Solution to
F.F. Chen's Plasma Physics

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Introduction

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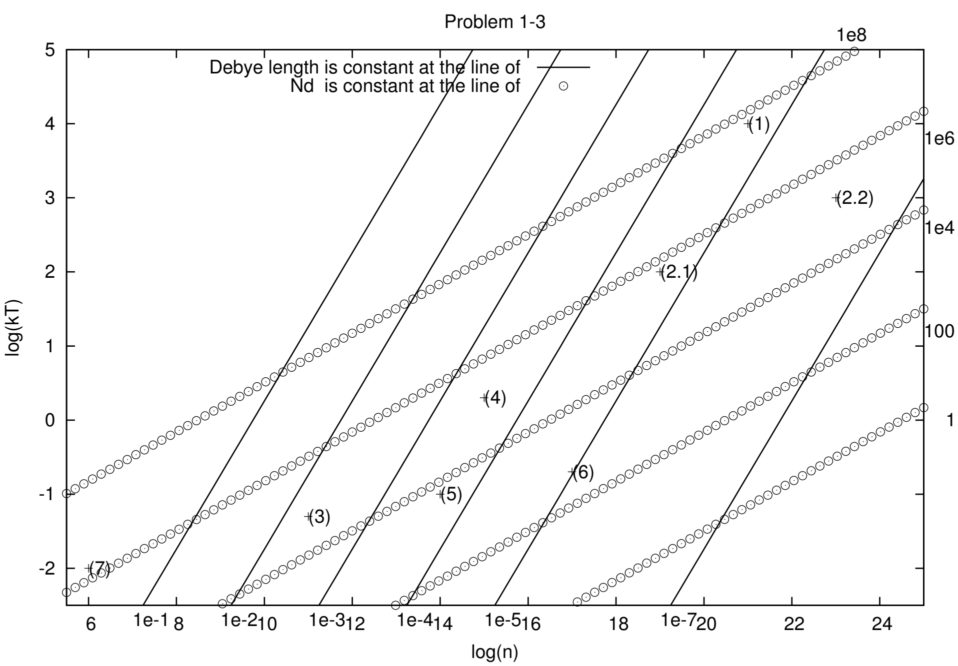


Figure 1.1: Label (1)(2.1)(2.2)(3)(4)(5)(6)(7) correspond to plasma in different condition as the problem describes.

According to the definition of the Debye Length

$$
\begin{aligned}
\lambda_D &= (\frac{\epsilon_0 k T_e}{n e^2})^{1/2} && (1.1)\\
\Rightarrow \log(\lambda_D) &= \frac{1}{2}\log(\frac{\epsilon_0}{e^2}) + \frac{1}{2}\log(kT_e) - \frac{1}{2}\log(n) && (1.2)\\
\Rightarrow \log(kT) &= \log(n) + 2\log(\frac{\lambda_D}{7430}) \qquad \text{(kT in eV)} && (1.3)
\end{aligned}
$$

Then we can draw the solid straight line in the Figure**??** with the Debye length as parameter ranged from 10^{-1} to 10^{-7}. Points on a certain solid line, named with *equi-Debye-length line*(analog to equipotential lines in electrostatic) , share a same Debye length. Similarily, with the given equation:

$$
\begin{aligned}
N_D &= \frac{4}{3}\pi\lambda_D^3 n && (1.4)\\
&= \frac{4}{3}\pi(7430\cdot\frac{kT}{n})^{\frac{3}{2}} \qquad \text{(kT in eV)} && (1.5)\\
\Rightarrow \frac{N_D}{2.7\times10^6} &= \frac{(kT)^{\frac{3}{2}}}{n^{\frac{1}{2}}} && (1.6)\\
\Rightarrow \log(\frac{N_D}{2.7\times10^6}) &= \frac{3}{2}\log(kT) - \frac{1}{2}\log(n) && (1.7)\\
\Rightarrow \log(kT) &= \frac{1}{3}\log(n) + \frac{2}{3}\log(\frac{N_D}{2.7\times10^6}) && (1.8)
\end{aligned}
$$

The dot lines in the Fig **??** can be named with *equi-*N_D lines, which means they share a same value of N_D in a dot lines.

As for the usage of this figure, take the point (4) for example, the point(4) falls in the region enclosed by two solid lines and two dots ones.The two solid lines respectively have the Debye length of 10^{-3}m and 10^{-4}m. And the dots lines have the particle numbers N_D of 10^4 and 10^6. That is tantamount to the fact that (4) has Debye length $10^{-4}m < \lambda_D < 10^{-3}m$ and number of particle $10^4 < N_D < 10^6$.

Recall for the criteria for plasmas, $N_D \ggg 1$ is automatically meet since the smallest number of particle is larger than 100.

Problem 1-8

The Debye length is

$$
\begin{aligned}
\lambda_D &= 69(\frac{T}{n})^{\frac{1}{2}} \qquad \text{(T in the unit of K)}\\
&= 69\times(\frac{5\times10^7}{10^{33}})^{\frac{1}{2}}\\
&= 1.54\times10^{-11}\text{m}
\end{aligned}
$$

Naturally, the number of particles contained in a Debye Sphere is :

$$\begin{aligned} N_D &= \frac{4}{3}\pi\lambda_D^3 \times n \\ &= \frac{4}{3}\pi \times (1.54)^3 \\ &\approx 15 \end{aligned}$$

Problem 1-9

Since protons and antiprotons have the same inertia, both of them are fixed. Assume that protons and antiprotons follows the Maxwellian distribution.

$$f(u) = Ae^{-(\frac{1}{2}mu^2+q\phi)/kT_e},$$

where q equals to e for protons while q equals to $-e$ for antiprotons. Moreover,

$$\begin{aligned} n_p(\phi \to \infty) &= n_\infty \\ n_{\bar{p}}(\phi \to \infty) &= n_\infty. \end{aligned}$$

Then we obtain

$$\begin{cases} n_p = n_\infty \mathbf{exp}(\frac{-e\phi}{kT}) \\ n_{\bar{p}} = n_\infty \mathbf{exp}(\frac{e\phi}{kT}) \end{cases}$$

The Poisson's Equation is

$$\varepsilon_0 \nabla^2\phi = \varepsilon_0 \frac{\partial^2 \phi}{\partial x^2} = -e(n_p - n_{\bar{p}})$$

With $e\phi/kT \ll 1$,

$$\varepsilon_0 \frac{\partial^2 \phi}{\partial x^2} = \frac{2n_\infty e^2 \phi}{kT}$$

$$\therefore \lambda_D = \sqrt{\frac{\varepsilon_0 kT}{2n_\infty e^2}} = 0.4879\mathbf{m}$$

Problem 1-10

Regrad it as an isotropic space, which means that Φ has no components ofθ or ϕ.

$$\begin{aligned} \Phi &= A\frac{e^{-kr}}{r} \\ \nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}\Phi) = \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial r^2} \\ \frac{\partial \Phi}{\partial r} &= -A\frac{kr+1}{r^2}e^{-kr} \\ \frac{\partial^2 \Phi}{\partial r^2} &= \frac{k^2r^2+2kr+2}{r^3}Ae^{-kr} \end{aligned}$$

$$\therefore \nabla^2\Phi = Ak^2\frac{e^{-kr}}{r}$$

$$\because \nabla^2\Phi = \frac{ne^2}{\varepsilon_0 k_B T}\Phi - q_0\delta(r-a),$$

where $q_0 = 4\pi\varepsilon_0 a\phi_0$.

$$\therefore \frac{ne^2}{\varepsilon_0 k_B T} = k^2 \Rightarrow k = \sqrt{\frac{ne^2}{\varepsilon k_B T}} \Rightarrow \lambda_D = \frac{1}{k} = \sqrt{\frac{\varepsilon_0 k_B T}{ne^2}}$$

Consider the boundary condition :

$$A\frac{e^{-ka}}{a} = \Phi_0 \Rightarrow A = \frac{\Phi_0 a}{e^{-ka}}$$

So,

$$\Phi = \begin{cases} \Phi_0, r \in (0, a] \\ \frac{\Phi_0 a}{e^{-\frac{a}{\lambda_D}}}\frac{e^{-\frac{r}{\lambda_D}}}{r}, r \in (r, +\infty) \end{cases} \tag{1.9}$$

Chapter 2

Motion of Particle

Problem 2-2

Since A=2, for deuterium ion,

$$\begin{aligned} m &= 2m_p = 3.34 \times 10^{-27} kg \\ q &= |e| = 1.60 \times 10^{-19} Coulomb. \end{aligned}$$

Assume that energy can be entirely converted to kinetic energy, then the momentum can be derived

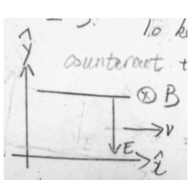
$$E = E_k = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE_k} = 1.46 \times 10^{-20} kg \cdot m/s$$

The Larmour radius

$$r_L = \frac{mv_\perp}{|q|B} = \frac{1.46 \times 10^{-20}}{5 \times 1.60 \times 10^{-19}} = 0.018\mathbf{m} \ll a = 0.6\mathbf{m}$$

So the Larmour radius satisfies the confined-ion condition.

