Solutions Manual to accompany An Introduction to Financial Markets

A Quantitative Approach

Version of December 12, 2018

Paolo Brandimarte

A Wiley-Interscience Publication

JOHN WILEY & SONS, INC.

New York / Chichester / Weinheim / Brisbane / Singapore / Toronto

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Preface

This solutions manual contains worked-out solutions to end-of-chapter problems in the book. Over time, I plan to add additional solved problems.

When useful, I will include hints about how software tools like R or MATLAB can be used. In fact, these tools have been used to carry out the required calculations, and there may be numerical differences in the results, if you use these environments, keeping the best numerical precision, rather than using paper and a pocket calculator, possibly introducing some rounding. Needless to say, the important point of these problems is conceptual, as they should support the understanding of the underlying financial concepts. Therefore, do not bother about small inconsistencies (which would be important in real life, where you have to stick with market conventions in rounding things!).

The manual is work-in-progress, so be sure to check back every now and then, to see whether a new version has been posted.

This version is dated December 12, 2018.

As usual, for comments, suggestions, and criticisms, my e-mail address is given below.

Paolo Brandimarte
`paolo.brandimarte@polito.it`

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Financial Markets: Functions, Institutions, and Traded Assets

1.1 SOLUTIONS

Problem 1.1 The expected returns are given by:

$$\begin{aligned}
\mu_1 &= 0.2 \times 0.03 + 0.2 \times 0.17 + 0.3 \times 0.28 + 0.2 \times 0.05 + 0.1 \times (-0.04) = 0.13 \\
\mu_2 &= 0.2 \times 0.09 + 0.2 \times 0.16 + 0.3 \times 0.10 + 0.2 \times 0.02 + 0.1 \times 0.16 = 0.10.
\end{aligned}$$

To find the standard deviations, we first compute

$$\begin{aligned}
\mathrm{E}[R_1^2] &= 0.2 \times 0.03^2 + 0.2 \times 0.17^2 + 0.3 \times 0.28^2 + 0.2 \times 0.05^2 + 0.1 \times (-0.04)^2 = 0.0301 \\
\mathrm{E}[R_2^2] &= 0.2 \times 0.09^2 + 0.2 \times 0.16^2 + 0.3 \times 0.10^2 + 0.2 \times 0.02^2 + 0.1 \times 0.16^2 = 0.0124.
\end{aligned}$$

Then

$$\begin{aligned}
\sigma_1 &= \sqrt{\mathrm{E}[R_1^2] - \mu_1^2} = \sqrt{0.0301 - 0.10^2} = 0.1151 \\
\sigma_2 &= \sqrt{\mathrm{E}[R_2^2] - \mu_2^2} = \sqrt{0.0124 - 0.13^2} = 0.0488.
\end{aligned}$$

Finally, the correlation is

$$\rho_{1,2} = \frac{\mathrm{Cov}(R_1, R_2)}{\sigma_1 \cdot \sigma_2} = \frac{\mathrm{E}[R_1 \cdot R_2] - \mu_1 \cdot \mu_2}{\sigma_1 \cdot \sigma_2}.$$

We need

$$\begin{aligned}
\mathrm{E}[R_1 \cdot R_2] &= 0.2 \times 0.03 \times 0.09 + 0.2 \times 0.17 \times 0.16 + 0.3 \times 0.28 \times 0.10 \\
&\quad + 0.2 \times 0.05 \times 0.02 + 0.1 \times (-0.04) \times 0.16 = 0.0139.
\end{aligned}$$

Therefore,

$$\rho_{1,2} = \frac{\mathrm{Cov}(R_1, R_2)}{\sigma_1 \cdot \sigma_2} = \frac{0.0139 - 0.13 \times 0.10}{0.1151 \times 0.0488} = 0.1674.$$

Problem 1.2 We sell (short) the 300 shares at €40 and we have to add the required margin to the posted assets:

$$\mathsf{A} = 300 \times 40 \times (1 + 0.50) = €18{,}000$$

The liabilities are 300 stock shares times price:

$$\mathsf{L} = 300 \times P$$

To find the critical price, we must consider the ratio of equity, $\mathsf{E} = \mathsf{A} - \mathsf{L}$ and the floating side, which is the liability side in this case:

$$\frac{\mathsf{E}}{\mathsf{L}} = \frac{18{,}000 - 300 \times P}{300 \times P} = 0.25$$

which yields

$$P_{\mathsf{lim}} = \frac{18{,}000}{300 \times 1.25} = €48.$$

As a reality check, this price is larger than the current one (€ 40).

Problem 1.3 The stock price and the payoff are given in the following table:

State	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
$S(T)$	20	25	30	35	40	45	50	55
$\max\{S(T) - 40, 0\}$	0	0	0	0	0	5	10	15

Since states are equiprobable, the expected payoff value is just

$$\frac{0 \times 5 + 5 + 10 + 15}{8} = 3.75.$$

The most important message here is that the expected value of a function is *not* the function of the expected value:

$$\max\big\{\mathrm{E}[S(T)] - 40, 0\big\} = \max\{37.5 - 40, 0\} = 0 \neq \mathrm{E}\big[\max\{S(T) - 40, 0\}\big].$$

Problem 1.4 To find the divisor, we divide the value of the market portfolio by the current index value, $I_0 = 118$:

$$D = \frac{50{,}000 \times 50 + 100{,}000 \times 30 + 80{,}000 \times 45}{118} = 77{,}118.64$$

The index value in the next three days is:

$$I_1 = \frac{50{,}000 \times 52 + 100{,}000 \times 28 + 80{,}000 \times 43}{77{,}118.64} = 114.63$$
$$I_2 = \frac{50{,}000 \times 48 + 100{,}000 \times 25 + 80{,}000 \times 40}{77{,}118.64} = 105.03$$
$$I_3 = \frac{50{,}000 \times 45 + 100{,}000 \times 30 + 80{,}000 \times 39}{77{,}118.64} = 108.53$$

2

Basic Problems in Quantitative Finance

2.1 SOLUTIONS

Problem 2.1 If you take the safe route, you earn \$100,000 for sure. The expected value of the fee you receive with the active strategy is

$$\begin{aligned}\mathrm{E}[X] = 0 \times \mathrm{P}\{R < 0\} + 50{,}000 \times \mathrm{P}\{0 \le R < 0.03\} \\ + 100{,}000 \times \mathrm{P}\{0.03 \le R < 0.09\} + 200{,}000 \times \mathrm{P}\{0.09 \le R\}.\end{aligned}$$

If you use statistical tables providing vales for the standard normal CDF $\Phi(z) \doteq \mathrm{P}\{Z \le z\}$, you should standardized the return thresholds, with $\mu = 0.08$ and $\sigma = 0.10$:

$$\begin{aligned} z_1 &= \frac{0.00 - 0.08}{0.10} = -0.8 \\ z_2 &= \frac{0.03 - 0.08}{0.10} = -0.5 \\ z_4 &= \frac{0.09 - 0.08}{0.10} = 0.1. \end{aligned}$$

Then, we find

$$\begin{aligned} \pi_1 &\doteq \mathrm{P}\{R < 0\} = \Phi(z_1) = 0.2119 \\ \pi_2 &\doteq \{0 \le R < 0.03\} = \Phi(z_2) - \Phi(z_1) = 0.0967 \\ \pi_3 &\doteq \mathrm{P}\{0.03 \le R < 0.09\} = \Phi(z_3) - \Phi(z_2) = 0.2313 \\ \pi_4 &\doteq \mathrm{P}(R \ge 0.09) = 1 - \pi_1 - \pi_2 - \pi_3 = 0.4602. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{E}[X] &= 50{,}000 \times 0.0967 + 100{,}000 \times 0.2313 + 200{,}000 \times 0.4602 = 120{,}000 \\ \mathrm{E}[X^2] &= 50{,}000^2 \times 0.0967 + 100{,}000^2 \times 0.2313 + 200{,}000^2 \times 0.4602 = 20{,}961{,}494{,}844.79 \\ \mathrm{std}[X] &= \sqrt{\mathrm{E}[X^2] - \mathrm{E}^2[X]} = \sqrt{20{,}961{,}494{,}844.79 - 120{,}000^2} \approx 81{,}000 \end{aligned}$$

If you are risk neutral, the active strategy would be preferred, but there is a considerable risk. If you are uncomfortable with standard deviation, you may also consider the probability of regretting a decision. On the one hand, if we take chances, the probability that the risky fee is less than the safe \$100,000 is

$$\mathrm{P}\{\text{risky fee} < 100{,}000\} = \pi_1 + \pi_2 = 0.3086.$$

On the other hand, if choose the safe alternative, we will regret our decision with a probability

$$\mathrm{P}\{\text{risky fee} > 100{,}000\} = \pi_4 = 0.4602.$$

Indeed, there is considerable uncertainty, and the choice is quite subjective. This may also depend on the context, i.e., is this fee our only source of income?

R Hint. The problem is very easy to solve with the help of R:

```
> p1=pnorm(0,mean=0.08,sd=0.1);p1
[1] 0.2118554
> p2=pnorm(0.03,mean=0.08,sd=0.1)-p1;p2
[1] 0.09668214
> p3=pnorm(0.09,mean=0.08,sd=0.1)-pnorm(0.03,mean=0.08,sd=0.1);p3
[1] 0.2312903
> p4=1-pnorm(0.09,mean=0.08,sd=0.1);p4
[1] 0.4601722
> probs=c(p1,p2,p3,p4)
> bonus=c(0,50,100,200)*1000
> m=sum(probs*bonus);m
[1] 119997.6
> stdev=sqrt(sum(probs*bonus^2)-m^2);stdev
[1] 81006.66
```

The slight differences with respect to the above solution are due to numerical roundoff.

Problem 2.2 Since we add (jointly) normal variables, the portfolio return will be normal, too:

$$\begin{aligned} R_p &= 0.4R_1 + 0.6R_2 \\ &= 0.4 \times 0.03 + 0.6 \times 0.04 + \big(0.4 \times 1.2 + 0.6 \times 0.8\big)R_m + 0.4\epsilon_1 + 0.6\epsilon_2 \\ &= 0.036 + 0.96R_m + 0.4\epsilon_1 + 0.6\epsilon_2. \end{aligned}$$

Then, since the specific risk factors have zero expected value,

$$\mathrm{E}[R_p] = 0.036 + 0.96 \times \mu_m = 0.036 + 0.96 \times 0.04 = 0.0744,$$

where $\mu_m = \mathrm{E}[R_m]$. To compute portfolio volatility $\sigma_p = \mathrm{std}(R_p)$, we take advantage of the lack of correlation among risk factors:

$$\begin{aligned} \sigma_p &= \sqrt{0.96^2\sigma_m^2 + 0.4^2\sigma_{\epsilon,1}^2 + 0.6^2\sigma_{\epsilon,2}^2} \\ &= \sqrt{(0.96 \times 0.25)^2 + (0.4 \times 0.3)^2 + (0.6 \times 0.4)^2} = 0.36. \end{aligned}$$

Note that, if you use the stock returns R_1 and R_2, you have to compute their covariance, since

$$\mathrm{Var}(R_p) = 0.4^2 \cdot \mathrm{Var}(R_1) + 2 \cdot 0.4 \cdot 0.6 \cdot \mathrm{Cov}(R_1, R_2) + 0.6^2 \cdot \mathrm{Var}(R_2).$$

Using standardization and standard normal tables (or statistical software), we find

$$\mathrm{P}\left\{Z \le \frac{0 - 0.0744}{0.36}\right\} = 1 - \Phi(0.2067) \approx 0.4181.$$

Problem 2.3 The current price of the zero is

$$P_z(0) = \frac{1000}{(1 + 0.043)^3} = \$881.3473$$

so that the value of the portfolio of 100 bonds, i.e., the value of the asset side is

$$\mathsf{A} = 100 \times P_z(0) = \$88{,}134.73.$$

The liability side is 50% of this value,

$$\mathsf{L} = \$44{,}067.365.$$

To find the critical bond price, we set the ratio of equity to the floating side (asset side) to the maintenance margin:

$$\frac{100 \times P_{\mathsf{lim}} - 44{,}067.365}{100 \times P_{\mathsf{lim}}} = 0.20 \qquad \Rightarrow \qquad P_{\mathsf{lim}} = \$550.8421,$$

which is lower than initial one, as we are in trouble when the value of our assets drops. The corresponding yield is found by solving

$$\frac{1000}{(1 + y^*)^3} = 550.8421 \qquad \Rightarrow \qquad y^* = 0.2199.$$

Given the problem data, we find a rather large value of yield, such that a margin call is quite unlikely. Anyway, we observe that if we buy bonds on margin, we are in trouble when interest rates rise. On the contrary, if we short-sell bonds, we are in trouble when interest rates drop, as this implies an increase in bond prices.

Problem 2.4 If we sell the asset and close the (long) futures positions at T_H, the cash flow will be

$$S(T_H) + \phi_1 \cdot \big[F_1(T_H) - F_1(0)\big] + \phi_2 \cdot \big[F_2(T_H) - F_2(0)\big],$$

where the positions in the futures, maturing at $t = T_F$, are denoted by ϕ_1 and ϕ_2, respectively. They will be negative in the case of a short position (taking a short position may sound more natural when we have to sell an asset, but this depends on the involved correlations). Also note that, actually, the two maturities of the futures contracts need not be the same, and that we disregard marking-to-market.

The variance of this cash flow is

$$V = \sigma_s^2 + \phi_1^2\sigma_1^2 + \phi_2^2\sigma_2^2 + 2\phi_1\sigma_{s,1} + 2\phi_2\sigma_{s,2} + 2\phi_1\phi_2\sigma_{1,2},$$

where we denote variances of the three prices involved by σ_s^2, σ_1^2, and σ_2^2, respectively, and their covariances by $\sigma_{s,1}$, $\sigma_{s,2}$, and $\sigma_{1,2}$. Note that, in terms of variance, nothing would change if we consider the variation of the hedged portfolio value from $t = 0$ to $t = T_H$,

$$\delta S + \phi_1 \cdot \delta F_1 + \phi_2 \cdot \delta F_2,$$

where $\delta S \doteq S(T_H) - S(0)$, $\delta F_1 \doteq F_1(T_H) - F_1(0)$, and $\delta F_2 \doteq F_2(T_H) - F_2(0)$. We write the first-order optimality conditions for the minimization of variance (a convex problem),

$$\frac{\partial V}{\partial \phi_1} = 2\phi_1 \sigma_1^2 + 2\sigma_{s,1} + 2\phi_2 \sigma_{1,2} = 0,$$
$$\frac{\partial V}{\partial \phi_2} = 2\phi_2 \sigma_2^2 + 2\sigma_{s,2} + 2\phi_1 \sigma_{1,2} = 0.$$

Rearranging a bit yields a system of linear equations,

$$\phi_1 \sigma_1^2 + \phi_2 \sigma_{1,2} = -\sigma_{s,1},$$
$$\phi_1 \sigma_{1,2} + \phi_2 \sigma_2^2 = -\sigma_{s,2},$$

which may be solved, e.g., by using Cramer's rule:

$$\phi_1 = \frac{\begin{vmatrix} -\sigma_{s,1} & \sigma_{1,2} \\ -\sigma_{s,2} & \sigma_2^2 \end{vmatrix}}{\begin{vmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{vmatrix}} = -\frac{\sigma_{s,1} \cdot \sigma_2^2 - \sigma_{1,2} \cdot \sigma_{s,2}}{\sigma_1^2 \cdot \sigma_2^2 - \sigma_{1,2}^2},$$

$$\phi_2 = \frac{\begin{vmatrix} \sigma_1^2 & -\sigma_{s,1} \\ \sigma_{1,2} & -\sigma_{s,2}^2 \end{vmatrix}}{\begin{vmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{vmatrix}} = -\frac{\sigma_{s,2} \cdot \sigma_1^2 - \sigma_{1,2} \cdot \sigma_{s,1}}{\sigma_1^2 \cdot \sigma_2^2 - \sigma_{1,2}^2}.$$

NOTE: A common mistake is to ignore the correlation between the two hedging instruments and use the familiar minimum variance hedging ratio. This is wrong in general but, if the two futures prices are uncorrelated, i.e., $\sigma_{1,2} = 0$, we find two decoupled equations, whose solution (apart from a change in sign) is in fact the same as in the case of a short position in a single futures contract.

Problem 2.5 To begin with, we need the probability distribution of the portfolio return R_p, which is normal with expected value

$$\mu_p = 0.057,$$

since risk factors have zero expected value, and standard deviation

$$\sigma_p = \sqrt{(3.4 \times 0.1)^2 + (-2.6 \times 0.12)^2 - 2 \times 0.48 \times 3.4 \times 2.6 \times 0.1 \times 0.12 + 0.2^2} = 0.3887.$$

Hence, for the first question (assuming we need to standardize and use the old statistical tables...)

$$\mathrm{P}\{R_p > 0.025\} = \mathrm{P}\left\{Z > \frac{0.025 - 0.057}{0.3887}\right\} = \mathrm{P}\{Z > -0.0823\} = \Phi(0.0823) = 0.5328.$$

For the second question, loss has normal distribution with expected value

$$\mu_L = -1{,}000{,}000 \times 0.057 = -57{,}000,$$

and standard deviation

$$\sigma_L = 1{,}000{,}000 \times 0.3887 = 388{,}700.$$

The quantile at probability level 95% is

$$\mathrm{V@R}_{0.95} = \mu_l + z_{0.95}\sigma_L = -57{,}000 + 1.6449 \times 388{,}700 = \$582{,}372.6.$$

This is the absolute V@R, which is reduced by the positive expected profit. If you use R, you get a slightly different result due to numerical roundoff.

Problem 2.6 The fair option value is

$$f_0 = e^{-rT} \cdot (\pi_u f_u + \pi_d f_d),$$

and the Laue of the process f_t/B_t at time $t = 0$ is just

$$\frac{f_0}{B_0} = \frac{e^{-rT} \cdot (\pi_u f_u + \pi_d f_d)}{1}.$$

At time $t = T$

$$\mathrm{E}_{\mathbb{Q}_n}\left[\frac{f_T}{B_T}\right] = \frac{\mathrm{E}_{\mathbb{Q}_n}[f_T]}{B_T} = \frac{\pi_u f_u + \pi_d f_d}{e^{rT}} = \frac{f_0}{B_0}.$$

Problem 2.7 This is essentially a pricing problem that we may solve by replication. We have are three assets with linearly independent tradeoffs, which are able to span any payoff in $\mathbb{R}^3$. The portfolio replicating the payoff of the insurance contract is found by solving the following system of linear equations:

$$\begin{aligned}
1 \cdot h_1 + 3 \cdot h_2 + 1.2 \cdot h_3 &= 0,\\
3 \cdot h_1 + 1 \cdot h_2 + 1.2 \cdot h_3 &= 0,\\
0 \cdot h_1 + 0 \cdot h_2 + 1.2 \cdot h_3 &= 1,
\end{aligned}$$

where h_1, h_2, and h_3 denote the respective holdings in the three primary assets. Solving the system yields

$$h_1 = -\tfrac{1}{4}, \qquad h_2 = -\tfrac{1}{4}, \qquad h_3 = \tfrac{5}{6}.$$

Hence, the current value of the replicating portfolio is

$$1 \cdot h_1 + 1 \cdot h_2 + 1 \cdot h_3 = -\tfrac{1}{4} - \tfrac{1}{4} + \tfrac{5}{6} = \tfrac{1}{3} \approx 0.3333,$$

which must be the fair value of the insurance contract, by the law of one price.

There is no need to consider risk aversion, as the market is complete and we perfectly replicate the payoff, state by state.

It is worth noting that the two risky assets are shorted, which make sense, since we must profit from those short positions when the bad state occurs. Let us check how the replication works in each state, noting that the $\frac{5}{6}$ shares of the risk-free asset yield \$1 in any state:

1. In state ω_1, we need to buy 0.25 shares of both assets 1 and 2, to close the two short positions. The cash we need, $0.25 \times 1 + 0.25 \times 3 = \$1$, is provided by the risk-free asset. We break even, and the total payoff is zero.

2. In state ω_2, we need to buy 0.25 shares of both assets 1 and 2, to close the two short positions. The cash we need, $0.25 \times 3 + 0.25 \times 1 = \$1$, is provided by the risk-free asset. We break even, and the total payoff is zero.

3. In state ω_3, we do not need any cash to close the two short positions, since the two risky assets are worth nothing, and we replicate the payoff of \$1 with the cash provided by the risk-free asset.

2.2 ADDITIONAL PROBLEMS

Additional problem 2.1 Consider a market model with two states and two assets, where:[1]

$$B(0) = 55, \quad B(T, \omega_1) = 60, \quad B(T, \omega_2) = 60,$$
$$S(0) = 45, \quad S(T, \omega_1) = 45, \quad S(T, \omega_2) = 40.$$

- Is there an arbitrage opportunity?
- Is there an arbitrage opportunity if shortselling is forbidden?
- Is there an arbitrage opportunity if shortselling is allowed, but there is a 5

Solution
Using our notation,

$$\mathbb{V} = \begin{bmatrix} 55 \\ 45 \end{bmatrix} \qquad \mathbb{Z} = \begin{bmatrix} 60 & 45 \\ 60 & 45 \end{bmatrix}.$$

Asset B (say, a zero) is clearly risk-free, with holding period return

$$\frac{60-55}{55} = \frac{1}{11} = 9.091\%,$$

whereas asset S (say, a stock share) is risky and has a nonpositive return. In this static framework, where interest rate risk plays no role, we may also interpret asset B as a bank account with a risk-free holding period return of 9.091%. Indeed, when we account for the initial price, asset B dominates asset S, and there is a clear arbitrage opportunity. For instance, if we shortsell one stock share and invest the resulting cash in the bond, we hold a portfolio

$$\mathbf{h} = \begin{bmatrix} 45/55 \\ -1 \end{bmatrix},$$

with initial value 0 and strictly positive terminal value:

$$\mathbb{V}^\mathsf{T}\mathbf{h} = 0, \qquad \mathbb{Z}\mathbf{h} = \begin{bmatrix} 4.0909 \\ 9.0909 \end{bmatrix}.$$

Indeed, the cash grows to

$$\frac{45}{55} \cdot 60 = 49.0909$$

which is more than enough to close the short position in the stock (the price is 45 in state ω_1, the less favorable one). Using MATLAB, we may check the calculation and verify that the LP problem (2.30) of the book is unbounded below:

```
>> V = [55; 45];
>> ZZ = [60 45; 60 40];
>> h = [45/55; -1];
>> dot(V,h)
ans =
```

[1]This problem is adapted from Problems 2.7 and 2.8 of G. Campolieti, R.N. Makarov, *Financial Mathematics: A Comprehensive Treatment*, CRC Press, 2014.

```
    0
>> ZZ*h
ans =
    4.0909
    9.0909
>> h = linprog(V', -ZZ, zeros(2,1))
Problem is unbounded.
h =
     []
```

Using MATLAB again, we may easily verify that the state prices are not all positive:

```
>> Pi = ZZ'\V
Pi =
    1.6667
   -0.7500

>> h1 = ZZ\[1;0]
h1 =
   -0.1333
    0.2000
dot(V,h1)
ans =
    1.6667
>> h2 = ZZ\[0;1]
h2 =
    0.1500
   -0.2000
>> dot(V,h2)
ans =
   -0.7500
```

In the first MATLAB line, we solve the pricing equations:

$$B(T, \omega_1) \cdot \pi_1 + B(T, \omega_2) \cdot \pi_2 = B(0)$$
$$S(T, \omega_1) \cdot \pi_1 + S(T, \omega_2) \cdot \pi_2 = S(0)$$

which are the equality constraints in the dual of LP problem (2.30); note the transposition of matrix $\mathbb{Z}$. The dual variables π_1 and π_2 are state prices, i.e., the prices of a contingent claim paying 1 if the corresponding state occurs, 0 otherwise. We can find them by replication. The second MATLAB line solve the replicating equations

$$B(T, \omega_1) \cdot h_1 + S(T, \omega_1) \cdot h_2 = 1$$
$$B(T, \omega_2) \cdot h_1 + S(T, \omega_2) \cdot h_2 = 0,$$

which yield a replicating portfolio with holdings

$$h_1 = -0.1333, \quad h_2 = 0.2$$

and initial value

$$B(0) \cdot h_1 + S(0) \cdot h_2 = 1.667,$$

which is the state price π_1 (by the law of one price). The initial value of the portfolio for state ω_2 is negative, and in fact there is an arbitrage opportunity.

If we forbid shortselling, we cannot take advantage of the arbitrage opportunity, since we cannot shortsell the risky asset. In fact, if we add non-negativity constraints on the decision variables of the LP, then this is not unbounded anymore (and has zero value):

```
>> h = linprog(V', -ZZ, zeros(2,1),[],[],zeros(2,1))
Optimal solution found.
h =
     0
     0
```

With proportional transaction costs, when we shortsell, we obtain a smaller amount of cash,

$$45 \times 0.95 = 42.75,$$

which, invested at the risk-free rate, yields only

$$42.75 \times \frac{60}{55} = 46.64.$$

In state ω_2, to close the short position, we need to buy the risky asset for the following cash price:

$$45.05 = 47.75,$$

which is larger than what we have. Transaction costs preclude the execution of the arbitrage strategy.

3

Elementary Theory of Interest Rates

3.1 SOLUTIONS

Problem 3.1 In the problem statement, the choice of compounding is not specified. For illustration purposes, we use annual compounding in the solution. Continuous compounding would simplify some calculations, as we have seen in the book chapter.

