

Solutions Manual to Accompany

CLASSICAL GEOMETRY

Euclidean, Transformational,
Inversive, and Projective

I. E. LEONARD

J. E. LEWIS

A. C. F. LIU

G. W. TOKARSKY

WILEY

**Solutions Manual to Accompany
Classical Geometry**

Solutions Manual to Accompany
CLASSICAL GEOMETRY
Euclidean, Transformational,
Inversive, and Projective

I. E. Leonard

J. E. Lewis

A. C. F. Liu

G. W. Tokarsky

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Canada

WILEY

Copyright © 2014 by John Wiley & Sons, Inc. All rights reserved.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4470, or on the web at www.copyright.com. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, (201) 748-6011, fax (201) 748-6008, or online at <http://www.wiley.com/go/permission>.

Limit of Liability/Disclaimer of Warranty: While the publisher and author have used their best efforts in preparing this book, they make no representation or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives or written sales materials. The advice and strategies contained herein may not be suitable for your situation. You should consult with a professional where appropriate. Neither the publisher nor author shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

For general information on our other products and services please contact our Customer Care Department within the United States at (800) 762-2974, outside the United States at (317) 572-3993 or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print, however, may not be available in electronic formats. For more information about Wiley products, visit our web site at www.wiley.com.

Library of Congress Cataloging-in-Publication Data:

Leonard, I. Ed., 1938– author.

Solutions manual to accompany classical geometry : Euclidean, transformational, inversive, and projective / I. E. Leonard, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada, J.E. Lewis, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada, A.C.F. Liu, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada, G.W. Tokarsky, Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Canada.

pages cm

ISBN 978-1-118-90352-0 (pbk.)

I. Geometry. I. Lewis, J. E. (James Edward) author. II. Liu, A. C. F. (Andrew Chiang-Fung) author. III. Tokarsky, G. W., author. IV. Title.

QA445.L46 2014

516—dc23

2013042035

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

CONTENTS

PART I EUCLIDEAN GEOMETRY

| | | |
|----------|--------------------------------------|-----------|
| 1 | Congruency | 3 |
| 2 | Concurrency | 11 |
| 3 | Similarity | 17 |
| 4 | Theorems of Ceva and Menelaus | 27 |
| 5 | Area | 31 |
| 6 | Miscellaneous Topics | 43 |

PART II TRANSFORMATIONAL GEOMETRY

| | | |
|-----------|---|------------|
| 7 | Euclidean Transformations | 57 |
| 8 | The Algebra of Isometries | 69 |
| 9 | The Product of Direct Isometries | 81 |
| 10 | Symmetry and Groups | 97 |
| 11 | Homotheties | 107 |
| 12 | Tessellations | 117 |

PART III INVERSIVE AND PROJECTIVE GEOMETRIES

| | | |
|-----------|---|------------|
| 13 | Introduction to Inversive Geometry | 127 |
| 14 | Reciprocation and the Extended Plane | 137 |
| 15 | Cross Ratios | 145 |
| 16 | Introduction to Projective Geometry | 153 |

PART I

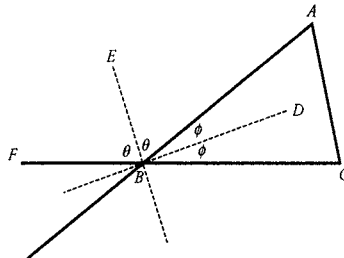
EUCLIDEAN GEOMETRY

CHAPTER 1

CONGRUENCY

1. Prove that the internal and external bisectors of the angles of a triangle are perpendicular.

Solution. Let BD and BE be the angle bisectors, as shown in the diagram below.



Then

$$\angle EBD = \angle EBA + \angle DBA = \frac{\angle FBA}{2} + \frac{\angle CBA}{2} = \frac{\angle FBA + \angle CBA}{2} = 90.$$

3. Let P be a point inside $\triangle ABC$. Use the Triangle Inequality to prove that $AB + BC > AP + PC$.

Solution. Extend AP to meet BC at D . Using the Triangle Inequality,

$$AB + BD > AD = AP + PD$$

so that

$$AB + BD + DC > AP + PD + DC.$$

Since

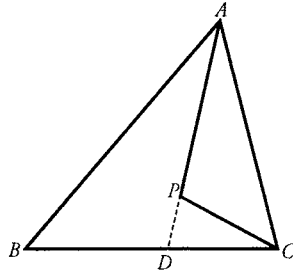
$$BD + DC = BC$$

and

$$PD + DC > PC,$$

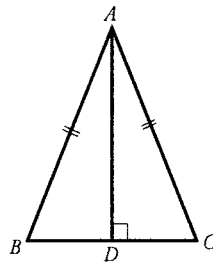
we have

$$AB + BC > AP + PC.$$



5. Given the isosceles triangle ABC with $AB = AC$, let D be the foot of the perpendicular from A to BC . Prove that AD bisects $\angle BAC$.

