SOLUTIONS/HINTS TO THE EXERCISES FROM COMPLEX ANALYSIS BY STEIN AND SHAKARCHI

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Abstract. This contains the solutions or hints to many of the exercises from the Complex Analysis book by Elias Stein and Rami Shakarchi.

I worked these problems during the Spring of 2006 while I was taking a Complex Analysis course taught by Andreas Seeger at the University of Wisconsin - Madison. I am grateful to him for his wonderful lectures and helpful conversations about some of the problems discussed below.

Contents

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1. Chapter 1. Preliminaries to Complex Analysis

Exercise 1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(1) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$.
(2) $1/z = \overline{z}$.
(3) $\mathrm{Re}(z) = 3$.
(4) $\mathrm{Re}(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.
(5) $\mathrm{Re}(az + b) > 0$ where $a, b \in \mathbb{C}$.
(6) $|z| = \mathrm{Re}(z) + 1$.
(7) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Solution 1.

(1) It is the line in the complex plane consisting of all points that are an equal distance from both z_1 and z_2. Equivalently the perpendicular bisector of the segment between z_1 and z_2 in the complex plane.
(2) It is the unit circle.
(3) It is the line where all the numbers on the line have real part equal to 3.
(4) In the first case it is the open half plane with all numbers with real part greater than c. In the second case it is the closed half plane with the same condition.
(5)
(6) Calculate $|z|^2 = x^2 + y^2 = (x+1)^2 = x^2 + 2x + 1$. So we are left with $y^2 = 2x + 1$. Thus the complex numbers defined by this relation is a parabola opening to the "right".
(7) This is a line.

Exercise 2. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z = (z_1, y_1)$ and $W = (x_2, y_2)$, then

$$\langle Z, W \rangle = x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot, \cdot)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}.$$

The term Hermitian is used to describe the fact that $(\cdot, \cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = \overline{(w, z)} \text{ for all } z, w \in \mathbb{C}.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}\left[(z, w) + (w, z)\right] = \mathrm{Re}(z, w),$$

where we use the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

Solution 2. This is a straightforward calculation

$$\begin{aligned}\frac{1}{2}\left[(z, w) + (w, z)\right] =& \frac{1}{2}\left(z\overline{w} + w\overline{z}\right) = \mathrm{Re}(z, w)\mathrm{Re}(z\overline{w}) \\ =& \frac{1}{2}\left((z_1 + z_2 i)(w_1 + iw_2) + (w_1 + iw_2)(z_1 - iz_2)\right) = z_1 w_1 + z_2 w_2.\end{aligned}$$

Exercise 3. With $\omega = se^{i\phi}$, where $s \geq 0$ and $\phi \in \mathbb{R}$, solve the equation $z^n = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

Solution 3.$z^n = se^{i\phi}$ implies that $z = s^{\frac{1}{n}} e^{i\left(\frac{\phi}{n}+\frac{2\pi ik}{n}\right)}$, where $k = 0, 1, \cdots, n-1$ and $s^{\frac{1}{n}}$ is the real nth root of the positive number s. There are n solutions as there should be since we are finding the roots of a degree n polynomial in the algebraically closed field $\mathbb{C}$.

Exercise 4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(1) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$ or $z = w$.
(2) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.
(3) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

Solution 4. Suppose, for a contradiction, that $i \succ 0$, then $-1 = i \cdot i \succ 0 \cdot i = 0$. Now we get $-i \succ -1 \cdot i \succ 0$. Therefore $i - i \succ i + 0 = i$. But this contradicts our assumption. We obtain a similar situation in the case $0 \succ i$. So we must have $i = 0$. But then for all $z \in \mathbb{C}$ we have $z \cdot i = z \cdot 0 = 0$ Repeating we have $z = 0$ for all $z \in \mathbb{C}$. So this relation would give a trivial total ordering.

Exercise 5. A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(1) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$ and let γ denote a curve in Ω joining ω_1 to ω_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = \omega_1$ and $z(1) = \omega_2$, and let
$$t^* = \sup_{0 \le t \le 1} \{t : z(s) \in \Omega_1 \text{ for all } 0 \le s < t\}.$$
Arrive at a contradiction by considering the point $z(t^*)$.
(2) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when Ω is open. For instance, we may take all curves to be continuous, or simply polygonal lines.

Solution 5. Following the first part, assume for a contradiction that $z(t^*) \in \Omega_1$. Since Ω_1 is open there exists a ball $B(z(t^*), \delta) \subset \Omega_1$. Now by assumption $z(t^* + \epsilon) \in \Omega_2$. Thus $|z(t^* + \epsilon) - z(t^*)| > \delta$ for all $\epsilon > 0$. But this is a contradiction since z is smooth.

Define Ω_1 and Ω_2 as in the problem. First to see that Ω_1 is open let $z \in \Omega_1$. Then since Ω is open and $z \in \Omega$ we know that there exists a ball $B(z, \delta \subset \Omega$. We claim that this ball is actually inside of Ω_1. If we prove this claim then we have established that Ω_1 is open. Let $s \in B(z, \delta)$ and consider $f : [0,1] \to \mathbb{C}$ given by $f(t) = st + z(1-t)$. Then $|f(t) - z| = t|s - z| < \delta$. So the image of f is contained in $B(z, \delta) \subset \Omega$. By concatenating the paths from w to z and z to s we see that $s \in \Omega_1$.

Finally we will prove that Ω_2 is also open. Suppose that Ω_2 is not open. Then for there is some $z \in \Omega_2$ such that every ball around z contains a point of Ω_1. So that $B(z, \delta) \subset \Omega$ is one such ball, with $s \in \Omega_1 \cap B(z, \delta)$. Then as in the previous paragraph we can use the straight line path to connect

z to s and the path has imagine inside $B(z, \delta) \subset \Omega$. Therefore, w is path connected to s which is path connected to z. Therefore, by concatenating paths, we see that $z \in \Omega_1$ which contradicts the definition of Ω_2. So Ω_2 must be open.

Now Ω_1 is non-empty since $w \in \Omega_1$. Therefore, by connectedness, $\Omega_2 = \emptyset$.

Remark. This argument works in any metric space.

Exercise 6. Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The **connected component** (or simply the **component**) of Ω containing z is the set $\mathcal{C}_z$ of all points w in Ω that can be joined to z by a curve entirely contained in Ω.

(1) Check first that $\mathcal{C}_z$ is open and connected. Then, show that $w \in \mathcal{C}_z$ defines an equivalence relation, that is (i) $z \in \mathcal{C}_z$, (ii) $w \in \mathcal{C}_z$ implies $z \in \mathcal{C}_w$, and (iii) if $w \in \mathcal{C}_z$ and $z \in \mathcal{C}_\zeta$, then $w \in \mathcal{C}_\zeta$.
Thus Ω is the union of all its connected components, and two components are either disjoint or coincide.
(2) Show that Ω can have only countably many distinct connected components.
(3) Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

Solution 6.

