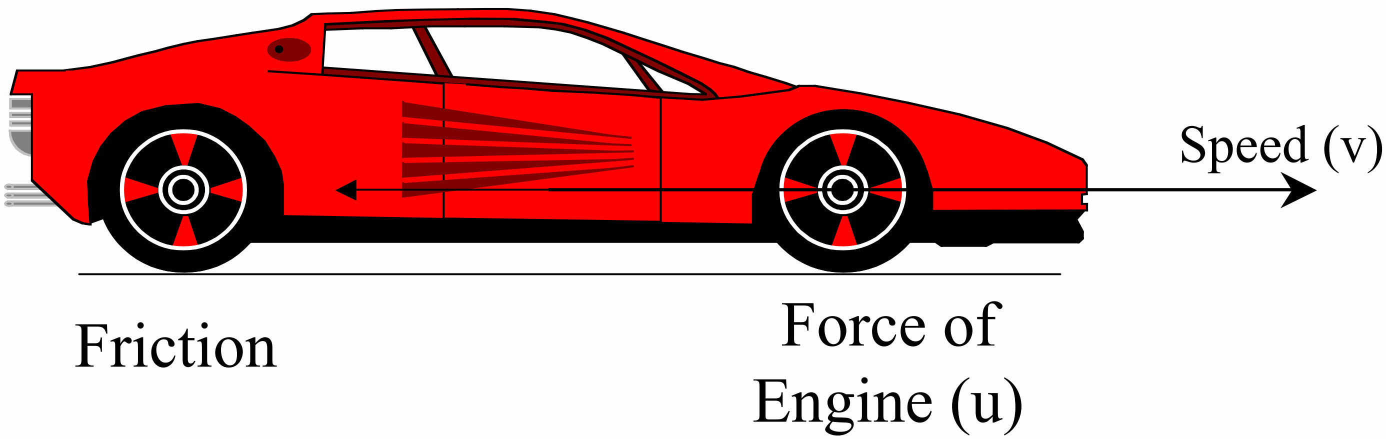
$$5\frac{dy}{dt} + 4y = 2, \;\; y(0) = 1$$

Result

$$y(t) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\frac{4}{5}t}$$

Solution of ODEs

Cruise Control Example



$$\dot{v} = \frac{1}{m_{car}} u - \frac{b}{m_{car}} v$$

➢ *Taking the Laplace transform of the ODE* yields (recalling the Laplace transform is a linear operator)

$$sV(s) = \frac{1}{m_{car}} U(s) - \frac{b}{m_{car}} V(s)$$

Solution of ODEs

- Isolate $V(s)$

$$(s + \frac{b}{m_{car}})V(s) = \frac{1}{m_{car}}U(s)$$

and solve

$$V(s) = \frac{\frac{1}{m_{car}}}{(s+\frac{b}{m_{car}})}U(s)$$

  - ➢ If the input is kept constant

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ c & \text{otherwise} \end{cases}$$

its Laplace transform

$$U(s) = \frac{c}{s}$$

  - ➢ Leading to

$$V(s) = \frac{\frac{c}{m_{car}}}{s(s+\frac{b}{m_{car}})}$$

Solution of ODEs

- Solve by inverse Laplace transform: (tables)

$$V(s) = \frac{\frac{c}{m_{car}}}{s(s+\frac{b}{m_{car}})} \longrightarrow v(t) = \frac{c}{b}(1 - e^{-\frac{m_{car}}{b}t})$$

➢ Solution is obtained by a getting the inverse Laplace transform from a table

➢ Alternatively we can use partial fraction expansion to compute the solution using simple inverse transforms