Problem 2-3



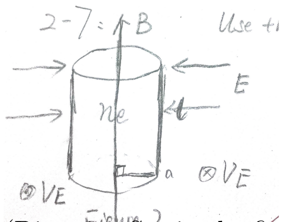
To keep a equibrillium in the $\hat{y}$ direction, the electric force should conteract the Lorentz force

$$Eq = qvB$$

$$\Rightarrow E = vB = 10^6 \mathbf{V/m}$$

Problem 2-7

Apply the Gauss Law to obtain the magnitude of electric field at r=a



$$E \cdot l \cdot 2\pi a = n_e l \cdot \pi a^2/\varepsilon_0$$

$$\therefore E = \frac{e n_e a}{2\varepsilon} = 9.04 \times 10^3 V/m$$

$$v_E = \frac{E}{B} = 4.52 \times 10^3 \mathbf{m/s}$$

(Direction:See in the figure)

Problem 2-10

The mass of a deutron is

$$m_d = 1875 \mathbf{MeV/c^2}$$

The kinetic energy

$$E_k = \frac{1}{2} m_d v^2 \Rightarrow v = 1.386 \times 10^6 \mathbf{m/s}$$

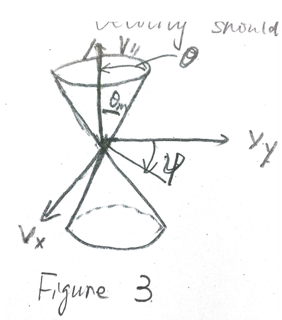
$$v_\perp = v \cos 45^\circ = 9.8 \times 10^5 \mathbf{m/s}$$

$$r_L \frac{m_d v_\perp}{qB} = 0.03m$$

Problem 2-11

$$R_m = 4 \Rightarrow \frac{1}{\sin^2 \theta_m} = 4 \Rightarrow \sin \theta_m = \frac{1}{2}$$

$$\therefore \theta_m = \frac{\pi}{6}$$



Since the velocity is isotropic distribution, the direction of velocity should distribute uniformly

$$\mathbf{d}\Omega = \sin \theta \mathbf{d}\theta \mathbf{d}\varphi$$

The total solid angle for a sphere is

$$\Omega_{total} = 4\pi$$

The solid angle for loss cone is

$$\Omega_{loss} = 2 \int_0^{\frac{\pi}{6}} \sin \theta \mathbf{d}\theta \int_0^{2\pi} \mathbf{d}\varphi = (1 - \frac{\sqrt{3}}{2}) 4\pi$$

$\therefore$ the fraction of the trapped is $\frac{\sqrt{3}}{2}$.

Problem 2-15

Define the displacement in polarization direction is x_p.
So the work done by the electric field is

$$W = q\vec{E} \cdot \vec{x_P}$$

The energy gain rate is

$$\frac{\mathbf{d}W}{\mathbf{d}t} = qE\frac{\mathbf{d}x_p}{\mathbf{d}t} = qEv_p.$$

The change of kinetic energy

$$\frac{\mathbf{d}E_k}{\mathbf{d}t} = \frac{\mathbf{d}}{\mathbf{d}t}(\frac{1}{2}mv_E^2) = mv_E\frac{\mathbf{d}v_E}{\mathbf{d}t}$$

Without loss of generality, let the v_E in the same direction of E

$$\begin{aligned}
v_E &= -\frac{E}{B} \\
\therefore \frac{\mathbf{d}v_E}{\mathbf{d}t} &= -\frac{1}{B}\frac{\mathbf{d}E}{\mathbf{d}t} \\
\therefore \frac{\mathbf{d}E_k}{\mathbf{d}t} &= mv_E(-\frac{1}{B}\frac{\mathbf{d}E}{\mathbf{d}t}) = qEv_p \\
v_p &= \frac{m}{qB}\frac{1}{B}\frac{\mathbf{d}E}{\mathbf{d}t}
\end{aligned}$$

Replace $\pm\frac{1}{\omega_c} = \frac{m}{qB}$ Thus,

$$v_p = \pm\frac{1}{\omega_c B}\frac{\mathbf{d}E}{\mathbf{d}t}$$

Problem 2-16

a) The Larmor frequency of electron is

$$\omega_e = \frac{eB}{m} = \frac{1.6 \times 10^{-19} \times 1}{9.109 \times 10^{-31}} = 1.76 \times 10^{11}\mathbf{rad/sec}$$

b) the Larmor frequency of ion is

$$\omega_i = \frac{eB}{m_i} = \frac{1.6 \times 10^{-19} \times 1}{1.67 \times 10^{-27}} = 9.58 \times 10^{7}\mathbf{rad/sec}$$

Since $\omega_0 = 10^9\mathbf{rad/sec}$, $\omega_e \gg \omega_0 \gg \omega_i$. The motion of electron is adiabatic, while that of ion not.

Problem 2-17

Since $\mu = \frac{mv_\perp^2}{2B}$ is conservative under this condition, it is easy to derive:

$$\frac{mv_\perp^2}{2B} = \frac{mv_\perp'^2}{2B'}$$

where $\frac{1}{2}mv_\perp^2 = 1\mathbf{keV}, B = 0.1T, B' = 1T$, so

$$\frac{1}{2}mv_\perp'^2 = 10\mathbf{keV}.$$

When collision happens, the direction of motion distorts, so $v_\perp = v_\parallel$. Then the kinetic energy is

$$\frac{1}{2}mv_\perp''^2 = 5\mathbf{keV}$$

Implement the adiabatic characteristic of μ, we know that

$$\frac{mv_\perp''^2}{2B'} = \frac{mv_\perp'''^2}{2B'} \Rightarrow \frac{1}{2}mv_\perp'''^2 = 0.5\mathbf{keV}$$

Finally, the energy is

$$E = E_\perp + E_\parallel = \frac{1}{2}mv_\perp'''^2 + \frac{1}{2}mv_\parallel''^2 = 5.5keV$$

Problem 2-18

a) At a certain moment, we calculate the motion in one periodic circular motion to certify the invariance of μ. In this period, we assume that the Larmour radius does not change with minor deviation of magnetic field-B. So

$$s = \pi r_L^2 = \pi\frac{m^2v_\perp^2}{q^2B^2}$$

$$\frac{\mathbf{d}}{\mathbf{d}t}\Phi = \frac{\mathbf{d}B}{\mathbf{d}t} \cdot S = \pi\frac{m^2v_\perp^2}{q^2B^2}\frac{\mathbf{d}B}{\mathbf{d}t} = \varepsilon(\mathbf{induced-potential})$$

So the change of the energy of the particle in one period is

$$\delta W = q\varepsilon = \pi\frac{m^2v_\perp^2}{qB^2}\frac{\mathbf{d}B}{\mathbf{d}t}$$

Within this period of gyration, the change of magnetic field is

$$\Delta B = \frac{\mathbf{d}B}{\mathbf{d}t}\tau = \frac{\mathbf{d}B}{\mathbf{d}t}\frac{2\pi m}{qB}$$