(1) (i) the trivial path works. (ii) Running the path in reverse works. (iii) We have a path from ζ to z and from z to w. Concatenating the paths gets the job done.
(2) The set of all elements of the form $q + iq'$ where $q, q' \in \mathbb{Q}$ is countable. Each component contains a point of the form $q + iq'$, since each $\mathcal{C}_z$ is open, we can be seen from the previous exercise.
(3) If K is compact then it is closed and bounded. So it is contained in an open disc with bounded radius and center the origin. So then the complement of that open disc is contained in Ω. Then if Ω is not connected it must have a component contained in the large disc. But thus it is bounded. So we see that Ω can have at most one unbounded component.

Exercise 7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(1) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that
$$\left| \frac{w - z}{1 - \overline{w}z} \right| < 1 \text{ if } |z| < 1 \text{ and } |w| < 1,$$
and also that
$$\left| \frac{w - z}{1 - \overline{w}z} \right| = 1 \text{ if } |z| = 1 \text{ or } |w| = 1.$$
(2) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping
$$F : z \mapsto \frac{w - z}{1 - \overline{w}z}$$
satisfies the following conditions
   (a) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.
   (b) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.
   (c) $|F(z)| = 1$ if $|z| = 1$.

(d) $F : \mathbb{D} \to \mathbb{D}$ is bijective.

Solution 7. (a) Suppose that $|w| < 1$ and $|z| = 1$, then we have

$$|\frac{w-z}{1-\overline{w}z}| = |\frac{w-z}{\overline{z}-\overline{w}}| = 1,$$

since $|frac1\overline{z}| = 1$. Since $|w| < 1$ we see that the function $f(z) := \frac{w-z}{1-\overline{w}z}$ is holomorphic in $\mathbb{D}$. Thus by the maximum modulus principle it satisfies $|f(z)| < 1$ in $\mathbb{D}$ because it is non-constant. A straightforward calculation can also give the result.

(b) We already showed that $F(\mathbb{D}) \subset \mathbb{D}$. Clearly, $F(0) = w$ and $F(w) = 0$. Also from (a) we had $F(\partial\mathbb{D}) \subseteq \partial\mathbb{D}$. Bijective can be shown by computing F^{-1}.

Exercise 8. Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

This is the complex version of the chain rule.

Solution 8. We see have

$$\frac{\partial 11}{\partial 111} gz \frac{\partial 11}{\partial 111} fz =$$

Exercise 9. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log(z) = \log(r) + i\theta \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$. Here the second logarithm is the standard real valued one.

Solution 9.

Exercise 10. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \Delta,$$

where Δ is the **Laplacian**

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Solution 10.

Exercise 11. Use exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

Solution 11.

Exercise 12. Consider the function defined by

$$f(x+iy)=\sqrt{|x||y|},\text{ where } x,y\in\mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Exercise 13. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(1) $\mathrm{Re}(f)$ is constant
(2) $\mathrm{Im}(f)$ is constant;
(3) $|f|$ is constant;

one can conclude that f is constant.

Exercise 14. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k=\sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0=0$. Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_nb_n = a_NB_N - a_MB_{M-1} - \sum_{n=M}^{N-1}(a_{n+1}-a_n)B_n.$$

Exercise 15. **Abel's theorem.** Suppose $\sum_{n=1}^\infty a_n$ converges. Prove that

$$\lim_{r\to 1, r<1}\sum_{n=1}^\infty r^n a_n = \sum_{n=1}^\infty a_n.$$

Exercise 16. Determine the radius of convergence of the series $\sum_{n=1}^\infty a_n z^n$ when:

(1) $a_n=(\log n)^2$
(2) $a_n=n!$
(3) $a_n=\frac{n^2}{4^n+3n}$
(4) $a_n=(n!)^3/(3n)!$
(5) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha,\beta,\gamma;z)=1+\sum_{n=1}^\infty \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)}z^n.$$

Here $\alpha,\beta\in\mathbb{C}$ and $\gamma\neq 0,-1,-2,\cdots$
(6) Find the radius of convergence of the Bessel function of order r:

$$J_r=\left(\frac{z}{2}\right)^r\sum_{n=0}^\infty\frac{(-1)^n}{n!(n+r)!}\left(\frac{z}{2}\right)^{2n},$$

where r is a positive integer.

Exercise 17. Show that if $\{a_n\}_{n=0}^\infty$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=L,$$

then

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}}.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

Exercise 18. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

Exercise 19. Prove the following:

(1) The power series $\sum nz^n$ does not converge on any point of the unit circle.
(2) The power series $\sum z^n/n^2$ converges at every point of the unit circle
(3) The power series $\sum z^n/n$ converges at every point of the unit circle except $z = 1$.

2. Chapter 2. Cauchy's Theorem and Its Applications

3. Chapter 3. Meromorphic Functions and the Logarithm

4. Chapter 4. The Fourier Transform

Solution 1.

5. Chapter 5: Entire Functions

Solution 1. We follow the proof of Jensen's formula that is given in the book. We keep step 1 exactly the same. Following step 2, we have set $g = f/\psi_1 \cdot \psi_N$, then g is holomorphic and bounded near each z_j. So it suffices to prove the theorem for Blaschke factors and for bounded functions that vanish nowhere. Functions that vanish nowhere are treated in Step 3. It remains to show the result for Blaschke factors. We have

$$\log|\psi_\alpha(0)| = \log|\alpha| = \log|\alpha| + \frac{1}{2\pi}\int_0^{2\pi} \log|\psi_\alpha(e^{i\theta})|d\theta,$$

since $|\psi_\alpha(z)| = 1$ for $z \in \partial\mathbb{D}$.

Solution 2. (a) Noting that $|z^n| = O(e^{z^\epsilon})$ for all $\epsilon > 0$ we see that the order of growth of p is 0. (b) Clearly the order of growth is n. (c) The order of growth is e^{e^z} is infinite. Since $|z|^k = O(e^z)$ we know that there is no k and constants A and B, such that $e^{e^{|z|}} \leq Ae^{Bz^k}$.

Solution 3.