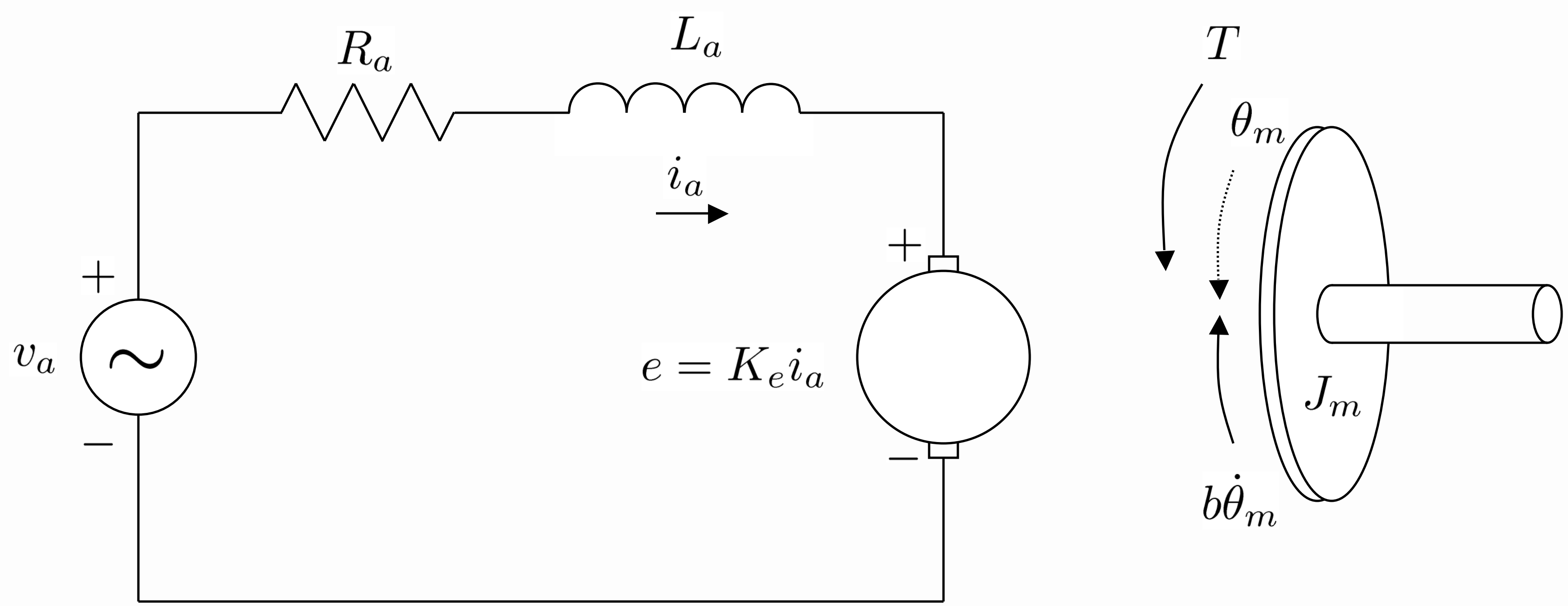
$$V(s) = \frac{\frac{c}{m_{car}}}{s(s+\frac{b}{m_{car}})} = \frac{A}{s} + \frac{B}{s+\frac{b}{m_{car}}} \longrightarrow A = \frac{c}{b},\ B = -\frac{c}{b}$$

$$\mathcal{L}^{-1}\left[V(s)\right] = \mathcal{L}^{-1}\left[\frac{A}{s}\right] + \mathcal{L}^{-1}\left[\frac{B}{s+\frac{b}{m_{car}}}\right]$$

$$= A + Be^{-\frac{b}{m_{car}}}$$

Solution of Linear ODEs

- DC Motor



- System dynamics describes (negligible inductance)

$$\ddot{\theta}_m + \left(\frac{b}{J_m} + \frac{K_e K_t}{J_m R_a}\right)\dot{\theta}_m = \frac{K_t}{J_m R_a} v_a$$

- Expressing in terms of angular velocity $\omega(t) = \dot{\theta}(t)$

$$\dot{\omega}_m + \left(\frac{b}{J_m} + \frac{K_t K_e}{J_m R_a}\right)\omega_m = \frac{K_t}{J_m R_a} v_a$$

  - Taking Laplace Transforms

$$s\Omega(s) + \left(\frac{b}{J_m} + \frac{K_e K_t}{J_m R_a}\right)\Omega(s) = \frac{K_t}{J_m R_a} V_a(s)$$

  - Solving

$$\frac{\Omega(s)}{V_a(s)} = \frac{\frac{K_t}{J_m R_a}}{s + \left(\frac{b}{J_m} + \frac{K_e K_t}{J_m R_a}\right)}$$

  - Note that this function can be written as

$$\frac{\Omega(s)}{V_a(s)} = \frac{K}{\tau s + 1}$$

Laplace Transform

Assume $v_a(t) = \sin \omega t$ then the transfer function gives directly

$$Q(s) = \frac{K}{\tau s+1} \frac{\omega}{s^2+\omega^2}$$

Cannot invert explicitly, but if we can find $A_0,\ A_1,\ B$ such that

$$\frac{A_1 s + A_0}{s^2+\omega^2} + \frac{B}{\tau s+1} = \frac{K}{\tau s+1} \frac{\omega}{s^2+\omega^2}$$

we can invert using tables.