Let us find the term structure in terms of annually compounded rates, by inverting the bond pricing formula for a zero,

$$Z(0,T) = \frac{F}{\big[1 + r_1(0,T)\big]^T} \qquad \Rightarrow \qquad r_1(0,T) = \left[\frac{F}{Z(0,T)}\right]^{1/T} - 1,$$

which gives

$$\begin{aligned}
r_1(0,1) &= \left[\frac{1000}{947.87}\right] - 1 = 5.5\%, \\
r_1(0,2) &= \left[\frac{1000}{885.81}\right]^{1/2} - 1 = 6.25\%, \\
r_1(0,3) &= \left[\frac{1000}{815.15}\right]^{1/3} - 1 = 7.05\%, \\
r_1(0,4) &= \left[\frac{1000}{757.22}\right]^{1/4} - 1 = 7.2\%.
\end{aligned}$$

The forward rates are found from the no-arbitrage relationship

$$\big[1 + r_1(0,T)\big]^T = \big[1 + r_1(0,T-1)\big]^{T-1} \cdot \, \big[1 + f_1(0,T-1,T)\big].$$

Inverting the above condition, we find:

$$\begin{aligned}
f_1(0,0,1) &\equiv r_1(0,1) = 5.5\%, \\
f_1(0,1,2) &= \frac{\left[1 + r_1(0,2)\right]^2}{\left[1 + r_1(0,1)\right]} - 1 = 7.01\%, \\
f_1(0,2,3) &= \frac{\left[1 + r_1(0,3)\right]^3}{\left[1 + r_1(0,2)\right]^2} - 1 = 8.67\%, \\
f_1(0,3,4) &= \frac{\left[1 + r_1(0,4)\right]^4}{\left[1 + r_1(0,3)\right]^3} - 1 = 7.65\%.
\end{aligned}$$

The Macauley duration of a zero maturing in two years is two years. If we use annual compounding again, we have the approximate relationship

$$\frac{\delta P}{P} \approx -\frac{1}{1 + y_1} \cdot D_{\mathsf{mac}} \cdot \delta y_1.$$

In our case, the current value of the portfolio is $\Pi = N \cdot Z(0,2)$, where N is the number of bonds, and loss is

$$L = -\delta\Pi \approx \Pi \cdot \frac{D_{\mathsf{mac}}}{1 + y_1} \cdot \delta y_1.$$

Since $\text{V@R}_{0.95}$ is the quantile of loss at probability level 0.95, we have

$$\text{V@R}_{0.95} = z_{0.95} \cdot \sigma_L,$$

where

$$\sigma_L \approx \Pi \cdot \frac{D_{\mathsf{mac}}}{1 + y_1} \cdot \sigma_{y_1} = 100{,}000 \times \frac{2}{1 + 0.0625} \times 0.01 \approx 1882.35.$$

Hence,

$$\text{V@R}_{0.95} = 1.6449 \cdot 1882.35 = 3096.28.$$

In this specific case, we might also find the "worst" yield with confidence level 0.95,

$$y^* = 0.0625 + 1.6449 \times 0.01 = 7.89\%,$$

and reprice the bond

$$P_z^* = \frac{1000}{(1 + 0.0789)^2} = 859.0879.$$

Therefore, the exact value-at-risk is just the number of bonds times the loss on each bond:

$$\frac{100{,}000}{885.81} \times \big(885.81 - 859.0879\big) = 3016.69,$$

which is smaller than the first-order approximation due to the convexity effect. Clearly, this exact approach applies to this very simple case, but not when multiple assets and risk factors are involved.

Problem 3.2 Using continuously compounded spot rates, the bond price would be

$$P_c = 50 \cdot e^{-r(0,1)\cdot 1} + 50 \cdot e^{-r(0,2)\cdot 2} + 1050 \cdot e^{-r(0,3)\cdot 3},$$

but, since we use continuous compounding, we may express the spot rates as averages of the forward rates:

$$\begin{aligned} r(0,1) &= f(0,0,1), \\ r(0,2) &= \frac{f(0,0,1) + f(0,1,2)}{2}, \\ r(0,3) &= \frac{f(0,0,1) + f(0,1,2) + f(0,2,3)}{3}. \end{aligned}$$

Thus, we may just use the forward rates directly and find

$$P_c = 50 \cdot e^{-0.037} + 50 \cdot e^{-(0.037+0.045)} + 1050 \cdot e^{-(0.037+0.045+0.051)} = \$1013.49.$$

What we actually do is very simple to interpret: We successively discount from year t to year $t-1$ by the forward rate $f(0, t-1, t)$.

Problem 3.3 We should solve the equation

$$102 = \frac{6}{1+y_1} + \frac{106}{(1+y_1)^2},$$

which we rewrite as the polynomial equation

$$106z^2 + 6z - 102 = 0,$$

by the variable substitution

$$z = \frac{1}{1+y_1}.$$

We find the two roots

$$\frac{-3 \pm \sqrt{3^2 + 106 \times 102}}{106} = \begin{cases} -1.0097 \\ 0.9531 \end{cases}.$$

The second root yields the positive solution

$$y_1 = \frac{1}{0.9531} - 1 = 4.93\%.$$

Problem 3.4 A portfolio consisting of the risky bond and the insurance is equivalent to a risk-free bond, whose price is

$$P_c = \frac{90}{0.07} + \frac{90}{0.07^2} + \frac{1090}{0.07^3} = €1052.49.$$

If we buy the risky bond and pay for the insurance, the total cash outflow is is

$$960 + 200 = €1160 > 1052.49.$$

Hence, the insurance is too expensive. If we hold the corporate bond and wonder about buying the insurance, we would be better off by selling the risky bond and buying a risk-free one, paying only the difference, $1052.49 - 960 = €92.49$.

Problem 3.5 The expected value of the portfolio is

$$\mathrm{E}\big[\Pi(\omega)\big] = \mathrm{E}\big[10P_z(\omega) + 20P_c(\omega)\big] = 10 \cdot \mathrm{E}\big[P_z(\omega)\big] + 20 \cdot \mathrm{E}\big[P_c(\omega)\big],$$

where $P_z(\omega)$ and $P_c(\omega)$ are the random prices of the zero and the coupon-bearing bond, respectively. The expected price of the zero is

$$\mathrm{E}\big[P_z(\omega)\big] = 0.2 \times \frac{1000}{1.043^3} + 0.5 \times \frac{1000}{1.035^3} + 0.3 \times \frac{1000}{1.028^3} = €903.3888.$$

The expected price of the coupon-bearing bond is

$$\mathrm{E}\big[P_c(\omega)\big] = 0.2 \times \left(\frac{40}{1.031} + \frac{1040}{1.038^2}\right) + 0.5 \times \left(\frac{40}{1.032} + \frac{1040}{1.033^2}\right) + 0.3 \times \left(\frac{40}{1.030} + \frac{1040}{1.029^2}\right)$$
$$= €1013.8081.$$

Hence, the expected value of the portfolio is

$$\mathrm{E}\big[\Pi(\omega)\big] = 10 \times 903.3888 + 20 \times 1013.8081 = \$29{,}310.05.$$

The fundamental message here is that we need the expected value of nonlinear functions of the interest rates, which is *not* the function of the expected value. It is a *wrong* procedure to find the expected rates and then price the bonds.

Furthermore, we are considering instantanoues changes in the underlying risk factors. We might wish to account for the passage of time as well.

Problem 3.6 As first step, we need the (annually compounded) spot rates:

$$r_1(0,1) = f_1(0,0,1) = 3\%$$
$$r_1(0,2) = \sqrt{1.03 \times 1.04} - 1 = 3.4988\%$$
$$r_1(0,3) = (1.03 \times 1.04 \times 1.05)^{1/3} - 1 = 3.9968\%.$$

In passing, we notice that the spot rates are quite close to the arithmetic averages of the forward rates (3.5% and 4%, respectively), but differ a bit because we are assuming annual, rather than continuous compounding.

Let us find the two current bond prices, given the current spread:

$$P_1(2.3\%) = \frac{1000}{(1 + 0.039968 + 0.023)^3} = €832.6059$$
$$P_2(2.3\%) = \frac{40}{1 + 0.03 + 0.023} + \frac{1040}{(1 + 0.034988 + 0.023)^2} = €967.1069.$$

Since we have a single risk factor, to find the exact value-at-risk, we may consider the "worst" spread s^* with confidence level 99%. The distribution of the additive shock on the spread is uniform, so

$$s^* = 0.023 + \big[-0.01 + 0.99 \cdot [0.02 - (-0.01)]\big] = 0.0427.$$

The new bond prices are (neglecting the passage of time):

$$P_1(4.27\%) = \frac{1000}{(1 + 0.039968 + 0.0427)^3} = €787.9781$$
$$P_2(4.27\%) = \frac{40}{1 + 0.03 + 0.0427} + \frac{1040}{(1 + 0.034988 + 0.0427)^2} = €932.7514.$$

Hence, the worst case loss with confidence level 99% is

$$\mathrm{V@R}_{0.99} = 53{,}000 \times \frac{832.6059 - 787.9781}{832.6059} + 93{,}000 \times \frac{967.1069 - 932.7514}{967.1069} = €6144.54.$$

Problem 3.7 A floater featuring a spread δ pays a coupon

$$C_i = F \cdot \frac{r_2(T_{i-1}, T_i) + \delta}{2}$$

at time T_i. By the linearity of pricing, we may decompose this bond into the sum of a floater without a spread and an annuity paying $F \cdot \delta/2$ every six months:

$$P_f(t, T; \delta) = P_f(t, T) + \frac{\delta \cdot F}{2} \sum_{i=1}^{m} Z(t, T_i),$$

where $P_f(t, T)$ is the price of a floater with no spread and the bond matures at time $T = T_m$.

Problem 3.8 A reverse floater pays a coupon

$$C_i = \max\left\{0,\ F \cdot \frac{S - r_2(T_{i-1}, T_i)}{2}\right\}$$

at time T_i, where S is given reference rate. If S is not large enough, we cannot rule out the possibility that $S - r_2(T_{i-1}, T_i) < 0$. If we neglect this, using the decomposition trick of Problem 3.7, we may decompose the reverse floater into the difference of bonds. In this case, however, we must pay attention to the nominal values. When we subtract the price $P_f(t, T)$ of the floater (with no spread), we are implicitly subtracting the nominal value. Thus, we should subtract $P_f(t, T)$ from the price $P_c(t, T; S)$ of a coupon bearing bond with coupon rate S, which cancels the two nominals; then, we must add the price of a zero maturing at $T = T_m$, with face value F:

$$P_{\mathsf{revf}}(t, T; S) = FZ(t, T) + P_c(t, T : S) - P_f(t, T).$$

The nonlinear coupon rate introduces an optionality component, which requires the pricing machinery of interest rate derivatives.

4

Forward Rate Agreements, Interest Rate Futures, and Vanilla Swaps

4.1 SOLUTIONS

Problem 4.1 To price the bond, we need the rates for 6 and 12 months. We already know $r(0, 0.5) = 0.047$, with continuous compounding. To find $r(0, 1)$, it is convenient to work with continuous compounding, which requires transforming the forward rates implied by the futures quotes to continuously compounded rates (keep in mind that we are neglecting the difference between forward and futures rates). The quotes give

$$f_4(0, 6/12, 9/12) = (100 - 94.9)/100 = 0.051,$$
$$f_4(0, 9/12, 1) = (100 - 94.5)/100 = 0.055.$$

Note that the futures contract maturing in six months provides us with the rate applying to a quarter starting in six months. Hence,

$$f(0, 6/12, 9/12) = 4 \log\big(1 + 0.051/4\big) = 5.068\%$$
$$f(0, 9/12, 1) = 4 \log\big(1 + 0.055/4\big) = 5.463\%.$$

To find the spot rate for a maturity of 1 year, we may directly consider the growth factor

$$e^{r(0,1)\cdot 1} = e^{r(0,0.5)\cdot 0.5} \cdot e^{f(0,0.5,0.75)\cdot 0.25} \cdot e^{f(0,0.75,1)\cdot 0.25},$$

which implies

$$r(0, 1) = \frac{0.047}{2} + \frac{0.05068}{4} + \frac{0.5463}{4} = 4.983\%.$$

Hence, the bond price is

$$P_c = 3 \times e^{-0.047\times 0.5} + 103 \times e^{-0.04983\times 1} = 100.92.$$

Problem 4.2 Since default occurs at month 28, the floating rate was reset four months ago. There are four payments left, at months 30, 36, 42, and 48 with respect to the start date. With respect to the date of default, the first payment will occur in two months, and we may price the fixed-rate bond as follows:

$$P_{\mathsf{fixed}} = 0.4e^{-0.05\times 2/12} + 0.4e^{-0.05\times 8/12} + 0.4e^{-0.05\times 14/12} + 20.4e^{-0.05\times 20/12} = 19.9298,$$

where we measure money in \$ millions (2% of 20 is 0.4, i.e., \$400,000). The next payment on the floating leg will be \$700,000; hence, the value of the floating-rate leg is

$$P_{\mathsf{float}} = e^{-0.05\times 2/12} \cdot \big(0.7 + 20\big) = 20.5282.$$

From the bank's viewpoint, which receives floating rate, the value of the swap is

$$P_{\mathsf{float}} - P_{\mathsf{fixed}} = 20.5282 - 19.9298 = 0.5984,$$

corresponding to a loss of \$598,400.

We may also use decomposition into FRAs. The net cash flow for the FRA corresponding to the next payment, in two months, is

$$\mathsf{CF}_2 = 20 \times \big(0.035 - 0.02\big) = 0.3,$$

in \$ millions. Since the term structure is flat, the forward rates are just 5% with continuous compounding, corresponding to

$$2 \times \left(e^{0.05/2} - 1\right) = 5.063\%$$

with semiannual compounding. Hence, the net cash flows are "forecasted" as

$$\mathsf{CF}_8 = \mathsf{CF}_{14} = \mathsf{CF}_{20} = 20 \times \big(0.025315 - 0.02\big) = 0.1063.$$

By discounting net cash flows, we find

$$0.3 \times e^{-0.05\times 2/12} + 0.1063 \times \left(e^{-0.05\times 8/12} + e^{-0.05\times 14/12} + e^{-0.05\times 20/12}\right) = 0.5984,$$

as before.

5

Fixed-Income Markets

5.1 SOLUTIONS

Problem 5.1 We notice that the price of the callable bond, despite the large coupon rate with respect to prevailing rates, is below par, reflecting the value of the call option. To find the value of the call option, we need the price of the corresponding noncallable bond, which in turn requires an estimate of the risk-free rate $r(0,2)$ (the other rates are given). We consider 3.5% as the swap rate for a swap spanning two years (this is the average of bid and ask rates for that maturity; in the problem text, I have used "offer" rather than "ask" by mistake).

Hence, assuming a notional of 100, the fixed payment for the swap is 3.5/2 every six months. We find $r(0,2)$ by solving the following equation:

$$\frac{3.5}{2}e^{-0.023\times 0.5}+\frac{3.5}{2}e^{-0.028\times 1}+\frac{3.5}{2}e^{-0.032\times 1.5}+\left(100+\frac{3.5}{2}\right)e^{-r(0,2)\times 2}=100,$$

which gives $r(0,2)=3.4846\%$. The price of the noncallable bond is, assuming $F=100$,

$$P=5e^{-0.023\times 0.5}+5e^{-0.028\times 1}+5e^{-0.032\times 1.5}+105e^{-0.034846\times 2}=112.5020.$$

Hence, the value of the call is

$$112.5020-97.12=€15.3820.$$

Problem 5.2 When an investor buys a puttable bond, she is buying a bundle consisting of the put option and the plain bond. Unlike the case of the callable bond, the puttable bond is more expensive than the plain one, as the investor is buying the put option (rather than writing the call option embedded in a callable bond). Hence, the value of the put is

$$V_{\mathsf{put}}=P_{\mathsf{putable}}-P_{\mathsf{plain}}.$$

6

Interest Rate Risk Management

6.1 SOLUTIONS

Problem 6.1 The present value of the liability, at the current level of yield, is

$$L(6\%) = \frac{10{,}000}{1.06^5} = \$7472.58,$$

and the bond price, assuming nominal value $F = \$10{,}000$, is

$$P(6\%) = \frac{700}{0.06} \cdot \left(1 - \frac{1}{1.06^6}\right) + \frac{10{,}000}{1.06^6} = \$10{,}491.73.$$

The value of equity is

$$E = N \times P - L,$$

which is zero if we hold

$$N = \frac{7472.58}{10{,}491.73} = 0.7122$$

bonds (we assume asset divisibility).

We may evaluate equity under a different yield. If yield goes up by 100 basis points, we have

$$L(7\%) = \frac{10{,}000}{1.07^5} = \$7129.86,$$

$$P(7\%) = \frac{700}{0.07} \cdot \left(1 - \frac{1}{1.07^6}\right) + \frac{10{,}000}{1.07^6} = \$10{,}000.00,$$

$$E = 0.7122 \times 10{,}000.00 - 7129.86 = \$(-7.5092).$$

If yield goes down by 100 basis points, we have

$$L(5\%) = \frac{10{,}000}{1.05^5} = \$7835.26,$$

$$P(5\%) = \frac{700}{0.05} \cdot \left(1 - \frac{1}{1.05^6}\right) + \frac{10{,}000}{1.05^6} = \$11{,}015.14,$$

$$E = 0.7122 \times 11{,}015.14 - 7835.26 = \$10.1083.$$

The hedge is not working too bad, even though it is not quite perfect. To see why, let us calculate the duration of the bond. If we use the definition,

$$D = \frac{1}{10{,}491.73} \cdot \left(\frac{700}{1.06} + \frac{2 \cdot 700}{1.06^2} + \frac{3 \cdot 700}{1.06^3} + \frac{4 \cdot 700}{1.06^4} + \frac{5 \cdot 700}{1.06^5} + \frac{6 \cdot 10{,}700}{1.06^6} \right) = 5.1242.$$

We may use the analytical formula

$$D_{\mathsf{mac}} = 1 + \frac{1}{0.06} + \frac{6 \cdot (0.06 - 0.07) - (1 + 0.06)}{0.07 \cdot \left[(1 + 0.06)^6 - 1\right] + 0.06},$$

which gives the same result. Thus, the bond duration does not match the duration of the liability exactly, but it gets quite close, which explains the rather good performance.

We might also assess performance by simulating cash flows over time, but to do so we need a future interest rate scenario. If we assume, as we did, an immediate jump in yield without any further change, we would obtain an equivalent result. The generation of stochastic interest rate scenarios would require a dynamic model and a Monte Carlo simulation. This would also allow us to deal with a proper term structure, but requires modeling assumptions and may be computationally demanding.

Problem 6.2 The prices of the two bonds are, respectively,

$$P_1 = \frac{1000}{1.04^7} = 759.92,$$
$$P_2 = \frac{50}{1.04} + \frac{50}{1.04^2} + \frac{1050}{1.04^3} = 1027.75,$$

and the Macauley durations are

$$D_{\mathsf{mac},1} = 7,$$
$$D_{\mathsf{mac},2} = \frac{1}{1027.75} \cdot \left(1 \cdot \frac{50}{1.04} + 2 \cdot \frac{50}{1.04^2} + 3 \cdot \frac{1050}{1.04^3} \right) = 2.8615.$$

Let us find the weights of the two bonds in the portfolio, matching the duration of the liability:

$$w_1 + w_2 = 1,$$
$$7w_1 + 2.8615w_2 = 5,$$

which yields

$$w_1 = 0.5167, \qquad w_2 = 0.4833.$$

Since the present value of the liability is

$$L = \frac{20{,}000}{1.04^5} = €16{,}438.54,$$

we should buy

$$N_1 = \frac{0.5167 \times 16{,}438.54}{759.92} = 11.18, \qquad N_2 = \frac{0.4833 \times 16{,}438.54}{1027.75} = 7.73$$

bonds of each type. Clearly, some rounding is needed, and the hedge will have to be rebalanced after the coupon-bearing matures (actually, earlier than that, since rates are going to change).

The duration of a coupon-bearing bond maturing in six years would be

$$D_{\mathsf{mac},3} = \frac{1 \cdot \frac{50}{1.04} + 2 \cdot \frac{50}{1.04^2} + 3 \cdot \frac{50}{1.04^3} + 4 \cdot \frac{50}{1.04^4} + 5 \cdot \frac{50}{1.04^5} + 6 \cdot \frac{1050}{1.04^6}}{\frac{50}{1.04} + \frac{50}{1.04^2} + \frac{50}{1.04^3} + \frac{50}{1.04^4} + \frac{50}{1.04^5} + \frac{1050}{1.04^6}} = 5.3489,$$

and the corresponding system,

$$\begin{aligned} w_1 + w_3 &= 1, \\ 7w_1 + 5.3489w_3 &= 5, \end{aligned}$$

would yield

$$w_1 = -0.2113, \qquad w_3 = 1.2113.$$

Since these two bond durations do not bracket the target duration, we should sell a bond short, which may be expensive and not easy to accomplish over a long time span.

Problem 6.3 The prices of the two bonds are, respectively,

$$\begin{aligned} P_1(3.5\%) &= \frac{1000}{1.035^8} = 759.41, \\ P_2(3.5\%) &= \frac{40}{1.035} + \frac{40}{1.035^2} + \frac{40}{1.035^3} + \frac{1040}{1.035^4} = 1018.37, \end{aligned}$$

where we emphasize the dependence on the interest rate. The two Macauley durations are

$$\begin{aligned} D_{\mathsf{mac},1} &= 8, \\ D_{\mathsf{mac},2} &= \frac{1}{1018.37} \cdot \left(1 \cdot \frac{40}{1.035} + 2 \cdot \frac{40}{1.035^2} + 3 \cdot \frac{40}{1.035^3} + 4 \cdot \frac{1040}{1.035^4}\right) = 3.7774. \end{aligned}$$

Let us find the weights of the two bonds in the portfolio, matching the duration of the liability:

$$\begin{aligned} w_1 + w_2 &= 1, \\ 8w_1 + 3.7774w_2 &= 6, \end{aligned}$$

which yields

$$w_1 = 0.5264, \qquad w_2 = 0.4736.$$

The present value of the liability is

$$L(3.5\%) = \frac{30{,}000}{1.035^6} = €24{,}405.02,$$

and the number of bonds in the immunized portfolio are

$$N_1 = \frac{0.5264 \times 24{,}405.02}{759.41} = 16.92, \qquad N_2 = \frac{0.4736 \times 24{,}405.02}{1018.37} = 11.35.$$

By construction, equity is zero at the current level of yield:

$$E(3.5\%) = N_1 \cdot P_1(3.5\%) + N_2 \cdot P_2(3.5\%) - L(3.5\%) = 0.$$

If there is an upshift of 50 basis points, we find:

$$
\begin{aligned}
P_1(4\%) &= \frac{1000}{1.04^8} = 730.69 \\
P_2(4\%) &= \frac{40}{1.04} + \frac{40}{1.04^2} + \frac{40}{1.04^3} + \frac{1040}{1.04^4} = 1000, \\
L(4\%) &= \frac{30{,}000}{1.04^6} = 23{,}709.44, \\
E(4\%) &= N_1 \cdot P_1(4\%) + N_2 \cdot P_2(4\%) - L(4\%) = €1.29.
\end{aligned}
$$

The hedge worked well, and equity is slightly increased because of a convexity effect.

If there is a downshift of 50 basis points, we find:

$$
\begin{aligned}
P_1(3\%) &= \frac{1000}{1.03^8} = 789.41 \\
P_2(3\%) &= \frac{40}{1.03} + \frac{40}{1.03^2} + \frac{40}{1.03^3} + \frac{1040}{1.03^4} = 1037.17, \\
L(3\%) &= \frac{30{,}000}{1.03^6} = 25{,}124.53, \\
E(3\%) &= N_1 \cdot P_1(3\%) + N_2 \cdot P_2(3\%) - L(3\%) = €1.37.
\end{aligned}
$$

NOTE ABOUT CONVEXITY. In this problem, since we are using discretely compounded rates, we cannot use the formulas for convexity that are given in the book, as they apply to continuous compounding. Derivatives of exponential are nice, but here we have to use the following reasoning:

$$
\begin{aligned}
P(y_1) &= \sum_t \frac{C_t}{(1+y_1)^t} \\
P'(y_1) &= -\sum_t \frac{tC_t}{(1+y_1)^{t+1}} \\
P''(y_1) &= \sum_t \frac{t(t+1)C_t}{(1+y_1)^{t+2}} = \frac{1}{(1+y_1)^2} \sum_t \frac{t(t+1)C_t}{(1+y_1)^t},
\end{aligned}
$$

which leads to the following expression for bond convexity:

$$
C = \frac{1}{P(y_1) \cdot (1+y_1)^2} \cdot \sum_t \frac{t(t+1)C_t}{(1+y_1)^t}
$$

In this case, the convexity of a zero is

$$
\frac{T(T+1)}{(1+y_1)^2},
$$

rather than just T^2, as is the case with continuous compounding. Here, we avoid introducing different definitions in the vein of Macauley or modified duration. Also note that when matching duration, we may use either duration, as this just implies multiplying both sides of the duration matching equation by the same factor. It is also possible to verify that the convexity of a linear combination of bonds is the corresponding linear combination of convexities (if we apply the same yield to all bonds).

Let us introduce the third bond, with the following price, duration, and convexity:

$$
\begin{aligned}
P_3(3.5\%) &= \frac{1000}{1.035^3} = 901.94,\\
D_{\mathsf{mac},3} &= 3,\\
C_3 &= \frac{3\times 4}{1.035^2} = 11.2021.
\end{aligned}
$$

The convexities of the first two bonds are:

$$
\begin{aligned}
C_1 &= \frac{8\times 9}{1.035^2} = 67.2128\\
C_2 &= \frac{1}{1018.37\times 1.035^2}\cdot\left(1\cdot 2\cdot\frac{40}{1.035} + 2\cdot 3\cdot\frac{40}{1.035^2} + 3\cdot 4\cdot\frac{40}{1.035^3} + 4\cdot 5\cdot\frac{1040}{1.035^4}\right)\\
&= 17.2887.
\end{aligned}
$$

To match both duration and convexity, we solve the linear system[1]

$$
\begin{aligned}
&w_1 + w_2 + w_3 = 1,\\
&8w_1 + 3.7774w_2 + 3w_3 = 6,\\
&67.2128w_1 + 17.2887w_2 + 11.2021w_3 = \frac{6\cdot 7}{1.035^2} = 39.2074,
\end{aligned}
$$

which gives

$$
w_1 = 0.2678,\qquad w_2 = 2.1364,\qquad w_3 = -1.4042,
$$

and

$$
\begin{aligned}
N_1 &= \frac{0.2678\times 24{,}405.02}{759.41} = 8.61,\\
N_2 &= \frac{2.1364\times 24{,}405.02}{1018.37} = 51.20,\\
N_3 &= \frac{-1.4042\times 24{,}405.02}{901.94} = -38,
\end{aligned}
$$

where we should note the short position in the third bond.

In the first scenario, the price of the zero maturing in three years is 889.00, and equity is -0.0034. In the second scenario, the price of the zero maturing in three years is 915.14, and equity is 0.0036. Indeed, we observe a more stable equity, even though the advantage is questionable in this specific case, especially given the need for a short position in a bond (which could be synthesized by interest rate derivatives, though).

Problem 6.4 **Note:** This is the same as Problem 3.6 and was included by mistake. The only difference is the level of confidence in value-at-risk, which here is 97%, rather than 99%. We repeat the solution here, for the sake of convenience.

[1] In the convexity matching equation, we could multiply everything by 1.035^2 to be somewhat more consistent with the duration matching equation. This is inconsequential.