So after a period, the energy is

$$\begin{aligned} E_k' &= \frac{1}{2}mv_\perp^2 + \delta W = (1 + \frac{2\pi m}{qB^2}\frac{\mathbf{d}B}{\mathbf{d}t})\frac{1}{2}mv_\perp^2 \\ B' &= B + \Delta B = (1 + \frac{2\pi m}{qB^2}\frac{\mathbf{d}B}{\mathbf{d}t})B \end{aligned}$$

$$\mu' = \frac{E_k'}{B'} = \frac{E_0}{B_0} = \mu$$

So μ is invariant for both ion and electron.
b) Assume that $\frac{1}{2}mv_\perp^2 = KT$, $\frac{1}{2}mv_\parallel^2 = KT$ at initial moment. Considering

$$\mu = \frac{\frac{1}{2}mv_\perp^2}{B} = \frac{\frac{1}{2}mv_\perp'^2}{B'},$$

and $B = \frac{1}{3}B'$, so $\frac{1}{2}mv_\perp'^2 = 3KT$. Furthermore, we assume that $v_\parallel$ remains unchanged,

$$\frac{1}{2}mv_\parallel^2 = KT$$

$$kT_\perp = 3Kev, kT_\parallel = 1KeV$$

Problem 2-19

The magnetic field is $B = \frac{\mu_0 I}{2\pi r}$, so the gradient is :

$$\nabla B = -\frac{\partial B}{\partial r} = \frac{\mu_0 I}{2\pi r^2}$$

Apply the equipartition theorem of Maxwellian gas, the total energy is

$$\varepsilon = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) = \frac{3}{2}kT$$

$$\therefore \varepsilon_\perp = \frac{1}{2m}(p_x^2 + p_y^2) = kT$$

The perpendicular velocity is :

$$\therefore v_\perp = \sqrt{\frac{2kT}{m}}$$

$$\therefore r_L = \frac{mv_\perp}{eB}$$

$$\therefore v_{\nabla B} = \frac{1}{2}v_\perp r_L \frac{\nabla B}{B} = \frac{kT}{eBr}, \textbf{Upwards}$$

$$\because v_\parallel = \sqrt{\frac{kT}{m}} \Rightarrow v_R = \frac{mv_\parallel^2}{eBr} = \frac{kT}{eBr}, \textbf{Upwards}$$

$$\therefore v_{R+\nabla B} = \frac{2kT}{eBr}$$

Area of the top surface of the cyclinder is :

$$s = \pi[(R + \frac{a}{2})^2 - (R - \frac{a}{2})^2] = 2\pi Ra$$

The number of charged particle, which hit the surface s with in time of $\mathbf{d}t$ is:

$$N = n \cdot v \cdot s \cdot \mathbf{d}t.$$

So the hit rate is

$$\frac{\mathbf{d}N}{\mathbf{d}t} = n \cdot v \cdot s.$$

Since $R \gg a$, the drift velocity with $[R - \frac{a}{2}, R + \frac{a}{2}]$ region can be considered as *uniform.* So the accumulation rate is

$$R_{acc} = n \cdot v_{R+\nabla B} \cdot s \cdot e = \frac{4\pi kTna}{B} = 20\mathbf{Coulomb/S}$$

Problem 2-20

a) $v_\parallel^2 + v_\perp^2$ is invariant because of the conservation of energy. At $z = 0$, $B_z = B_0$, $v_\perp^2 = \frac{2}{3}v^2$,$v_\parallel^2 = \frac{1}{3}v^2$,

$$\mu = \frac{mv_\perp^2}{2B_0} = \frac{mv^2}{2B_0}.$$

When the electron reflects, $v_\parallel = 0$, then $v_\perp^2 = v^2$

$$\mu = \frac{mv^2}{3B_0} \Rightarrow B_z = \frac{3}{2}B_0$$

$$\Rightarrow 1 + \alpha^2 z^2 \Rightarrow z = \pm\frac{\sqrt{2}}{2\alpha}$$

b)

$$\because \mu = \frac{mv_\perp^2}{2B_z} \Rightarrow \mu B_0(1 + \alpha^2 z^2) = \frac{1}{2}mv_\perp^2$$

$$\therefore v_\parallel^2 = v^2 - v_\perp^2 = \frac{\mu_0 B}{m} - \frac{2\mu_0 B}{m}\alpha^2 z^2$$

$$\therefore (\frac{\mathbf{d}z}{\mathbf{d}t})^2 = \frac{\mu_0 B}{m}(1 - 2\alpha^2 z^2)$$

$$\frac{\mathbf{d}z}{\mathbf{d}t} = \sqrt{\frac{\mu_0 B_0}{m}}\sqrt{1 - 2\alpha^2 z^2}$$

$$\Rightarrow z = \frac{\sqrt{2}}{2\alpha}\sin(\sqrt{\frac{2\mu_0 B_0}{m}}\alpha t + \phi)$$

And this equation can describe the trajectory of the particle.
c) It is apparent that the gyration frequency is $\sqrt{\frac{2\mu_0 B_0}{m}}\alpha$ from the the equation of motion.
d) Claim: $\theta = \sqrt{\frac{2\mu_0 B_0}{m}}\alpha t + \phi$, $z = \frac{\sqrt{2}}{2\alpha}\sin\theta$

$$J = \int_a^b v_\parallel \mathbf{d}z$$

where $a = -\frac{\sqrt{2}}{2\alpha}, b = \frac{\sqrt{2}}{2\alpha}$. Thus,

$$\theta_a = -\frac{\pi}{2}, \theta_b = \frac{\pi}{2}$$

$$J = \int_a^b v_\parallel \mathbf{d}z = \int v_\parallel v_\parallel \mathbf{d}t = \frac{1}{\alpha}\sqrt{\frac{m}{2\mu B_0}}\int_{\theta_a}^{\theta_b} v_\parallel^2 \mathbf{d}\theta$$

where

$$\begin{aligned}
\mathbf{d}t &= \frac{1}{\alpha}\sqrt{\frac{m}{2\mu B_0}}\mathbf{d}\theta, \\
v_\parallel^2 &= (\frac{\mathbf{d}z}{\mathbf{d}t})^2 = \frac{\mu B_0}{m}\sin^2\theta.
\end{aligned}$$

$$\begin{aligned}
\therefore J &= \frac{1}{\alpha}\sqrt{\frac{m}{2\mu B_0}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mu B_0}{m}\sin^2\theta \mathbf{d}\theta \\
&= \frac{\pi}{2\alpha}\sqrt{\frac{\mu B_0}{2m}} = \textbf{constant}
\end{aligned}$$

Problem 2-21

a) According to the Ampere theorem, the magnetic field can be obtained

$$2\pi r \cdot B = \mu_0 I \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

In cylinder coordinate:

$$\nabla B = -\frac{\partial B}{\partial r} = \frac{\mu_0 I}{2\pi r^2}$$

$$\therefore v_{\nabla B} = \frac{1}{2}v_\perp r_L \frac{\nabla B}{B},$$

where $r_L = \frac{mv_\perp}{eB}, v_{\perp 0} = v_\perp$.

$$\therefore v_{\nabla B} = \frac{1}{2}v_{\perp 0}\frac{mv_{\perp 0}}{eB}\frac{1}{r} = \frac{\pi m v_{\perp 0}^2}{\mu_0 I e}$$

Besides,

$$\begin{aligned} v_R &= \frac{mv_\parallel^2}{e}\frac{1}{R_cB} \\ &= \frac{mv_\parallel^2}{e}\frac{1}{R_c\cdot\frac{\mu_0 I}{2\pi R_C}} \\ &= \frac{2\pi m v_\parallel^2}{\mu_0 I e} \end{aligned}$$

$$\therefore \vec{v} = \vec{v_R} + \vec{v_{\nabla B}} = \frac{3\pi m v_{\parallel 0}^2}{\mu_0 I e}(-\hat{z})$$

$$(v_{\parallel 0} = v_{\perp 0})$$

Chapter 3

Wave in Plasma

Problem 4-5

According to the dispersion relation

$$\omega^2 = \omega_p^2 + \frac{3KT_e}{m}k^2$$

Now we calculate the unknown value in the equation

$$\omega_p \approx 2\pi\sqrt{n} = 5.62 \times 10^9 rad/sec$$

$$\omega = 2\pi f = 6.908 \times 10^9 rad/sec$$

$$\frac{3kT_e}{m} = \frac{3 \times 100 \times 1.6E{-}19}{9.109 \times 10^{-31}} = 5.27 \times 10^{13}$$

$$\therefore k^2 = \frac{\omega^2 - \omega_p^2}{\frac{3kT_e}{m}} = 3.05 \times 10^5 \Rightarrow k = 552.377$$

$$\Rightarrow \lambda = \frac{2\pi}{k} = 1.14 \times 10^{-2} m$$

Problem 4-9

The critical density is

$$n_0 = \frac{m\varepsilon_0\omega^2}{e^2} = 1.12 \times 10^{15} m^{-3}$$

Problem 4-16

If the motion of ion is neglected, the dispersion relation of electron is

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

1) The resonance of X-wave is found by setting $k \to \infty$. So the dispersion relation can be rewrite into

$$c^2k^2 = \omega^2 - \omega_p^2\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$$