Need <u>*Partial Fraction Expansion*</u> to deal with such functions

Linear ODEs

We deal with rational functions of the form $r(s) = \frac{p(s)}{q(s)}$ where degree of $q(s) >$ degree of $p(s)$

$q(s)$ is called the characteristic polynomial of the function $r(s)$

The roots of $q(s) = 0$ *are the* ***poles*** *of the function* $r(s)$

Theorem:

Every polynomial $q(s)$ with real coefficients can be factored into the product of only two types of factors

- powers of linear terms $(s+b)^n$ and/or
- powers of irreducible quadratic terms $(s^2 + d_1 s + d_0)^m$

Partial fraction Expansions

1. $q(s)$ has real and distinct factors

$$q(s) = \prod_{i=1}^{n} (s + b_i)$$

expand as

$$r(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i}$$

2. $q(s)$ has real but repeated factor

$$q(s) = (s + b)^n$$

expanded

$$r(s) = \frac{\alpha_1}{s+b} + \frac{\alpha_2}{(s+b)^2} + \cdots + \frac{\alpha_n}{(s+b)^n}$$

Partial Fraction Expansion

Heaviside expansion

For a rational function of the form

$$r(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{\prod_{i=1}^{n}(s+b_i)} = \sum_{i=1}^{n} \frac{\alpha_i}{s+b_i}$$

Constants are given by

$$\alpha_i = (s+b_i)\frac{p(s)}{q(s)}\bigg]_{s=-b_i}$$

Partial Fraction Expansion

Example

$$r(s) = \frac{s+2}{s^3+10s^2+29s+20}$$

The polynomial

$$q(s) = s^3 + 10s^2 + 29s + 20$$

has roots

$$s = -1, \; s = -4 \;, s = -5$$

It can be factored as

$$q(s) = (s+1)(s+4)(s+5)$$

By partial fraction expansion

$$r(s) = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} + \frac{\alpha_3}{s+5}$$

Partial Fraction Expansion

By Heaviside

$$\alpha_1 = \left.\frac{s+2}{(s+4)(s+5)}\right]_{s=-1} = \frac{1}{12} \qquad \alpha_2 = \left.\frac{s+2}{(s+1)(s+5)}\right]_{s=-4} = \frac{2}{3}$$

$$\alpha_3 = \left.\frac{s+2}{(s+1)(s+4)}\right]_{s=-4} = -\frac{3}{4}$$

- $r(s)$ becomes

$$r(s) = \frac{1}{12(s+1)} + \frac{2}{3(s+4)} - \frac{3}{4(s+5)}$$

- By inverse laplace

$$\mathcal{L}^{-1}\left[r(s)\right] = \frac{1}{12}e^{-t} + \frac{2}{3}e^{-4t} - \frac{3}{4}e^{-5t}$$

Partial Fraction Expansion

Heaviside expansion

For a rational function of the form

$$r(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s+b)^n} = \frac{\alpha_1}{s+b} + \frac{\alpha_2}{(s+b)^2} + \cdot + \frac{\alpha_n}{(s+b)^n}$$

Constants are given by $\quad \alpha_n = (s+b)^n \frac{p(s)}{q(s)}\Big]_{s=-b}$

$$\alpha_{n-1} = \frac{d}{ds}\left((s+b)^n \frac{p(s)}{q(s)}\right)\Big]_{s=-b}$$

$$\vdots$$

$$\alpha_1 = \frac{d^{n-1}}{ds^{n-1}}\left((s+b)^n \frac{p(s)}{q(s)}\right)\Big]_{s=-b}$$