Solution 4. For (a) see the hint. For (b): let $r = \min 1, t$ and $R = \max 1, t$. So there are eight (m, n) such that $r \leq |nit + m| \leq R$, namely $(0, \pm 1), (\pm 1, 0), (\pm 1, \pm 1)$. There are 16 (m, n) such that $2r \leq |nit + m| \leq 2R$. In general there are $8k$ tuples with $kr \leq |nit + m| \leq kR$. Thus with

$$S(k) := \sum_{(m,n) \text{ with } \max(m,n) \leq k} \frac{1}{|int + m|^\alpha},$$

we have

$$\frac{8}{R^\alpha} \sum_{j=1}^{k} \frac{1}{j^{\alpha-1}} \leq S(k) \leq \frac{8}{r^{\alpha-1}} \sum_{j=1}^{k} \frac{1}{j^{\alpha-1}}.$$

Solution 5. To show entire let $C_R(z_0)$ be a circle of radius R centered at z_0. Then we have

$$\begin{aligned}\int_{C_1(z_0)} F_\alpha(z)dz &= \int_0^{2\pi}\int_{-\infty}^{\infty} e^{-|t|^\alpha + 2\pi i z_0 t + e^{i\theta}2\pi i t} dt ie^{i\theta} d\theta \\ &= \int_{-\infty}^{\infty} e^{-|t|^\alpha + 2\pi i z_0 t} \int_0^{2\pi} e^{2\pi i e^{i\theta} t} ie^{i\theta} d\theta dt \\ &= 0.\end{aligned}$$

We can switch the order of integration by Fubini's theorem since the integrand is L^1. The final equality is established since $\int_{C(z_0)} e^{2\pi i z t} dz = 0$. This shows that the function is holormophic in any disc by Morera's theorem. Thus it is entire.

Solution 6. By the product formula for sine we have

$$\sin(\pi/2) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(1 - \frac{1}{4n^2}\right) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{(2n+1)(2n-1)}{(2n)(2n)}\right).$$

Solution 7.

Solution 8.

Solution 9.

Solution 10.(a) $e^z - 1$ is order 1 and has zeros precisely when $z = 2\pi i n$ for $n \in \mathbb{Z}$. Thus we have

$$e^z - 1 = e^{Az+B} z \prod_{n\neq 0} \left(1 - \frac{z}{2\pi i n}\right) e^{z/2\pi i n} = e^{Az+B} z \prod_{n\geq 1} \left(1 + \frac{z^2}{4\pi^2 n^2}\right).$$

Multiplying this equation by $e^{-z/2}$ we are left with two expressions both of which must be odd. Thus we see that we must have $A = 1/2$. Also considering $\lim_{z\to 0}(e^z - 1)/z = 1$ we see that we must have $B = 0$.

(b) $\cos(\pi z)$ is also order 1 and has zeros at $\frac{2n+1}{2}$ for all $n \in \mathbb{Z}$. Thus we are left with

$$\cos(\pi z) = e^{Az+B} \prod_{n\in\mathbb{Z}} \left(1 - \frac{2z}{(2n+1)}\right) e^{2z/(2n+1)} = e^{Az+B} \prod_{n\geq 1} \left(1 - \frac{4z^2}{(2n+1)^2}\right).$$

Since cosine is even we see that $A = 0$. Also letting $z = 0$ we see that we must have $B = 0$.

Solution 11. By Hadamard's theorem if it omits a and b, then we have $f(z) - a = e^{p(z)}$ and $f(z) - b = e^{q(z)}$ for polynomials p and q. Then $e^{p(z)} - e^{q(z)} = C$ for some constant C. Letting z tend to infinity we see that the leading terms of the polynomials p and q must be the same. Say it is $a_n z^n$. Then, considering the limit as z tends to infinity of

$$e^{p(z)-a_n z^n} - e^{q(z)-a_n z^n} = Ce^{-a_n z^n},$$

we see that the next leading terms are also equal. Proceeding by induction shows that $p = q$, but then this is a contradiction since it would imply that $b = a$. But we assume they are distinct.

Solution 12. If f has finite order and never vanishes then we have $f(z) = Ce^{p(z)}$ for some polynomial, this follows from Hadamard's theorem. So then $f'(z) = Cp'(z)e^{p(z)}$. Since we assume that the derivatives are never 0, we see that p' is a constant function.

Solution 13. $e^z - z$ is entire of order 1. So $e^z - z = e^{Az+B} \prod_n \left(1 - \frac{z}{a_n}\right) e^{z/a_n}$. If there are only finitely many zeros then we have $e^z - z = e^{\tilde{A}z+\tilde{B}} Q(z)$ for some polynomial Q. But then

$$Q(z) = \frac{e^z - z}{e^{\tilde{A}z+\tilde{B}}} = O\left(e^{(1-\tilde{A})z}\right).$$

This is only possible if Q is a constant and $1 - \tilde{A} = 0$. In that case we have $z = Ce^z$ for some constant $C = 1 - e^{\tilde{B}}$, but this is clearly not possible.

Solution 14. Let $k < \rho < k + 1$, and say we have only finitely many zeros, then as in the previous problem, $F(z) = e^{p(z)}Q(z)$ for some polynomial p of degree at most k and Q a polynomial. Now F has order of growth less than or equal to k, since Q has order of growth 0. This contradicts the assumption that F has order of growth ρ.

Solution 15. Every meromorphic function, f, is holomorphic with poles at some sequence $p_1, p_2, \ldots$. Then there is an entire function with zeros precisely at the p_j with desired multiplicity. Call this function g. Then fg has no poles and is holomorphic except possibly at the p_j, but since there are no poles at the p_j, we see that we have an entire function, say h. Then $f = g/h$. For the second part of the problem construct two entire functions and take their quotient.

Solution 16.

6. Chapter 6. The Gamma and Zeta Functions

Solution 1. We have

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod_{n\geq 1}\left(1+\frac{s}{n}\right)e^{-s/n} = \lim_{N\to\infty} e^{s\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)} s\prod_{n=1}^{N}\frac{n+s}{n}e^{-s/n} = \lim_{N\to\infty}\frac{e^{\log N^{-s}}}{N!}s(s+1)\cdots(s+N).$$

Solution 2. The easier part is to deduce the identity from the product expansion of sine and the desired identity from this problem. To do so first divide by $(1+a+b)$, we want to make the substation $a = s-1$ and $b = -s$. But to do so we must substitute $a = s-\delta$ and $b = -s$ and then let δ tend to 1 and use the fact that $\Gamma(x)/x \to 1$ as $x \to 0$. We are then left with

$$\frac{1}{(1-s)s}\prod_{n>1}\frac{n(n-1)}{(n-s)(n-1-s)} = \frac{1}{(s-1)s}\prod_{n>1}\frac{1}{(1-s/n)\,(1+s/(n-1))}.$$

Thus we may deduce that

$$(\Gamma(s)\Gamma(1-s))^{-1} = s\prod_{n\geq 1}\left(1-\frac{s}{n}\right)\prod_{n>1}\left(1+\frac{s}{n+1}\right) = s\prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right) = \frac{\sin(\pi s)}{\pi}.$$

This is the desired result.