Differentiate both sides of the equation

$$2kc^2\mathbf{d}k = 2\omega\mathbf{d}\omega - \frac{2\omega(\omega_p^2 - \omega_h^2}{\omega^2 - \omega_h^2}\omega_p^2\mathbf{d}\omega$$

So the group velocity is

$$v_g = \frac{\mathbf{d}w}{\mathbf{d}k} = \frac{kc^2}{\omega[1 + \frac{\omega_c^2\omega_p^2}{(\omega^2 - \omega_h^2)^2}]}$$

When $k \to \infty$,which implies that $\omega = \omega_h$.
So $1 + \frac{\omega_c^2\omega_p^2}{(\omega^2 - \omega_h^2)^2} \to \infty, k \to \infty$ and $1 + \frac{\omega_c^2\omega_p^2}{(\omega^2 - \omega_h^2)^2}$ has a higher order than k. Thus, $v_g = 0$ at resonance point.
2) The cut-off X-wave is found by setting k=0, then

$$1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2} \Rightarrow \omega = \omega_R \textbf{ or } \omega_L$$

At this point, $\omega(1 + \frac{\omega_c^2\omega_p^2}{(\omega^2 - \omega_h^2)^2})$ is a finite value. As a result

$$v_g = \frac{kc^2}{\omega[1 + \frac{\omega_c^2\omega_p^2}{(\omega^2 - \omega_h^2)^2}]}|_{k=0} = 0$$

$\therefore$ **Q.E.D**.

Problem 4-18

For L-wave,

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2/\omega^2}{1 + \omega_c/\omega}$$

And

$$\omega_p^2 = \frac{n_0e^2}{m\varepsilon_0}, \omega_c = \frac{eB}{m}$$

The cut-off is found when $k = 0$. With the provided condtion $f = 2.8GHz, B_0 = 0.3T$, the critical density is

$$\begin{aligned} n_0 &= \frac{m\varepsilon_0}{e^2}[(2\pi f)^2 + 2\pi f\frac{eB}{m}] \\ &= 3.89 \times 10^{17}\mathbf{m}^{-3} \end{aligned}$$

Problem 4-23

R-wave

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2(1-\omega_c/\omega_p)}$$

L-wave

$$k^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2(1+\omega_c/\omega_p)}$$

Thus

$$k_R = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2/\omega^2}{1-\frac{\omega_c}{\omega}}} \approx \frac{\omega}{c}(1-\frac{1}{2}\frac{\omega_p^2/\omega^2}{1-\frac{\omega_c}{\omega}}) \approx \frac{\omega}{c}[1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}(1+\frac{\omega_c}{\omega})]$$

$$k_L = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2/\omega^2}{1+\frac{\omega_c}{\omega}}} \approx \frac{\omega}{c}(1-\frac{1}{2}\frac{\omega_p^2/\omega^2}{1+\frac{\omega_c}{\omega}}) \approx \frac{\omega}{c}[1-\frac{1}{2}\frac{\omega_p^2}{\omega^2}(1-\frac{\omega_c}{\omega})]$$

The difference of the phase is twice of the Faraday rotation angle

$$\begin{aligned} \Delta\theta &= \frac{1}{2}\frac{180^\circ}{\pi}(k_L - k_r) \\ &= \frac{180^\circ}{2\pi}\frac{1}{c}(\frac{1}{4\pi^2})\frac{e^2}{m\varepsilon_0}\frac{e}{m}B(Z)n(z) \\ &= \frac{90^\circ}{\pi}\frac{1}{c}(\frac{1}{4\pi^2})(\frac{\lambda_0}{c})^2\frac{e^3}{m^2\varepsilon_0}B(z)n(z) \\ &= \frac{90^\circ}{\pi}\frac{e^3}{4\pi^2 m^2\varepsilon_0 c^3}B(z)n(z)\lambda_0^2(\mathbf{degree}) \end{aligned}$$

And

$$\frac{90^\circ}{\pi}\frac{e^3}{4\pi^2 m^2\varepsilon_0 c^3} = 1.5\times 10^{-11}\mathbf{degree} = 2.62\times 10^{-13}\mathbf{rad}$$

In conclusion

$$\theta = \frac{180^\circ}{2\pi}\int_0^L (k_L - k_R)\mathbf{d}z = 1.5\times 10^{-11}\lambda_0\int_0^L B(z)n(z)\mathbf{d}z$$

(in the unit of degree)

Problem 4-26

a)

$$v_a = \frac{B}{\sqrt{\mu_0 \rho}} = 2.18 \times 10^8 m/s$$

b) The Alfvén wave represents for phase velocity. And phase velocity did not carry information. So it does not mean that wave can travel faster than light.

Problem 4-27

$$\rho = n_0 M = 1.67 \times 10^{-19}$$
$$v_A = \frac{B}{\sqrt{\mu_0 \rho}} = 2.18 \times 10^4 m/s$$

Problem 4-37

a) Consider the elastical collision from ion, the equation of eletron's motion

$$m\frac{\partial v_e}{\partial t} = -eE - mv_e\nu$$

Linearize the equation

$$-\mathbf{i}m\omega v_e = -eE - mv_e\nu \Rightarrow j_1 = -n_0 e v_e = -\frac{n_0 e^2 E_1}{\mathbf{i}m(\omega + \mathbf{i}\nu)}$$

The equation of the transverse wave is

$$(\omega^2 - c^2k^2)E_1 = -\mathbf{i}\omega j_1/\varepsilon$$

Insert j_1 into the wave equation, we get

$$(\omega^2 - c^2k^2)E_1 = \frac{\omega}{\omega + \mathbf{i}\nu}\omega_p^2 E_1$$

$$\Rightarrow \omega^2 - c^2k^2 = \frac{\omega\omega_p^2}{\omega + \mathbf{i}\nu}$$

$$\Rightarrow \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega + \mathbf{i}\nu)}$$

$\therefore$ **Q.E.D.**

b) Apply the previous result

$$c^2k^2 = \omega^2 - \frac{\omega_p^2}{1 + \mathbf{i}\frac{\nu}{\omega}}$$

When $\nu \ll \omega$ or $\frac{\nu}{\omega} \ll 1$,then

$$(1+\mathbf{i}\frac{\nu}{\omega})^{-1} \approx 1-\mathbf{i}\frac{\nu}{\omega}$$

So the dispersion relation turns to

$$c^2k^2 = \omega^2 - \omega_p^2 + \mathbf{i}\frac{\nu\omega_p^2}{\omega}$$

Assume that $\omega = a + b\mathbf{i}$, then

$$\begin{aligned} \omega_p^2 + c^2k^2 &= (a^2-b^2) + 2ab\mathbf{i} + \mathbf{i}\frac{\omega_p^2\nu}{a+b\mathbf{i}} \\ &= (a^2-b^2) + \frac{\omega_p^2 v_b}{a^2+b^2} + \mathbf{i}(2ab + \frac{\omega_p^2\nu a}{a^2+b^2}) \end{aligned}$$

The image part should be zero, that is

$$\mathbf{i}b = \frac{-\omega_p^2 v}{2(a^2+b^2)}\mathbf{i} = Im(\omega)\mathbf{i}$$