As a first step, we need the (annually compounded) spot rates:

$$r_1(0,1) = f_1(0,0,1) = 3\%$$
$$r_1(0,2) = \sqrt{1.03 \times 1.04} - 1 = 3.4988\%$$
$$r_1(0,3) = (1.03 \times 1.04 \times 1.05)^{1/3} - 1 = 3.9968\%.$$

The two current bond prices, given the current spread, are

$$P_1(2.3\%) = \frac{1000}{(1 + 0.039968 + 0.023)^3} = €832.6059$$
$$P_2(2.3\%) = \frac{40}{1 + 0.03 + 0.023} + \frac{1040}{(1 + 0.034988 + 0.023)^2} = €967.1069.$$

Since we have a single risk factor, to find the exact value-at-risk, we may consider The distribution of the additive shock on the spread is uniform, so the "worst" spread s^* with confidence level 97% is

$$s^* = 0.023 + \big[-0.01 + 0.97 \cdot [0.02 - (-0.01)]\big] = 0.0421.$$

The new bond prices are (neglecting the passage of time):

$$P_1(4.21\%) = \frac{1000}{(1 + 0.039968 + 0.0421)^3} = €789.2896$$
$$P_2(4.21\%) = \frac{40}{1 + 0.03 + 0.0421} + \frac{1040}{(1 + 0.034988 + 0.0421)^2} = €933.7702.$$

Hence, the worst case loss with confidence level 97% is

$$\text{V@R}_{0.97} = 53{,}000 \times \frac{832.6059 - 789.2896}{832.6059} + 93{,}000 \times \frac{967.1069 - 933.7702}{967.1069} = €5963.09.$$

Problem 6.5 This problem requires quite a bit of calculations, and it is best solved using a tool like R, Excel, or MATLAB. Here we use MATLAB, but any other tool will do.

As a first step we find the features of the bond, whose interest rate risk we want to hedge.

```
% Coupon-bearing bond, maturing in 5 years, semiannual coupons
F=10000;
c=0.05;
r = 0.03;     % flat rate 3%
cf = [repmat(c/2,9,1); 1+c/2]*F; % cash flows from bond
times = 0.5*(1:10)';             % cash flow times
df = exp(-times*r);              % discount factors
P = dot(cf,df);              % bond price
Pdur = sum(times.*df.*cf)/P; % bond duration
Pdolldur = Pdur*P;           % dollar duration
Pfolio = P   % current value of unhedged portfolio
```

We find:

- Bond price $P_0 = 10{,}911.25$, the current value of the unhedged portfolio
- Bond duration $D = 4.5118$

- Bond dollar duration $D^{\$} = 49{,}229.34$

Then we find the features of the hedging instrument, a zero maturing in six months:

```
% zero maturing in six months, our hedging instrument
Fzero = 10000;
Pzero = Fzero*df(1);
Hdur = 0.5;
Hdolldur = Hdur*Pzero;
phi = - Pdolldur/Hdolldur;
```

- Zero price $Z_0 = 9851.12$
- Zero duration $D_z = 0.5$
- Zero dollar duration $D_z^{\$} = 4925.56$

The hedge requires a position

$$\phi = -\frac{D^{\$}}{D_z^{\$}} = -9.9947$$

in the hedging instrument.

Let us apply the parallel shift, from 3% to 4%:

```
% What if we have a parallel shift of 100 basis points ?
rnew = r + 0.01;
dfnew = exp(-times*rnew);
Pnew = dot(cf,dfnew);
Pzeronew = Fzero*dfnew(1);
PfolioNew = Pnew + phi * (Pzeronew - Pzero)
retPar = (PfolioNew-Pfolio)/Pfolio*100
```

The new value of the hedged portfolio is

$$10{,}430.59 - 9.9947 \times (9801.99 - 9851.12) = 10{,}921.65,$$

and the return is

$$\frac{10{,}921.65 - 10{,}911.25}{10{,}911.25} = 0.0954\%.$$

We have a small gain, and the hedged worked well.

Now, let us apply the nonparallel shift:

```
% Now a nonparallel shift
rnew = r + linspace(0,0.02,10)';
dfnew = exp(-times.*rnew);
Pnew = dot(cf,dfnew);
Pzeronew = Fzero*dfnew(1);
PfolioNew = Pnew + phi * (Pzeronew - Pzero)
retNONPar = (PfolioNew-Pfolio)/Pfolio*100
```

The new value of the hedged portfolio is

$$10{,}013.43 - 9.9947 \times (9851.12 - 9851.12) = 10{,}013.43,$$

and the return is

$$\frac{10{,}013.43 - 10{,}911.25}{10{,}911.25} = -8.2283\%.$$

We have a large loss, which is not quite surprising: The relevant rate for the zero did not change at all, so the hedging instrument is ineffective.

Let us repeat, using a zero maturing in three years as the hedging instrument:

```
% CHANGE THE ZERO: maturity 3 years
Fzero = 10000;
Pzero = Fzero*df(6);
Hdur = 3;
Hdolldur = Hdur*Pzero;
phi = - Pdolldur/Hdolldur;
rnew = r + linspace(0,0.02,10)';
dfnew = exp(-times.*rnew);
Pnew = dot(cf,dfnew);
Pzeronew = Fzero*dfnew(6);
PfolioNew = Pnew + phi * (Pzeronew - Pzero)
retNONPar = (PfolioNew-Pfolio)/Pfolio*100
```

The new value of the hedged portfolio (note the different hedging ratio) is

$$10{,}013.43 - 1.7955 \times (8839.69 - 9139.31) = 10{,}551.41,$$

and the return is

$$\frac{10{,}551.41 - 10{,}911.25}{10{,}911.25} = -3.2979\%.$$

We have a loss again, since we hedge a single risk factor, but this second hedge is definitely more effective than the previous one.

Let us repeat, using a zero maturing in five years as the hedging instrument:

```
% CHANGE THE ZERO: maturity 5 years
Fzero = 10000;
Pzero = Fzero*df(10);
Hdur = 5;
Hdolldur = Hdur*Pzero;
phi = - Pdolldur/Hdolldur;
rnew = r + linspace(0,0.02,10)';
dfnew = exp(-times.*rnew);
Pnew = dot(cf,dfnew);
Pzeronew = Fzero*dfnew(10);
PfolioNew = Pnew + phi * (Pzeronew - Pzero)
retNONPar = (PfolioNew-Pfolio)/Pfolio*100
```

The new value of the hedged portfolio (note the different hedging ratio) is

$$10{,}013.43 - 1.1439 \times (7788.01 - 8607.08) = 10{,}950.39,$$

and the return is

$$\frac{10{,}950.39 - 10{,}911.25}{10{,}911.25} = 0.3588\%.$$

Given the increased duration of the zero, now its drop in price is large enough to compensate the loss on the coupon-bearing bond.

7

Decision-Making under Uncertainty: The Static Case

7.1 SOLUTIONS

Problem 7.1 Let us denote the risky (multiplicative) gain by $\widetilde{R}$, the risk-free gain by $R_f = 1 + r_f$, where r_f is the holding period riskless return, and write the future wealth as a function of q:

$$\widetilde{W}_T = q \cdot \widetilde{R} + \big(W_0 - q\big) \cdot R_f = q \cdot (\widetilde{R} - R_f) + W_0 R_f.$$

Note that we are using multiplicative gains to avoid terms like $(1 + \cdots)$, but we could use returns as well. If the utility function is $u(x) = -e^{-\alpha x}$, the expected utility is

$$\begin{aligned}
\mathrm{E}\big[u\big(\widetilde{W}_T\big)\big] &= -\pi_u e^{-\alpha \cdot \left[q \cdot (R_u - R_f) + W_0 R_f\right]} - \pi_d e^{-\alpha \cdot \left[q \cdot (R_d - R_f) + W_0 R_f\right]} \\
&= -e^{-\alpha \cdot W_0 R_f} \cdot \left\{ \pi_u e^{-\alpha \cdot \left[q \cdot (R_u - R_f)\right]} + \pi_d e^{-\alpha \cdot \left[q \cdot (R_d - R_f)\right]} \right\}.
\end{aligned}$$

This is the product of a factor depending on initial wealth W_0, but not q, and a factor depending on q, but not initial wealth W_0. When we apply the first-order optimality condition with respect to q, initial wealth will not play any role in finding the optimal q^*. Thus, our allocation does not depend on initial wealth, which sounds rather weird. It is hard to believe that we have an exponential utility function, even though it is used in the academic literature for the sake of mathematical convenience.

Problem 7.2 If we invest the initial wealth W_0, the terminal wealth is

$$\widetilde{W}_T = \sum_{i=1}^{n} x_i \widetilde{R}_i,$$

where $\widetilde{R}_i$ is the random multiplicative gain (one plus holding period return) for asset i, and x_i is the amount allocated to asset i, subject to the constraint

$$\sum_{i=1}^{n} x_i = W_0.$$

The first investor has $W_0 = 1$ and solves the optimization problem

$$\max \mathrm{E}\left[a\sum_i x_i\widetilde{R}_i - \frac{b}{2}\left(\sum_i x_i\widetilde{R}_i\right)^2\right]$$

subject to the constraint $\sum_i x_i = 1$.

The second investor has $W_0 = K$ and solves the optimization problem

$$\max \mathrm{E}\left[a\sum_i y_i\widetilde{R}_i - \frac{b}{2}\left(\sum_i y_i\widetilde{R}_i\right)^2\right]$$

subject to the constraint $\sum_i y_i = K$, where we use different decision variables to avoid confusion. If we rewrite the constraint for the second investor as

$$\sum_i \frac{y_i}{K} = 1$$

we may substitute variables $x_i = y_i/K$ and rewrite her problem as

$$\max K \cdot \mathrm{E}\left[a\sum_i x_i\widetilde{R}_i - \frac{bK}{2}\left(\sum_i x_i\widetilde{R}_i\right)^2\right].$$

The leading K multiplying the expected value is irrelevant, but we see that the risk aversion coefficient is related to bK, rather than b, implying a different risk aversion for the second investor, which results in a different portfolio. If the second investor had a coefficient or risk aversion

$$b' = \frac{b}{K},$$

she would find the same portfolio as the first investor. This example shows that some caution is needed when we use quadratic utility and take portfolio *weights* as decision variables, rather than allocations expressed as monetary amounts.

Problem 7.3 We have to find an indifference price, i.e., an insurance premium c such that you, the decision maker, are indifferent between insuring and not insuring your property. If you pay for insurance, your wealth $100{,}000 - c$ is certain, otherwise it is random. We should solve the equation

$$\log(100{,}000 - c) = 0.95 \cdot \log 100{,}000 + 0.04 \cdot \log 50{,}000 + 0.01 \cdot \log 1 = 11.3701,$$

which yields

$$c = 100{,}000 - e^{11.3701} = \$13{,}312.$$

This is the maximum price you should be willing to pay.

Problem 7.4 Let σ_{L1}, σ_{L2}, and σ_{L1+L2} denote the standard deviations of loss for the two random variables and their sum, respectively. Given confidence level $1 - \alpha$, we have

$$\begin{aligned}
\mathrm{V@R}_{1-\alpha}(L_1) &= z_{1-\alpha} \cdot \sigma_{L1}\\
\mathrm{V@R}_{1-\alpha}(L_2) &= z_{1-\alpha} \cdot \sigma_{L2}\\
\mathrm{V@R}_{1-\alpha}(L_1 + L_2) &= z_{1-\alpha} \cdot \sigma_{L1+L2}.
\end{aligned}$$

Given the correlation $\rho \leq 1$ between L_1 and L_2, we also know that

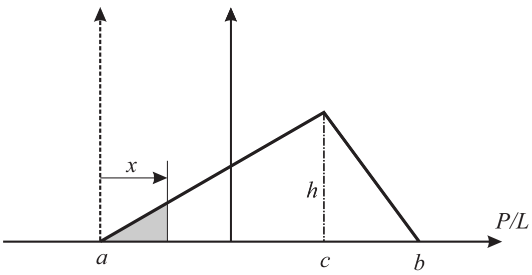
$$\begin{aligned}\sigma_{L1+L2} &= \sqrt{\sigma_{L1}^2 + 2\rho \cdot \sigma_{L1} \cdot \sigma_{L2} + \sigma_{L2}^2} \leq \sqrt{\sigma_{L1}^2 + 2\sigma_{L1} \cdot \sigma_{L2} + \sigma_{L2}^2} \\ &= \sqrt{\left(\sigma_{L1} + \sigma_{L2}\right)^2} = \sigma_{L1} + \sigma_{L2}.\end{aligned}$$

Hence,

$$\mathrm{V@R}_{1-\alpha}(L_1 + L_2) \leq \mathrm{V@R}_{1-\alpha}(L_1) + \mathrm{V@R}_{1-\alpha}(L_2).$$

Problem 7.5 A common mistake is to use expected value and standard deviation of a generic distribution and apply the formula for a normal distribution to find V@R. This is certainly not even recommended as an approximation, when dealing with a skewed distribution like the triangular distribution we consider here.

We should find, on the tail of the PDF corresponding to loss, a triangle with area 0.05. To this aim, we may imagine shifting the origin to the point $a = -75{,}000$ of maximum loss and consider an increasing line (going through the new origin) with slope depending on the height of the triangle (corresponding to the mode c). This is illustrated in the following figure, showing the profit/loss distribution:



- The point corresponding to the maximum loss, $a = -75{,}000$, is the origin of the shifted axis.
- Corresponding to the mode $c = 40{,}000$, we show the height h of the triangular PDF.
- We must find a shift x, such that the shaded area must be 5%.

In order to find the height h, we use the fact that the total area of the PDF is 1, hence:

$$\tfrac{1}{2} \cdot h \cdot \left[b - a\right] = \tfrac{1}{2} \cdot h \cdot \left[55{,}000 - (-75{,}000)\right] = 1 \qquad \Rightarrow \qquad h = \frac{1}{65{,}000}.$$

Now let us imagine an increasing line $y = mx$ departing from the new origin, corresponding to $a = -75{,}000$, reaching height h for the point $c = 40{,}000$. We need the slope m of this line, which joins the new origin and the point $(c - a, h)$. Hence:

$$h = m(c - a) \qquad \Rightarrow \qquad m = \frac{h}{40{,}000 - (-75{,}000)} = \frac{1}{115{,}000 \times 65{,}000}.$$

The area of the shaded triangle corresponding to x is

$$\tfrac{1}{2} m x^2,$$

and we need x such that this area is 5%:

$$\tfrac{1}{2} m x^2 = 0.05 \qquad \Rightarrow \qquad x = \sqrt{\frac{2 \times 0.05}{m}} = \sqrt{\frac{2 \times 0.05}{115{,}000 \times 65{,}000}} = 27{,}340.45.$$

This amount should be transformed back to the original coordinates (which means that, in terms of loss, it should be subtracted from the maximum loss), and the value-at-risk is:

$$\mathrm{V@R}_{0.95} = 75{,}000 - 27{,}340.45 = €47{,}659.55.$$

What happens if, on the contrary, we use the formula for a normal distribution? To deal with loss, we flip the PDF, so that now $a = -55$, $b = 75$, and $c = -40$ (where we express monetary amounts in thousands of dollars, for the sake of convenience). Using the formulae for the triangular distribution, we find

$$\mu_L = \frac{-55 - 40 + 75}{3} = -6.6667$$

and

$$\sigma_L = \sqrt{\frac{75^2 + (-40)^2 + (-55)^2 + 75 \times 40 + 75 \times 55 - 55 \times 40}{18}} = 29.0354,$$

so that value-at-risk with confidence level 95% is

$$1000 \cdot \big(-6.6667 + 1.6449 \times 29.0354\big) = €41{,}092.35,$$

with an underestimation error, since we neglect the negative skew of the profit/loss distribution.

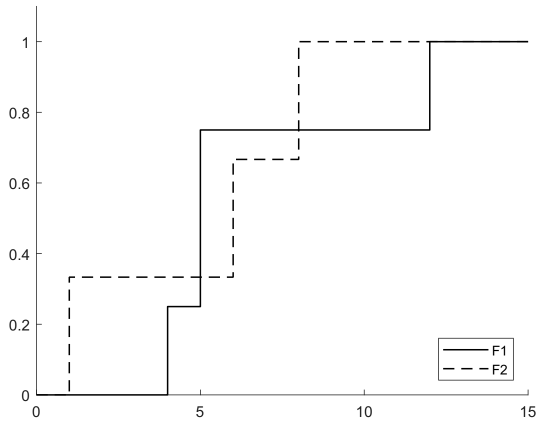
Problem 7.6 The CDF for the first random variable is the piecewise constant function

$$F_1(x) = \begin{cases} 0 & \text{if } x < 4 \\ 0.25 & \text{if } 4 \le x < 5 \\ 0.75 & \text{if } 5 \le x < 12 \\ 1 & \text{if } x \ge 12, \end{cases}$$

with jumps corresponding to points where the probability mass is located. For the second random variable, we have

$$F_1(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{3} & \text{if } 1 \le x < 6 \\ \frac{2}{3} & \text{if } 6 \le x < 8 \\ 1 & \text{if } x \ge 8. \end{cases}$$

[In the book, I wrote probabilities as 0.33, so that they do not add up to 1, but they are meant to be 1/3.] The figure below shows that there is no first-order stochastic dominance, as the plots of the two CDFs cross each other.

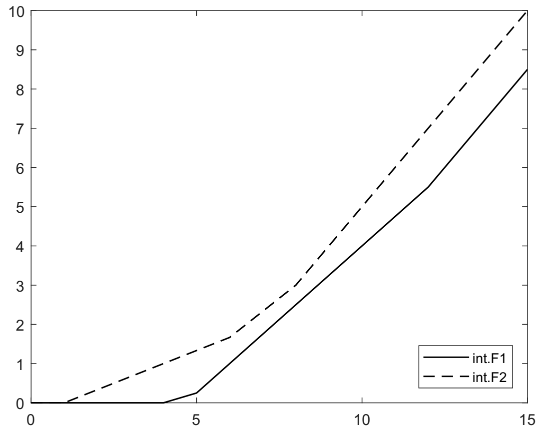


The integrated CDFs are two continuous piecewise linear functions:

$$\widetilde{F}_1(x) = \begin{cases} 0 & \text{if } x < 4 \\ 0.25 \cdot (x-4) & \text{if } 4 \le x < 5 \\ 0.75 \cdot (x-5) + 0.25 & \text{if } 5 \le x < 12 \\ (x-12) + 5.5 & \text{if } x \ge 12, \end{cases}$$

$$\widetilde{F}_2(x) = \begin{cases} 0 & \text{if } x < 1 \\ \frac{1}{3} \cdot (x-1) & \text{if } 1 \le x < 6 \\ \frac{2}{3} \cdot (x-6) + \frac{5}{3} & \text{if } 6 \le x < 8 \\ (x-8) + 3 & \text{if } x \ge 8, \end{cases}$$

If we plot the functions, as shown below, or we evaluate them at the breakpoints, we see that $\widetilde{F}_2(x) \ge \widetilde{F}_1(x)$, with strict inequality for $x > 1$.



Hence, the first investment dominates the second one in the sense of second-order stochastic dominance. For instance, if we assume logarithmic utility, we find

$$0.25 \cdot \log 4 + 0.5 \cdot \log 5 + 0.25 \cdot \log 12 = 1.7725,$$
$$\frac{1}{3} \cdot \log 1 + \frac{1}{3} \cdot \log 6 + \frac{1}{3} \cdot \log 8 = 1.2904.$$

8

Mean–Variance Efficient Portfolios

8.1 SOLUTIONS

Problem 8.1 Let us denote the expected return and the volatility of the risky asset by μ and σ, respectively, whereas μ_p and σ_p refer to the overall portfolio. We consider the case in which there is a positive risk premium, i.e., $\mu > r_f$, so that short-selling does not make sense and the weight of the risky asset satisfies $w \geq 0$.

If we set $w < 1$, we are investing a fraction $1 - w$ of portfolio in the risk-free asset, so that (as we show in the book):

$$\begin{aligned}
\tilde{r}_p &= w\tilde{r} + (1 - w)r_f = w(\tilde{r} - r_f) + r_f, \\
\mu_p &= w(\mu - r_f) + r_f, \\
\sigma_p &= w\sigma, \\
\mu_p &= \sigma_p \cdot \frac{\mu - r_f}{\sigma} + r_f.
\end{aligned}$$

Thus, the CAL for $w < 1$ is a line with slope given by the Sharpe ratio corresponding to r_f and an intercept given by r_f itself.

If $w = 1$, we just invest in the risky asset, and we find a single point $\mu_p = \mu$ and $\sigma_p = \sigma$.

If we set $w > 1$, the weight of the risk-free asset is negative, and we are *borrowing* at the rate $r_f^B > r_f$:

$$\begin{aligned}
\tilde{r}_p &= w(\tilde{r} - r_f^B) + r_f^B, \\
\mu_p &= w(\mu - r_f^B) + r_f^B, \\
\sigma_p &= w\sigma, \\
\mu_p &= \sigma_p \cdot \frac{\mu - r_f^B}{\sigma} + r_f^B.
\end{aligned}$$

Thus, the CAL for $w < 1$ is a line with a smaller slope

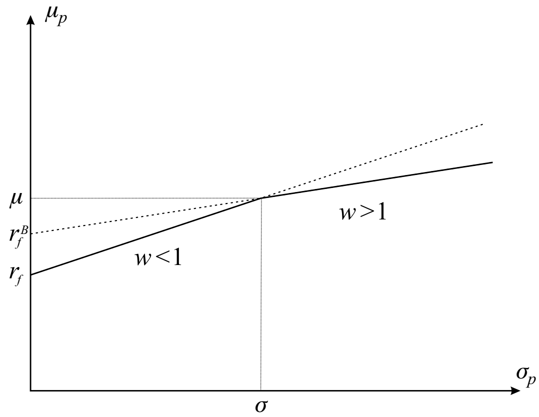
$$\frac{\mu - r_f^B}{\sigma} < \frac{\mu - r_f}{\sigma},$$

but a larger intercept

$$r_f^B > r_f.$$

We assume that the risk premium is positive ($\mu - r_f^B > 0$) when borrowing, too. Otherwise, if $\mu < r_f^B$, the slope of the second portion of the CAL is negative, and the portfolios there would not be efficient.

The two lines intersect at the point corresponding to $w = 1$, as shown in the following picture, where the resulting CAL is shown as a continuous line:

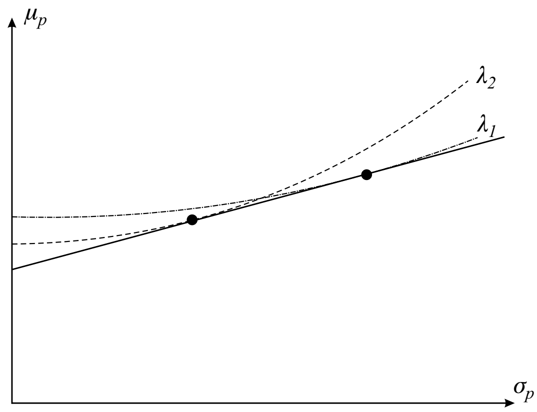


This has an interesting implication in terms of asset allocation, as a function of the risk aversion coefficient λ. On the mean–risk plane, if risk is measured by standard deviation, the level curves of the mean–variance function (for a given expected “utility” level U) are portions of a parabola with equation

$$\mu_p = U + \frac{\lambda}{2}\sigma_p^2,$$

whose symmetry axis is the vertical coordinate axis. The value U is the ordinate where the parabola crosses the vertical axis, and it may be interpreted as a certainty equivalent return, i.e., a riskless rate that would provide us with the same “utility” as the risky portfolios on the parabola.

The optimal portfolio, in the standard case, corresponds to a tangency point between the CAL and the level curve with the highest attainable utility, as shown below:



In the picture, $\lambda_2 > \lambda_1$.

For each λ, we have a unique optimal portfolio, and if we change risk aversion, the solution will change continuously, according to the formula:

$$w^* = \frac{\mu - r_f}{\lambda \sigma^2}.$$

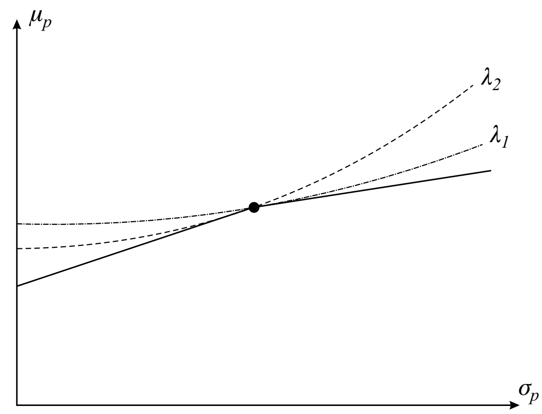
In the case of a different borrowing rate, the above formula applies only to the case where $w^* < 1$, which requires

$$\lambda > \frac{\mu - r_f}{\sigma^2} \doteq \lambda_{\min}.$$

The case $w^* < 1$ requires

$$\lambda < \frac{\mu - r_f^B}{\sigma^2} \doteq \lambda_{\max}.$$

Since there is a gap between $\lambda_{\min}$ and $\lambda_{\max}$, the solution $w^* = 1$ is obtained for a *range* of coefficients λ, rather than a single one. This is illustrated in the figure below, and it is due to the kinky point where the two lines intersect.



As an aside observation, we note that in linear programming, too, a solution (vertex of a polyhedron) may be stable for a range of coefficients of the objective function. This is due to the fact that the feasible set is a “kinky” polyhedron, which is nonsmooth at vertices.

Problem 8.2 We have to show that the general formula

$$\mathbf{w}^*_{\min} = \frac{\mathbf{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}^{\mathsf{T}}\mathbf{\Sigma}^{-1}\mathbf{i}}.$$

yields

$$w_1 = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}, \qquad w_2 = 1 - w_1,$$

in the case of two assets.

The covariance matrix is

$$\mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

and its inverse is

$$\mathbf{\Sigma}^{-1} = \frac{1}{\Delta} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

where we use

$$\Delta \doteq \sigma_1^2\sigma_2^2 - \rho^2\sigma_1\sigma_2$$

to denote the determinant of $\mathbf{\Sigma}$. Then,

$$\mathbf{\Sigma}^{-1}\mathbf{i} = \frac{1}{\Delta}\begin{bmatrix}\sigma_2^2 - \rho\sigma_1\sigma_2\\ \sigma_1^2 - \rho\sigma_1\sigma_2\end{bmatrix}$$

and

$$\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i} = \frac{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}{\Delta}.$$

We may note, in passing, that a vector–matrix product like $\mathbf{i}^\mathsf{T}\mathbf{A}\mathbf{i}$ amounts to adding all of the elements of $\mathbf{A}$.

Thus, when taking the ratio

$$\frac{\mathbf{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i}},$$

the factors Δ cancel each other, and we find

$$\mathbf{w}^* = \begin{bmatrix}\dfrac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\\ \dfrac{\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}\end{bmatrix}.$$

Problem 8.3 Let us consider the weight of asset 1,

$$w_1^* = \frac{1}{\lambda(1-\rho^2)}\left(\frac{\pi_1}{\sigma_1^2} - \rho\frac{\pi_2}{\sigma_1\sigma_2}\right),$$

which may be rewritten as

$$w_1^* = K \cdot \frac{\gamma - \rho}{1-\rho^2},$$

where we define

$$K \doteq \frac{\pi_2}{\lambda\sigma_1\sigma_2(1-\rho^2)},$$
$$\gamma \doteq \frac{\pi_1\sigma_2}{\pi_2\sigma_1}.$$

The leading factor K includes a risk premium and can be positive or negative. The factor γ is essentially the ratio of Sharpe ratios of the two assets and, in principle, it can take any value, positive or negative, larger and smaller than 1.