To prove the identity use Lemma 1.2 on page 161 and induction to establish that

$$\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)} = \left(N!\prod_{n=1}^{N}\frac{(a+b+n)}{(a+n)(b+n)}\right)\frac{\Gamma(a+N+1)\Gamma(b+N+1)}{N!\Gamma(a+b+N+1)}. \tag{6.1}$$

Now let a', b' be integers such that $a' \leq a \leq a'+1$, $b' \leq b \leq b'+1$, $a'+b' \leq a+b < a'+b'+1$. Then we have the inequalities

$$\frac{(a'+N+1)!(b'+N+1)!}{N!\,(a'+b'+N+2)!} \leq \frac{\Gamma(a+N+1)\Gamma(b+N+1)}{N!\Gamma(a+b+N+1)} \leq \frac{(a'+N+2)!(b'+N+2)!}{N!\,(a'+b'+N+1)!}.$$

We claim that for any A and B non-negative integers,

$$\lim_{N\to\infty}\frac{(A+N)!(B+N)!}{N!(A+B+N)!} = 1.$$

With this in hand or a slight modification of this we can easily see that letting N go to infinity in (6.1), gives the result. To prove the necessary limit use Stirling's approximation to the factorial and standard analysis. One useful limit to know is

$$\left(1+\frac{\alpha}{N^2}\right)^N \to 1,$$

as $N \to \infty$.

It occured to me later that exercise 1 can be used prove this result fairly easily so long as you can prove existence of the infinity product.

Solution 3. Notice that $\prod_{j=1}^{m}(2j+1) = \frac{(2m+1)!}{2^m m!}$. Thus Wallis's formula immediately gives

$$\frac{\pi}{2} = \lim_{n\to\infty}\frac{2^{2n}(n!)^2}{(2m+1)!^2/[2^{2n}(n!)^2(2n+1)]}.$$

Simplifying and taking square-roots gives the desired identity.

I am not sure how to deduce the identity of the Gamma function. It seems like an argument similar to the one used in problem 2 should work, however I can't get to seem the details to work out.

Solution 4. We have $a_n(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{1}{n}$.

Solution 5. This follows immediately from using the fact that $\overline{\Gamma(\sigma+it)} = \Gamma(\sigma - it)$.

Solution 6. Notice that $\gamma + \log(n) = \sum_{m=1}^{n} \frac{1}{m} + o(1)$. Thus

$$1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} - \frac{1}{2}\left(\log(n) + \gamma\right) = \sum_{m=1}^{2n} \frac{(-1)^m}{m} \to \log(2).$$

Solution 7. For (a) follow the hint to obtain

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) =& \int_0^\infty \int_0^\infty t^{\alpha-1}s^{\beta-1}e^{-t-s}dtds\\ =& \int_0^\infty \int_0^1 (ur)^{\beta-1}(u(1-r))^{\alpha-1}e^{-u}udrdu\\ =&\Gamma(\alpha+\beta)B(\alpha,\beta).\end{aligned}$$

For (b) use

$$B(\alpha,\beta) = \int_0^\infty \left(\frac{u}{1+u}\right)^{\alpha-1}\left(\frac{1}{1+u}\right)^{\beta-1}\frac{du}{(1+u)^2}.$$

Solution 8. Follow the hint to get

$$J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1/2)\sqrt{\pi}}\int_{-1}^1 \sum_{n=0}^\infty \frac{(ixt)^n}{n!}(1-t^2)^{\nu-1/2}dt.$$

Switching the integral and sum we see that we need to compute

$$\int_{-1}^1 t^n(1-t^2)^{\nu-1/2}dt.$$

This is 0 when n is odd because it is an odd function. In the case $n = 2m$, $m \in \mathbb{Z}$, we have

$$\int_{-1}^1 t^{2m}(1-t^2)^{\nu-1/2}dt = 2\int_0^1 t^{2m}(1-t^2)^{\nu-1/2}dt = \frac{\Gamma(\nu+1/2)\Gamma(m+1/2)}{\Gamma(\nu+m+1)},$$

by the previous problem with the change of variables $u = t^2$. From this we see that we need to prove something like

$$\frac{1}{2^{2m}m!} = \frac{\Gamma(m+1/2)}{(2m)!\sqrt{\pi}}.$$

Solution 9. Use the facts that

$$\frac{1}{(1-zt)^\alpha} = \sum_{n=0}^\infty \frac{\alpha\cdots(\alpha+n-1)}{n!}t^nz^n$$

for $|tz| < 1$ and

$$\frac{\Gamma(n+\beta)}{\Gamma(\beta)} = (\beta+n-1)\cdots(\beta+1)\beta.$$

Solution 10. Follow the hint. For (b) notice that

$$\mathcal{M}(\sin)(0) = \int_0^\infty \sin(t)t^{-1}dt = \lim_{z\to 0} \Gamma(z)\sin(\pi z/2) = \frac{\pi}{2}$$

and

$$\mathcal{M}(\sin)(-1/2) = \int_0^\infty \sin(t)t^{-3/2}dt = \Gamma(-1/2)\sin(-\pi/4) = (-2\sqrt{\pi})(-\frac{1}{\sqrt{2}}) = \sqrt{2\pi}.$$

Solution 12. For (a) Follow the hint and notice that We essentially have

$$\frac{1}{|\Gamma(s)|} \geq k!/\pi \geq C\left(k/e\right)^k \geq Ce^{k\log k}$$

when $s = -k-1/2$, k a positive integer. So we have essentially, $\frac{1}{|\Gamma s|} \geq Ce^{c|s|\log|s|}$ for some constants c and C.

For (b) notice that if we had such an F, then $F(s)\Gamma(s) = e^{P(z)}$ for some polynomial P of degree at most 1. But this would imply that $\frac{1}{|\Gamma(s)|} = O\left(e^{c|s|}\right)$. Which we know from part (a) is a contradiction.

Solution 13. We have

$$\log(1/Gamma(s)) = -\gamma s + \log s + \sum_{n=1}^\infty \left(-\frac{s}{n} + \log\left(1+\frac{s}{n}\right)\right).$$

Hence we have

$$\frac{d}{ds}\log\frac{1}{\Gamma(s)} = -\frac{d}{ds}\log(\Gamma(s)) = -\gamma + \frac{1}{s} + \sum_{n=1}^\infty\left(\frac{1}{s+n} - \frac{1}{n}\right).$$

Where we are allowed to pass the derivative inside the sum because of absolute convergence. Differentiating again we have

$$\left(\frac{d}{ds}\right)^2 \log\Gamma(s) = \frac{1}{s^2} + \sum_{n=1}^\infty \frac{1}{(n+s)^2},$$

which is justified again by absolute convergence.

Notice that Γ'/Γ is holomorphic away from $s = 0, -1, -2, \ldots$ so it derivative is also holomorphic in this region.

Solution 14. (a) follows from

$$\frac{d}{dx}\int_x^{x+1} \log\Gamma(t)dt = \log\Gamma(x+1) - \log\Gamma(x) = \log(x\Gamma x) - \log\Gamma x = \log x.$$

Integrating $\log x$ gives the result.

Since Γt is increasing for $t \geq 1$, we see that $\log\Gamma(t)$ is increasing in that range. Thus

$$\log\Gamma x \leq \int_x^{x+1}\log\Gamma(t)dt \leq \log\Gamma(x+1) = \log x + \log\Gamma(x).$$

Therefore we have

$$1 - \frac{1}{\log x} + \frac{c}{x\log x} \leq \frac{\log\Gamma(x)}{x\log x} \leq 1 - \frac{\log x + x + c}{x\log x}.$$

Letting $x \to \infty$ gives the result.