So the damping rate:

$$\gamma = -Im(\omega) = \frac{\omega_p^2\nu}{2|\omega|^2}$$

c) With previous conclusion of a), we get

$$k^2 = \frac{\omega^2-\omega_p^2}{c^2} + \mathbf{i}\frac{\omega_p^2\nu}{c^2\omega}$$

Let $k = e + d\mathbf{i}$, then $k^2 = e^2 - d^2 + 2ed\mathbf{i}$, we get

$$\begin{cases} e^2 - d^2 &= \frac{\omega^2-\omega_p^2}{c^2} \\ 2ed &= \frac{\omega_p^2\nu}{c^2\omega} \end{cases}$$

By solving the simultaneous equations, it is *easy* (Uh huh!)to obtain that

$$\delta = \frac{1}{Im(k)} = \frac{1}{d} = (\frac{2c}{\nu}\cdot\frac{\omega^2}{\omega_p^2})(1-\frac{\omega_p^2}{\omega^2})^{1/2}$$

Problem 4-38

The loss part of energy will heat up the electron to oscillation. So we firstly need to derive the vibration motion due to the microwave. The wave-length of the microwave is $\lambda = 0.3m$ So the frequency is

$$\omega = \frac{2\pi c}{\lambda} = 6.28\times10^9 rad/sec$$

The collision frequency is

$$\nu = n_n \bar{\sigma\nu} = 10^2/sec$$

So the equation of motion of electron is

$$m\frac{\partial v_e}{\partial t} = -eE - mv_e\nu$$

Linearize the equation to be

$$\mathbf{i}\omega m v_e = -eE - \nu m v_e \Rightarrow v_e = \frac{eE}{\mathbf{i}m(\omega + \mathbf{i}\nu)}$$

Then the dispersion raltion is

$$k^2 = \frac{\omega^2 - \omega_p^2}{c^2} + \frac{\omega_p^2\nu}{\omega_c^2}\mathbf{i}$$

$$k = Re(k) + Im(k)\mathbf{i}$$

And microwave in plasma is

$$E = E_0 e^{\mathbf{i}(kr-\omega t)} = E_0 e^{\mathbf{i}(Re(k)r-\omega t)} e^{-Im(k)r}$$

The term $e^{-Im(k)r}$ means the wave decay when penetrate the ionosphere. The ratio of outgoing energy v.s. incident energy is

$$A = \frac{E_{out}^2}{E_{ini}^2} = (\frac{e^{-Im(k)R_1}}{e^{-Im(k)R_2}})^2 = e^{-2Im(k)(R_1-R_2)}$$

And $R_1 - R_2 = 100km = 10^5 m$

$$\omega_p^2 = \frac{n_e e^2}{\varepsilon_0 m} = 3.17 \times 10^{14}$$

$$Im(k) = \frac{\nu}{2c}\frac{\omega_p^2}{\omega^2}\frac{1}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} = 1.34 \times 10^{-12}$$

$$\therefore A = e^{-2.69\times 10^{-7}} = 0.999999732$$

So the loss fraction is

$$1 - A \approx 0$$

Problem 4-39

a) When $k \to \infty$, the resonance happens, which indicates:

$$2\omega^2(1-\frac{\omega_p^2}{\omega^2}) - \omega_c^2\sin^2\theta \pm \omega_c\left[\omega_c^2\sin^4\theta + 4\omega^4(1-\frac{\omega_p^2}{\omega^2})^2\cos^2\theta\right]^{1/2} = 0$$

$$
\begin{aligned}
&\Rightarrow \quad 2\omega^2 - 2\omega_p^2 - \omega_c^2 \sin^2\theta = \pm\omega_c[\omega_c^2 \sin^4\theta + 4\omega^2(1-\frac{\omega_p^2}{\omega^2})^2\cos^2\theta]^{1/2} \\
&\Rightarrow \quad (2\omega^2 - 2\omega_p^2 - \omega_c^2\sin^2\theta)^2 = \omega_c^4\sin^4\theta + 4\omega^2\omega_c^2(1-\frac{\omega_p^2}{\omega^2})^2\cos^2\theta \\
&\Rightarrow \quad 4\omega^4 + 4\omega_p^4 + 4\omega_c^4\sin^4\theta - 8\omega^2\omega_p^2 - 4\omega^2\omega_c^2\sin^2\theta + 4\omega_p^2\omega_c^2\sin^2\theta \\
&= \quad \omega_c^4\sin^4\theta + 4\omega^2\omega_c^2(1-2\frac{\omega_p^2}{\omega^2}+\frac{\omega_p^4}{\omega^4})\cos^2\theta \\
&\Rightarrow \quad 4\omega^6 + 4\omega_p^4\omega^2 - 8\omega^4\omega_p^2 - 4\omega^4\omega_c^2\sin^2\theta + 4\omega^2\omega_p^2\omega_c^2\sin^2\theta \\
&= \quad 4\omega_c^2\omega^4\cos^2\theta - 8\omega^2\omega_c^2\omega_p^2\cos^2\theta + 4\omega_c^2\omega_p^4\cos^2\theta \\
&\Rightarrow \quad \omega^6 + \omega_p^4\omega^2 - 2\omega^4\omega_p^2 - \omega_c^2\omega^4 + \omega^2\omega_p^2\omega_c^2 + \omega_c^4\omega_p^2\cos^2\theta - \omega_c^2\omega_p^4\cos^2\theta = 0 \\
&\Rightarrow \quad \omega^6 - 2(2\omega_p^2+\omega_c)\omega^4 + \omega^2\omega_p^2(\omega_p^2+\omega_c^2+\omega_c^2\cos^2\theta) - \omega_c^2\omega_p^4\cos^2\theta = 0 \\
&\Rightarrow \quad (\omega^2-\omega_p^2)(\omega^4 - \omega_h^2\omega^2 + \omega_p^2\omega_c^2\cos^2\theta) = 0, (\omega_h^2 = \omega_p^2 + \omega_c^2) \\
&\Rightarrow \quad \omega^2 = \frac{1}{2}[\omega_h^2 \pm \sqrt{\omega^4 - 4\omega_p^2\omega_c^2\cos^2\theta}]\mathbf{or}\omega^2 = \omega_p^2
\end{aligned}
$$

b) When$k \to 0$ the cut-off happens, and this means that

$$2\omega_p^2(1-\frac{\omega_p^2}{\omega^2}) = 2\omega^2(1-\frac{\omega_p^2}{\omega^2}) - \omega_c^2\sin^2\theta \pm \omega_c[\omega_c^2\sin^4\theta + 4\omega^2(1-\frac{\omega_p^2}{\omega^2}^2\cos^2\theta]^{1/2},$$

and $\omega_p = \omega$ meets the requirement. So the cut off frequency is ω_p.

Problem 4-40

Plasma frequenc in the slab is

$$\omega_p^2 = \frac{n_0 e^2}{m\varepsilon_0}$$

When $B_0 \parallel E_1$, it is an ordinary wave in plasma Dispersion relation is

$$\omega^2 = \omega_p^2 + c^2k^2$$

$$(\frac{2\pi}{\lambda})^2 = \frac{n_0e^2}{m\varepsilon_0} + c^2k^2$$

so the wave vector is

$$k = 543.72\mathbf{m}^{-1}$$

Say, the nubmer of wavelength is

$$N = \frac{d}{\lambda} = \frac{d}{\frac{2\pi}{k}} = 8.6$$

When $B_0 \perp E_1$, is is an extraordianry wave, the dispersion relation is

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2}$$

$$\begin{aligned}
\omega_h^2 &= \omega_p^2 + \omega_c^2 \\
\omega_c &= \frac{eB}{m} = 1.88\times 10^{11}\mathbf{rad/s} \\
\omega_p^2 &= \frac{n_0e^2}{m\varepsilon_0} = 8.9\times 10^{21}(\mathbf{rad/s})^2 \\
\omega_h^2 &= 4.432\times 10^{22}(\mathbf{rad/s})^2 \\
\omega &= \frac{2\pi c}{\lambda} = 1.88\times 10^{11}\mathbf{rad/s}
\end{aligned}$$

$$\therefore k = 832.2\mathbf{m}^{-1}$$

$$N = \frac{d}{\lambda} = 13.2$$

Problem 4-41

In the equilibrium state, there is no magnetic or electric field.