The factor $\gamma - \rho$ can be zero if $|\gamma| < 1$. For $\rho \to \pm 1$ the weight goes to $\pm\infty$. Apart from this, to analyze the behavior of the weight, we need the derivative of the function

$$\tilde{w} = \frac{\gamma - \rho}{1-\rho^2}$$

with respect to ρ:

$$\frac{d\tilde{w}}{d\rho} = -\frac{\rho^2 - 2\gamma\rho + 1}{(1-\rho^2)^2}.$$

The sign of the derivative depends on the roots of the quadratic function $\rho^2 - 2\gamma\rho + 1$. These roots are

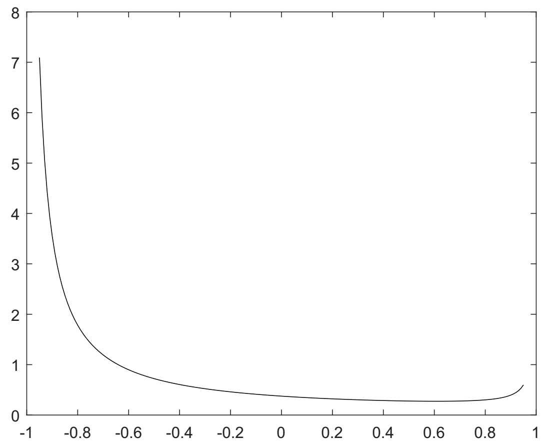
$$\gamma \pm \sqrt{\gamma^2 - 1},$$

and they are real if $\gamma \geq 1$. The product of the roots, in any case, is 1, which means that they have the same sign and that if one is larger than 1 (in absolute value), the other one is less than 1 (in absolute value). Hence, at most one root is compatible with a coefficient of correlation, and there the weight may switch from a decreasing function of ρ to an increasing one (or vice versa).

The whole behavior depends on γ and the sign of the risk premia. Let us see a couple of examples.

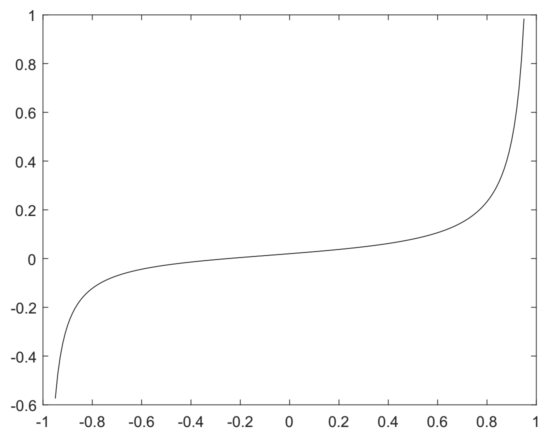
- Case 1: $\pi_1 = 0.03$, $\sigma_1 = 0.2$, $\pi_2 = 0.04$, $\sigma_2 = 0.3$, and $\lambda = 2$. Actually, the coefficient of risk aversion λ scales weights up and down, but is not relevant in terms of the sensitivity analysis we are interested in. Here, $\gamma = 1.1250$, and the two roots of the quadratic equation are 1.6404 and 0.6096.

  The plot of w_1^* for $\rho \in [-0.95, 0.95]$ is as follows:



  The weight goes to $+\infty$ for limit correlations. The weight switches from decreasing to increasing for $\rho = 0.6096$. The plot looks flat, but this is only because of the extreme weight values taken for extreme correlations.

- Case 2: $\pi_1 = 0.01$, $\sigma_1 = 0.5$, $\pi_2 = -0.04$, and $\sigma_2 = 0.5$. For these (rather weird) values, we find $\gamma = -0.2500$, and the two roots of the quadratic equation are complex conjugates. The corresponding weight behavior has no switch from increasing to decreasing (or vice versa) and is illustrated below:



Problem 8.4 To solve the problem

$$\begin{aligned} \max \quad & \boldsymbol{\mu}^\mathsf{T}\mathbf{w} - \frac{\lambda}{2}\mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w} \\ \text{s.t.} \quad & \mathbf{i}^\mathsf{T}\mathbf{w} = 1, \end{aligned}$$

we associate a Lagrange multiplier ν with the equality constraint and build the Lagrangian function

$$\mathcal{L}(\mathbf{w}, \nu) = \boldsymbol{\mu}^\mathsf{T}\mathbf{w} - \frac{\lambda}{2}\mathbf{w}^\mathsf{T}\boldsymbol{\Sigma}\mathbf{w} + \nu \cdot (1 - \mathbf{i}^\mathsf{T}\mathbf{w}).$$

The stationarity condition yields a system of linear equations,

$$\nabla_{\mathbf{w}}\mathcal{L}(\mathbf{w}, \nu) = \boldsymbol{\mu} - \lambda\boldsymbol{\Sigma}\mathbf{w} - \nu\mathbf{i} = \mathbf{0},$$

which may be solved for $\mathbf{w}^*$:

$$\mathbf{w}^* = \frac{1}{\lambda}\boldsymbol{\Sigma}^{-1}\big(\boldsymbol{\mu} - \nu\mathbf{i}\big). \tag{8.1}$$

To find the multiplier ν, we plug the optimal weights into the equality constraint of the optimization problem,

$$\begin{aligned} & \mathbf{i}^\mathsf{T}\mathbf{w}^* = 1 \\ \Rightarrow \qquad & \mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \nu\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i} = \lambda \\ \Rightarrow \qquad & \nu = \frac{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \lambda}{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}} \\ \Rightarrow \qquad & \nu = \frac{\alpha - \lambda}{\beta}, \end{aligned}$$

where, for the sake of convenience, we define the following scalars:

$$\begin{aligned} \alpha &\doteq \mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}, \\ \beta &\doteq \mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}. \end{aligned}$$

Then, we plug this back into the optimal weights:

$$\begin{aligned} \mathbf{w}^* &= \frac{1}{\lambda}\left[\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{\alpha - \lambda}{\beta}\boldsymbol{\Sigma}^{-1}\mathbf{i}\right] \\ &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\beta} + \frac{1}{\lambda}\frac{\beta\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \alpha\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\beta} \\ &= \frac{\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}} + \frac{1}{\lambda}\frac{\big(\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}\big)\,\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \big(\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\big)\,\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}}. \end{aligned}$$

To find the corresponding expected return we have just to write

$$\begin{aligned} \mu^* &= \boldsymbol{\mu}^\mathsf{T}\mathbf{w}^* \\ &= \frac{\boldsymbol{\mu}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\beta} + \frac{1}{\lambda}\frac{\beta\boldsymbol{\mu}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \alpha\boldsymbol{\mu}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}}{\beta} \\ &= \frac{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}} + \frac{1}{\lambda}\frac{\big(\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}\big)\big(\boldsymbol{\mu}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\big) - \big(\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\big)^2}{\mathbf{i}^\mathsf{T}\boldsymbol{\Sigma}^{-1}\mathbf{i}}. \end{aligned}$$

Note that, since the covariance matrix is symmetric, its inverse is, too. Hence, the transpose of $\mathbf{\Sigma}^{-1}$ is just $\mathbf{\Sigma}^{-1}$ itself.

Finding variance is equally easy, even though a bit more tedious:

$$\begin{aligned}(\sigma^*)^2 &= (\mathbf{w}^*)^\mathsf{T}\mathbf{\Sigma}\mathbf{w} \\ &= \frac{1}{\beta^2}\left[\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1} + \frac{1}{\lambda}\left(\beta\boldsymbol{\mu}^\mathsf{T}\mathbf{\Sigma}^{-1} - \alpha\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\right)\right]\cdot\mathbf{\Sigma}\cdot\left[\mathbf{\Sigma}^{-1}\mathbf{i} + \frac{1}{\lambda}\left(\beta\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \alpha\mathbf{\Sigma}^{-1}\mathbf{i}\right)\right].\end{aligned}$$

Let us introduce the scalar

$$\gamma \doteq \boldsymbol{\mu}^\mathsf{T}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}.$$

The calculation involves the following four products

$$\begin{aligned}&\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{i} = \mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i} = \beta, \\ &\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}\left(\beta\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \alpha\mathbf{\Sigma}^{-1}\mathbf{i}\right) = \beta\alpha - \alpha\beta = 0, \\ &\left(\beta\boldsymbol{\mu}^\mathsf{T}\mathbf{\Sigma}^{-1} - \alpha\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\right)\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\mathbf{i} = \beta\alpha - \alpha\beta = 0, \\ &\left(\beta\boldsymbol{\mu}^\mathsf{T}\mathbf{\Sigma}^{-1} - \alpha\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\right)\mathbf{\Sigma}\left(\beta\mathbf{\Sigma}^{-1}\boldsymbol{\mu} - \alpha\mathbf{\Sigma}^{-1}\mathbf{i}\right) = \beta^2\gamma - \beta\alpha^2 - \alpha^2\beta + \alpha^2\beta.\end{aligned}$$

By putting the whole mess together, we find

$$\begin{aligned}(\sigma^*)^2 &= \frac{1}{\beta^2}\left[\beta + \frac{1}{\lambda^2}\left(\beta^2\gamma - \beta\alpha^2\right)\right] \\ &= \frac{1}{\beta} + \frac{1}{\lambda^2}\frac{\beta\gamma - \alpha^2}{\beta} \\ &= \frac{1}{\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i}} + \frac{1}{\lambda^2}\frac{\left(\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i}\right)\left(\boldsymbol{\mu}^\mathsf{T}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right) - \left(\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\boldsymbol{\mu}\right)^2}{\mathbf{i}^\mathsf{T}\mathbf{\Sigma}^{-1}\mathbf{i}}.\end{aligned}$$

Problem 8.5 We have to solve the linear system

$$\begin{bmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} = \begin{bmatrix}\pi_1 \\ \pi_2\end{bmatrix}$$

where x_1, x_2 are the two pseudoweights, to be normalized, and $\pi_1 = \mu_1 - r_f$, $\pi_2 = \mu_2 - r_f$ are the two risk premia. Using, e.g., Cramer's rule, we immediately find

$$\begin{aligned}x_1 &= \frac{\pi_1\sigma_2^2 - \pi_2\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1\sigma_2}, \\ x_2 &= \frac{\pi_2\sigma_1^2 - \pi_1\rho\sigma_1\sigma_2}{\sigma_1^2\sigma_2^2 - \rho^2\sigma_1\sigma_2}.\end{aligned}$$

Using normalization,

$$w_1 = \frac{x_1}{x_1 + x_2} = \frac{\pi_1\sigma_2^2 - \pi_2\rho\sigma_1\sigma_2}{\pi_1\sigma_2^2 + \pi_2\sigma_1^2 - (\pi_1 + \pi_2)\rho\sigma_1\sigma_2},$$

and $w_2 = 1 - w_1$.

Problem 8.6 The portfolio return, over the two time periods, is

$$\tilde{r}_p = w\tilde{r}_{[t,t+1]} + (1 - w)r_{[t,t+1],f},$$

where w is the weight of the risky asset, and $\tilde{r}_{[t,t+1]}$ and $r_{[t,t+1],f}$ are the holding period returns, over the two consecutive time buckets, for the risky and the risk-free assets, respectively. Expected utility is

$$\begin{aligned}
&\mathrm{E}\big[w\tilde{r}_{[t,t+1]} + (1-w)r_{[t,t+1],f}\big] - \lambda \cdot \mathrm{Var}\big[w\tilde{r}_{[t,t+1]} + (1-w)r_{[t,t+1],f}\big] \\
&\qquad = w\mathrm{E}\big[\tilde{r}_{[t,t+1]}\big] + (1-w)r_{[t,t+1],f} - \lambda w^2 \cdot \mathrm{Var}\big[\tilde{r}_{[t,t+1]}\big].
\end{aligned}$$

The first-order optimality condition yields the familiar result

$$w^* = \frac{\mathrm{E}\big[\tilde{r}_{[t,t+1]}\big] - r_{[t,t+1],f}}{2\lambda \cdot \mathrm{Var}\big[\tilde{r}_{[t,t+1]}\big]}.$$

Thus, we need the expected return and variance of the risky asset over the holding period. The correct calculation is based on the multiplication of single-period gains:

$$\tilde{r}_{[t,t+1]} = (1+\tilde{r}_t)\cdot(1+\tilde{r}_{t+1}) - 1 = \tilde{r}_t + \tilde{r}_{t+1} + \tilde{r}_t \cdot \tilde{r}_{t+1}.$$

This involves a rather annoying product of random variables, which may be neglected if we assume that single-period returns are small enough. Using the assumed model, we find

$$\begin{aligned}
\tilde{r}_{[t,t+1]} &\approx \tilde{r}_t + \tilde{r}_{t+1} = \big(a + b\tilde{r}_{t-1} + \epsilon_t\big) + \big(a + b\tilde{r}_t + \epsilon_{t+1}\big) \\
&= a + b\tilde{r}_{t-1} + \epsilon_t + a + b\cdot\big(a + b\tilde{r}_{t-1} + \epsilon_t\big) + \epsilon_{t+1} \\
&= 2a + ab + b\cdot(b+1)\cdot\tilde{r}_{t-1} + (b+1)\cdot\epsilon_t + \epsilon_{t+1},
\end{aligned}$$

where $\tilde{r}_{t-1}$ is known from the previous time bucket $t-1$, and ϵ_t and ϵ_{t+1} are independent variables with expected value zero and standard deviation σ_ϵ.

Therefore, in the formula for the optimal weight w^* we have to plug the following expressions:

$$\begin{aligned}
\mathrm{E}\big[\tilde{r}_{[t,t+1]}\big] &= 2a + ab + b\cdot(b+1)\cdot\tilde{r}_{t-1}, \\
\mathrm{Var}\big[\tilde{r}_{[t,t+1]}\big] &= (b+1)^2\cdot\sigma_\epsilon^2 + \sigma_\epsilon^2, \\
r_{[t,t+1],f} &= (1+r_f)^2 - 1.
\end{aligned}$$

9

Factor Models

Problem 9.1 The first step is to find the portfolio return as a function of portfolio weights:

$$\begin{aligned}
\tilde{r}_p &= w\tilde{r}_a + (1-w)\tilde{r}_b \\
&= w \cdot (0.14 + 0.8\tilde{r}_M + \tilde{\epsilon}_a) + (1-w) \cdot (0.08 + 1.2\tilde{r}_M + \tilde{\epsilon}_b) \\
&= (0.08 + 0.06w) + (1.2 - 0.4w)\tilde{r}_M + w\tilde{\epsilon}_a + (1-w)\tilde{\epsilon}_b,
\end{aligned}$$

where w is the weight of asset a. Note that we have expressed the portfolio return in terms of uncorrelated risk factors, so we need not worry about the covariance between $\tilde{r}_a$ and $\tilde{r}_b$.

Then, the variance of the portfolio return is

$$\sigma_p^2 = (1.2 - 0.4w)^2\sigma_M^2 + w^2\sigma_{\epsilon a}^2 + (1-w)^2\sigma_{\epsilon b}^2.$$

To find its minimum, we write the first-order optimality condition:

$$\begin{aligned}
&\frac{d\sigma_p^2}{dw} = -2 \cdot 0.4 \cdot (1.2 - 0.4w) \cdot 0.2^2 + 2w \cdot 0.3^2 - 2 \cdot (1-w) \cdot 0.25^2 = 0 \\
&\Rightarrow \qquad 0.3178w = 0.1634 \\
&\Rightarrow \qquad w = 0.5142.
\end{aligned}$$

Problem 9.2 The BIG mistake you do not want to make is to compute the expected return, and then evaluate the corresponding bonus. This would be the function of the expected value, whereas we want the expected value of a (nonlinear) function mapping random return to bonus. The correct calculation is

$$\mathrm{E}[B] = 200{,}000 \cdot \mathrm{P}\{\tilde{r}_p \geq 9\%\} + 100{,}000 \cdot \mathrm{P}\{\tilde{r}_p \geq 12\%\},$$

where B is the bonus and $\tilde{r}_p$ is the portfolio return. Since every risk factor is normal (more precisely, we deal with a jointly normal multivariate distribution), we need the expected value and the standard deviation of the random return. Let r_f be the risk-free holding period return and β_{ij} be the risk exposure of asset i to factor j. The portfolio return is

$$\begin{aligned}
\tilde{r}_p &= \big(w_0 r_f + w_1\alpha_1 + w_2\alpha_2\big) + \big(w_1\beta_{11} + w_2\beta_{21}\big) \cdot F_1 + \big(w_1\beta_{12} + w_2\beta_{22}\big) \cdot F_2 + w_1\tilde{\epsilon}_1 + w_2\tilde{\epsilon}_2 \\
&= \alpha_p + \beta_{p1}F_1 + \beta_{p2}F_2 + w_1\tilde{\epsilon}_1 + w_2\tilde{\epsilon}_2,
\end{aligned}$$

where $w_0 = 0.4$ and $w_1 = w_2 = 0.3$ are the asset weights, and α_p, β_{p1}, and β_{p2} denote the portfolio alpha and beta exposures. Plugging numbers, we find

$$\alpha_p = 0.024, \qquad \beta_{p1} = 0.12, \qquad \beta_{p2} = 0.45,$$

and the expected return is

$$\mu_p = \alpha_p + \beta_{p1}\mu_{F1} + \beta_{p2}\mu_{F2} = 0.024 + 0.12 \times 0.03 + 0.45 \times 0.13 = 0.0861.$$

The standard deviation is

$$\begin{aligned}\sigma_p &= \sqrt{\beta_{p1}^2\sigma_{F1}^2 + \beta_{p2}^2\sigma_{F2}^2 + 2\rho\beta_{p1}\sigma_{F1}\beta_{p2}\sigma_{F2} + w_1^2\sigma_{\epsilon 1}^2 + w_2^2\sigma_{\epsilon 2}^2} \\ &= \sqrt{(0.12 \cdot 0.3)^2 + (0.45 \cdot 0.4)^2 + 2 \cdot 0.68 \cdot 0.12 \cdot 0.3 \cdot 0.45 \cdot 0.4 + (0.3 \cdot 0.4)^2 + (0.3 \cdot 0.5)^2} \\ &= 0.2818,\end{aligned}$$

where we have to account for the correlation ρ between the two systematic factors.

Hence, the expected value of the bonus is

$$\begin{aligned}\mathrm{E}[B] &= 200{,}000 \cdot \big(1 - \mathrm{P}\{\tilde{r}_p \le 9\%\}\big) + 100{,}000 \cdot \big(1 - \mathrm{P}\{\tilde{r}_p \le 12\%\}\big) \\ &= 200{,}000 \cdot \big(1 - 0.05055211\big) + 100{,}000 \cdot \big(1 - 0.5478772\big) \\ &= €144{,}108.1.\end{aligned}$$

We use statistical software (or statistical tables, with unavoidable roundoff errors) to find the required probabilities. Note that, since the involved random variables are continuous, we may write $\mathrm{P}\{\tilde{r} \ge \gamma\} = 1 - \mathrm{P}\{\tilde{r} \le \gamma\}$, without bothering too much about strict inequalities.

Problem 9.3 The return of each asset is modeled as

$$\tilde{r}_i = \alpha_i + \beta_i \tilde{r}_M + \tilde{\epsilon}_i,$$

where

$$\begin{aligned}&\alpha_i = 0.02, \qquad i = 1, \ldots, 10, \\ &\alpha_i = -0.02, \qquad i = 11, \ldots, 20, \\ &\beta_i = 1, \qquad i = 1, \ldots, 20.\end{aligned}$$

The portfolio profit/loss is

$$\begin{aligned}\widetilde{G}_p &= \frac{1{,}000{,}000}{10} \cdot \sum_{i=1}^{10} \big(0.02 + \tilde{r}_m + \tilde{\epsilon}_i\big) - \frac{1{,}000{,}000}{10} \cdot \sum_{i=11}^{20} \big(-0.02 + \tilde{r}_m + \tilde{\epsilon}_i\big) \\ &= \frac{1{,}000{,}000}{10} \cdot \sum_{i=1}^{20} 0.02 + \frac{1{,}000{,}000}{10} \left(\sum_{i=1}^{10} \tilde{\epsilon}_i - \sum_{i=11}^{20} \tilde{\epsilon}_i \right) \\ &= 40{,}000 + \tilde{\epsilon}_p,\end{aligned}$$

where

$$\tilde{\epsilon}_p \doteq \frac{1{,}000{,}000}{10} \left(\sum_{i=1}^{10} \tilde{\epsilon}_i - \sum_{i=11}^{20} \tilde{\epsilon}_i \right),$$

is a random variable with the following expected value and standard deviation:

$$\mu_{\epsilon p} = 0,$$
$$\sigma_{\epsilon p} = \sqrt{20} \cdot 100{,}000 \cdot 0.3 = \$134{,}164.$$

Note that we cannot define return, as the initial capital is zero.

This is a long–short portfolio unrelated with systematic risk. If we compare specific risk $\sigma_{\epsilon p}$ against the expected profit, \$40,000, we notice that the portfolio is not quite well diversified.

If we increase the total number of assets to 50 (25 long, 25 short), the position in each stock share is ±40,000, and

$$\sigma_{\epsilon p} = \sqrt{50} \cdot 40{,}000 \cdot 0.3 = \$84{,}853.$$

If we increase the total number of assets to 100 (50 long, 50 short), the position in each stock share is ±20,000, and

$$\sigma_{\epsilon p} = \sqrt{100} \cdot 20{,}000 \cdot 0.3 = \$60{,}000.$$

Note that when the number of stocks increases by a factor 5 (from 20 to 100), the standard deviation decreases by a factor $\sqrt{5}$, from \$134,164 to \$60,000.

Problem 9.4 We may work with either portfolio weights or monetary amounts. Let us use weights

$$w_1 = \frac{100}{350} = 0.2857, \qquad w_2 = \frac{250}{350} = 0.7143.$$

The annual expected return is

$$\begin{aligned}
\mu_{p,y} &= w_1\alpha_1 + w_2\alpha_2 + \big(w_1\beta_1 + w_2\beta_2\big) \cdot \mu_M \\
&= 0.2857 \cdot 0.007 - 0.7143 \cdot 0.003 + \big(0.2857 \cdot 1.1 + 0.7143 \cdot 0.8\big) \cdot 0.07 \\
&= 0.0618567.
\end{aligned}$$

The annual variance is

$$\begin{aligned}
\sigma_{p,y}^2 &= \big(w_1\beta_1 + w_2\beta_2\big)^2 \cdot \sigma_M^2 + \big(w_1\sigma_{\epsilon 1}\big)^2 + \big(w_2\sigma_{\epsilon 2}\big)^2 \\
&= \big(0.2857 \cdot 1.1 + 0.7143 \cdot 0.8\big)^2 \cdot 0.37^2 + \big(0.2857 \cdot 0.22\big)^2 + \big(0.7143 \cdot 0.31\big)^2 \\
&= 0.160378.
\end{aligned}$$

To find $\mathrm{V@R}_{0.99}$, we need the corresponding quantile for the standard normal, $z_{0.99} = 2.3263$.

The annual value-at-risk is

$$\begin{aligned}
\mathrm{V@R}_{0.99,y} &= 350{,}000 \cdot \big(-\mu_{p,y} + z_{0.99}\sigma_{p_y}\big) \\
&= 350{,}000 \cdot \big(-0.0618567 + 2.3263 \cdot \sqrt{0.160378}\big) = \$304{,}416.6,
\end{aligned}$$

where we change the sign of $\mu_{p,y}$ to switch from profit to loss. This value is not quite reassuring.

To find the daily value-at-risk, we assume (given the square-root rule):

$$\mu_{p,d} \approx 0, \qquad \sigma_{p,d} = \sqrt{\frac{\sigma_{p,y}^2}{250}} = 0.02532809,$$

where we scale annual volatility to daily volatility assuming that there are 250 trading days in one year. Then

$$\mathrm{V@R}_{0.99,d} = 350{,}000 \cdot z_{0.99} \cdot \sigma_{p,d} = 350{,}000 \cdot 2.3263 \cdot 0.02532809 = \$20{,}622.68.$$

Problem 9.5 The optimization of the risk-adjusted expected return may be expressed as

$$\min -\boldsymbol{\mu}^{\mathsf{T}}\mathbf{w} + \frac{\lambda}{2}\mathbf{w}^{\mathsf{T}}\boldsymbol{\Sigma}\mathbf{w},$$

where we flip sign to be more compatible with the treatment in the chapter on optimization methods. We usually interpret w_i as a portfolio weight, but for a dollar-neutral portfolio this interpretation fails because the overall wealth invested is zero. Indeed, rather than the usual constraint $\sum_i w_i = 1$, we should write

$$\sum_{i=1}^{n} w_i = 0,$$

to enforce **dollar-neutrality**. We should interpret the decision variables as monetary amounts invested. Since we hold both long and short positions, we may also write

$$\sum_{i\in\mathcal{L}} w_i = \sum_{i\in\mathcal{S}} w_i,$$

where $\mathcal{L}$ is the subset of assets for which we hold a long position, and $\mathcal{S}$ is the subset of shorted assets. The total wealth W_0 invested in the long side of the portfolio is matched by a corresponding shorted amount. We might interpret w_i, $i \in \mathcal{L}$, as weights of the long side, with respect to W_0, but this is a bit pointless. The amount W_0 can be scaled up and down at will, while preserving dollar-neutrality.

Then, the usual expression for a multifactor model,

$$\tilde{r}_p = \sum_{i=1}^{n} w_i \tilde{r}_i = \sum_{i=1}^{n} w_i \alpha_i + \sum_{i=1}^{n}\sum_{k=1}^{m} w_i \beta_{ik} \tilde{F}_k + \sum_{i=1}^{n} w_i \tilde{\epsilon}_i,$$

should be interpreted in terms of monetary profit/loss, rather than return. Nevertheless, we may express the portfolio beta with respect to factor k as

$$\beta_{pk} = \sum_{i=1}^{n} w_i \beta_{ik}.$$

Then, we may enforce **beta-neutrality** by the constraints

$$\sum_{i=1}^{n} w_i \beta_{ik} = 0, \qquad \forall k = 1, \ldots, m.$$

Again, scaling pseudoweights w_i up and down is irrelevant in terms of systematic risk exposure.