Solution 15. Use the suggested substitution and then change variables $t = nx$ to get $\int_0^\infty x^{s-1}e^{-nx}dx = \frac{1}{n^s}\int_0^\infty t^{s-1}e^{-t}dt = \frac{\Gamma(s)}{n^s}$. After substituting the Taylor series the change of integral and summation is justified by using absolute convergence and the fact that for $Re(s) > 1$ the integral $\int_0^1 x^{s-1}dx$ converges.

Solution 16. Proceed as in the proof they give and follow the hint.

Solution 17. First observe that $\int_0^\infty f(x)x^{s-1}dx$ is holomorphic for $Re(s) > 1$ because $\int_0^1 x^{s-1}dx$ converges for these s and for $x \to \infty$ f decays faster than any polynomial.

Next we may apply the same argument to

$$I(s) = \frac{(-1)^k}{\Gamma(s+k)} \int 0^\infty f^{(k)}(x)x^{s+k-1}dx$$

to see that $I(S)$ is holomorphic in the region $Re(s) > k-1$. This formula is justified by integration by parts.

Finally using this formula for $k = n+1$ gives

$$I(-n) = (-1)^{n+1} f^{(n)}(0).$$

There seems to be a typo in the book.

7. Chapter 7: The Zeta Function and Prime Number Theorem

Solution 1. To obtain convergence use summation by parts to obtain

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \frac{A_N}{N^s} + \sum_{n=1}^{N} A_n \left(n^{-s} - (n+1)^{-s}\right).$$

Then $|n^{-s} - (n+1)^{-s}| \leq Cn^{Re(s)+1}$. Thus letting $N \to \infty$ we see that

$$|\sum_{n=1}^{N} A_n \left(n^{-s} - (n+1)^{-s}\right)| \leq B \sum_{n=1}^{N} n^{Re(s)+1}$$

will converge for $Re(s) > 0$. As the hint indicates we can show absolute convergence on any closed half-plane $\{z : Re(z) \geq \delta > 0\}$ thus showing that the series defines a holomorphic function on the open half-plane $\{z : Re(z) > 0\}$.

Solution 2. The sequences are bounded thus $B_N = \sum_{n=1}^{N} b_n$ and $A_N = \sum_{n=1}^{N} a_n$ satisfy $|A_N| \leq AN$ and $|B_N| \leq BN$ where A and B are the bounds for the respective sequences. Using this and an argument as in the first exercise we see that the series define holomorphic functions for $Re(s) > 1$. The produce formula is easily verified by first considering the partial sums and then letting them tend to infinity. To prove the convergence for the sequence $\sum_{n\geq 1} \frac{c_n}{n^s}$ in the range $Re(s) > 1$, we use the estimate $|c_n| \leq ABd(n)$. Now $d(n) \leq c\log(n)$ for some constant c. With this in hand we apply partial summation again just as in the previous exercise.

(b) The first desired identity is a direct consequence since $a_n = b_n = 1$ for that identity. For the second identity notice that $\zeta(s-a) = \sum_{n\geq 1} \frac{n^a}{n^s}$. In this case we don't have boundedness of the sequences under consideration, but we can apply the same argument so long as we assume that $Re(s) > Re(a) + 1$, since we will need this to get the convergence in our partial summation trick.

Solution 3. We actually prove (b) first. For $n > 1$, write $n = p_1^{e_1} \cdots p_m^{e_m}$. Notice that

$$\begin{aligned}\sum_{k|n} \mu(k) =& \mu(1) + \mu(p_1) + \cdots \mu(p_m) + \mu(p_1p_2) + \cdots + \mu(p_1 \cdots p_m)\\ =& 1 - \binom{m}{1} + \binom{m}{2} + \cdots (-1)^m \binom{m}{m}\\ =& (1-1)^m = 0.\end{aligned}$$

Now to prove (a) we prove that $\zeta(s) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1$. To do this we apply exercise 2, part (a) directly and use the identity we just proved.

Solution 4.

Solution 5. (a) By the alternating series test, the sum converges for real $s > 0$. More generally, the partial sums of $a_n = (-1)^{n+1}$ are bounded, thus we may apply exercise 1 of this chapter to get $\tilde{\zeta}$ is holomorphic in $Re(s) > 0$.

(b) Use the identity

$$\zeta(s) + \tilde{\zeta}(s) = 2 \sum_{n=1}^{\infty} \frac{1}{2^s n^s} = 2^{1-s}\zeta(s),$$

which is justified in $Re(s) > 1$ by absolute convergence of the series involved.

(c) Since $\tilde{\zeta}$ is holomorphic for $Re(s) > 0$ and ζ extends to a holomorphic function in all of $\mathbb{C}$ we know they must agree on $Re(s) > 0$. Additionally $\tilde{\zeta}$ does not vanish for $s \in (0,1)$ because it is an alternating sum. If $\zeta(0) = 0$ then by the functional equation

$$\zeta(s)\Gamma(s/2)\pi^{-s/2} = \zeta(1-s)\Gamma((1-s)/2)\pi^{-(1-s)/2},$$

we see that if $\zeta(0) = 0$, then $\zeta(1)$ is finite because Γ has a simple pole at 0, but this contradicts the fact that ζ has a simple pole at $s = 1$.

Solution 6. In the case $0 \leq 0 < 1$ shift the contour to $c \to +\infty$. In the case $a > 1$ shift the contour to $c \to -\infty$ and use the residue theorem to pick up the desired value because of the pole. Use the principal value in the case $a = 1$.

Solution 7. For real s use the definition of ξ to see that it is real. Also for $s > 0$ we can see it easily from $\tilde{\zeta}$ defined in exercise 5. For s with real part $1/2$, we use the functional equation and the facts that $\zeta(1/2 - it) = \overline{\zeta(1/2 + it)}$, which can be observed from $\tilde{\zeta}$ in exercise 5, and the fact that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$, which follows from the definition of Γ.

Solution 8. Follow the hints.

Solution 9.

Solution 10. Integration by parts gives

$$\int_2^x \frac{dt}{\log(t)^m} = \frac{x}{\log(x)^m} - \frac{2}{(\log 2)^m} + m\int_2^x \frac{dt}{(\log t)^{m+1}}.$$

Additionally, we have

$$\int_2^x \frac{dt}{(\log t)^m} = \left(\int_2^{\sqrt{x}} + \int_{\sqrt{x}}^x\right) \frac{dt}{(\log t)^m} \leq C\sqrt{x} + C'\frac{1}{(\log\sqrt{x})^m}(x - \sqrt{x}) \leq \tilde{C}\frac{x}{(\log x)^m}.$$

Using these calculations with induction gives the results we wish to obtain.