Partial Fraction Expansion

Example

$$r(s) = \frac{s+2}{s^3+9s^2+24s+16}$$

The polynomial

$$q(s) = s^3 + 9s^2 + 24s + 16$$

has roots

$$s = -1, \; s = -4 \;, s = -4$$

It can be factored as

$$q(s) = (s+1)(s+4)^2$$

By partial fraction expansion

$$r(s) = \frac{\alpha_1}{s+1} + \frac{\alpha_2}{s+4} + \frac{\alpha_3}{(s+4)^2}$$

Partial Fraction Expansion

By Heaviside

$$\alpha_1 = \left.\frac{s+2}{(s+4)^2}\right]_{s=-1} = \frac{1}{9} \qquad \alpha_3 = \left.\frac{s+2}{(s+1)}\right]_{s=-4} = \frac{2}{3}$$

$$\alpha_3 = \left.\frac{d}{ds}\left(\frac{s+2}{(s+1)}\right)\right]_{s=-4} = -\left.\frac{1}{(s+1)^2}\right]_{s=-4} = -\frac{1}{9}$$

- $r(s)$ becomes

$$r(s) = \frac{1}{9(s+1)} - \frac{1}{9(s+4)} + \frac{2}{3(s+4)^2}$$

- By inverse laplace

$$\mathcal{L}^{-1}\left[r(s)\right] = \frac{1}{9}e^{-t} - \frac{1}{9}e^{-4t} + \frac{2}{3}te^{-4t}$$

3. $q(s)$ has an irreducible quadratic factor

$$q(s) = (s^2 + d_1 s + d_0)$$

- Gives a pair of complex conjugates if $d_1^2 < 4d_0$

$$s = -\frac{d_1}{2} \pm \frac{1}{2}\sqrt{d_1^2 - 4d_0}$$

- Can be factored in two ways

  a) $r(s)$ is factored as

$$r(s) = \frac{A}{s+a+bj} + \frac{B}{s+a-bj}$$

  b) or as

$$r(s) = \frac{As+B}{s^2+d_1 s+d_0} = \frac{As+B}{(s+a)^2+b^2}$$

Partial Fraction Expansion

Heaviside expansion

For a rational function of the form

$$r(s) = \frac{p(s)}{q(s)} = \frac{p(s)}{(s^2 + d_1 s + d0)} = \frac{A}{s + a + bj} + \frac{B}{s + a - bj}$$

Constants are given by

$$A = (s + a + bj)\frac{p(s)}{q(s)}\bigg]_{s=-a-bj}$$

$$A = (s + a - bj)\frac{p(s)}{q(s)}\bigg]_{s=-a+bj}$$

Partial Fraction Expansion

Example
$$r(s) = \frac{s+1}{s^2+s+1}$$

The polynomial
$$q(s) = s^2 + s + 1$$

has roots
$$s = -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2} j$$

It can be factored as
$$q(s) = \left(s + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} j\right)\left(s + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} j\right)$$

By partial fraction expansion
$$r(s) = \frac{\alpha_1}{s+\frac{1}{2}-\frac{\sqrt{3}}{2}j} + \frac{\alpha_2}{s+\frac{1}{2}+\frac{\sqrt{3}}{2}j}$$

Partial Fraction Expansion

By Heaviside,

$$\alpha_1 = \left(s + \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}j\right) \frac{s+1}{s^2+s+1}\Bigg]_{s=-\frac{1}{2}+\frac{\sqrt{3}}{2}j} \qquad \alpha_2 = \left(s + \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}j\right) \frac{s+1}{s^2+s+1}\Bigg]_{s=-\frac{1}{2}-\frac{\sqrt{3}}{2}j}$$

$$= \frac{s+1}{\left(s+\frac{1}{2}+\frac{\sqrt{3}}{2}j\right)}\Bigg]_{s=-\frac{1}{2}+\frac{\sqrt{3}}{2}j} \qquad = \frac{s+1}{\left(s+\frac{1}{2}-\frac{\sqrt{3}}{2}j\right)}\Bigg]_{s=-\frac{1}{2}-\frac{\sqrt{3}}{2}j}$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{6}j \qquad = \frac{1}{2} + \frac{\sqrt{3}}{6}j$$

which yields $\quad r(s) = \dfrac{\frac{1}{2}-\frac{\sqrt{3}}{6}j}{s+\frac{1}{2}-\frac{\sqrt{3}}{2}j} + \dfrac{\frac{1}{2}+\frac{\sqrt{3}}{6}j}{s+\frac{1}{2}+\frac{\sqrt{3}}{2}j}$