Now, we may take advantage of beta-neutrality to simplify the objective function. In fact, the only risk exposure is due to specific risk. Let us consider the familiar expressions

$$\mathrm{Var}(\tilde{r}_i) = \sum_{k=1}^{m}\sum_{q=1}^{m} \beta_{ik}\sigma_{kp}\beta_{iq} + \sigma_{\epsilon i}^2,$$

$$\mathrm{Cov}(\tilde{r}_i, \tilde{r}_j) = \sum_{k=1}^{m}\sum_{q=1}^{m} \beta_{ik}\sigma_{kp}\beta_{iq},$$

where σ_{kp} is the covariance between systematic factors $\tilde{F}_k$ and $\tilde{F}_q$ and $\sigma^2_{\epsilon i}$ is the variance of the specific factor $\tilde{\epsilon}_i$ (we assume a diagonal model, thus specific factors are mutually uncorrelated). If we collect betas β_{ik} into matrix $\mathbf{B} \in \mathbb{R}^{n\times m}$, systematic factor covariances into matrix $\mathbf{\Sigma}_F \in \mathbb{R}^{m\times m}$, and specific variances into the diagonal matrix $\mathbf{S} \in \mathbb{R}^{n\times n}$, all of the above reads

$$\mathbf{\Sigma} = \mathbf{B}\mathbf{\Sigma}_F\mathbf{B}^{\mathsf{T}} + \mathbf{S}.$$

It may be useful to associate the covariance between returns of assets i and j with the product among row i of $\mathbf{B}$, the square matrix $\mathbf{\Sigma}_F$, and colum j of $\mathbf{B}^{\mathsf{T}}$. Now, beta-neutrality implies

$$\mathbf{B}^{\mathsf{T}}\mathbf{w} = \mathbf{0}.$$

Therefore variance of profit/loss boils down to

$$\mathbf{w}^{\mathsf{T}}\mathbf{\Sigma}\mathbf{w} = \mathbf{w}^{\mathsf{T}}\big(\mathbf{B}\mathbf{\Sigma}_F\mathbf{B}^{\mathsf{T}} + \mathbf{S}\big)\mathbf{w} = \mathbf{w}^{\mathsf{T}}\mathbf{S}\mathbf{w},$$

a diagonal quadratic form.

We may write the resulting model as follows:

$$\begin{aligned} \min \quad & -\sum_{i=1}^{n} \mu_i w_i + \frac{\lambda}{2}\sum_{i=1}^{n} w_i^2 \sigma^2_{\epsilon i} \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i = 0 \\ & \sum_{i=1}^{n} w_i \beta_{ik} = 0, \qquad \forall k = 1, \ldots, m. \end{aligned}$$

To solve the problem, we may associate a Lagrange multiplier ν_0 with the dollar-neutrality constraint and m multipliers ν_k with dollar-neutrality constraints. The Lagrangian function is

$$\mathcal{L} = -\sum_{i=1}^{n} \mu_i w_i + \frac{\lambda}{2}\sum_{i=1}^{n} w_i^2 \sigma^2_{\epsilon i} + \nu_0 \sum_{i=1}^{n} w_i + \sum_{k=1}^{m} \nu_k \cdot \left(\sum_{i=1}^{n} w_i \beta_{ik} \right),$$

with stationarity conditions

$$\frac{\partial \mathcal{L}}{\partial w_i} = -\mu_i + 2\lambda w_i \sigma^2_{\epsilon i} + \nu_0 + \sum_{k=1}^{m} \nu_k \beta_{ik} = 0, \qquad i = 1, \ldots, n.$$

These n linear equations, together with the $m+1$ neutrality conditions, allow to find the $n+m+1$ unknown variables.

We have claimed that scaling the portfolio up and down does not affect dollar- and beta-neutrality. However, as we have seen in Problem 7.2, the risk aversion parameter does interact with the size of portfolio positions. Thus, we might wish to use a constraint like $w_1 = 1$ as an anchor to explore the impact of risk aversion.

10

Equilibrium Models: CAPM and APT

Problem 10.1 Whatever utility function we assume, finding a portfolio with a given expected return, in this case, just requires solving the following equation:

$$w_1 \cdot 0.06 + (1 - w_1) \cdot 0.04 = 0.03 \qquad \Rightarrow \qquad w_1 = -0.5.$$

The weight of the second index is $w_2 = 1.5$. Since the target return is outside the range of the two expected returns of the two indexes, short-selling the index with the larger expected value is required.

The volatility of the resulting portfolio would be

$$\sigma_p = \sqrt{(-0.5 \cdot 0.15)^2 + (1.5 \cdot 0.1)^2} = 16.77\%.$$

If every investor behaves in the same way, they would try to short-sell the first index, but this is impossible if no one is holding the index. More generally, at market equilibrium, we cannot have only short positions on an asset. In fact, the capitalization of the first market accounts for 25% of the total, which could happen if every investor takes a short position in it.

CAPM assumes that investors are mean–variance optimizers. This is consistent with investors featuring a quadratic utility, not a logarithmic utility. Alternatively, mean–variance optimization is consistent with utility maximization under a joint normality assumption. Returns are assumed to be independent in this case, but they cannot be normal, as kurtosis is larger than 3. However, mean–variance optimization would be consistent with logarithmic utility with a suitably heavy-tailed and symmetric distributions within the family of elliptical distributions. A Student's t distribution with $\nu > 4$ degrees of freedom has excess kurtosis

$$\frac{6}{\nu - 4}.$$

Thus, such a distribution with $\nu = 7$ would have kurtosis 5, like the second index. A kurtosis of 7 is not obtained for an integer ν, but such a distribution can be conceived for the first market (and merged with the independent t for the second index to yield a joint elliptical distribution), so we may not really say that CAPM cannot hold.

Problem 10.2 To solve the problem, we need the current fair price of a Joint stock share, since

$$S_J(0) = S_E(0) + S_{PM}(0) \qquad \Rightarrow \qquad S_E(0) = S_J(0) - 30.$$

Using CAPM, we find

$$\mu_J = r_f + \beta(\mu_M - r_f) = 0.05 + 2 \cdot 0.10 = 0.25.$$

Hence,

$$S_J(0) = \frac{\mathrm{E}[S_J(1)]}{1 + \mu_J} = \frac{100}{1 + 0.25} = 80,$$

which implies

$$S_E(0) = 80 - 30 = 50.$$

Problem 10.3 We need expected returns, volatilities, the covariance between $\tilde{r}_i$ and $\tilde{r}_M$, etc.:

$$\begin{aligned}
\mu_i &= \sum_{k=1}^{5} \pi_k \tilde{r}_i(\omega_k) = 0.2 \cdot 0.03 + 0.2 \cdot 0.17 + 0.3 \cdot 0.28 + 0.2 \cdot 0.05 + 0.1 \cdot (-0.04) = 0.13, \\
\mu_M &= 0.2 \cdot 0.09 + 0.2 \cdot 0.16 + 0.3 \cdot 0.10 + 0.2 \cdot 0.02 + 0.1 \cdot 0.16 = 0.10, \\
\sigma_i^2 &= \sum_{k=1}^{5} \pi_k \tilde{r}_i^2(\omega_k) = 0.2 \cdot 0.03^2 + 0.2 \cdot 0.17^2 + 0.3 \cdot 0.28^2 + 0.2 \cdot 0.05^2 + 0.1 \cdot (-0.04)^2 - 0.13^2 = 0.0132, \\
\sigma_i &= \sqrt{0.0132} = 0.1151, \\
\sigma_M &= 0.0488, \\
\sigma_{iM} &= \sum_{k=1}^{5} \pi_k \tilde{r}_i(\omega_k) \tilde{r}_M(\omega_k) - \mu_i \mu_M = 9.4 \cdot 10^{-4}, \\
\rho_{iM} &= \frac{\sigma_{iM}}{\sigma_i \sigma_M} = 0.1675, \\
\beta_{iM} &= \frac{\sigma_{iM}}{\sigma_M^2} = \frac{9.4 \cdot 10^{-4}}{0.0488^2} = 0.395.
\end{aligned}$$

The CAPM formula implies

$$r_f = \frac{\mu_i - \beta_{iM} \cdot \mu_M}{1 - \beta_{iM}} = \frac{0.13 - 0.395 \cdot 0.1}{1 - 0.395} = 14.96\%.$$

This risk-free rate is surprisingly large, and even larger than the expected return from asset i and the market. Needless to say, this is due to the fictional character of the input data, and it may be understood by noting that asset i is not quite risky, in terms of exposure to systematic risk, as its beta is fairly small (the correlation with the market is small, too). However, asset i has a large expected return, which may only be obtained if the risk-free rate itself is large.

Problem 10.4 We have

$$\lambda_0 = r_f = 0.04.$$

Then, we have to solve the system of linear equations

$$\begin{aligned}
1.5\lambda_1 - 0.9\lambda_2 + 2\lambda_3 &= 0.085 - 0.04, \\
0.5\lambda_1 + 1.2\lambda_2 + 0.6\lambda_3 &= 0.128 - 0.04, \\
-0.1\lambda_1 + 0.4\lambda_2 - 0.3\lambda_3 &= 0.049 - 0.04,
\end{aligned}$$

which yields

$$\lambda_1 = 0.02, \quad \lambda_2 = 0.05, \quad \lambda_3 = 0.03.$$

The system may be solved in any way you like, but Cramer's rule using determinants is probably the easiest way to do it manually. The equilibrium expected return of a portfolio with unit exposure to F_1, i.e., such that $\beta_{p1} = 1$ and $\beta_{p2} = \beta_{p3} = 0$, is

$$\lambda_0 + \beta_{p1}\lambda_1 = 0.04 + 0.02 = 6\%.$$

Problem 10.5 We have to solve the system of linear equations

$$\begin{aligned}
1.6\lambda_1 - 0.8\lambda_2 &= 0.095 - 0.04,\\
0.5\lambda_1 + 1.3\lambda_2 &= 0.117 - 0.04,
\end{aligned}$$

which yields

$$\lambda_1 = 0.0537, \quad \lambda_2 = 0.0386.$$

The equilibrium expected return of a portfolio with the same exposure to systematic risks as portfolio C should be

$$0.04 - 0.1\lambda_1 + 0.4\lambda_2 = 0.0501,$$

which is larger than the value 4.1% in the table. Using portfolios A and B, as well as the risk-free asset, we may build a portfolio (say, D) with the same systematic risk exposure as C, but a larger expected return (0.0501). This is obtained by finding the respective weights w_A, w_B, and w_0 (of the risk-free asset) as follows:

$$\begin{aligned}
&1.6w_A + 0.5w_B = -0.1,\\
&-0.8w_A + 1.3w_B = 0.4,\\
&w_A + w_B + w_0 = 1,
\end{aligned}$$

where we note that the betas of the risk-free asset are clearly zero. This yields

$$w_A = -0.1331, \quad w_B = 0.2258, \quad w_0 = 0.9073.$$

If we take a long position in portfolio D and an offsetting short position in C, where the exact monetary amount of each position is irrelevant, we find a dollar-neutral and beta-neutral portfolio, with return

$$w_A \cdot \mu_A + w_B \cdot \mu_B + w_0 \cdot r_f - \mu_C = -0.1331 \cdot 0.095 + 0.2258 \cdot 0.117 + 0.9073 \cdot 0.04 - 0.041 = 0.91\%,$$

and no systematic risk, since (by construction)

$$\begin{aligned}
w_A \cdot \beta_{A1} + w_B \cdot \beta_{A1} - \beta_{C1} &= -0.1331 \cdot 1.6 + 0.2258 \cdot 0.5 + 0.1 = 0,\\
w_A \cdot \beta_{A2} + w_B \cdot \beta_{A2} - \beta_{C2} &= 0.1331 \cdot 0.8 + 0.2258 \cdot 1.3 - 0.4 = 0.
\end{aligned}$$

If we disregard specific risk (as well as model and estimation risks), the above return is risk-free and can be obtained with a zero net investment, an arbitrage opportunity.

11

Modeling Dynamic Uncertainty

11.1 SOLUTIONS

Problem 11.1 The stochastic differential equation

$$dX_t = 0.5\,dt + 2\,dW_t,$$

as we have seen in the book, defines a generalized Wiener process. Straightforward integration yields

$$X_T = X_0 + 0.5T + 2 \cdot (W_T - W_0).$$

Therefore, if $T = 4$ and $X_0 = x$, we find that the capital X_4 after four years is normally distributed, with expected value

$$\mu = x + 0.5 \cdot 4 = x + 2,$$

and standard deviation

$$\sigma = 2 \cdot \sqrt{4} = 4.$$

The probability that X_4 is negative is found by standardization (if you use tables)

$$\mathrm{P}\{X_4 < 0\} = \mathrm{P}\left\{Z < \frac{0-(x+2)}{4}\right\} = \Phi\left(-\frac{x+2}{4}\right).$$

This probability should be no more than 5%, which requires (in \$ millions)

$$-\frac{x+2}{4} \leq z_{0.05} = -1.6449 \qquad \Rightarrow \qquad x \geq 4 \cdot 1.6449 - 2 = 4.579415.$$

Hence, the initial capital should be at least \$4,579,415.

Problem 11.2 This is again a generalized Wiener process, with piecewise constant coefficients. The increment in the first time period is normally distributed:

$$S_3 - S_0 \sim \mathsf{N}\big(2 \cdot 3, 3^2 \cdot 3\big),$$

where we use the notation $\mathsf{N}(\mu, \sigma^2)$, so the second parameter is variance, not standard deviation. The increment in the second time period is normally distributed, too:

$$S_6 - S_3 \sim \mathsf{N}\big(3 \cdot 3, 4^2 \cdot 3\big).$$

Since increments are independent, we are summing two independent normal variables, which gives another normal with

$$\mathrm{E}\big[X_6 \,|\, X_0 = 5\big] = 5 + 6 + 9 = 20$$

and

$$\mathrm{Var}\big(X_6\big) = 27 + 48 = 75.$$

Problem 11.3 Since $e^{Y_t^2} \geq 1$, we cannot obtain a GBM, which features a distribution ranging over the half-line $[0, +\infty)$. We apply Itô's lemma to the function

$$X_t = F(Y_t) = e^{Y_t^2},$$

where Y_t satisfies the equation

$$dY_t = 3Y_t\, dt + 7Y_t\, dW_t.$$

We collect the partial derivatives

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial Y_t} = 2Y_t e^{Y_t^2}, \quad \frac{\partial^2 F}{\partial Y_t^2} = 2e^{Y_t^2} + 4Y_t^2 e^{Y_t^2}.$$

Note that the partial derivative with respect to time is zero, since F does not depend directly on time. We find

$$\begin{aligned} dX_t &= \left[6Y_t^2 \cdot e^{Y_t^2} + 49Y_t^2 \cdot \left(e^{Y_t^2} + 2Y_t^2 e^{Y_t^2}\right)\right] dt + 14Y_t^2 e^{Y_t^2}\, dW_t \\ &= \left(55Y_t^2 + 98Y_t^4\right) e^{Y_t^2}\, dt + 14Y_t^2 e^{Y_t^2}\, dW_t. \end{aligned}$$

Using substitutions

$$e^{Y_t^2} = X_t, \qquad Y_t^2 = \log X_t,$$

we may rewrite the equation in terms of X_t:

$$dX_t = \left(55 \log X_t + 98 \log^2 X_t\right) X_t\, dt + 14X_t \log X_t\, dW_t.$$

As expected, this equation is not in the GBM form $dX = \mu X\, dt + \sigma X\, dW$.

Problem 11.4 We apply Itô's lemma to the function

$$Y_t \equiv F(S_t, t) = (S_t - 10)^2 \cdot e^{-2t}$$

where S_t satisfies the equation

$$dS_t = 2S_t\, dt + 3S_t\, dW_t.$$

Note that, in this case, there is an explicit dependence on time. We collect the partial derivatives

$$\frac{\partial F}{\partial t} = -2\big(S_t - 10\big)^2 e^{-2t}, \quad \frac{\partial F}{\partial S_t} = 2\big(S_t - 10\big)e^{-2t}, \quad \frac{\partial^2 F}{\partial S_t^2} = 2e^{-2t}.$$

Therefore, the resulting SDE is

$$
\begin{aligned}
dY_t &= \left[-2\big(S_t - 10\big)^2 e^{-2t} + 2\big(S_t - 10\big)e^{-2t} \cdot 2S_t + \frac{1}{2} \cdot 2e^{-2t} \cdot 9S_t^2\right] dt + 3S_t \cdot 2\big(S_t - 10\big)e^{-2t}\, dW_t \\
&= \big(11S_t^2 - 200\big)e^{-2t}dt + 6S_t\big(S_t - 10\big)e^{-2t}dW_t.
\end{aligned}
$$

We may express the equation in terms of Y_t by the variable substitution $S_t = 10 + \sqrt{Y_t e^{2t}}$. Clearly, this is not a GBM.

Problem 11.5 The events $\{W_5 - W_2 \geq 0\}$ and $\{W_1 \leq 0\}$ are independent, since they refer to increments of a standard Wiener process on disjoint time intervals. Furthermore, these increments are normally distributed with expected value zero, so

$$
\mathrm{P}\big\{(W_5 - W_2 \geq 0) \cap (W_1 \leq 0)\big\} = \mathrm{P}\big\{W_5 - W_2 \geq 0\big\} \cdot \mathrm{P}\big\{W_1 \leq 0\big\} = 0.5 \times 0.5 = 0.25.
$$

We may take advantage of independence when dealing with variance as well. Using the standard equality $\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}^2[X]$, we write

$$
\mathrm{Var}\big[(W_5 - W_2) \cdot W_1\big] = \mathrm{E}\big[(W_5 - W_2)^2 \cdot W_1^2\big] - \mathrm{E}^2\big[(W_5 - W_2) \cdot W_1\big].
$$

Using independence, we have

$$
\mathrm{E}\big[(W_5 - W_2) \cdot W_1\big] = \mathrm{E}\big[W_5 - W_2\big] \cdot \mathrm{E}\big[W_1\big] = 0 \times 0 = 0.
$$

By the same token, recalling that $\mathrm{Var}(W_t - W_s) = t - s$,

$$
\begin{aligned}
\mathrm{E}\big[(W_5 - W_2)^2 \cdot W_1^2\big] &= \mathrm{E}\big[(W_5 - W_2)^2\big] \cdot \mathrm{E}\big[W_1^2\big] \\
&= \Big\{\mathrm{Var}\big(W_5 - W_2\big) + \underbrace{\mathrm{E}^2\big[W_5 - W_2\big]}_{=0}\Big\} \cdot \Big\{\mathrm{Var}\big(W_1\big) + \underbrace{\mathrm{E}^2\big[W_1\big]}_{=0}\Big\} \\
&= (5 - 2) \times 1 = 3.
\end{aligned}
$$

Problem 11.6 An easy way to solve the problem is to represent $S_a(T)$ and $S_b(T)$ explicitly as lognormal variables depending on a standard normal:

$$
\begin{aligned}
S_a(T) &= S_a(0) \cdot \exp\left[\left(\mu_a - \frac{\sigma_a^2}{2}\right) \cdot T + \sigma_a \sqrt{T}\tilde{\epsilon}\right], \\
S_b(T) &= S_b(0) \cdot \exp\left[\left(\mu_b - \frac{\sigma_b^2}{2}\right) \cdot T + \sigma_b \sqrt{T}\tilde{\epsilon}\right],
\end{aligned}
$$

where $\tilde{\epsilon}$ is the *same* standard normal variable, since the two price processes are driven by the same standard Wiener process. Then we take advantage of properties of the exponential function and write

$$
S_a(T) \cdot S_b(T) = S_a(0) \cdot S_b(0) \cdot \exp\big[\mu_{ab} + \sigma_{ab}\tilde{\epsilon}\big],
$$

where

$$
\begin{aligned}
\mu_{ab} &\doteq \left(\mu_a - \frac{\sigma_a^2}{2}\right)T + \left(\mu_b - \frac{\sigma_b^2}{2}\right)T, \\
\sigma ab &\doteq \sigma_a \sqrt{T} + \sigma_b \sqrt{T}.
\end{aligned}
$$

Note that we **cannot** write $\mathrm{E}\big[S_a(T)\cdot S_b(T)\big] = \mathrm{E}\big[S_a(T)\big]\cdot\mathrm{E}\big[S_b(T)\big]$, since the two prices are not independent; on the contrary, they are perfectly correlated. We observe that $S_a(T)\cdot S_b(T)$ is lognormal. By recalling that

$$X \sim \mathsf{N}\big(\mu,\sigma\big) \quad \Rightarrow \quad \mathrm{E}\big[e^X\big] = e^{\mu+\sigma^2/2},$$

we may write

$$\mathrm{E}\big[S_a(T)\cdot S_b(T)\big] = S_a(0)\cdot S_b(0)\cdot \exp\left[\mu_{ab} + \frac{\sigma_{ab}^2}{2}\right].$$

By a similar token,

$$\frac{S_a(T)}{S_b(T)} = \frac{S_a(0)}{S_b(0)}\cdot \exp\left[\mu_{a/b} + \sigma_{a/b}\tilde{\epsilon}\right],$$

where

$$\mu_{a/b} \doteq \left(\mu_a - \frac{\sigma_a^2}{2}\right)T - \left(\mu_b - \frac{\sigma_b^2}{2}\right)T,$$
$$\sigma a/b \doteq \sigma_a\sqrt{T} - \sigma_b\sqrt{T}.$$

Then

$$\mathrm{E}\left[\frac{S_a(T)}{S_b(T)}\right] = \frac{S_a(0)}{S_b(0)}\cdot \exp\left[\mu_{a/b} + \frac{\sigma_{a/b}^2}{2}\right].$$

Problem 11.7 We may apply Itô's lemma to the function

$$Y_t = F(X_t) = e^{X_t},$$

where X_t satisfies the equation

$$dX_t = 0\,dt + 1\,dW_t,$$

which means that $X_t \equiv W_t$. The usual drill with derivatives yields

$$\frac{\partial F}{\partial t} = 0, \quad \frac{\partial F}{\partial X_t} = e^{X_t} = Y_t, \quad \frac{\partial^2 F}{\partial X_t^2} = e^{X_t} = Y_t.$$

Hence, the stochastic differential equation for Y_t is

$$dY_t = \left[\frac{\partial F}{\partial t} + 0\cdot\frac{\partial F}{\partial X_t} + \frac{1}{2}\cdot 1\cdot\frac{\partial^2 F}{\partial X_t^2}\right]dt + 1\cdot\frac{\partial F}{\partial X_t}\cdot dW_t = 0.5Y_t\,dt + Y_t\,dW_t.$$

This is a GBM, but not a martingale, as the drift is not zero.

For the second question, what we want is

$$\mathrm{P}\big\{e^{W_{10}} > 150 \mid W_5 = 3\big\} = \mathrm{P}\big\{W_{10} > \log 150 \mid W_5 = 3\big\}.$$

Since the increment from $t=0$ to $t=5$ has been $W_5 = 3$ and increments are independent, the distribution of $W_{10} = (W_{10} - W_5) + (W_5 - W_0)$, conditional on $W_5 - W_0 = 3$, is $\mathsf{N}(3,5)$. What we need, therefore, is the probability that a random variable $S \sim \mathsf{N}(3,5)$ is larger than $\log 150 = 5.0106$:

$$\mathrm{P}\{S > 5.0106\} = \mathrm{P}\left\{\frac{S-3}{\sqrt{5}} > \frac{5.0106-3}{\sqrt{5}}\right\} = 1 - \Phi\left(\frac{5.0106-3}{\sqrt{5}}\right) = 1 - 0.8157 = 0.1843.$$

Problem 11.8 We might use an extension of Itô's lemma to cope with multidimensional processes, but in this case, by taking logs, we may transform the multiplicative form $Y(t) = S_1(t) \cdot S_2(t)$ into the additive form

$$\log Y(t) = \log S_1(t) + \log S_2(t).$$

By using Itô's lemma, we find the equations for the log processes:

$$d \log S_1(t) = \left(\mu_1 - \frac{\sigma_1^2}{2}\right) dt + \sigma_1 \, dW(t),$$
$$d \log S_2(t) = \left(\mu_2 - \frac{\sigma_2^2}{2}\right) dt + \sigma_2 \, dW(t),$$

which (due to additivity of stochastic integrals) may be added to give

$$d\big(\log S_1(t) + \log S_2(t)\big) = d \log \big(S_1(t) \cdot S_2(t)\big) = \left(\mu_1 + \mu_2 - \frac{\sigma_1^2 + \sigma_2^2}{2}\right) dt + (\sigma_1 + \sigma_2) \, dW(t).$$

By using Itô's lemma again, switching back from log-prices to prices, we find

$$\begin{aligned} dY(t) &= d\big(S_1(t) \cdot S_2(t)\big) \\ &= \left(\mu_1 + \mu_2 - \frac{\sigma_1^2 + \sigma_2^2}{2} + \frac{(\sigma_1 + \sigma_2)^2}{2}\right) Y(t) \, dt + (\sigma_1 + \sigma_2) Y(t) \, dW(t) \\ &= (\mu_1 + \mu_2 + \sigma_1 \sigma_2) \, Y(t) \, dt + (\sigma_1 + \sigma_2) Y(t) \, dW(t). \end{aligned}$$

Note that we have always used the same driving standard Wiener process, since the two processes were assumed to be perfectly correlated. This assumption does not sound realistic if the two processes are, as in the second question, a stock share price on the US market and an exchange rate. In such a case, we really need to extend Itô's lemma to cope with multidimensional processes (which is done in books on stochastic calculus).

Problem 11.9 We apply Itô's lemma to the process $B_t = e^{-y_t \cdot (T-t)}$, the price at time t of a zero with unit face value, maturing at time T, with yield y_t:

$$\begin{aligned} \frac{\partial B_t}{\partial t} &= y_t e^{-y_t \cdot (T-t)} = y_t B_t, \\ \frac{\partial B_t}{\partial y_t} &= -(T-t) e^{-y_t \cdot (T-t)} = -(T-t) B_t, \\ \frac{\partial^2 B_t}{\partial y_t^2} &= (T-t)^2 e^{-y_t \cdot (T-t)} = (T-t)^2 B_t, \end{aligned}$$

where we note the explicit dependence of B_t on time t. Therefore,

$$\begin{aligned} dB_t &= \left[\frac{\partial B_t}{\partial t} + \alpha \cdot (\overline{y} - y_t) \frac{\partial B_t}{\partial y_t} + \frac{1}{2} \sigma^2 y_t^2 \frac{\partial^2 B_t}{\partial y_t^2}\right] dt + \sigma y_t \frac{\partial B_t}{\partial y_t} \, dW_t \\ &= \left[y_t - \alpha \cdot (\overline{y} - y_t) \cdot (T-t) + \frac{1}{2} \sigma^2 y_t^2 (T-t)^2\right] B_t dt - \sigma y_t (T-t) B_t \, dW_t. \end{aligned}$$

We notice that the volatility coefficient $\sigma y_t (T-t)$ goes to zero when time t approaches maturity T. This makes sense, as the bond price volatility should decrease when time-to-maturity goes to zero. We may also recall that, indeed, the term $(T-t)$ is the duration of a zero-coupon bond.

12

Forward and Futures Contracts

12.1 SOLUTIONS

Problem 12.1 The equilibrium forward price should be

$$F_0 = S_0 \cdot e^{(r_d - r_f)T} = 1.2 \cdot e^{(0.024-0.031)/2} = €1.1958.$$

Note that we are taking the viewpoint of a Eurozone investor, so the domestic rate is the interest rate on euro and we consider the price of 1 GBP in euro. The quoted price is larger, so it is convenient to *sell* GBP forward, after buying them spot now. A possible strategy is:

- Borrow €1000 now; after six months, the outstanding debt will be $1000 \cdot e^{0.024/2} =$ €1012.07.
- Buy $1000/1.2 = 833.33$ GBP spot and invest them for six months; we will end up with $833.33 \cdot e^{0.031/2} = 846.35$ GBP.
- Take a short position to sell 846.35 GBP forward at the forward price.
- After six months we sell GBP forward, earning $846.35 \cdot 1.22 = 1032.55$ EUR. After closing our debt, the resulting profit is $1032.55 - 1012.07 =$ €20.48.