Solution. Referring to the diagram, the two right triangles ADB and ADC have a common side and equal hypotenuses, so they are congruent by **HSR**. Consequently, $\angle BAD \cong \angle CAD$.



7. D is a point on BC such that AD is the bisector of $\angle A$. Show that

$$\angle ADC = 90 + \frac{\angle B - \angle C}{2}.$$

Solution. Referring to the diagram, $2\theta + \beta + \gamma = 180$, which implies that

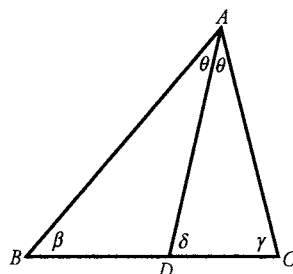
$$\theta = 90 - \frac{\beta + \gamma}{2}.$$

From the Exterior Angle Theorem, we have

$$\delta = \theta + \beta,$$

so that

$$\delta = 90 - \frac{\beta + \gamma}{2} + \beta = 90 + \frac{\beta - \gamma}{2}.$$



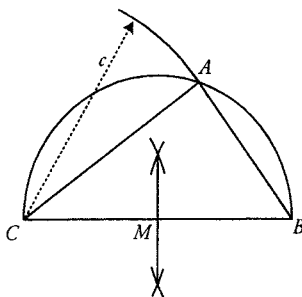
9. Construct a right triangle given the hypotenuse and one side.

Solution. We construct a right triangle ABC given the hypotenuse BC and the length c of side AC .

Construction.

- (1) Construct the right bisector of BC , yielding M , the midpoint of BC .
- (2) With center M , draw a semicircle with diameter BC .
- (3) With center C and radius equal to c , draw an arc cutting the semicircle at A .

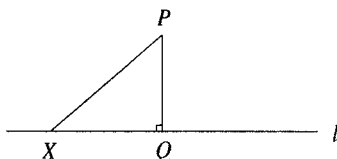
Then ABC is the desired triangle.



Justification. $\angle BAC$ is a right angle by Thales' Theorem.

11. Let Q be the foot of the perpendicular from a point P to a line l . Show that Q is the point on l that is closest to P .

Solution. Let X be any point on l with $X \neq Q$, as in the figure below.



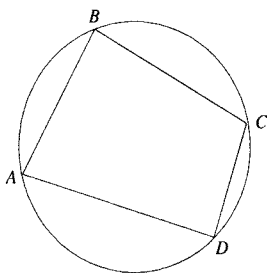
By Pythagoras' Theorem, we have

$$PX^2 = PQ^2 + XQ^2 > PQ^2,$$

and therefore $PX > PQ$.

13. Let $ABCD$ be a simple quadrilateral. Show that $ABCD$ is cyclic if and only if the opposite angles sum to 180° .

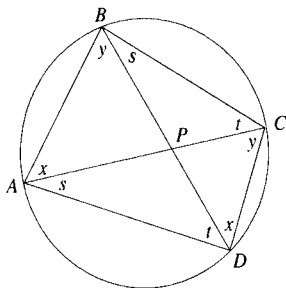
Solution. We will show that the simple quadrilateral $ABCD$ can be inscribed in a circle if and only if $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$.



Note that we only have to show that $\angle A + \angle C = 180$, since if this is true, then

$$\angle B + \angle D = 360 - (\angle A + \angle C) = 360 - 180 = 180.$$

Suppose first that the quadrilateral $ABCD$ is cyclic. Draw the diagonals AC and BD and let P be the intersection of the diagonals, then use Thales' Theorem to get the angles as shown.

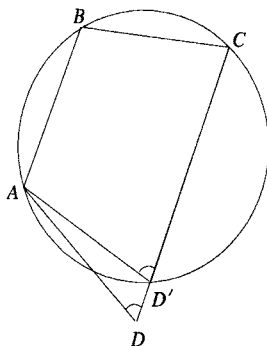


Since the sum of the internal angles in $\triangle ABC$ is 180, then

$$x + y + s + t = (x + s) + (y + t) = 180.$$

That is, $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$, so that opposite angles are supplementary.