Taking the inverse laplace

$$\mathcal{L}^{-1}\left[r(s)\right] = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}j\right) e^{-\frac{1}{2}t} e^{\frac{\sqrt{3}}{2}tj} + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}j\right) e^{-\frac{1}{2}t} e^{-\frac{\sqrt{3}}{2}tj}$$

Partial Fraction Expansion

The inverse laplace

$$\mathcal{L}^{-1}\left[r(s)\right] = \left(\frac{1}{2} - \frac{\sqrt{3}}{6}j\right)e^{-\frac{1}{2}t}e^{\frac{\sqrt{3}}{2}tj} + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}j\right)e^{-\frac{1}{2}t}e^{-\frac{\sqrt{3}}{2}tj}$$

Can be re-arranged to

$$\mathcal{L}^{-1}\left[r(s)\right] = e^{-\frac{1}{2}t}\left(\frac{e^{\frac{\sqrt{3}}{2}tj}+e^{-\frac{\sqrt{3}}{2}tj}}{2} - \frac{\sqrt{3}}{3}j\,\frac{e^{\frac{\sqrt{3}}{2}tj}-e^{-\frac{\sqrt{3}}{2}tj}}{2}\right)$$

$$\mathcal{L}^{-1}\left[r(s)\right] = e^{-\frac{1}{2}t}\left(\frac{e^{\frac{\sqrt{3}}{2}tj}+e^{-\frac{\sqrt{3}}{2}tj}}{2} + \frac{\sqrt{3}}{3}\,\frac{e^{\frac{\sqrt{3}}{2}tj}-e^{-\frac{\sqrt{3}}{2}tj}}{2j}\right)$$

$$\mathcal{L}^{-1}\left[r(s)\right] = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right)$$

Partial Fraction Expansion

Example

$$r(s) = \frac{s+1}{s^2+s+1}$$

The polynomial $q(s) = s^2 + s + 1$

has roots $s = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$

It can be factored as ($a = \frac{1}{2},\ b = \frac{\sqrt{3}}{2}$)

$$r(s) = \frac{A(s+a)}{(s+a)^2+b^2} + \frac{Bb}{(s+a)^2+b^2}$$

Solving for A and B,

$$r(s) = \frac{A(s+a)}{(s+a)^2+b^2} + \frac{Bb}{(s+a)^2+b^2} = \frac{s+1}{(s+a)^2+b^2}$$

Partial Fraction Expansion

Equating similar powers of s in,

$$r(s) = \frac{A(s+a)}{(s+a)^2+b^2} + \frac{Bb}{(s+a)^2+b^2} = \frac{s+1}{(s+a)^2+b^2}$$

yields

$$\begin{aligned} A &= 1 \\ Aa + Bb &= 1 \end{aligned}$$

hence

$$B = \frac{1-a}{b} = \frac{1}{2}\frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Giving

$$r(s) = \frac{(s+a)}{(s+a)^2+b^2} + \frac{\sqrt{3}}{3}\frac{b}{(s+a)^2+b^2}$$

Taking the inverse laplace

$$\mathcal{L}^{-1}\left[r(s)\right] = e^{-\frac{1}{2}t}\left(\cos\frac{\sqrt{3}}{2}t + \frac{\sqrt{3}}{3}\sin\frac{\sqrt{3}}{2}t\right)$$

Partial Fraction Expansions

Algorithm for Solution of ODEs

- Take Laplace Transform of both sides of ODE
- Solve for

$$r(s) = \frac{p(s)}{q(s)}$$

- Factor the characteristic polynomial $q(s)$
  - Find the roots (roots or poles function in Matlab)
  - Identify factors and multiplicities
- Perform partial fraction expansion
- Inverse Laplace using Tables of Laplace Transforms

Partial Fraction Expansion

- For a given function

$$r(s) = \frac{p(s)}{q(s)}$$

- The polynomial $q(s)$ has three distinct types of roots
  - Real roots
    - $s = -b_i$ yields exponential terms
    - $s = 0$ yields constant terms
  - Complex roots
    - $s = a_i \pm b_i j$ yields exponentially weighted sinusoidal signals
    - $s = \pm b_i j$ yields pure sinusoidal signal
- A lot of information is obtained from the roots of $q(s)$