Problem 12.2 We need the forward prices of the four days in the scenario:

$$\begin{aligned}
F_0 &= 1.13 \cdot e^{(0.02-0.03)\cdot 120/360} = 1.1262, \\
F_1 &= 1.15 \cdot e^{(0.02-0.03)\cdot 119/360} = 1.1462, \\
F_2 &= 1.13 \cdot e^{(0.02-0.03)\cdot 118/360} = 1.1662, \\
F_3 &= 1.11 \cdot e^{(0.02-0.03)\cdot 117/360} = 1.1064,
\end{aligned}$$

where we assume that interest rates are constant and that a year consists of 12 months of 30 days. Since you hold a long position for 150,000 GBP, the three cash flows, at the end of days 1, 2, and 3 are (in EUR):

$$\begin{aligned}150{,}000 \times (F_1 - F_0) &= 150{,}000 \times (1.1462 - 1.1262) = 2994.79,\\ 150{,}000 \times (F_2 - F_1) &= 150{,}000 \times (1.1662 - 1.1462) = 2994.96,\\ 150{,}000 \times (F_3 - F_2) &= 150{,}000 \times (1.1064 - 1.1662) = -8965.94.\end{aligned}$$

If we neglect the time value of money, the total loss is

$$150{,}000 \times (1.1064 - 1.1262) = €(-2976.19).$$

The initial deposit on the margin account was

$$150{,}000 \times 1.1262 \times 0.25 = €42{,}233.96.$$

Therefore, the return (on equity) has been

$$-\frac{2976.19}{42{,}233.96} = -7.047\%.$$

The smaller the initial deposit, the larger the percentage profit/loss, because of the leverage effect.

Problem 12.3 Let h be the hedging ratio, i.e., the number of index futures we need for each unit (share) of the portfolio. The variation of the value of a hedged portfolio share is

$$\delta H = \delta S + hM_F \cdot \delta F,$$

where δS is the variation of the value of a portfolio share, δF is the variation in the index futures price, and M_F is the multiplier converting the index futures price to cash flows (we neglect margin-to-market mechanics, in terms of time value of daily cash flows, as well as margin requirements). The return on the hedged portfolio is

$$r_H = \frac{\delta S + hM_F \cdot \delta F}{S_0},$$

where we just divide by the initial portfolio share value S_0, since there is no initial cost when entering into a futures contract (we neglect margin issues).

The returns on the portfolio and the return on the market portfolio are

$$r = \frac{\delta S}{S_0}, \qquad r_M = \frac{\delta I}{I_0},$$

respectively, where I_0 is the initial value of the index, and δI is its variation. The beta of the unhedged portfolio is

$$\beta = \frac{\mathrm{Cov}(r, r_M)}{\sigma_M^2} = \mathrm{Cov}\left(\frac{\delta S}{S_0}, \frac{\delta I}{I_0}\right) \cdot \frac{1}{\sigma_M^2},$$

where σ_M^2 is the variance of the market return. The target beta of the hedged portfolio is β^*, which means

$$\beta_H = \frac{\mathrm{Cov}(r_H, r_M)}{\sigma_M^2} = \beta^* \qquad \Rightarrow \qquad \mathrm{Cov}(r_H, r_M) = \beta^* \cdot \sigma_M^2.$$

Now we can write

$$\begin{aligned}\operatorname{Cov}(r_H, r_M) &= \operatorname{Cov}\left(\frac{\delta S + hM_F \cdot \delta F}{S_0}, \frac{\delta I}{I_0}\right)\\ &= \operatorname{Cov}\left(\frac{\delta S}{S_0}, \frac{\delta I}{I_0}\right) + \operatorname{Cov}\left(\frac{hM_F \cdot \delta F}{S_0}, \frac{\delta I}{I_0}\right)\\ &= \beta\sigma_M^2 + \frac{hM_F \cdot F_0}{S_0} \cdot \operatorname{Cov}\left(\frac{\delta F}{F_0}, \frac{\delta I}{I_0}\right).\end{aligned}$$

If the variations in the futures index price track the variations in the index, we may write

$$\frac{\delta F}{F_0} \approx \frac{\delta I}{I_0} \qquad \Rightarrow \qquad \operatorname{Cov}\left(\frac{\delta F}{F_0}, \frac{\delta I}{I_0}\right) \approx \sigma_M^2.$$

Hence, we find

$$\begin{aligned}\beta\sigma_M^2 + \frac{hM_F \cdot F_0}{S_0} \cdot \sigma_M^2 &= \beta^* \cdot \sigma_M^2,\\ h &= \frac{S_0}{M_F \cdot F_0} \cdot (\beta^* - \beta).\end{aligned}$$

If we consider an amount Q_A of portfolio shares, we may rewrite the above condition in terms of number of futures contracts for the whole portfolio,

$$N = \frac{V_A}{V_F} \cdot (\beta^* - \beta),$$

where $V_A = Q_A S_0$ is the current value of the assets, and $M_F \cdot F_0$ is the current monetary futures price of the index. Again, we find the number of futures contracts in terms of values, rather than quantities. If we really want to hedge, we aim at reducing β, so $\beta^* < \beta$ and $N < 0$. Hence, we should hold a short position in index futures. If, on the contrary, we want to increase exposure to systematic risk within a speculation strategy, $N > 0$ and we should hold a long position in index futures.

13

Option Pricing: Complete Markets

13.1 SOLUTIONS

Problem 13.1 The replication portfolio for the call is found by solving the system

$$42\Delta + e^{0.08/12} \cdot \Psi = 3,$$
$$38\Delta + e^{0.08/12} \cdot \Psi = 0,$$

which yields

$$\Delta = 0.75, \qquad \Psi = -28.5 \cdot e^{-0.08/12} = -28.3106.$$

The call option price is

$$C_0 = 0.75 \times 40 - 28.3106 = 1.6894.$$

The call option writer collects the option premium and borrows an amount 28.3106, whose sum, 30, is used to purchase 0.75 stock shares. At maturity, in the up scenario:

- The call is in-the-money, 0.25 additional stock shares must be purchased, with cost $0.25 \times 42 = 10.5$, in order to have a whole share for the option holder.
- The strike price, 39, is collected.
- The outstanding debt, 28.5, is repaid.
- Since $10.5 + 28.5 = 39$, the call writer breaks even.

In the down scenario:

- The call is out-of-the-money, and the 0.75 shares are sold, collecting a cash amount $0.75 \times 38 = 28.5$.
- The outstanding debt, 28.5, is repaid using the cash.

In the case of the put (with same strike and maturity), we solve

$$42\Delta + e^{0.08/12} \cdot \Psi = 0,$$
$$38\Delta + e^{0.08/12} \cdot \Psi = 1,$$

which yields

$$\Delta = -0.25, \qquad \Psi = 10.5 \cdot e^{-0.08/12} = 10.4302.$$

The put option price is

$$P_0 = -0.25 \times 40 + 10.4302 = 0.4302.$$

The put option writer collects the option premium and shorts 0.25 shares, investing the resulting cash at the risk-free rate. At maturity, in the up scenario:

- The risk-free investment yields 10.5.
- The put is out-of-the-money, 0.25 stock shares are purchased with cost $0.25 \times 42 = 10.5$, to close the short position.
- The put writer breaks even.

In the down scenario:

- The risk-free investment yields 10.5.
- The put is in-the-money, so the writer has to buy a stock share for 39.
- The short position in the stock is closed using a fraction 0.25 of the stock share. The remaining 0.75 shares are sold for $0.75 \times 38 = 28.5$.
- Since $28.5 + 10.5 = 39$, the put writer breaks even.

Problem 13.2 This option trading strategy is called a *bull spread*, as it is a bet on the increase of the underlying asset price. The most intuitive way to create this payoff is by taking a long position in a call with strike K_1 and a short position in a call with strike $K_2 > K_1$. All options are European-style.

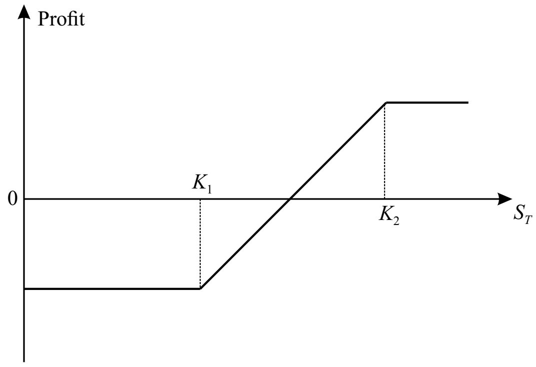
This may be checked in the following table:

Condition	Payoff from call 1	Payoff from call 2	Total payoff
$S_T < K_1$	0	0	0
$K_1 \le S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$K_2 \le S_T$	$S_T - K_1$	$K_2 - S_T$	$K_2 - K_1$

Since the call option price is a monotonically decreasing function of the strike, as it may be easily checked by no-arbitrage, the initial value of the portfolio is

$$C_0^e(K_1) - C_0^e(K_2) \ge 0,$$

where $C_0^e(K)$ is the initial price of a call with strike K. Hence, we have to pay an amount of money at time 0 (negative cash flow), hoping to receive a positive payoff at maturity. The plot of profit requires shifting the payoff down by an amount corresponding to the cost of the strategy, as we show below.



Taking the time value of money into account, we should shift the payoff down by an amount

$$\left[C_0^e(K_1) - C_0^e(K_2)\right] \cdot e^{rT}.$$

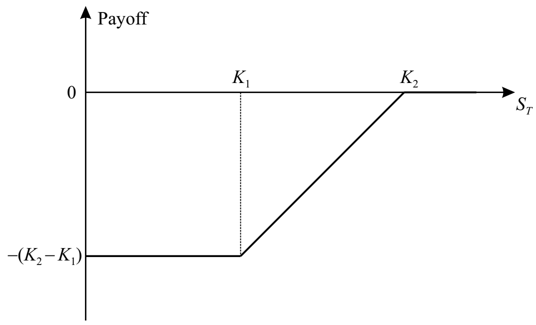
By using put–call parity, we may create the same payoff by using put options and cash. The equations

$$C_0^e(K_1) = P_0^e(K_1) + S_0 - K_1 e^{-rT},$$
$$C_0^e(K_2) = P_0^e(K_2) + S_0 - K_2 e^{-rT},$$

imply

$$C_0^e(K_1) - C_0^e(K_2) = P_0^e(K_1) - P_0^e(K_2) + \left(K_2 - K_1\right)e^{-rT}.$$

If we just use the put options, taking a long position in a put with strike K_1 and a short position in a put with strike K_2, we find the following payoff:



Again, we may check in tabular form:

Condition	Payoff from put 1	Payoff from put 2	Total payoff
$S_T < K_1$	$K_1 - S_T$	$S_T - K_2$	$-\left(K_2 - K_1\right)$
$K_1 \le S_T < K_2$	0	$S_T - K_2$	$S_T - K_2$
$K_2 \le S_T$	0	0	0

This payoff has the same form as the bull spread created with call options, but it is shifted down by $K_2 - K_1$. If we only use the put options, however, the initial cash flow is positive, since

$$P_0^e(K_1) - P_0^e(K_2) \le 0.$$

Therefore, we receive a positive amount of cash from the initial trade, whereas the terminal payoff is nonpositive. If we shift the payoff up, we find a profit similar to the profit diagram obtained with calls.

Problem 13.3 Note that the slopes of the lines where the payoff is not constant are exactly -1 and $+1$. This payoff may be decomposed into a bull spread (the portion on the right) and a bear spread (the portion on the left).

Therefore we may:

- Buy a call with strike 90, which creates the slope $+1$ on the interval $[90, 160]$.
- Write a call with strike 160, which sets the slope to zero on the interval $[160, +\infty)$.
- Buy a put with strike 70, which creates the slope -1 on the interval $[20, 70]$.
- Write a put with strike 20, which sets the slope to zero on the interval $[0, 20]$.

Problem 13.4 For any real variable X, we may write

$$X = X^+ - X^- = \max\{X, 0\} - \max\{-X, 0\}.$$

This decomposes the variable into the difference of its positive and negative parts, where $X^+ \cdot X^- = 0$.

Therefore, if S_T is the underlying asset price at maturity and F_0 is the forward price for delivery at time T, we have

$$S_T - F_0 = \max\big\{S_T - F_0, 0\big\} - \max\big\{F_0 - S_T, 0\big\}.$$

Therefore, the long position in the forward may be synthesized by a long position in a call and a short position in a put, both European-style, maturing at T, with strike F_0.

By the way, this shows that the call and the put option have the same price, if the strike is the forward price. We may also see this using put–call parity, relating the option prices at time 0:

$$C_0^e - P_0^e = S_0 - F_0 e^{-rT} = \big(S_0 e^{rT} - F_0\big)\, e^{-rT} = 0,$$

by spot–forward parity.

If we consider options with a strike $K \neq F_0$, we may write

$$S_T - F_0 = \big(K - F_0\big) + \big(S_T - K\big)^+ - \big(K - S_T\big)^-,$$

which implies that we should buy a zero with face value $\big(K - F_0\big)$ maturing in T. In other words, if $K > F_0$, we should invest a cash amount $\big(K - F_0\big)e^{-rT}$ at time 0; otherwise, if $K < F_0$, we should borrow the same amount of cash. This corresponds to the non-zero value of a forward with delivery price $K \neq F_0$.

Problem 13.5 Let us check put–call parity:

$$\begin{aligned} P_0^e + S_0 &= 4 + 60 = 64 \\ C_0^e + Ke^{-rT} &= 12 + 55e^{-0.05 \cdot 0.75} = 64.98. \end{aligned}$$

Put–call parity is violated, so there is an arbitrage opportunity. We should buy the cheap portfolio and short the expensive one:

- Buy a put option and a stock share.
- Write a call option and borrow \$52.98, which will amount to \$55 in nine months.

The net cash flow is positive, \$0.98, yielding an immediate profit. At maturity:

- If the stock price is larger than \$55, the call is exercised by the holder, and we sell her the stock share we own, receiving \$55 that we need to repay debt. The put option expires worthless.
- If the stock price is smaller than \$55, the call is not exercised by the holder. We sell the stock share using the put, obtaining \$55 that we need to repay debt.

In any scenario, we break even and the net cash flow at maturity is zero.

Problem 13.6 Consider the following portfolios:

1. **Portfolio 1**: One long position in a call, a cash amount D invested in a bank account, and a zero with face value K.
2. **Portfolio 2**: One long position in a put plus one stock share.

We assume a constant risk-free rate, so that a bank account with rate r and a zero with yield r are essentially equivalent.

At maturity, the value of portfolio 1 is

$$\max\left\{S_T - K, 0\right\} + De^{rT} + K = \max\left\{S_T, K\right\} + De^{rT}.$$

The value of portfolio 2 is

$$\max\left\{K - S_T, 0\right\} + S_T + De^{rT} = \max\left\{K, S_T\right\} + De^{rT},$$

where dividends are reinvested at rate r whenever they are received. Note that D is the present value of the dividend cash flow stream at time $t = 0$, which must be shifted forward to time $t = T$. Since the two portfolios have the same value in every state of the world, by the law of one price, the following extended put–call parity must hold:

$$P_0^e + S_0 - D = C_0^e + Ke^{-rT}.$$

Problem 13.7 As a first step, we calibrate the binomial lattice, where the time step $\delta t = 1/3$ consists of four months, the discount factor for each time step is $D = e^{-r\cdot\delta t} = e^{-0.03/3} = 0.99$, $u = 1.15$, $d = 0.9$, and the risk-neutral probabilities are

$$\pi_u = \frac{e^{0.03/3} - 0.9}{1.15 - 0.9} = 0.4402, \quad \pi_d = 1 - \pi_u = 0.5598.$$

We may denote the lattice nodes as follows:

- Time 0: N_0
- Time 1, i.e., $t = 1 \cdot \delta t$: N_1^u, N_1^d
- Time 2, i.e., $t = 2 \cdot \delta t$: N_2^{uu}, N_2^{ud}, N_2^{dd}
- Time 3, i.e., $t = 3 \cdot \delta t \equiv T$: N_3^{uuu}, N_3^{uud}, N_3^{udd}, N_3^{ddd}

The stock prices and the corresponding call payoffs at the four terminal nodes are:

$$\begin{aligned}
S_T^{uuu} &= S_0 \cdot u^3 = 45.6262 && \Rightarrow && C_T^{uuu} = 15.6262,\\
S_T^{uud} &= S_0 \cdot u^2 d = 35.7075 && \Rightarrow && C_T^{uud} = 5.7075,\\
S_T^{udd} &= S_0 \cdot ud^2 = 27.9450 && \Rightarrow && C_T^{udd} = 0,\\
S_T^{ddd} &= S_0 \cdot d^3 = 21.8700 && \Rightarrow && C_T^{ddd} = 0.
\end{aligned}$$

Backward recursion yields, at time $t = 2 \cdot \delta t$:

$$\begin{aligned}
C_2^{uu} &= 0.99 \cdot \big[0.4402 \cdot 15.6262 + 0.5598 \cdot 5.7075\big] = 9.9735,\\
C_2^{ud} &= 0.99 \cdot \big[0.4402 \cdot 5.7075 + 0.5598 \cdot 0\big] = 2.4874,\\
C_2^{dd} &= 0.99 \cdot \big[0.4402 \cdot 0 + 0.5598 \cdot 0\big] = 0.
\end{aligned}$$

By the same token,

$$\begin{aligned}
C_1^{u} &= 0.99 \cdot \big[0.4402 \cdot 9.9735 + 0.5598 \cdot 2.4874\big] = 5.7253,\\
C_1^{d} &= 0.99 \cdot \big[0.4402 \cdot 2.4874 + 0.5598 \cdot 0\big] = 1.0841,\\
C_0 &= 0.99 \cdot \big[0.4402 \cdot 5.7253 + 0.5598 \cdot 1.0841\big] = 3.0960.
\end{aligned}$$

The estimate of the call price is $C_0 = \$3.096$.

In order to answer the second question, we need the following deltas:

$$\begin{aligned}
\Delta_1^u &= \frac{C_2^{uu} - C_2^{ud}}{S_0 \cdot u^2 - S_0 \cdot ud} = \frac{9.9735 - 2.4874}{39.6750 - 31.0500} = 0.8679,\\
\Delta_2^{uu} &= \frac{C_3^{uuu} - C_3^{uud}}{S_3^{uuu} - S_3^{uud}} = \frac{15.6262 - 5.7075}{45.6262 - 35.7075} = 1.
\end{aligned}$$

In fact, on the sample path (up, up, down), we are at node N_1^u at time δt, where we should hold $\Delta_1^u = 0.8679$ shares; then, at time $2 \cdot \delta t$, we are at node N_2^{uu}, where we should hold one stock share. No calculation is actually needed, as the option will be in-the-money for sure at maturity, conditional on being at node N_2^{uu}; so we should hold one share. Hence, we should adjust delta from 0.8679 to 1 and buy

$$1 - 0.8679 = 0.1321$$

additional stock shares.

Problem 13.8 We first calibrate the lattice:

$$\begin{aligned}
u &= e^{\sigma\sqrt{\delta t}} = e^{0.50 \cdot \sqrt{0.25}} = 1.284,\\
d &= \frac{1}{d} = 0.7788,\\
\pi_u &= \frac{e^{r \cdot \delta t} - d}{u - d} = \frac{e^{0.03 \cdot 0.25} - 0.7788}{1.284 - 0.7788} = 0.4527.
\end{aligned}$$

The discount factor is $e^{-0.03 \cdot 0.25} = 0.9925$.

The lattice for the underlying asset price is

50.0000	64.2013	82.4361	105.8500
	38.9400	50.0000	64.2013
		30.3265	38.9400
			23.6183

The corresponding lattice for the option prices is:

$$\begin{array}{cccc} 14.3394 & 6.2138 & 0 & 0 \\ & 21.2585 & 11.4395 & 0 \\ & & 29.6735 & 21.0600 \\ & & & 36.3817 \end{array}$$

The last time layer consists of the option payoffs; the put option is out-of-the-money for large stock prices.

Then we roll backwards to time 2:

- At node N_2^{uu} we clearly have $P_2^{uu} = 0$, since the put is out-of-the-money at both successor nodes.
- At node N_2^{ud}, the put is in the money, with intrinsic value $I_2^{ud} = 60 - 50 = 10$. The continuation value is

$$0.9925 \cdot \big[0.4527 \cdot 0 + (1 - 0.4527) \cdot 21.06\big] = 11.4395,$$

  which is larger than the intrinsic value, so it is not optimal to exercise and $P_2^{ud} = 11.4395$.
- At node N_2^{dd}, the put is in the money, with intrinsic value $I_2^{dd} = 60-30.3265 = 29.6735$. The continuation value is

$$0.9925 \cdot \big[0.4527 \cdot 21.06 + (1 - 0.4527) \cdot 36.3817\big] = 29.2252,$$

  which is smaller than the intrinsic value, so it is optimal to exercise and $P_2^{dd} = 29.6735$.

By a similar token:

$$\begin{aligned}
P_1^u &= \max\big\{60 - 64.2013, 0.9925 \cdot \big[0.4527 \cdot 0 + (1 - 0.4527) \cdot 11.4395\big] \\
&= \max\{-4.2013, 6.2138\} = 6.2138, \\
P_1^d &= \max\big\{60 - 38.9400, 0.9925 \cdot \big[0.4527 \cdot 11.4395 + (1 - 0.4527) \cdot 29.6735\big] \\
&= \max\{21.06, 21.2585\} = 21.2585, \\
P_0 &= \max\big\{60 - 50, 0.9925 \cdot \big[0.4527 \cdot 6.2138 + (1 - 0.4527) \cdot 21.2585\big] \\
&= \max\{10, 14.3394\} = 14.3394.
\end{aligned}$$

The calibration of the model of the underlying asset price dynamics does not depend on the option. Hence, we could use it for an Asian-style option as well. However, we cannot recombine the lattice, and should use a binomial tree. In practice, we may use a multinomial tree only to price a Bermudan-style Asian option, with a limited set of exercise opportunities. The European-style case may be tackled by random Monte Carlo sampling. The American-style case, when several observations define the average, is quite challenging, but we may use approximate dynamic programming.

Problem 13.9 There is a **typo** (sorry!) in the book, as the payoff of this option is $S_T - S_{\min}$, rather than $S_{\max} - S_{\min}$.

We calibrate the lattice first:

$$\begin{aligned}
u &= e^{\sigma\sqrt{\delta t}} = e^{0.35\sqrt{0.25}} = 1.1912, \\
d &= \frac{1}{u} = 0.8395, \\
\pi_u &= \frac{e^{r\cdot\delta t} - d}{u - d} = \frac{e^{0.05\cdot 0.25} - 0.8395}{1.1912 - 0.8395} = 0.4921.
\end{aligned}$$

We may use a binomial tree, but we cannot recombine, as the option is path-dependent. Hence, we evaluate the payoff for each sample path as follows:

Scenario	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$S_{\min}$	Payoff
1: *uuu*	50	59.56	70.95	84.52	50	34.52
2: *uud*	50	59.56	70.95	59.56	50	9.56
3: *udu*	50	59.56	50	59.56	50	9.56
4: *udd*	50	59.56	50	41.97	41.97	0
5: *duu*	50	41.97	50	59.56	41.97	17.59
6: *dud*	50	41.97	50	41.97	41.97	0
7: *ddu*	50	41.97	35.23	41.97	35.23	6.74
8: *ddd*	50	41.97	35.23	29.58	29.58	0

To find the option price, we need to compute and discount the expected payoff, considering the risk-neutral probability of each path:

$$L_0 = e^{-0.05 \cdot 0.75} \cdot \big[0.4921^3 \cdot 34.52 + 0.4921^2 \cdot (1 - 0.4921) \cdot (9.56 + 9.56 + 17.59) \\ + 0.4921 \cdot (1 - 0.4921)^2 \cdot 6.74\big] = 9.1359.$$

If we consider an option with payoff $S_{\max} - S_{\min}$, the procedure is similar:

Scenario	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$S_{\min}$	$S_{\max}$	Payoff
1: *uuu*	50	59.56	70.95	84.52	50	84.52	34.52
2: *uud*	50	59.56	70.95	59.56	50	70.95	9.56
3: *udu*	50	59.56	50	59.56	50	59.56	9.56
4: *udd*	50	59.56	50	41.97	41.97	59.56	17.59
5: *duu*	50	41.97	50	59.56	41.97	59.56	17.59
6: *dud*	50	41.97	50	41.97	41.97	50	8.03
7: *ddu*	50	41.97	35.23	41.97	35.23	50	14.77
8: *ddd*	50	41.97	35.23	29.58	29.58	50	20.42

Rather unsurprisingly, we find a quite expensive option:

$$SW_0 = e^{-0.05 \cdot 0.75} \cdot \big[0.4921^3 \cdot 34.52 + 0.4921^2 \cdot (1 - 0.4921) \cdot (9.56 + 9.56 + 17.59) \\ + 0.4921 \cdot (1 - 0.4921)^2 \cdot (17.59 + 8.03 + 14.77) + (1 - 0.4921)^3 \cdot 20.42\big] = 15.8267.$$

Needless to say, the accuracy of a three-step binomial tree is not quite adequate. Some analytical formulas are available for lookback options in the case of continuous monitoring, and Monte Carlo methods could be easily applied for discrete monitoring or more realistic models than GBM, in the case of European-style options. We should also mention that it is possible to adapt recombining binomial lattices to price lookback options, as shown in an online technical note supplementing the textbook by John Hull; this may come in very handy to price American-style lookbacks.

Problem 13.10 At time $t = T_1$ we must make a decision by comparing the values of call and put options, when the stock price is $S_1 = S(T_1)$ and time-to-maturity is $T_2 - T_1$. To answer the first question, we apply put–call parity at time $t = T_1$:

$$P_1 + S_1 = C_1 + Ke^{-r \cdot (T_2 - T_1)},$$

which yields a critical price $S^* = Ke^{-r\cdot(T_2-T_1)}$. When $S_1 > S^*$, we should choose the call; when $S_1 < S^*$, we should choose the put (indeed, the as-you-like-option is also called *chooser* option).