$$B_0 = 0, E_0 = 0, T_e = 0$$

For the electromagnetic wave, the relevant Maxwell Equation is

$$\begin{aligned}
\nabla\times E_1 &= -\dot{B}_1 \\
c^2\nabla\times B_1 &= \dot{E}_1
\end{aligned}$$

$$\rightarrow (\omega^2 - c^2k^2)E_1 = -\mathbf{i}\omega j_1/\varepsilon_0$$

And

$$\begin{aligned}
j_1 &= n_{0+}Zev_{0+} - n_{0-}ev_{0-} \\
&= Zn_{0+}e(v_{0+} - v_{0-})
\end{aligned}$$

The equation of motion is ($T_e = 0$)

$$\begin{aligned}
M_+\frac{\partial v_{0+}}{\partial t} &= ZeE_1 \Rightarrow v_{0+} = -\frac{-ZeE_1}{\mathbf{i}M_+\omega} \\
M_-\frac{\partial v_{0-}}{\partial t} &= -eE_1 \Rightarrow v_{0-} = -\frac{eE_1}{\mathbf{i}M_-\omega}
\end{aligned}$$

$$\therefore j_1 = -Zn_{0+}e(\frac{ZeE_1}{\mathbf{i}M_+\omega} + \frac{eE_1}{\mathbf{i}M_-\omega}) = -\frac{Zn_{0+}e^2}{\mathbf{i}\omega}(\frac{Z}{M_+} + \frac{1}{M_-})E$$

THe dispersion relation is

$$\omega^2 - c^2k^2 = \frac{Z^2e^2n_{0+}}{M_+\varepsilon_0} + \frac{e^2n_{0-}}{M_-\varepsilon_0}$$

Problem 4-42

Boltzman relation is

$$n_{e1} = n_0 \frac{e\phi_1}{kT_e}$$

The plasma approximation

$$Zn_{A1} + n_{H1} = n_{e1}$$

The equation of continuity

$$\begin{aligned} i\omega n_{A1} &= n_A \mathbf{i} k v_{A1} \\ i\omega n_{H1} &= n_H \mathbf{i} k v_{H1} \end{aligned}$$

And the equation of motion

$$M_A(-\mathbf{i}\omega)v_{A1} = M_A \frac{\partial v_{A1}}{\partial t} = ZeE_1 = Ze(-\mathbf{i}k\phi_1)$$

$$M_H(-\mathbf{i}\omega)v_{H1} = M_H b \frac{\partial v_{H1}}{\partial t} = eE_1 = e(-\mathbf{i}k\phi_1)$$

Then we get

$$\frac{\omega^2}{k^2} = \frac{kT_e}{n_0}\left(\frac{Z^2 n_A}{M_A} + \frac{n_H}{M_H}\right)$$

The phase velocity is

$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{kT_e}{n_0}\left(\frac{Z^2 n_A}{M_A} + \frac{n_H}{M_H}\right)}$$

Problem 4-43

The Poisson Equation

$$\varepsilon_0 \nabla \cdot E_1 = n_{1+}e - Zn_{1-}e$$

And the continuity equation

$$\begin{aligned} n_{1+} &= \frac{k}{\omega} n_0 v_{1+} \\ n_{1-} &= \frac{k}{\omega} n_0 v_{1-} \end{aligned}$$

Since $kT = 0, B_0 = 0$, there is no collision and magnetic term in the equation of motion.

$$\begin{aligned} M_+ \frac{\partial v_{+1}}{\partial t} &= eE_1 \Rightarrow v_{+1} = \frac{-eE_1}{\mathbf{i}\omega M_+} \\ M_- \frac{\partial v_{-1}}{\partial t} &= -ZeE_1 \Rightarrow v_{-1} = \frac{ZeE_1}{\mathbf{i}\omega M_-} \end{aligned}$$

Then we get

$$\mathbf{i}\varepsilon_0 k E_1 = \frac{k}{\omega} n_0 \frac{-e^2 E_1}{\mathbf{i}\omega M_+} - Z\frac{k}{\omega} n_0 \frac{Ze^2 E_1}{i\omega M_-}$$

That is

$$\omega = \sqrt{\frac{n_{0+}e^2}{M_+\varepsilon_0} + \frac{Z^2 n_{0-} e^2}{M_-\varepsilon_0}}$$

Problem 4-45

Assume that the ionosphere is extremly cold, we can presume that

$$kT_i = 0$$

So the sonic ion wave velocity is

$$v_s = \sqrt{\frac{kT_e}{M}}$$

And the Alfvenic velocity is

$$v_A = \frac{B}{\sqrt{\mu_0 M n_0}}$$

And it indicates that super sonic wave is not super-Alfvenic. That is ,

$$v > \sqrt{\frac{kT_e}{M}} \to T_e < \frac{Mv^2}{k} = 1.2 \times 10^7 \mathbf{K}.$$

This is the upper limit of temperature.

$$v < \frac{B}{\sqrt{\mu_0 M n_0}} \to n_0 < \frac{B}{\mu_0 M v^2} = 4.76 \times 10^{11}/\mathbf{m}^3.$$

This is the upper limit of density.

Problem 4-49

a)

$$\frac{\omega_p^2}{\omega^2 - \omega_c^2} + \frac{\Omega_p^2}{\omega^2 - \Omega_c^2} + \frac{(\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2} - \frac{\Omega_c}{\omega}\frac{\Omega_p^2}{\omega^2-\Omega_c^2})^2}{1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} - \frac{\Omega_p^2}{\omega^2-\Omega_p^2}} \tag{3.1}$$

$$\begin{aligned}
&(\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2 - \omega_c^2} - \frac{\Omega_c}{\omega}\frac{\Omega_p^2}{\omega^2 - \Omega_c^2})^2 \\
= \; & \frac{\omega_c^2}{\omega^2}\frac{\omega_p^4}{(\omega^2 - \omega_c^2)^2} + \frac{\Omega_c^2}{\omega^2}\frac{\Omega_p^4}{(\omega^2 - \Omega_c^2)^2} - \frac{2\omega_c\Omega_c}{\omega^2}\frac{\omega_p^2\Omega_p^2(\omega^2 - \omega_c^2)(\omega^2 - \Omega_c^2)}{(\omega^2 - \omega_c^2)^2(\omega^2 - \Omega_c^2)^2} \\
= \; & \frac{\omega_c^2\omega_p^4(\omega^2 - \Omega_c^2)^2 + \Omega_c^2\Omega_p^4(\omega^2 - \omega_c^2)^2 - 2\omega_c\Omega_c\omega_p^2\Omega_p^2(\omega^2 - \omega_c^2)(\omega^2 - \Omega_c^2)}{\omega^2(\omega^2 - \omega_c^2)^2(\omega^2 - \Omega_c^2)^2}
\end{aligned}$$

the denominator of 3.1 is:

$$\begin{aligned} & (1-\frac{\omega_p^2}{\omega^2-\omega_c^2}-\frac{\Omega_p^2}{\omega^2-\Omega_c^2})\cdot\omega^2(\omega^2-\omega_c^2)^2(\omega^2-\Omega_c^2)^2 \\ = & \ \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)-\omega_p^2(\omega^2-\Omega_c^2)-\Omega_p^2(\omega^2-\omega_c^2)] \end{aligned}$$

Now let's calculate the numerator of Eq3.1
Let:

$$\begin{aligned} \Delta &= (\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)-\omega_p^2(\omega^2-\Omega_c^2)-\Omega_p^2(\omega^2-\omega_c^2) \\ &= \omega^4-\omega^2\Omega_c^2-\omega^2\omega_c^2+\omega_c^2\Omega_c^2-\omega_p^2\omega^2+\omega_p^2\Omega_c^2-\Omega_p^2\omega^2+\omega_c^2\Omega_p^2 \end{aligned}$$