Solution 11. (iv) implies (iii) implies (ii) all appear in the text. Assume (ii). Partial summation gives

$$\phi(x) = \log(x)\pi(x) + \sum_{y \leq x} \pi(y)\left(\log(y+1) - \log(y)\right).$$

Notice that

$$|\sum_{y\leq x} \pi(y)\left(\log(y+1) - \log(y)\right)| \leq C\sum_{y\leq x}\frac{\pi(y)}{y} \leq C\sum_{y\leq x^\epsilon} 1 + C'\sum_{x^\epsilon < y \leq x}\frac{1}{\log(y)} \leq Cx^\epsilon + C'\frac{x}{\log x}.$$

Thus we see that $\phi(x) \sim x$. We will show that (i) implies (iii) and that (iii) implies (iv). This finishes the result. Assume (iii). Then

$$\psi_1(x) = \int_1^x \psi(u)du = \int_1^{x^\epsilon}\psi(u)du + \int_{x^\epsilon}^x \psi(u)du = O(x^{2\epsilon}\log x) + \frac{1}{2}x^2 + O(x^{2\epsilon}) + o(x^2).$$

Assume (i) then we have $\sum_{p^n \leq x}\log(p) = \sum_{p \leq x^{1/n}}\log(p) \sim x^{1/n}$. So we have

$$\psi(x) \sim x + x^{1/2} + \cdots + x^{1/n} + \cdots$$

Solution 12. We have $\pi(p_n) = n \sim \frac{p_n}{\log p_n}$. Taking the logarithm of both sides we see that $\log n \sim \log p_n - \log\log p_n$. Hence $\log n \sim \log p_n$, because as $n \to \infty$, $\frac{\log\log p_n}{\log p_n} \to 0$. Returning to our first expression for n we have

$$n \sim \frac{p_n}{\log p_n} \sim \frac{p_n}{\log n}.$$

This gives

$$n \log n \sim p_n,$$

which is the desired result.

8. Chapter 8: Conformal Mappings

Solution 1. Suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then write $f(z) - f(z_0) = a(z - z_0)^k + G(z)$ for z in a disc around z_0, $a \neq 0$, and $k \geq 2$, with G vanishing to order at least $k + 1$. We wish to prove that for some disc D around z_0 f is one-to-one. We proceed as in the proof of Proposition 1.1 and pick w small so that $|G(z)| < |a(z - z_0)^k - w|$ for all z in some ball of readius ϵ. We want to pick ϵ so small that $f'(z)$ is non-zero for all $z \in B(z_0, \epsilon) \setminus \{z_0\}$ and so that $|G(z)| < |a(z - z_0)^k - w|$ for all $z \in B(z_0, \epsilon)$. Then we can apply Rouche's theorem to see that $f(z) - f(z_0) - w$ has two roots and that they are distinct.

To obtain the converse direction we will wish to exploit the fact that if it is not locally bijective then we can find two different continuous paths γ_1 and γ_2 from $[0, 1]$ to $\mathbb{C}$ such that $\gamma_1(0) = \gamma_2(0) = z_0$ and for each $\epsilon \in [0, 1]$, $f(\gamma_1(\epsilon)) = f(\gamma_2(\epsilon))$, but $\gamma_1(\epsilon) \neq \gamma_2(\epsilon)$ for any $\epsilon > 0$.

Solution 3. $f : U \to V$ is a conformal equivalence and let γ_1 and γ_2 be two paths from $z_0 \to z_1$ in V. Then use the conformal equivalence to pull back the paths to paths in U, then find a homotopy in U and then push them forward with f to a homotopy in V.

Solution 4. Follow the hint. First use F given in the chapter to find a map from $\mathbb{D} \to H$, then shift H down with the map $z \mapsto z - i$ and then compose with squaring. Recall that squaring will double the argument and thus fill out the entire plane. So the map is $z \mapsto (F(z) - i)^2$.

Solution 5. From the hint we know that the solutions to $f(z) = w$ are $z = -w \pm \sqrt{w^2 - 1}$. Let these solutions be z_+ and z_-. Since there product is 1, we know that one of them lies inside of $\overline{\mathbb{D}}$. Suppose z^* is that solution, then since $w \in H$ we wish to prove that $z^* \in \overline{D} \cap H$. To see this notice that

$$Im(f(z)) = -1/2(Im(z) + Im(1/z) = -1/2 \left(r + \frac{1}{r} \right) \sin(\theta)$$

where $z = re^{i\theta}$. Therefore, if $f(z) = w \in H$, then $Im(f(z)) > 0$, so we see that for $z^* \in \overline{\mathbb{D}}$, we have $r \leq 1$, thus $\sin(\theta) > 0$, so $\theta \in (0, \pi)$. We also observe from this argument that we must have $z^* \in \mathbb{D}$, thus $z^* \in \mathbb{D} \cap H$.

Solution 8. Using the hint we have $F_1(z) = \frac{z-1}{z+1}$, $F_2(z) = \log(z)$, $F_3(z) = e^{-i\pi/2} z - \frac{\pi}{2}$, $F_4(z) = \sin(z)$, and $F_5(z) = z - 1$.

Solution 9. u is the real part of a holomorphic function on $\mathbb{D}$, thus u is harmonic. A calculation shows that for $|z| = 1$, we have $\overline{z} = \frac{1}{z}$, a straight forward calculation gives

$$2Re\left(\frac{i + z}{i - z}\right) = \frac{i + z}{i - z} + \overline{\frac{i + z}{i - z}} = \frac{i + z}{i - z} + \frac{i + z}{-i + z} = 0.$$

Solution 10. Let $T : \mathbb{D} \to H$ be given by $T(z) = i\frac{1+z}{1-z}$. Then we see that $F \circ T : \mathbb{D} \to \mathbb{D}$ and it is holomorphic in $\mathbb{D}$, with $F(T(0)) = F(i) = 0$. Thus by the Schwarz Lemma we have $|F(T(z))| \leq |z|$, for $z \in \mathbb{D}$. Then we have

$$|F(z)| = |F \circ T \circ T^{-1}(z)| \leq |T^{-1}(z)| = |\frac{z - i}{z + i}|.$$

Solution 11. Define $\tilde{f} : \mathbb{D} \to \mathbb{D}$ by $\tilde{f}(z) = \frac{1}{M} f\left(\frac{1}{R}z\right)$. Then $\tilde{f}(0) \in \mathbb{D}$ unless $|f(0)| = M$, in which case f is constant by the maximum modulus theorem. Define $T : \mathbb{D} \to \mathbb{D}$ by $T(z) = \frac{z - \tilde{f}(0)}{1 - \overline{\tilde{f}(0)}}$. Then $T \circ \tilde{f} : \mathbb{D} \to \mathbb{D}$ and $T \circ \tilde{f}(0) = 0$. Thus the Schwarz lemma gives

$$|\frac{\tilde{f}(z) - \tilde{f}(0)}{1 - \overline{\tilde{f}(0)}\tilde{f}(z)}| \leq |z|.$$

Simplifying and substituting $\tilde{f}(z) = \frac{1}{M} f\left(\frac{1}{R}z\right)$.