It turns out that a chooser option may be priced analytically in the GBM world, but let us apply the idea to the binomial lattice, which is calibrated as usual:

$$u = e^{0.31\cdot\sqrt{0.25}} = 1.168, \quad d = \frac{1}{u} = 0.856, \quad \pi_u = \frac{e^{-0.06\cdot 0.25} - 0.856}{1.168 - 0.856} = 0.51,$$
$$D = e^{-0.06\cdot 0.25} = 0.9851.$$

The lattice for the stock price is

$$\begin{array}{ccc} & & 74.99 \\ & 64.22 & \\ 55 & & 55 \\ & 47.10 & \\ & & 40.34 \end{array}$$

We find the values of the call and put options at time $T_1 = 0.25$ by rolling back the respective lattices,

$$\begin{array}{ccc} & & 19.99 \\ & 10.04 & \\ C_0 & & 0 \\ & 0 & \\ & & 0 \end{array} \qquad\qquad \begin{array}{ccc} & & 0 \\ & 0 & \\ P_0 & & 0 \\ & 07.08 & \\ & & 14.66 \end{array}$$

The values of the call and put options at time $t = 0$ are not relevant. In this case, the choice is trivial, since we choose the one option that is in-the-money at each node. The chooser option value at time $t = 0$

$$V = 0.9851 \cdot \big[0.51 \cdot 10.04 + (1 - 0.51) \cdot 07.08\big] = 8.46.$$

Problem 13.11 The real-world drift is irrelevant. The call delta is

$$\Delta_{\mathsf{C}} = \Phi\big(d_1\big),$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\log(37/40) + (0.06 + 0.30^2/2)/3}{0.30\sqrt{1/3}} = -0.2480.$$

Hence, using tables or any software with basic statistical functions, we find

$$\Delta_{\mathsf{C}} = \Phi\big(-0.2480\big) = 0.4021.$$

Problem 13.12 The real-world drift is irrelevant. The put delta is

$$\Delta_{\mathsf{P}} = \Delta_{\mathsf{C}} - 1 = \Phi\big(d_1\big) - 1,$$

where

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\log(47/35) + (0.03 + 0.45^2/2)\cdot 5/12}{0.45\sqrt{5/12}} = 1.2032.$$

Hence, using tables or any software with basic statistical functions, we find

$$\Delta_{\mathsf{P}} = \Phi\big(1.2032\big) - 1 = -0.1145.$$

The delta of the portfolio is

$$\Delta_{\mathsf{portf}} = 100 \cdot \Delta_{\mathsf{P}} + h,$$

where h is the position in stock shares, and the delta of the stock share itself is just $\Delta_{\mathsf{S}} = 1$. Setting the portfolio delta to zero, we find

$$h = -100 \cdot \Delta_{\mathsf{P}} = 11.45.$$

To immunize the risk of a *long* position in put options, we should hold a long position in stock shares, which will compensate the drop in the put price, if the stock share price goes up.

Problem 13.13 As a first step, let us find the deltas and gammas for the three options (referred to as option a, b, and c, respectively). We need the terms d_1:

$$d_1^a = \frac{\log(30/27) + (0.03 + 0.25^2/2)/4}{0.25 \cdot \sqrt{1/4}} = 0.9654,$$
$$d_1^b = \frac{\log(30/30) + (0.03 + 0.25^2/2)/3}{0.25 \cdot \sqrt{1/3}} = 0.1415,$$
$$d_1^c = \frac{\log(30/28) + (0.03 + 0.25^2/2)/2}{0.25 \cdot \sqrt{1/2}} = 0.7760.$$

Then, we find deltas:

$$\Delta^a = \Phi(0.9654) - 1 = -0.1672,$$
$$\Delta^b = \Phi(0.1415) = 0.5562,$$
$$\Delta^c = \Phi(0.7760) = 0.7811.$$

Note the negative delta of the put and the large delta of the last call, which is currently in-the-money. We also find gammas, which we need for the second question:

$$\Gamma^a = \frac{\phi(0.9654)}{40 \cdot 0.45 \cdot \sqrt{1/4}} = 0.0668,$$
$$\Gamma^b = \frac{\phi(0.1415)}{40 \cdot 0.45 \cdot \sqrt{1/3}} = 0.0912,$$
$$\Gamma^c = \frac{\phi(0.7760)}{40 \cdot 0.45 \cdot \sqrt{1/2}} = 0.0964.$$

Denoting the holding of stock shares by h_s, the value of the portfolio is

$$h_s \cdot S - 1000 \cdot P^a + 500 \cdot C^b - 1500 \cdot C^c,$$

with delta

$$h_s - 1000 \cdot \Delta^a + 500 \cdot \Delta^b - 1500 \cdot \Delta^c,$$

and gamma

$$-1000 \cdot \Gamma^a + 500 \cdot \Gamma^b - 1500 \cdot \Gamma^c.$$

To set delta to zero, we need to hold

$$h_s = 1000 \cdot (-0.1672) - 500 \cdot 0.5562 + 1500 \cdot 0.7811 = 726.395$$

stock shares, which we may round to 726.

If we want a gamma-neutral portfolio, this cannot be achieved by using stock shares, as their gamma is zero. We must use nonlinear instruments, i.e., other options, or a change in the current position in options. If we change the holding of the last call option, however, this will change delta as well, undoing the delta-neutrality we have just achieved. We find the new holding in shares and call options by solving a system of two equations:

$$\begin{aligned} h_s + h_c \cdot \Delta^c &= 1000 \cdot \Delta^a - 500 \cdot \Delta^b, \\ h_c \cdot \Gamma^c &= 1000 \cdot \Gamma^a - 500 \cdot \Gamma^b. \end{aligned}$$

This system is in upper triangular form. We first find h_c to get gamma right,

$$h_c = \frac{1000 \cdot 0.0668 - 500 \cdot 0.0912}{0.0964} = 219.37.$$

We should hold a long position in the second call, buying $1500 + 219.37$ options to offset the current short position. Then, we adjust delta:

$$h_s = 1000 \cdot (-0.1672) - 500 \cdot 0.5562 - 219.37 \cdot 0.7811 = -616.65.$$

Now, we should hold a short position in the stock.

Problem 13.14 The portfolio value is

$$V = 1000C - 1500P,$$

and its vega is

$$\frac{\partial V}{\partial \sigma} = 1000\frac{\partial C}{\partial \sigma} - 1500\frac{\partial P}{\partial \sigma}.$$

Using the formula

$$\mathcal{V} = \phi(d_1) S_t \sqrt{\tau},$$

we find that the two vegas are:

$$\begin{aligned} \mathcal{V}_C &= 50 \cdot \sqrt{0.5} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{0.1806^2}{2}\right\} = 13.8767, \\ \mathcal{V}_P &= 50 \cdot \sqrt{\frac{2}{12}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \exp\left\{-\frac{0.8555^2}{2}\right\} = 5.6479. \end{aligned}$$

Then, the portfolio vega is

$$1000 \cdot 13.8767 - 1500 \cdot 5.6479 = 5404.79.$$

Since this is positive, we are long vega: An increase in volatility will increase the value of the portfolio.

Problem 13.15 The formula of vega is the same for call and put options, and the deltas of a call and a put differ by a constant. Hence, vanna is the same for call and put options.

Let us recall the formulas for call delta and vega:

$$\begin{aligned}\Delta &= \Phi(d_1),\\ \mathcal{V} &= \phi(d_1)S_t\sqrt{\tau},\\ d_1 &= \frac{\log(S_t/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}},\end{aligned}$$

where $\tau = T - t$ is time-to-maturity.

If we consider vanna as the partial derivative of vega with respect to S_t, we have:

$$\begin{aligned}\frac{\partial^2 C}{\partial S_t \partial\sigma} &= \frac{\partial C}{\partial S_t}\left[\phi(d_1)S_t\sqrt{\tau}\right] = \sqrt{\tau}\cdot\left[\phi(d_1)+S_t\cdot\phi(d_1)\cdot\big(-d_1\big)\cdot\frac{1}{S_t\sigma\sqrt{\tau}}\right]\\ &= \sqrt{\tau}\cdot\phi(d_1)\left[1-\frac{d_1}{S_t\sigma\sqrt{\tau}}\right] = \frac{\phi(d_1)}{\sigma}\cdot\left[\sigma\sqrt{\tau}-d_1\right] = -\frac{\phi(d_1)}{\sigma}\cdot d_2,\end{aligned}$$

where we use $d_2 = d_1 - \sigma\sqrt{\tau}$ and the fact that the PDF of a standard normal is just an exponential function

$$\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

to which we apply the chain rule to find derivatives of a composite function.

We may also go the other way around:

$$\begin{aligned}\frac{\partial^2 C}{\partial\sigma\partial S_t} &= \frac{\partial C}{\partial\sigma}\Phi(d_1) = \phi(d_1)\cdot\frac{\partial d_1}{\partial\sigma} = \phi(d_1)\cdot\frac{\sigma\tau\cdot\sigma\sqrt{\tau}-\sigma\sqrt{\tau}d_1\cdot\sqrt{\tau}}{\sigma^2\tau}\\ &= \phi(d_1)\cdot\frac{\sigma\tau-d_1}{\sigma} = -\frac{\phi(d_1)}{\sigma}\cdot d_2.\end{aligned}$$

Problem 13.16 We may decompose this option as a portfolio of binary call options. A binary call option pays \$1 if $S_T \geq K$, 0 otherwise, and its price is

$$B(K) = e^{-rT}\cdot \mathrm{P}_{\mathbb{Q}_n}\big\{S_T\geq K\big\} = e^{-rT}\cdot\Phi\big[d_2(r,K)\big],$$

where $d_2(r,K)$, evaluated with risk-neutral drift r and strike K, is the familiar formula

$$d_2 = \frac{\log(S_0/K)+\big(r-\sigma^2/2\big)T}{\sigma\sqrt{T}}.$$

We should:

- Buy 5 binaries with strike 50 to create the value 5 in the range $50 \leq S_T < 60$.
- Buy 5 more binaries with strike 60 to create the value 10 in the range $60 \leq S_T < 70$.
- Write 9 binaries with strike 70 to create the value 1 in the range $70 \leq S_T$.

Hence, the price is

$$5B(50)+5B(60)-9B(70) = 5\cdot 0.4747+5\cdot 0.1684-9\cdot 0.0426 = 2.8320.$$

Problem 13.17 The put payoff is €10 if $S_T = 40$, and it is €20 if $S_T = 30$. Thus, we are looking for the *real-world* probability

$$\mathrm{P}\{30 \le S_T \le 40\} = \mathrm{P}\{S_T \ge 30\} - \mathrm{P}\{S_T \ge 40\}.$$

We recall that, in the risk-neutral world,

$$\mathrm{P}_{\mathbb{Q}_n}\{S_T \ge K\} = \Phi\big(d_2(r, K)\big),$$

where we evaluate d_2 using the risk-neutral drift r. This is what we need, e.g., to price a binary option with strike K (see Problems 13.16 and 13.18). However, here we need probabilities under the real-world measure, with drift μ. Let us define the function

$$\Psi(\mu, K) \doteq \Phi\big(d_2(\mu, K)\big),$$

where $d_2(\mu, K)$ is the d_2 term evaluated with real drift μ and strike K. Thus, for instance,

$$d_2(0.10, 30) = \frac{\log(40/30) + (0.1 - 0.40^2/2)/2}{0.4 \cdot \sqrt{0.5}} = 1.0525,$$

and

$$\mathrm{P}\{S_T \ge 30\} = \Psi(0.10, 40) = \Phi(1.0525) = 0.8537.$$

Note that we are not using the actual option strike (50) as K, but the relevant stock price (30), for which the payoff is 20. Then, repeating for a "strike" 40, we find the desired probability as

$$\mathrm{P}\{30 \le S_T \le 40\} = \Psi(0.10, 30) - \Psi(0.10, 40) = 0.8537 - 0.5141 = 0.3396.$$

More generally, we should realize that we must use the risk-neutral measure for pricing, but we need the real measure for other purposes, most notably risk measurement.

Problem 13.18 The initial step in the profit diagram may be obtained by buying 5 binary calls with strike 10. To increase slope from 0 to 1, we buy a vanilla call with strike 20, and to bring it back to zero we write a call with strike 30. Finally, to set the portfolio value to zero, for $S_T > 35$, we sell 15 binaries with strike 35. Thus, the value of the option is

$$5B(10) + C(20) - C(30) - 15B(35),$$

where $C(K)$ is the BSM formula for a call with strike K, and $B(K) = e^{-rT}\Phi\big(d_2(K)\big)$ is the price of a digital (binary) call. We recall that $\Phi\big(d_2\big)$ is the *risk-neutral* probability that $S_T > K$, which is what we need to *price* the option; see Problem 13.17 for a different use of the formula.

Problem 13.19 Let us denote the time-to-maturity of the option by $\tau \equiv T - t$. At time $t = 0$, this is just $\tau = T$, but it is better to be more general (especially when we need to answer the second question).

By risk-neutral pricing, the option price is

$$f(S_t, t) = e^{-r(T-t)} \cdot \mathrm{E}_{\mathbb{Q}_n}\big[S_T^2 \,|\, S_t\big].$$

There are a few ways to evaluate the required expectation under the GBM model.

One possibility is to write the stochastic differential equation for S_t^2, which is easily found by using Itô's lemma:

$$dY_t = \left(2r + \sigma^2\right) Y_t\, dt + 2\sigma Y_t\, dW_t,$$

where $Y_t = S_t^2$. We know that, for a GBM following the equation $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$, $\mathrm{E}\left[S_T \,|\, S_t\right] = S_t e^{\mu(T-t)}$. Hence,

$$\mathrm{E}_{\mathbb{Q}_n}\left[S_T^2 \,|\, S_t\right] = S_t^2 e^{\left(2r+\sigma^2\right)\cdot\left(T-t\right)}.$$

An alternative way to find the same expression is by rewriting $S_T^2 \mid S_t$ as the square of a lognormal variable:

$$\begin{aligned} S_T^2 \,|\, S_t &= \left\{ S_t \cdot \exp\left[\left(r - \frac{\sigma^2}{2}\right) \cdot (T-t) + \sigma\sqrt{T-t}\cdot\epsilon\right]\right\}^2 \\ &= S_t^2 \cdot \exp\left[\left(2r - \sigma^2\right) \cdot (T-t) + 2\sigma\sqrt{T-t}\cdot\epsilon\right]. \end{aligned}$$

Using the formula for the expected value of a lognormal variable,[1] we find again

$$\mathrm{E}_{\mathbb{Q}_n}\left[S_T^2 \,|\, S_t\right] = S_t^2 \cdot \exp\left[\left(2r - \sigma^2\right) \cdot (T-t) + 2\sigma^2 \cdot (T-t)\right] = S_t^2 e^{\left(2r+\sigma^2\right)\cdot\left(T-t\right)}.$$

Therefore

$$f(S_t, t) = S_t^2 e^{\left(r+\sigma^2\right)\cdot\left(T-t\right)}.$$

Let us check that this expression satisfies the BSM equation:

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\left(r + \sigma^2\right) \cdot S_t^2 e^{\left(r+\sigma^2\right)\cdot\left(T-t\right)} = -\left(r + \sigma^2\right) \cdot f, \\ \frac{\partial f}{\partial S_t} &= 2S_t e^{\left(r+\sigma^2\right)\cdot\left(T-t\right)} = 2\frac{f}{S_t}, \\ \frac{\partial^2 f}{\partial S_t^2} &= 2e^{\left(r+\sigma^2\right)\cdot\left(T-t\right)} = 2\frac{f}{S_t^2}. \end{aligned}$$

By plugging these derivatives into the BSM equation, we find

$$\frac{\partial f}{\partial t} + rS_t \frac{\partial f}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 f}{\partial S_t^2} = -\left(r + \sigma^2\right) \cdot f + 2rS_t \frac{f}{S_t} + \frac{1}{2}\sigma^2 S_t^2 \cdot 2\frac{f}{S_t^2} = rf.$$

Problem 13.20 Given the shape of the PDE, we may consider the underlying stochastic process as a GBM following the stochastic differential equation

$$dX_t = aX_t\, dt + bX_t\, dW_t.$$

Hence, we may express the random variable X_T, conditional on $X_t = x$, as

$$X_T = x \cdot \exp\left\{\left(a - \frac{b^2}{2}\right) \cdot (T-t) + b\sqrt{T-t}\cdot\epsilon\right\},$$

[1] If $X \sim \mathsf{N}(\nu, \xi^2)$, the expected value of the lognormal variable $Y = e^X$ is $e^{\nu+\xi^2/2}$.

where $\epsilon \sim \mathsf{N}(0,1)$ as usual. The log in the function defining the terminal condition allows considerable simplifications when taking the conditional expectation, where only ϵ is random, with expected value 0:

$$\begin{aligned}
\mathrm{E}\big[\log\big(X_T^4\big) + k \,|\, X_t = k\big] &= \mathrm{E}\left[\log\left(\left[x \cdot \exp\left\{\left(a - \frac{b^2}{2}\right) \cdot (T-t) + b\sqrt{T-t} \cdot \epsilon\right\}\right]^4\right) + k\right] \\
&= \log\big(x^4\big) + 4 \cdot \mathrm{E}\left[\left(a - \frac{b^2}{2}\right) \cdot (T-t) + b\sqrt{T-t} \cdot \epsilon\right] + k \\
&= \log\big(x^4\big) + 4 \cdot \left(a - \frac{b^2}{2}\right) \cdot (T-t) + k.
\end{aligned}$$

Therefore, the solution is

$$V(x,t) = \log\big(x^4\big) + 4 \cdot \left(a - \frac{b^2}{2}\right) \cdot (T-t) + k.$$

This solution does satisfy the terminal condition:

$$V(x,T) = \log\big(x^4\big) + 4 \cdot \left(a - \frac{b^2}{2}\right) \cdot (T-T) + k = \log\big(x^4\big) + k.$$

Furthermore, to check that it satisfies the PDE, we find the partial derivatives

$$\frac{\partial V}{\partial t} = -4 \cdot \left(a - \frac{b^2}{2}\right), \qquad \frac{\partial V}{\partial x} = \frac{4}{x}, \qquad \frac{\partial^2 V}{\partial x^2} = -\frac{4}{x^2}.$$

Then, we plug them into the equation

$$\frac{\partial V}{\partial t} + ax\frac{\partial V}{\partial x} + \frac{1}{2}b^2x^2\frac{\partial^2 V}{\partial x^2} = 0,$$

and find

$$-4 \cdot \left(a - \frac{b^2}{2}\right) + ax \cdot \frac{4}{x} - \frac{1}{2}b^2x^2 \cdot \frac{4}{x^2} = 0,$$

Problem 13.21 Let us consider two strike prices K_1 and $K_2 > K_1$. Any strike between these values may be expressed as the convex combination

$$K_\lambda = \lambda K_1 + (1-\lambda)K_2, \qquad \lambda \in [0,1].$$

We want to show that the current call price $C(K)$, as a function of strike K, is convex, i.e., that

$$C\big(K_\lambda\big) \le \lambda C\big(K_1\big) + (1-\lambda)C\big(K_2\big), \qquad \forall \lambda \in [0,1].$$

Consider a portfolio consisting of a long position in λ calls with strike K_1 and $1-\lambda$ calls with strike K_2, and a short position in one call with strike K_λ. To prove convexity of $C(K)$, we must show that the current value of the portfolio is non-negative:

$$\lambda C\big(K_1\big) - C\big(K_\lambda\big) + (1-\lambda)C\big(K_2\big) \ge 0, \qquad \forall \lambda \in [0,1].$$

Let us consider the payoff of the portfolio for each possible scenario of the underlying asset price S_T:

Condition	Payoff call K_1	Payoff call K_λ	Payoff call K_2	Total payoff
$S_T < K_1$	0	0	0	0
$K_1 \le S_T < K_\lambda$	$\lambda\big(S_T - K_1\big)$	0	0	$\lambda\big(S_T - K_1\big)$
$K_\lambda \le S_T < K_2$	$\lambda\big(S_T - K_1\big)$	$-\big(S_T - K_\lambda\big)$	0	$(1-\lambda)(K_2 - S_T)$
$K_2 \le S_T$	$\lambda\big(S_T - K_1\big)$	$-\big(S_T - K_\lambda\big)$	$(1-\lambda)\big(S_T - K_2\big)$	0

Let us check the total payoff in each scenario:

- In the range $S_T < K_1$, no call is in-the-money, and the total payoff is clearly 0.
- In the range $K_1 \le S_T < K_\lambda$, the first option yields a payoff $\lambda\big(S_T - K_1\big) \ge 0$.
- In the range $K_\lambda \le S_T < K_2$, the total payoff is

$$\begin{aligned}\lambda\big(S_T - K_1\big) - \big(S_T - K_\lambda\big) &= K_\lambda - \lambda K_1 - (1-\lambda)S_T \\ &= (1-\lambda)K_2 - (1-\lambda)S_T = (1-\lambda)\big(K_2 - S_T\big),\end{aligned}$$

  which is positive in that range.
- Finally, in the range $K_2 \le S_T$, the total payoff is

$$\lambda\big(S_T - K_1\big) - \big(S_T - K_\lambda\big) + (1-\lambda)\big(S_T - K_2\big) = 0.$$

Since the payoff of the portfolio is never negative, by no-arbitrage, its initial value is non-negative as well, which proves the claim.

Problem 13.22 We assume a year consisting of 360 days, where each month in turn consists of 30 days.[2]

It is convenient to think of the portfolio as consisting of 1000 shares of a fund, with unit value \$100 (there is a **mistake** in the problem statement; we hold shares of the *fund*, not units of the *index*). Let us denote:

- The current price of each share of the fund by $S_0 = \$100$
- The current value of the index by $I_0 = 100$
- The variations in the share price and the index by δS and δI, respectively
- The price and delta of the put on the index by $P(I, \tau)$ and $\Delta(I, \tau)$, respectively, when the index value is I and time-to-maturity is τ

We use a single-index model such that

$$\frac{\delta S}{S_0} = \beta \frac{\delta I}{I_0},$$

[2]In this problem, we replicate and extend an example proposed in the HBS Business Case n. 9-201-071, *Pine Street Capital*, by G. Chacko.

where $\beta = 1.5$, we ignore specific risk, and $\alpha = 0$, which may make sense for a short holding period (as volatility may dominate drift on the short term). The variations in the share and put price, given a variation in the index, are given (to first-order approximation) by

$$\delta S = \beta \cdot \frac{S_0}{I_0} \cdot \delta I = \beta \delta I,$$
$$\delta P = \Delta \cdot \delta I,$$

where we use $S_0 = I_0 = 100$ in the first equation (neglecting units of measurement).

Since

$$I_0 = 100,\ K = 100,\ r = 0.05,\ \tau = 60/360,\ \sigma = 0.25,$$

the put option price and delta are

$$P(100, 60/360) = 3.6516, \quad \Delta(100, 60/360) = -0.4472,$$

when the index value is $I_0 = 100$.

Now we must allocate total wealth, \$100,000, to the fund and the put option, in order to achieve delta-neutrality. Let us denote the two holdings by h_s and h_p (number of shares and puts, respectively) and solve the following system of linear equations:

$$h_s \cdot S_0 + h_p \cdot P\big(S_0\big) = 100{,}000,$$
$$h_s \cdot \beta + h_p \cdot \Delta\big(S_0\big) = 0.$$

Plugging numerical values, we have

$$h_s \cdot 100 + h_p \cdot 3.6516 = 100{,}000,$$
$$h_s \cdot 1.5 - h_p \cdot 0.4472 = 0,$$

which yields

$$h_s = 890.89, \qquad h_p = 2988.07.$$

We may convert holdings in terms of wealth:

$$W_s = 890.89 \times 100 = \$89{,}088.81, \qquad W_p = 2988.07 \times 3.6516 = \$10{,}911.19.$$

Now, let us analyze instantaneous changes in the index:

- Scenario 1, instantaneous increase by 5%: The return on the stock shares is $1.5 \cdot 0.05 = 7.5\%$, and the new put price is $P(105, 60/360) = 1.8734$. Hence, the new wealth is

  $$890.89 \times 107.5 + 2988.07 \times 1.8734 = \$101{,}368.22,$$

  with a return of 1.37% due to a convexity effect.

- Scenario 2, instantaneous decrease by 5%: The return on the stock shares is $-1.5 \cdot 0.05 = -7.5\%$, and the new put price is $P(95, 60/360) = 1.8734$. Hence, the new wealth is

  $$890.89 \times 92.5 + 2988.07 \times 6.3809 = \$101{,}473.82,$$

  with a return of 1.47% due to a convexity effect.

In this case, the hedge seems to be working pretty well. However, what if we consider non-instantaneous changes, so that the theta effect cannot be neglected?

- Scenario 1, increase by 5% after one month: The return on the stock shares is $1.5 \cdot 0.05 = 7.5\%$, and the new put price is $P(105, 30/360) = 0.995$. Hence, the new wealth is

$$890.89 \times 107.5 + 2988.07 \times 0.995 = \$98{,}743.60,$$

  with a return of -1.26%.

- Scenario 2, decrease by 5% after one month: The return on the stock shares is $-1.5 \cdot 0.05 = -7.5\%$, and the new put price is $P(95, 30/360) = 5.669$. Hence, the new wealth is

$$890.89 \times 92.5 + 2988.07 \times 5.669 = \$99{,}346.55,$$

  with a return of -0.65%.

The hedge is less effective, due to the decay in the option price over time. When considering an extended period of time, we should also consider the effect of α in the single-index model.

14

Option Pricing: Incomplete Markets

This chapter does not include problems in the book, but I will try to add some in the (hopefully not so remote) future.

15

Optimization Model Building

15.1 SOLUTIONS

Problem 15.1 We want to show that, if S_1 and S_2 are convex sets, then

$$\mathbf{x}_a, \mathbf{x}_b \in S_1 \cap S_2 \quad \Rightarrow \quad \lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_1 \cap S_2,\ \forall \lambda \in [0,1].$$

Since S_1 and S_2 are convex sets,

$$\begin{aligned} \mathbf{x}_a, \mathbf{x}_b \in S_1 \quad &\Rightarrow \quad \lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_1,\ \forall \lambda \in [0,1], \\ \mathbf{x}_a, \mathbf{x}_b \in S_2 \quad &\Rightarrow \quad \lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_2,\ \forall \lambda \in [0,1]. \end{aligned}$$

If $\mathbf{x}_a, \mathbf{x}_b \in S_1 \cap S_2$, then $\mathbf{x}_a, \mathbf{x}_b \in S_1$ and $\mathbf{x}_a, \mathbf{x}_b \in S_2$. Given convexity of S_1 and S_2, this implies that both of the following conditions are true:

$$\begin{aligned} &\lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_1,\ \forall \lambda \in [0,1], \\ &\lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_2,\ \forall \lambda \in [0,1]. \end{aligned}$$

These, in turn, imply

$$\lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S_1 \cap S_2,\ \forall \lambda \in [0,1],$$

which proves the claim.

This does not apply to the union of convex sets. For instance, the two intervals $[0,1]$ and $[2,3]$ are convex subsets of the real line, but their union is not convex.