After the reduction of fraction to a common denominator, the numerator induced by $\frac{\omega_p^2}{\omega^2-\omega_c^2}$ is :

$$\mathbf{Term1}=\omega_p^2\omega^2(\omega^2-\Omega_c^2)\Delta.$$

And the numerator from $\frac{\Omega_p^2}{\omega^2-\Omega_c^2}$ is :

$$\mathbf{Term2}=\Omega_p^2\omega^2(\omega^2-\omega_c^2)\Delta$$

Numerator caused by $(\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}-\frac{\Omega_c}{\omega}\frac{\Omega_p^2}{\omega^2-\Omega_c^2})^2$ is

$$\mathbf{Term3}=\omega_c^2\omega_p^4(\omega^2-\Omega_c^2)^2+\Omega_c^2\Omega_p^4(\omega^2-\omega_c^2)^2-2\omega_c\Omega_c\omega_p^2\Omega_p^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)$$

Note that

$$\begin{aligned} \frac{\Omega_p^2}{\omega_c} &= \frac{\omega_p^2}{\omega_c}\Rightarrow\Omega_p^2\omega=\omega_p^2\Omega_c \\ &\Rightarrow\omega_c\Omega_c\omega_p^2\Omega_p^2=\omega_c^2\Omega_p^4=\omega_p^4\Omega_c^2 \end{aligned}$$

.

$$\begin{aligned} \mathbf{Term3} &= \omega_c^2\omega_p^4(\omega^2-\Omega_c^2)^2+\Omega_c^2\Omega_p^4(\omega^2-\omega_c^2)^2-\omega_p^4\Omega_c^2(\omega^2-\omega_c^2)(\omega-\Omega_c^2)-\omega_c^2\Omega_p^4(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2) \\ &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\frac{\omega_c^2\omega_p^4}{\omega^2}\frac{\omega^2-\Omega_c^2}{\omega^2-\omega_c^2}+\frac{\Omega_c^2\Omega_p^4}{\omega^2}\frac{\omega^2-\omega_c^2}{\omega^2-\Omega_c^2}-\frac{\omega_p^4\Omega_c^2}{\omega^2}-\frac{\omega_c^2\Omega_p^4}{\omega^2}] \\ \mathbf{Term1} &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\frac{\omega_p^2}{\omega^2-\omega_c^2}\Delta], \\ &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\omega_p^2(\omega^2-\Omega_c^2-\Omega_p^2)-\omega_p^4\frac{\omega^2-\Omega_c^2}{\omega^2-\omega_c^2}], \\ \mathbf{Term2} &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\frac{\Omega_p^2}{\omega^2-\Omega_c^2}\Delta] \\ &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\Omega_p^2(\omega^2-\omega_c^2-\omega_p^2)-\frac{\omega^2-\omega_c^2}{\omega^2-\Omega_c^2}\Omega_p^4], \\ \mathbf{Term1+2+3} &= \omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\Omega_p^2(\omega^2-\omega_c^2-\omega_p^2)+\omega_p^2(\omega^2-\Omega_c^2-\Omega_p^2) \\ &+ (\frac{\omega_c^2\omega_p^4}{\omega^2}-\omega_p^4)\frac{\omega^2-\Omega_c^2}{\omega^2-\omega_c^2}+(\frac{\Omega_c^2\Omega_p^4}{\omega^2}-\Omega_p^4)\frac{\omega^2-\omega_c^2}{\omega^2-\Omega_c^2}-\frac{\omega_p^4\Omega_c^2}{\omega^2}-\frac{\omega_c^2\Omega_p^4}{\omega^2}-\frac{\omega_c^2\Omega_p^4}{\omega^2}]. \end{aligned}$$

Among this ,

$$
\begin{aligned}
& (\frac{\omega_c^2\omega_p^4}{\omega^2}-\omega_p^4)\frac{\omega^2-\Omega_c^2}{\omega^2-\omega_c^2}+(\frac{\Omega_c^2\Omega_p^4}{\omega^2}-\Omega_p^4)\frac{\omega^2-\omega_c^2}{\omega^2-\Omega_c^2}-\frac{\omega_p^4\Omega_c^2}{\omega^2}-\frac{\omega_c^2\Omega_p^4}{\omega^2}-\frac{\omega_c^2\Omega_p^4}{\omega^2} \\
= \;& (\frac{\omega_c^2-\omega^2}{\omega^2})(\frac{\omega^2-\Omega_c^2}{\omega^2-\omega_c^2})\omega_p^4+(\frac{\Omega_c^2-\omega^2}{\omega^2})(\frac{\omega^2-\omega_c^2}{\omega^2-\Omega_c^2})\Omega_p^4-\frac{\Omega_c^2}{\omega^2}\omega_p^4-\frac{\omega_c^2}{\omega^2}\Omega_p 4 \\
= \;& -\omega_p^4+\frac{\Omega_c^2}{\omega^2}\omega_p^4-\Omega_p^4+\frac{\omega_c^2}{\omega^2}\Omega_p^4-\frac{\Omega_c^2}{\omega^2}\omega_p^4-\frac{\omega_c^2}{\omega^2}\Omega_p^4 \\
= \;& -\omega_p^4-\Omega_p^4.
\end{aligned}
$$

$\therefore$ $\mathbf{Term1+2+3}=\omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\Omega_p^2(\omega^2-\omega_c^2-\omega_p^2)+\omega_p^2(\omega^2-\Omega_c^2-\Omega_p^2)-\omega_p^4-\Omega_p^4]$

This is numerator of 3.1. So,

$$
\begin{aligned}
& \frac{\omega_p^2}{\omega^2-\omega_c^2}+\frac{\Omega_p^2}{\omega^2-\Omega_c^2}+\frac{(\frac{\omega_c}{\omega}\frac{\omega_p^2}{\omega^2-\omega_c^2}-\frac{\Omega_c}{\omega}\frac{\Omega_p^2}{\omega^2-\Omega_c^2})^2}{1-\frac{\omega_p^2}{\omega^2-\omega_c^2}-\frac{\Omega_p^2}{\omega^2-\Omega_p^2}} \\
= \;& \frac{\omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\Omega_p^2\omega^2-\Omega_p^2\omega_c^2-\Omega_p^2\omega_p^2+\omega_p^2\omega^2-\omega_p^2\Omega_c^2-\omega_p^2\Omega_p^2-\omega_p^4-\Omega_p^4]}{\omega^2(\omega^2-\omega_c^2)(\omega^2-\Omega_c^2)[\omega^4-\omega^2\Omega_c^2-\omega^2\omega_c^2+\omega_c^2\Omega_c^2-\omega_p^2\omega^2+\omega_p^2\Omega_c^2-\Omega_p^2\omega^2+\omega_c^2\Omega_p^2]} \\
= \;& \frac{\Omega_p^2\omega^2-\Omega_p^2\omega_c^2-\Omega_p^2\omega_p^2+\omega_p^2\omega^2-\omega_p^2\Omega_c^2-\omega_p^2\Omega_p^2-\omega_p^4-\Omega_p^4}{\omega^4-\omega^2\Omega_c^2-\omega^2\omega_c^2+\omega_c^2\Omega_c^2-\omega_p^2\omega^2+\omega_p^2\Omega_c^2-\Omega_p^2\omega^2+\omega_c^2\Omega_p^2}
\end{aligned}
$$

Let

$$\overline{\omega_p^2}=\omega_p^2+\Omega_p^2$$

And

$$
\begin{aligned}
& \frac{\overline{\omega_p^2}}{\omega^2-\omega_c\Omega_c+\frac{\omega^2(\omega_c-\Omega_c)^2}{\overline{\omega_p^2}-\omega^2+\omega_c\Omega_c}} \\
= \;& \frac{\overline{\omega_p^2}(\overline{\omega_p^2}-\omega^2+\omega_c\Omega_c)}{(\omega^2-\omega_c\Omega_c)(\overline{\omega_p^2}-\omega^2+\omega_c\Omega_c)+\omega^2(\omega_c-\Omega_c)^2} \\
= \;& \frac{(\omega_p^2+\Omega_p^2)(\omega_p^2+\Omega_p^2-\omega^2+\omega_c\Omega_c)}{(\omega^2-\omega_c\Omega_c)(\omega_p^2+\Omega_p^2-\omega^2+\omega_c\Omega_c)+\omega^2(\omega_c-\Omega_c)^2}.
\end{aligned}
\tag{3.2}
$$