Solution 12. If $f(0) = 0$ and we have $f(z_0) = z_0$, then we have $|f(z_0)| = |z_0|$ so we know by the Schwarz lemma f is a rotation, but $f(z) = e^{i\theta}z$ and since $f(z_0) = z_0$, so $\theta = 0$.

In the upper half-plane, the map $z \mapsto z+1$ is a conformal map and has no fixed points. Composing with a conformal map from $\mathbb{D} \to H$ and its inverse gives a conformal map on $\mathbb{D}$ to $\mathbb{D}$ that has no fixed points.

Solution 13. For the first part follow the hint. Divide both sides of the result from (a) to obtain

$$|\frac{f(z) - f(w)}{z - w}||\frac{1}{1 - \overline{f(w)}f(z)}| \leq \frac{1}{|1 - \overline{w}z|}.$$

Letting w tend to z gives the result.

Solution 14. Consider G defined by $z \mapsto i\frac{1-z}{1+z}$ which conformally maps $\mathbb{D} \to H$. Let f be conformal from H to $\mathbb{D}$, then $f \circ G : \mathbb{D} \to \mathbb{D}$. So we see that $f \circ G = \lambda \frac{z-a}{1-\overline{a}z}$ for some $a \in \mathbb{D}$ and λ with modulus 1. A straightfoward calculation shows that

$$f(z) = \lambda \frac{G^{-1}(z) - z}{1 - \overline{a}G^{-1}(z)} = \lambda \left(\frac{1+a}{1+\overline{a}}\right) \frac{z - \beta}{z - \overline{\beta}},$$

where $\beta = i\frac{1-a}{1+a} \in H$.

Solution 15. For the first part it is enough to consider the analogous problem on the circle. On the circle we know that the automorphisms take the form $\psi_a(z) := \lambda \frac{z-a}{1-\overline{a}z}$ for $a \in \mathbb{D}$ and $|\lambda| = 1$. So if there are three fixed points then we have three solutions to $\psi_a(z) = z$, but there are only two roots of a quadratic so we have a contradiction.

Solution 17. The first part follows since

$$\pi = Area(\mathbb{D}) = \int\int_{\psi(\mathbb{D})} dxdy = \int\int_{\mathbb{D}} |\psi_\alpha'(z)|^2 dxdy.$$

I don't know how to establish the second part.

Solution 18. Follow the proof and lemmas of Theorem 4.2

Solution 19. Follow hint, the result should be clear.

Solution 20. (a) Following the notation of Proposition 4.1 we have $A_1 = 0$, $A_2 - 1$, and $A_3 = \lambda$, when $\lambda > 1$, the other cases $0 < \lambda < 1$ and $\lambda < 0$ are handled similarly. In any case we have $\beta_1 = \beta_2 = \beta_3 = \beta_\infty = \frac{1}{2}$. To see that we have a rectangle it is enough to notice that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_\infty = \frac{\pi}{2}$.

In the second part, we have

$$\begin{aligned}\int_0^1 \frac{d\zeta}{\sqrt{\zeta(\zeta^2-1)}} =& \frac{1}{2}\int_0^1 (1-u)^{-1/2}u^{-3/4}du \\ =& \frac{1}{2}\frac{\Gamma(1/2)\Gamma(1/4)}{\Gamma(3/4)} \\ =& \frac{1}{2\sqrt{2\pi}}\Gamma(1/4)^2,\end{aligned}$$

where we use the calculation of the beta function and then the fact that $\Gamma(3/4)\Gamma(1/4) = \frac{\pi}{\sin(\pi/4)} = \pi\sqrt{2}$ and $\Gamma(1/2) = \sqrt{\pi}$. We need only to calculate the length of one of the perpendicular sides which can be done similarly.

Solution 21. (a) follows immediately from Proposition 4.1. For (d) we have the following similar calculation

$$\begin{aligned}\int_0^1 z^{-\beta_1}(1-z)^{-\beta_2}dz = \frac{\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(2-\beta_1-\beta_3)} =& \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\beta_3)} \\ =& \frac{\sin(\alpha_3\pi)\Gamma(\alpha_3)}{\pi}\Gamma(\alpha_1)\Gamma(\alpha_2).\end{aligned}$$

Where we use $\Gamma(\beta_3)\Gamma(1-\beta_3) = \frac{\pi}{\sin((1-\beta_3)\pi)}$.

A similar calculation with the change of variables $u = \frac{1}{z}$ can be used to compute the integral

$$\int_1^\infty z^{-\beta_1}(1-z)^{-\beta_2}dz = (-1)^{\beta_2}\frac{\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)}.$$

Similar calculations finish the problem.

Solution 22. This result is not suprising because of the correspondence between H and $\mathbb{D}$ that has $\mathbb{R}^* = \mathbb{R}\cup\{\infty\}$ and $\partial\mathbb{D}$.

9. Chapter 9: An Introduction to Elliptic Functions

Solution 1. (a) With the notation given in the problem,

$$n\omega_2 = \frac{np}{q}\omega_1 = \frac{1-mq}{q}\omega_1 = \frac{1}{q}\omega_1 - m\omega_1.$$

This gives us that

$$f\left(z + \frac{1}{q}\omega_1\right) = f\left(z + \frac{1}{q}\omega_1 - m\omega_1\right) = f\left(z + n\omega_2\right) = f\left(z\right).$$

This shows that $\frac{1}{q}\omega_1$ is a period of f and it is smaller than ω_2 and ω_1 in size, since $\frac{1}{q}\omega_1 = \frac{1}{p}\omega_2$.

(b) Follow the hint and use continuity.

Solution 2. The argument of Theorem 4.1 in Chapter 3 shows that

$$\int_{\partial P} z\frac{f'(z)}{f(z)}dz = 2\pi i\left(a_1 + \cdots + a_r - b_1 - \cdots - b_r\right).$$

This follows from the residue theorem. I am not sure how to use the hint to compute this integral in a second way so as to yield the desired result.

Solution 3. Let $r = \min(1, |\tau|)$ and $R = \max(1, |\tau|)$. Then there are $4n$ values of $\omega \in \Lambda$ such that $nr \leq |\omega| \leq nR$. This gives

$$\sum_{|\omega|\leq M} \frac{1}{|\omega|^2} \geq \sum_{n=1}^{N} \frac{4n}{n^2R^2} \sim \frac{4}{R^2}\log N.$$

We get similarly that

$$\sum_{|\omega|\leq M} \frac{1}{|\omega|^2} \leq \sum_{n=1}^{N} \frac{4n}{n^2r^2} \sim \frac{4}{r^2}\log N.$$

Where the M is chosen appropriately. So we see that the sum must diverge.

Just be more careful with this argument to get the second result.

Solution 5. If we establish convergence then since it is an infinite product it can only vanish at a point where one of the terms vanishes, thus it has simple zeros at all the periods and does not vanish anywhere else.