Problem 15.2 We want to show that

$$\mathbf{x}_a, \mathbf{x}_b \in S \quad \Rightarrow \quad \lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S,\ \forall \lambda \in [0,1],$$

where $S = \{\mathbf{x} \in \mathbb{R}^n \,|\, g(\mathbf{x}) \leq 0\}$. Given the definition of S, the conditions $\mathbf{x}_a, \mathbf{x}_b \in S$ mean

$$g(\mathbf{x}_a) \leq 0, \quad g(\mathbf{x}_b) \leq 0.$$

Given convexity of function $g(\cdot)$, for any $\lambda \in [0,1]$, we have

$$g\big[\lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b\big] \le \lambda g(\mathbf{x}_a) + (1-\lambda) g(\mathbf{x}_b) \le 0,$$

which implies

$$\lambda \mathbf{x}_a + (1-\lambda)\mathbf{x}_b \in S, \ \forall \lambda \in [0,1],$$

proving the claim.

As an example, the subset of $\mathbb{R}^2$

$$x_1^2 + x_2^2 - 4 \le 0,$$

defined by the convex function $g(\mathbf{x}) = x_1^2 + x_2^2 - 4$, is convex: It is a circle of radius 2, including both the circumference (boundary) and the interior. However, the constraint

$$x_1^2 + x_2^2 - 4 = 0,$$

corresponding to the boundary circumference, does not define a convex set. Also the constraint

$$x_1^2 + x_2^2 - 4 \ge 0,$$

does not define a convex set, as it gives the (open) region outside the circle.

Problem 15.3 The L_1 norm is defined as

$$\|\mathbf{x}\|_1 \doteq \sum_{i=1}^{n} |x_i| .$$

Its dual norm $\|\cdot\|_*$ is

$$\|\mathbf{u}\|_* \doteq \max\big\{\mathbf{u}^\mathsf{T}\mathbf{x} : \|\mathbf{x}\|_1 \le 1\big\}.$$

More explicitly, we are dealing with the LP problem

$$\begin{aligned} \max \quad & \sum_{i=1}^{n} u_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} |x_i| \le 1. \end{aligned}$$

The constraint implies that $x_i \in [-1,1]$. Let i^* be the subscript corresponding to the largest u_i in absolute value (breaking ties arbitrarily). Let us rule out the trivial case $u_{i^*} = 0$, where whatever norm is just zero. Clearly, the objective function is maximized by setting the corresponding x_{i^*} to $+1$, if $u_{i^*} > 0$, or to -1, if $u_{i^*} < 0$, and $x_i = 0$ for $i \ne i^*$. Then, the optimal value of the objective is

$$\max_{i=1,\dots,n} |u_i| \equiv \|\mathbf{u}\|_\infty,$$

proving the claim.

Problem 15.4 Let us denote the weights of the *new* tracking portfolio by w_i, associated with binary variables δ_i, set to 1 when $w_i \neq 0$. These weights are collected into vector $\mathbf{w}$, whereas $\mathbf{w}^b$ collects the weights of the benchmark to be tracked. Let us denote the weights of the *current* portfolio by w_i^0.

Since we allow short-selling, we cannot rely on the fact that weights are non-negative. Hence, the link between weights and binary variables must we written as

$$w_i \leq M\delta_i, \qquad w_i \geq -M\delta_i,$$

for a suitable constant M (e.g., the max absolute value of a weight in the portfolio; choosing $M = 1$ would work in practice, but we might try to choose a smaller value, in order to improve bounds from the convex relaxation).

To limit turnover and transaction costs, we should include a penalty term in the objective function, of the form

$$\omega \cdot \sum_{i=1}^{n} |\, w_i - w_i^0 \,|,$$

where ω is a penalty coefficient. To express the penalty in linear form, we introduce non-negative variables w_i^+ and w_i^- and rewrite the penalty term, subject to an additional constraint:

$$\begin{aligned}
&\omega \cdot \sum_{i=1}^{n} \left(w_i^+ + w_i^-\right), \\
&w_i^+ + w_i^- = w_i - w_i^0, \qquad \forall i.
\end{aligned}$$

Let us introduce an uncertainty set $\mathcal{U}$, collecting covariance matrices $\boldsymbol{\Sigma}_k$, $k \in \mathcal{U}$. The objective function would be

$$\min_{\mathbf{w}} \left\{ \max_{k \in \mathcal{U}} \left(\mathbf{w} - \mathbf{w}^b\right)^{\mathsf{T}} \boldsymbol{\Sigma}_k \left(\mathbf{w} - \mathbf{w}^b\right) \right\} + \omega \cdot \sum_{i=1}^{n} \left(w_i^+ + w_i^-\right).$$

We may recast the min–max part in a computationally viable way by introducing the auxiliary variable z, and solving the following problem:

$$\begin{aligned}
\min \quad & z + \omega \cdot \sum_{i=1}^{n} \left(w_i^+ + w_i^-\right) \\
\text{s.t.} \quad & z \geq \left(\mathbf{w} - \mathbf{w}^b\right)^{\mathsf{T}} \boldsymbol{\Sigma}_k \left(\mathbf{w} - \mathbf{w}^b\right), \qquad k \in \mathcal{U}, \\
& w_i \leq M\delta_i, \qquad i = 1 \ldots, n, \\
& w_i \geq -M\delta_i, \qquad i = 1 \ldots, n, \\
& w_i^+ + w_i^- = w_i - w_i^0, \qquad i = 1 \ldots, n, \\
& \sum_{i=1}^{n} \delta_i \leq C_{\max}, \\
& \sum_{i=1}^{n} w_i = 1, \\
& \delta_i \in \{0, 1\},\ w_i^+, w_i^- \geq 0,
\end{aligned}$$

where $C_{\max}$ is the maximum portfolio cardinality. This is a mixed-integer QCQP. The continuous relaxation is a convex optimization problem.

Problem 15.5 The current price of the underlying asset, $S_i = S(t_i)$, is clearly a state variable but, since the option is path-dependent, this is not sufficient. The simplest idea is to consider the cumulated sum of prices observed so far as another state variable:

$$I_i = \sum_{j=1}^{i} S(t_j).$$

We might also consider the average or the intrinsic value.

Let us set $\delta t = T/M$ as the discretized time step. The state transition equations, are

$$S_{i+1} = S_i \cdot \exp\left[\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} \cdot \epsilon_{i+1}\right],$$
$$I_{i+1} = I_i + S_i \cdot \exp\left[\left(r - \frac{\sigma^2}{2}\right)\delta t + \sigma\sqrt{\delta t} \cdot \epsilon_{i+1}\right],$$

with initial conditions $S_0 = S(0)$ and $I_0 = 0$. The random variable ϵ_{i+1} is the disturbance, i.e., the risk factor.

The control variable is binary, related to the exercise or continuation decision, which may be left implicit in the choice between two values. In fact, the value function is just the (undiscounted) continuation value (other choices are possible) and the recursion is:

$$V_i(S_i, I_i) = \max\left\{\frac{1}{i}I_i - K,\ e^{-r\cdot\delta t} \cdot \mathrm{E}\big[V_{i+1}(S_{i+1}, I_{i+1}) \,|\, S_i, I_i\big]\right\}.$$

To be precise, there would be another state variable, binary, which is initially set to 0 and then is set to 1 in case we exercise the option, and then stays there, preventing further exercise (the control variables are constrained by states in general). In our case, we may neglect this, but we could not in the case of options with multiple exercise opportunities. Such options are traded, e.g., on energy markets.

Problem 15.6 If we want to tackle the problem by dynamic programming, we need to formalize state variables, control variables, and disturbances (risk factors). In some cases, the choice is rather constrained and unique, in other cases, there may be room for sensible alternatives. In specifying these ingredients, we are forced to clarify our hidden modeling assumptions. In this example a natural choice is the following:

State variables:

- Available wealth W_t at time instants $t = 0, 1, \ldots, T$, i.e., the beginning of each time interval. Initial state W_0 is given. The terminal wealth W_T will define the utility of bequest $q(W_T)$.
- Employment state $\lambda_t \in \mathcal{L} = \{\alpha, \beta, \eta\}$. The current employment state λ_0 is known. As we discuss below, this state is related to the amount of labor income, in a way that must be clarified, as there may be alternative modeling assumptions.

Control variables:

- Consumption C_t, for time instants $t = 0, 1, \ldots, T-1$. This decision is constrained by available wealth, $0 \leq C_t \leq W_t$, if we do not allow borrowing money.
- Fraction of saved amount invested in the risky asset, $t = 0, 1, \ldots, T-1$. The constraint on this control variable is $\alpha_t \in [0, 1]$.

Risk factors:

- Return R_t from the risky asset, $t = 1, 2, \ldots, T$. The probability distribution is assumed to be independent of time and state. So, returns are a sequence of i.i.d. random variables.
- Labor income $L_{t+1}(\lambda_t)$, which is relevant for $t = 0, 1, \ldots, T-1$. There is a subtle modeling choice here. We assume that income is collected *during* a time period $(t-1, t)$ and is available at the end of that period. Furthermore, we assume that the employment state observed at time instant t defines the income L_{t+1} collected later, during the time period $(t, t+1)$, and available for consumption at time instant $t+1$. An alternative (and perfectly legitimate) modeling choice would be to assume that when we are at the beginning a time period, we have no idea about the labor income that we will receive during the time period. Available wealth W_0 includes the income during time period $(-1, 0)$; terminal wealth W_T will include labor income L_T, earned during the time period $(T-1, T)$, depending on employment state λ_{T-1}. The employment state λ_T is undefined. Labor income is assumed to be a deterministic function of the employment state variable, but the employment state (which follows a Markov chain), might be used to define a state-dependent probability distribution.

The state transition equation for wealth is

$$W_{t+1} = (W_t - C_t) \cdot [1 + r_f + \alpha_t (R_{t+1} - r_f)] + L_{t+1}, \qquad t = 0, 1, \ldots, T-1.$$

This is a discrete-time, continuous-state Markov process, partially controlled by control decisions C_t and α_t. The process of the employment state is a discrete-time Markov chain, with discrete states, governed by conditional probabilities

$$\pi_{ij} = \mathrm{P}\left\{\lambda_{t+1} = j \mid \lambda_t = i\right\}, \qquad i, j \in \mathcal{L},\ t = 0, 1, \ldots, T-2.$$

This evolution is assumed to be purely exogenous.

The value function is $V_t(W_t, \lambda_t)$, which satisfies the recursive functional equation

$$V_t(W_t, \lambda_t) = \max_{\alpha_t \in [0,1], C_t \in [0, W_t]} \left\{ u(C_t) + \beta \cdot \mathrm{E}\big[V_{t+1}(W_{t+1}, \lambda_{t+1}) \,|\, C_t, \alpha_t, \lambda_t\big] \right\}, \qquad t = 0, 1, \ldots, T-1,$$

with terminal condition

$$W_T(W_T, \lambda_T) = q(W_T),$$

where the employment state λ_T is actually irrelevant, and $\beta \in (0, 1]$ is a subjective discount factor. The expectation is conditional on the control choices C_t and α_t, as well as the employment state λ_t. If we do not consider utility from bequest, we may assume that the whole terminal wealth is consumed, so that $q(W_T) = u(W_T)$.

In order to tackle the problem by multistage stochastic programming with recourse, we have to define a scenario tree, which includes:

- A root node n_0, corresponding to time instant $t = 0$, where we have to make the here-and-now decision. The initial states are λ^{n_0} and W^{n_0}.
- A set of intermediate nodes, say $\mathcal{I}$, corresponding to time instants $t = 1, 2, \ldots, T-1$, where we observe realized return, collect income, and make the next consumption–saving decisions.

- A set of terminal nodes $\mathcal{S}$, corresponding to the last time instant $t = T$, where we collect the last labor income and measure the utility of bequest from terminal wealth W_T.

Each terminal node corresponds to a sample path. Each node n, with the exception of the root node n_0, has a unique parent (antecedent) node $a(n)$. Each node $n \in \mathcal{I} \cup \mathcal{S}$ is associated with an unconditional probability π^n. This unconditional probability is just the product of all the conditional probabilities along the path leading from n_0 to n. In order to discount utilities, we also use $\tau(n)$ to denote the time period of each note, so that we may discount the utility from consumption at node n as

$$\beta^{\tau(n)} u(C^n).$$

To generate the scenario tree, we need to generate sample paths of returns from the risky asset, as well as employment states. The state variables related to employment are exogenous, and if we assume, as we did with dynamic programming, that the state of employment is known at the beginning of a time interval, but the corresponding labor income is collected at the end, the income L^n associated with successor nodes of any node will be the same (we have a predictable stochastic process, just like a time-varying interest rate). Thus, for each node $n \in \mathcal{I} \cup \mathcal{S}$ we shall observe a return R^n for the risky asset, as well as an income I^n.

To avoid unnecessary nonlinearities, due to a product of control variables, we use S_r^n and S_f^n, for $n \in \{n_0\} \cup \mathcal{I}$, to denote the amounts saved and invested in the risky and risk-free asset, respectively.

The resulting model is

$$\begin{aligned}
\max \quad & u(C^{n_0}) + \sum_{n \in \mathcal{I}} \pi^n \beta^{\tau(n)} u(C^n) + \sum_{n \in \mathcal{S}} \pi^n \beta^{\tau(n)} q(W^n) \\
\text{s.t.} \quad & C^n + S_r^n + S_f^n = W^n, \qquad \forall n \in \{n_0\} \cup \mathcal{I}, \\
& W^n = S_r^{a(n)} \cdot \big(1 + R^n\big) + S_f^{a(n)} \cdot \big(1 + r_f\big) + L^n, \\
& C^n, S_r^n, S_f^n \geq 0.
\end{aligned}$$

Problem 15.7 We use subscripts $i = 1, \ldots, n$ to refer to assets, $k = 1, \ldots, m$ to refer to scenarios, and denote:

- The current holding of assets by h_i^0, so that current wealth is $\sum_{i=1}^n h_i^0 P_i^0$, where P_i^0 is the current price of each asset.
- The holding of assets after rebalancing by h_i, so that $W_T^k = \sum_{i=1}^n h_i P_i^k$ is wealth in scenario k, where P_i^k is the price of each asset in each scenario, at the end of the holding horizon.
- The total number of asset shares of type i bought/sold by b_i and s_i, respectively.
- The number of asset shares bought/sold on the platform by b_i^{p} and s_i^{p}, respectively, for each asset i.
- The *integer* number of lots bought/sold through the brokers by b_i^{bk} and s_i^{bk}, respectively.

To define the objective function, we need the expected wealth, which should not be smaller than the minimum target $W_{\min}$,

$$\overline{W} = \frac{1}{m} \sum_{k=1}^{m} W_T^k \geq W_{\min},$$

and the positive/negative deviations D_+^k and D_-^k with respect to the expected value, in each scenario. The deviations for each scenario satisfy the constraint

$$W_T^k - \overline{W} = D_+^k - D_-^k,$$

and MAD is

$$\frac{1}{m}\sum_{k=1}^{m}\left(D_+^k + D_-^k\right).$$

Note that we do not need to explictly enforce the complementarity restriction $D_+^k \cdot D_-^k = 0$, as this will be enforced by optimality when we minimize MAD.

Let us represent the percentage transaction cost by f_i and the fixed transaction cost of each lot by C_i, for assets $i = 1, \ldots, n$. Then, the budget constraint on the total transaction expenditure is

$$G = \sum_{i=1}^{n} f_i P_i^0\left(b_i^{\mathsf{p}} + s_i^{\mathsf{p}}\right) + \sum_{i=1}^{n} C_i\left(b_i^{\mathsf{bk}} + s_i^{\mathsf{bk}}\right) \le B,$$

where the first sum is related with the platform and the second one with the brokers.

We must also write inventory balance constraints for each asset (using the number L_i of assets in each lot):

$$\begin{aligned}
h_i &= h_i^0 + b_i - s_i,\\
b_i &= b_i^{\mathsf{p}} + L_i b_i^{\mathsf{bk}},\\
s_i &= s_i^{\mathsf{p}} + L_i s_i^{\mathsf{bk}}.
\end{aligned}$$

Finally, we also need a cash balance constraint equating cash flows in and out,

$$\sum_{i=1}^{n} s_i P_i^0 = \sum_{i=1}^{n} b_i P_i^0 + G,$$

i.e., what we gain by selling assets must cover what we need to buy other assets, including the total transaction expenditure. To summarize, the resulting model is:

$$\begin{aligned}
\min \quad & \frac{1}{m}\sum_{k=1}^{m}\left(D_+^k + D_-^k\right)\\
\text{s.t.} \quad & \overline{W} = \frac{1}{m}\sum_{k=1}^{m} W_T^k,\\
& \overline{W} \ge W_{\min},\\
& W_T^k - \overline{W} = D_+^k - D_-^k, \qquad k = 1, \ldots, m,\\
& G = \sum_{i=1}^{n} f_i P_i^0\left(b_i^{\mathsf{p}} + s_i^{\mathsf{p}}\right) + \sum_{i=1}^{n} C_i\left(b_i^{\mathsf{bk}} + s_i^{\mathsf{bk}}\right),\\
& G \le B,\\
& \sum_{i=1}^{n} s_i P_i^0 = \sum_{i=1}^{n} b_i P_i^0 + G,\\
& h_i = h_i^0 + b_i - s_i, \qquad \imath = 1, \ldots, n,\\
& b_i = b_i^{\mathsf{p}} + L_i b_i^{\mathsf{bk}}, \qquad \imath = 1, \ldots, n,\\
& s_i = s_i^{\mathsf{p}} + L_i s_i^{\mathsf{bk}}, \qquad \imath = 1, \ldots, n.
\end{aligned}$$

All variables are non-negative, and b_i^{bk} and s_i^{bk} are also required to be integer.

16

Optimization Model Solving

16.1 SOLUTIONS

Problem 16.1 We may use two alternative ways to prove concavity of the dual function. One is based on the straightforward application of the definition of concave function, the other one is based on a useful geometrical insight.

The dual function is defined as

$$w(\boldsymbol{\mu}) \doteq \min_{\mathbf{x}\in S} f(\mathbf{x}) + \boldsymbol{\mu}^{\mathsf{T}}\mathbf{g}(\mathbf{x}), \tag{16.1}$$

for $\boldsymbol{\mu} \geq \mathbf{0}$ (we consider only inequality constraints, but this is not really necessary; the reasoning below applies to equality constraints associated with unrestricted multipliers). Concavity means

$$w(\boldsymbol{\mu}_\lambda) \geq \lambda(\boldsymbol{\mu}_1) + (1-\lambda)(\boldsymbol{\mu}_2), \qquad \forall \lambda \in [0,1],$$

where

$$\boldsymbol{\mu}_\lambda = \lambda\boldsymbol{\mu}_1 + (1-\lambda)\boldsymbol{\mu}_2.$$

As a preliminary observation, note that, for any functions h_1 and h_2,

$$\min_{\mathbf{x}\in S} \big[h_1(\mathbf{x}) + h_2(\mathbf{x})\big] \;\geq\; \min_{\mathbf{x}\in S} h_1(\mathbf{x}) + \min_{\mathbf{x}\in S} h_2(\mathbf{x}),$$

since, when we optimize separately, we are not constrained to use the same $\mathbf{x}^*$. Then, under the condition $\lambda \geq 0$ and $1-\lambda \geq 0$,

$$\begin{aligned}
w(\boldsymbol{\mu}_\lambda) &= \min_{\mathbf{x}\in S} \Big\{ f(\mathbf{x}) + \big[\lambda\boldsymbol{\mu}_1 + (1-\lambda)\boldsymbol{\mu}_2\big]^{\mathsf{T}}\mathbf{g}(\mathbf{x}) \Big\} \\
&= \min_{\mathbf{x}\in S} \big\{ \lambda \cdot \big[f(\mathbf{x}) + \boldsymbol{\mu}_1^{\mathsf{T}}\mathbf{g}(\mathbf{x})\big] + (1-\lambda)\cdot\big[f(\mathbf{x}) + \boldsymbol{\mu}_2^{\mathsf{T}}\mathbf{g}(\mathbf{x})\big] \big\} \\
&\geq \lambda \cdot \min_{\mathbf{x}\in S} \big[f(\mathbf{x}) + \boldsymbol{\mu}_1^{\mathsf{T}}\mathbf{g}(\mathbf{x})\big] + (1-\lambda)\cdot \min_{\mathbf{x}\in S} \big[f(\mathbf{x}) + \boldsymbol{\mu}_2^{\mathsf{T}}\mathbf{g}(\mathbf{x})\big] \\
&= \lambda(\boldsymbol{\mu}_1) + (1-\lambda)(\boldsymbol{\mu}_2).
\end{aligned}$$

As an alternative approach, note that for any *fixed* $\mathbf{x} \in S$, the function

$$h_{\mathbf{x}}(\boldsymbol{\mu}) \doteq f(\mathbf{x}) + \boldsymbol{\mu}^{\mathsf{T}}\mathbf{g}(\mathbf{x})$$

is a linear affine function of $\boldsymbol{\mu}$. When we fix a vector $\boldsymbol{\mu}$ and optimize with respect to $\mathbf{x} \in S$, we pick the line with the smallest value. Hence, the dual function is the lower envelope of a (possibly infinite) family of linear affine function, which is a concave function (the upper envelope is convex).

This is a consequence of a general property of convex functions (see Boyd, Vandenberghe, Convex Optimization, pp. 80–81), stating that the pointwise maximum of a finite family of convex functions

$$H(\mathbf{x}) \doteq \min\big\{h_1(\mathbf{x}), h_2(\mathbf{x}), \ldots, h_m(\mathbf{x})\big\},$$

is convex. This may be extended to the pointwise supremum over an infinite collection of convex functions,

$$H(\mathbf{x}) \doteq \sup_{\mathbf{y} \in \mathcal{S}} f\big(\mathbf{x}, \mathbf{y}\big).$$

We are flipping everything upside down and applying the idea to the pointwise infimum of a family of concave functions. Linear affine functions are both convex and concave.

As we illustrate in Problems 16.2 and 12.3, this may result in a differentiable function or not, depending on the nature of set S and the involved functions.

Problem 16.2 If we really want to do it "by the book," we should associate three non-negative Lagrange multipliers μ_0, μ_1, and μ_2 with the inequality and the lower bound constraints, respectively, and build the Lagrangian function

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = x_1^2 + x_2^2 + \mu_0\big(4 - x_1 - x_2\big) - \mu_1 x_1 - \mu_2 x_2.$$

Stationarity with respect to primal variables yields

$$\frac{\partial \mathcal{L}}{\partial x_1} = 2x_1 - \mu_0 - \mu_1 = 0,$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = 2x_2 - \mu_0 - \mu_2 = 0.$$

Complementary slackness conditions are

$$\mu_0\big(4 - x_1 - x_2\big) = 0, \qquad \mu_1 x_1 = 0, \qquad \mu_2 x_2 = 0.$$

Case-by-case analysis is a bit annoying, but let's do it with respect to the second and third condition:

1. Case $\mu_1 > 0$, $\mu_2 > 0$: Then, we should have $x_1 = x_2 = 0$, but the inequality constraint is not satisfied.

2. Case $\mu_1 > 0$, $\mu_2 = 0$: Then, we should have $x_1 = 0$. The first stationarity condition becomes $\mu_0 = -\mu_1$, which, given non-negativity of the multipliers, can only be satisfied if $\mu_0 = \mu_1 = 0$, contradicting the assumption.

3. Case $\mu_1 = 0$, $\mu_2 > 0$: By symmetry, this is similar to the second case.

4. Case $\mu_1 = 0$, $\mu_2 = 0$: The two stationarity conditions imply $x_1 = x_2$ (take the difference of the two equations). If we assume $\mu_0 = 0$, then we should have $x_1 = x_2 = 0$,

but the inequality is not satisfied. If we assume $\mu_0 > 0$, the inequality must be active, and we find $x_1 = x_2 = 2$, with value 8.

So, the optimal value is $f^* = 8$, corresponding to the optimal solution $(2, 2)$.

A common trick to simplify the matter is assuming an "interior" solution, i.e., we neglect the lower bounds and assume $x_1, x_2 > 0$, which easily yields the above optimal solution, which is in fact interior. The advantage is that we have to deal with a single Lagrange multiplier, but of course we cannot exclude "corner" solutions a priori.

If we dualize the inequality, using a single multiplier $\mu \geq 0$, we obtain the dual function

$$\begin{aligned} w(\mu) &= \min_{x_1, x_2 \geq 0} \left\{ x_1^2 + x_2^2 + \mu(-x_1 - x_2 + 4) \right\} \\ &= \min_{x_1 \geq 0} \left\{ x_1^2 - \mu x_1 \right\} + \min_{x_2 \geq 0} \left\{ x_2^2 - \mu x_2 \right\} + 4\mu. \end{aligned}$$

The problem is decomposed into two independent subproblems. Since the two quadratic functions are convex, the first-order condition is sufficient for optimality. The optima with respect to x_1 and x_2 are obtained for

$$x_1^* = x_2^* = \frac{\mu}{2},$$

where $\mu \geq 0$. Hence, plugging this values into the dual function, we find the explicit form

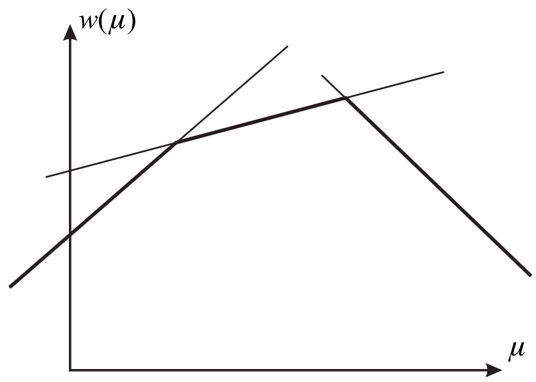
$$w(\mu) = 2 \cdot \left(\frac{\mu^2}{4} - \mu \cdot \frac{\mu}{2} \right) + 4\mu = -\frac{1}{2}\mu^2 + 4\mu.$$

This is a concave and differentiable function. The maximum of the dual function is obtained for $\mu^* = 4$, and we have $w(4) = f^* = 8$, verifying strong duality in this case.

Problem 16.3 By dualizing the inequality constraint with a multiplier $\mu \geq 0$, we obtain the dual function

$$w(\mu) = \min_{j=1,\ldots,m} \left\{ \mathbf{c}^\mathsf{T}\mathbf{x}^j + \mu\big(\mathbf{a}^\mathsf{T}\mathbf{x}^j - b\big) \right\},$$

which is the lower envelope of a finite family of affine functions, as shown in the following figure:



This is a concave function that should be maximized. Each line corresponds to a feasible solution $\mathbf{x}^j$. When two (or more) lines intersect for a given value of μ, there are multiple equivalent solutions. So, we have a nondifferentiability point when the relaxed problem has multiple optimal solutions. A solution $\mathbf{x}^j$ may be optimal for a range of values of the multiplier μ.

This case should contrasted with Problem 16.2, where there is an uncountable set of solutions in S and, due to strict convexity of the objective function, there is a unique solution for each relaxed problem as a function of μ. In that case, the dual function may be thought as the lower envelope of an infinite (uncountable) family of affine functions, and it is everywhere differentiable.