Recall

$$\Omega_p^2\omega_c=\omega_p^2\Omega_c$$

Then the numerator of Eq.3.2 is

$$
\begin{aligned}
& \omega_p^4+\omega_p^2\Omega_P^2-\omega^2\omega_p^2+\omega_p^2\omega_c\Omega_c+\Omega_p^2\omega_p^2+\Omega_p^4-\omega^2\Omega_p^2+\Omega_p^2\omega_c\Omega_c \\
= \;& \omega_p^4+\omega_p^2\Omega_P^2-\omega^2\omega_p^2+\Omega_p^2\omega_c^2+\Omega_p^2\omega_p^2+\Omega_p^4-\omega^2\Omega_p^2+\omega_p^2\Omega_c^2,
\end{aligned}
$$

which equals to that of Eq.3.1.
The denominator of Eq.3.2:

$$\begin{aligned}
& \omega^2\omega_p^2 + \omega^2\Omega_p^2 - \omega^4 + \omega^2\omega_c\Omega_c - \omega_p^2\omega_c\Omega_c + \Omega_p^2\omega_c\Omega_c \\
& +\omega^2\omega_c\Omega_c - \omega_c^2\Omega_c^2 + \omega^2\omega_c^2 - 2\omega^2\omega_c\Omega_c + \omega^2\Omega_c^2 \\
= \;& \omega^2\omega_p^2 + \omega^2\Omega_p^2 - \omega^4 + 0 - \Omega_p^2\omega_c^2 + \omega_p^2\Omega_c^2 + 0 - \omega_c^2\Omega_c^2 + \omega^2\omega_c^2 - 0 + \omega^2\Omega_c^2 \\
= \;& -\omega^4 + \omega^2\Omega_c^2 + \omega^2\omega_c^2 - \omega_c^2\Omega_c^2 + \omega_p^2\omega^2 - \omega_p^2\Omega_c^2 + \Omega_p^2\omega^2 - \omega_c^2\Omega_p^2,
\end{aligned}$$

which equals to that of Eq.3.2.
Therefore, they are identical.
Q.E.D.
b) (Incomplete solution)For cut-off, $k \to 0$

$$\begin{aligned}
& \Omega_p^2\omega^2 - \Omega_p^2\omega_c^2 - \Omega_p^2\omega_p^2 + \omega_p^2\omega^2 - \omega_p^2\Omega_c^2 - \omega_p^2\Omega_p^2 - \omega_p^4 - \Omega_p^4 \\
= \;& \omega^4 - \omega^2\Omega_c^2 - \omega^2\omega_c^2 + \omega_c^2\Omega_c^2 - \omega_p^2\omega^2 + \omega_p^2\Omega_c^2 - \Omega_p^2\omega^2 + \omega_c^2\Omega_p^2 \\
\Rightarrow \;& \omega^4 - (\Omega_c^2 + \omega_c^2 + 2\omega_p^2 + 2\Omega_p^2)\omega^2 + \omega_c^2\Omega_c^2 + 2\Omega_p^2\omega_c^2 + 2\omega_p^2\Omega_c^2 + 2\Omega_p^2\omega_p^2 + \omega_p^4 + \Omega_p^4 = 0
\end{aligned}$$

$$\omega^2 = \frac{1}{2}[(\Omega_c^2 + \omega_c^2 + 2\omega_p^2 + 2\Omega_p^2) \pm \sqrt{(\Omega_c^2 + \omega_c^2)^2 + 4(\Omega_p^2 - \omega_p^2)(\Omega_c^2 - \omega_c^2)}]$$

The left hand cut off frequency is

$$\omega_L^2 = \frac{1}{2}[(\Omega_c^2 + \omega_c^2 + 2\omega_p^2 + 2\Omega_p^2) - \sqrt{(\Omega_c^2 + \omega_c^2)^2 + 4(\Omega_p^2 - \omega_p^2)(\Omega_c^2 - \omega_c^2)}]$$

(c)

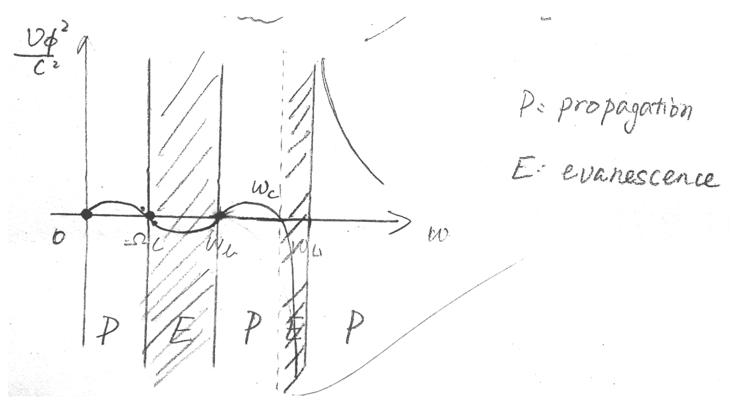
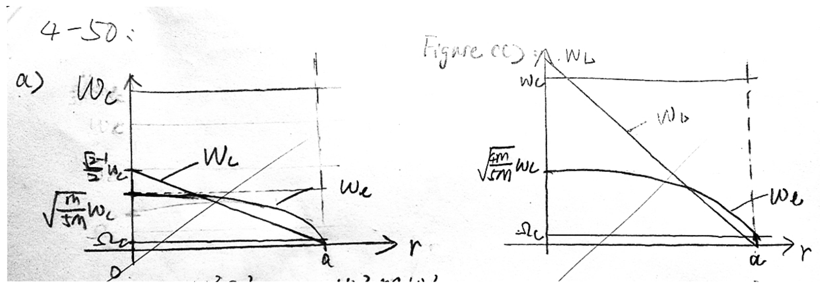


Figure 3.1: Figure for 4-49(c)

Problem 4-50

a)



b)

$$\omega_l^2 = \frac{\omega_c^2\Omega_p^2}{\omega_c^2+\omega_p^2} = \frac{\omega_c^2 m\omega_p^2}{M(\omega_c^2+\omega_p^2)},$$

$$\omega_L = \frac{1}{2}[-\omega_c + \sqrt{\omega_c^2+4\omega_p^2}],$$

$$\omega_p = (1-\frac{r}{a})\frac{\omega_c}{2}.$$

And

$$\begin{aligned}
\omega^2 &= \omega_c^2(r=0) = \frac{m}{M}\frac{\omega_c^2}{\omega_c^2+\frac{1}{4}\omega_c^2}\frac{1}{4}\omega_c^2 = \frac{m}{5M}\omega_c^2 \\
\omega_L(r=0) &= \frac{\sqrt{2}-1}{2}\omega_c \\
\omega_L(r=a) &= 0,
\end{aligned}$$

where $\omega_l < \omega < \omega_L$ lays in the evanescent layer.

$$\begin{aligned}
\omega &= \omega_l(r=0) = \sqrt{\frac{m}{5M}}\omega_c \\
\omega_l^2 &= \frac{m}{M}\frac{(1-\frac{r}{a})^2)\omega_c^2}{4+(1-\frac{r}{a})^2} \\
\omega_L &= \frac{1}{2}[-\omega_c + \omega_c(1+(1-\frac{r}{a})^{1/2})]
\end{aligned}$$

when $\omega > \omega_l \rightarrow r > 0$
when $\omega < \omega_L \rightarrow r < 0.975a$
Note

$$\frac{m}{M} = 1836$$

So the evanescent layer thickness is 0.975a.

c)

$$\begin{aligned}
\omega_p &= (1-\frac{r}{a})2\omega_c, \\
\omega^2 &= \omega_l^2(r=0) = \frac{m}{M}\,4\omega_c^2 = \frac{4m}{5M}\omega_c^2, \\
\omega_L &= \frac{\omega_c}{2}[-1+\sqrt{1+16(1-\frac{r}{a})^2}], \\
\omega_l^2 &= \frac{m}{M}\frac{16(1-\frac{r}{a})^2}{1+16(1-\frac{r}{a})^2}\omega_c^2.
\end{aligned}$$

Similarly, when $\omega > \omega_l, r > 0$; when $\omega < \omega_L, r < 0.927a$.
Conclusion: the higher center density is, the thicker evanescent layer will be.