Conversely, suppose that $\angle A + \angle C = 180$ (and therefore that $\angle B + \angle D = 180$ also) and let the circle shown on the following page be the circumcircle of $\triangle ABC$.



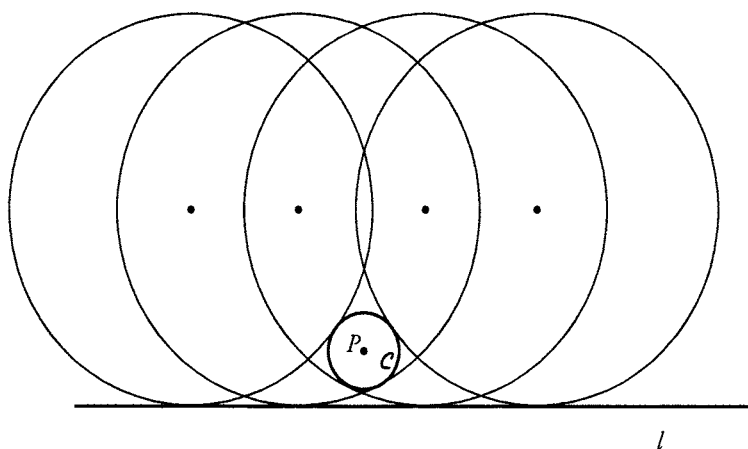
If the quadrilateral $ABCD$ is not cyclic, then the point D does not lie on this circumcircle. Assume that D lies outside the circle and let D' be the point where the line segment CD hits the circle. Since $ABCD'$ is a cyclic quadrilateral, $\angle B + \angle D' = 180$ and therefore $\angle D = \angle D'$, which contradicts the External Angle Inequality in $\triangle AD'D$.

If the point D is inside the circle, a similar argument leads to a contradiction of the External Angle Inequality.

Thus, if $\angle A + \angle C = 180$ and $\angle B + \angle D = 180$, then quadrilateral $ABCD$ is cyclic.

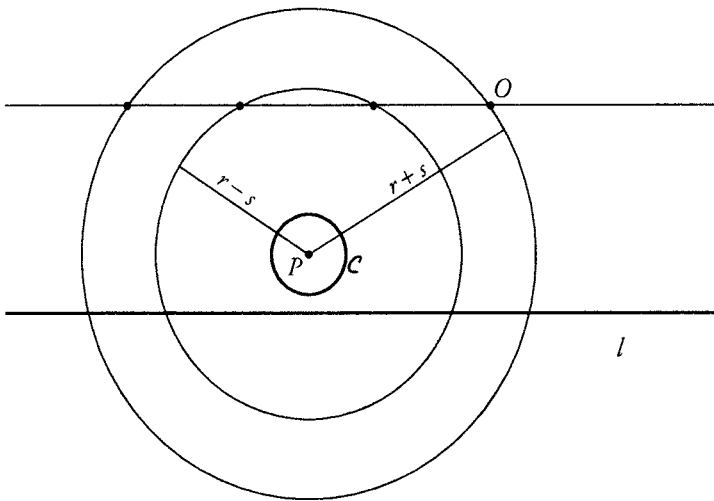
15. Given a circle $\mathcal{C}(P, s)$, a line l disjoint from $\mathcal{C}(P, s)$, and a radius r , ($r > s$), construct a circle of radius r tangent to both $\mathcal{C}(P, s)$ and l .

Note: The analysis figure indicates that there are four solutions.



Solution. We see that the centers of the circles lie on the following constructible loci:

- a line parallel to l at distance r from l
- a circle $\mathcal{C}(P, r + s)$ or a circle $\mathcal{C}(P, r - s)$



Since we are given the radius, the construction is reduced to finding the centers of the desired circles. We show how to construct one of them.

- (1) Construct a line m parallel to l at distance r from l on the same side of l as $\mathcal{C}(P, s)$.
- (2) Construct $\mathcal{C}(P, r + s)$.
- (3) Let $O = m \cap \mathcal{C}(P, r + s)$. Note that if m and $\mathcal{C}(P, r + s)$ do not intersect, there is no solution.
- (4) Construct $\mathcal{C}(O, r)$.