Differentiating term by term gives the desired formula for σ'/σ. Again differentiating term by term shows the desired formula for $\wp(z)$.

Solution 6. We have $(\wp')^2 = 4\wp^3 - a\wp - b$, hence differentiating both sides and dividing by $\wp'$ gives,

$$2\wp'' = 12\wp^2 - a.$$

Solution 7. We see that $4\sum_{m \text{ odd}} \frac{1}{m^2} = \frac{\pi^2}{\sin(\pi/2)^2} = \frac{\pi^2}{2}$. Use the fact that

$$\frac{1}{4}\sum_{m} \frac{1}{m^2} = \sum_{m \text{ even}} \frac{1}{m^2} = \left(\sum_{m} - \sum_{m \text{ odd}}\right)\frac{1}{m^2}$$

which results in $\sum_m = \frac{4}{3}\sum_{m \text{ odd}}$. Similar analysis works for $\zeta(4)$.

Solution 8.

$$E_4(it) = \sum_{n\neq 0} \frac{1}{n^4} + sum_n \sum_{m\neq 0} \frac{1}{n+imt} = 2\zeta(4) + sum_n \sum_{m\neq 0} \frac{1}{n+imt}$$

so we need to analysis the remaining sum. As $t \to \infty$ it is clear that the sum should go to 0, which is what we need to analyze and so. I am a bit confused about the rate of convergence to zero.

10. Chapter 10: Applications of Theta Functions

Solution 2. Multiply the recursive relation by x^n and sum from $n = 2$ up to infinity. This gives

$$F(x) - F_0 - F_1 x = x\left(F(x) - F_0\right) + x^2 F(x).$$

This proves the first part in a formal sense, to get convergence in a ball around 0 we will show that $F_n \sim c\alpha^n$ for some $\alpha > 0$, which is enough to get convergence in a ball around 0.

For (c) and (d) follow the partial fraction decomposition, then use the geometric series, but absolute convergence in a small disc we can combine the two sums. Then matching up the appropriate powers of x^n gives the result.

Solution 3. Do the same things as in (2) but in more generality. In the case $\alpha = \beta$ we just note that we have $U(x) = \frac{u_0}{(1-\alpha x)^2} + (u_1 - au_0)\frac{x}{(1-\alpha x)^2}$. Now notice that $\frac{1}{(1-x)^2} = \sum_{n\geq 1} nx^{n-1}$ to finish off the calculations.

Solution 4. We have $\prod_{n\geq 1}(1-x^n) = \sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2}$. So we see that

$$\left(\sum_{n\geq 0} p(n)x^n\right)\left(\sum_{k=-\infty}^{\infty}(-1)^k x^{k(3k+1)/2}\right) = 1.$$

Multiplying out and comparing powers of x^n on each side gives the desired formula

$$\sum_k (-1)^k p\left(n - \frac{k(3k+1)}{2}\right) = 0,$$

for $n \geq 1$.

Solution 5. Follow the hint. To prove the needed inequality notice that

$$mx^{m-1} < \frac{1-x^m}{1-x} = 1 + x + \cdots + x^{m-1} < m$$

for $x \in (0,1)$, since $x^j > x^{m-1}$ for $j < m-1$ in this range and $x^j < 1$.

Solution 6. Following the hint we have $\log(F(e^{-y})) \sim \frac{\pi^2}{6(1-e^{-y})} \sim e^{\frac{\pi^2}{6y}}$. Therefore we have $F(e^{-y}) \leq Ce^{\frac{\pi^2}{6y}}$, hence $p(n) \leq ce^{\frac{\pi^2}{6y}+ny}$. Taking $y = \frac{1}{n^{1/2}}$ gives the upper bound.

Solution 7. Both identities are a consequence of the triple product identity for the theta function. For the first identity take $\tau = \frac{1}{2}\alpha$ and $z = \frac{1}{4}\alpha$ and $x = e^{\pi i\alpha}$. For the second take $z = \frac{3}{4}\alpha + \frac{1}{2}$, $\tau = \frac{5}{2}\alpha$ and $x = e^{\pi i\alpha}$.

Solution 8. Since they are relatively prime both are not even. If a and b are both odd then $a^2 + b^2 \equiv 2 \pmod 4$, but all squares are 0 or 1 modulo 4. Following the hint write $\left(\frac{b}{2}\right)^2 = \frac{c-a}{2}\frac{c+a}{2}$. Notice that $\frac{c-a}{2}$ and $\frac{c+a}{2}$ must be relatively prime since the gcd must then divide b, but we assume a, b, and c to be relatively prime. Therefore we see that $\frac{c+a}{2}$ and $\frac{c-a}{2}$ are both squares, say m^2 and n^2 respectively. Then $c = m^2 + n^2$, $\frac{b}{2} = mn$, and $a = m^2 - n^2$, as desired.

For the last part notice that

$$(a^2+b^2)(c^2+d^2) = a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 = (ac+bd)^2 + (ad-bc)^2. \tag{10.1}$$

Solution 9. From the formula we know that $r_2(p) = 4(2-0) = 8$, when $p \equiv 1 \pmod 4$. For $q \equiv 3 \pmod 4$ we have

$$r_2(q^a) = 4((\lfloor \frac{a}{2} \rfloor + 1) - (\lfloor \frac{a-1}{2} \rfloor + 1)).$$

So when a is odd we see that $\lfloor \frac{a}{2} \rfloor = \lfloor \frac{a-1}{2} \rfloor$ and we have 0, otherwise it is non-zero.

From the displayed equation in the previous problem and the earlier parts of this problem we may deduce that when all the primes congruent to 3 modulo 4 appear to an even power n is representable as the sum of two squares. To get the other direction use the formula writing $n = p_1^{a_1} \cdots p_j^{a_j} q_1^{b_1} \cdots q_k^{b_k}$ where $p_m \equiv 1 \pmod 4$ and $q_m \equiv 3 \pmod 4$ are distinct primes.

Solution 10. For (a) notice that $r_2(q) = 0$ for all $q \equiv 3 \pmod 4$ and $r_2(5^k) = k+1$. For (b) first notice that $r_4(2^k) = 8 \times (1+2) = 24$ for all $k \geq 1$.

Solution 12. For the first part use exercise 11 For (b) we note that $\sigma_1^*(n) = \sigma_1(n) - 4\sigma_1(n/4)$. For (c) notice that it is enough to show that

$$\sum_{n\geq 1} \frac{q^n}{(1+(-1)^n q^n)^2} = \sum_{n \geq 1} \frac{q^n}{(1-q^n)^2} - 4 \sum_{n \geq 1} \frac{q^{4n}}{(1-q^{4n})^2}.$$

To do so consider the odd and even n in the left hand side separately. The odd terms match up with the odd terms from the first sum on the right hand side. To finish notice that

$$\frac{q^2}{(1-q^{2n})^2} - \frac{4q^{4n}}{(1-q^{4n})^2} = \frac{q^{2n}}{(1+q^{2n})^2}.$$

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