Solutions to

An Introduction to Manifolds

Chapter 2 - Manifolds

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Problem 5.1. (a) It is clear that h is bijective and that it restricts to a homeomorphism

$$]-c,0[\cup[0,d]\longrightarrow]-c,0[\cup[0,d].$$

Furthermore, if $0<\varepsilon<\min\{c,d\}$, then h restricts to a homeomorphism

$$]-\varepsilon,0[\cup\{A\}\cup]0,\varepsilon[\longrightarrow]-\varepsilon,\varepsilon[.$$

Thus h is a bijective local homeomorphism, hence a homeomorphism.

(b) It follows from (a) that S is locally Euclidean of dimension 1. Let $\mathscr{B}$ be a countable basis for $\mathbb{R}\setminus\{0\}\subset S$, and let $\mathscr{B}_A=\{I_A(-c,d)\mid c,d\in\mathbb{Q}^+\}$, $\mathscr{B}_B=\{I_B(-c,d)\mid c,d\in\mathbb{Q}^+\}$. Then $\mathscr{B}\cup\mathscr{B}_A\cup\mathscr{B}_B$ is a countable basis for S. Finally, note that S is not Hausdorff since $A,B\in S$ cannot be separated by disjoint open sets, as given $c,c',d,d'>0$, the point $p=\min\{d,d'\}/2\in S$ belongs to the intersection $I_A(-c,d)\cap I_B(-c',d')$.

Problem 5.2. Let M be the sphere with a hair (at q) in $\mathbb{R}^3$. Assume that $q \in M$ has a neighbourhood U homeomorphic to some open subset $V \subset \mathbb{R}^n$ for some n. Let $\phi\colon U \to V$ be a homeomorphism. Without loss of generality we may assume that U is a connected ball $B_\varepsilon(q) \cap M$. Then $U \setminus \{q\}$ is a disjoint union of two open connected subsets U_1 and U_2, where U_1 is homeomorphic to an interval $I \subset \mathbb{R}$ and U_1 is homeomorphic to a punctured ball $B' \subset \mathbb{R}^2$. Now, ϕ restricts to a homeomorphism $U_1 \to \phi(U_1)$. Since $U_1 \subset U$ is open, $\phi(U_1) \subset \mathbb{R}^n$ is open, so $n = 1$ by the theorem on invariance of dimension. On the other hand, ϕ also restricts to a homeomorphism $U_2 \to \phi(U_2)$, where $\phi(U_1) \subset \mathbb{R}^n$ is open, so $n = 2$. This contradiction shows that there is no such U and ϕ.

Problem 5.3. We have

$$\phi_4(U_{14}) = \phi_4(U_1 \cap U_4) = \{(x, z) \in \mathbb{R}^2 \mid x > 0, x^2 + z^2 < 1\}.$$

As

$$\phi_1 \circ \phi_4^{-1}(x, z) = \phi_1(x, \sqrt{1 - x^2 - z^2}, z) = (\sqrt{1 - x^2 - z^2}, z) \quad \text{for all } (x, z) \in \phi_4(U_{14}),$$

$\phi_1 \circ \phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$.

Similarly,

$$\phi_1(U_{16}) = \{(y, z) \in \mathbb{R}^2 \mid z < 0, y^2 + z^2 < 1\}$$

and

$$\phi_6 \circ \phi_1^{-1}(y, z) = \phi_1(\sqrt{1 - y^2 - z^2}, y, z) = (\sqrt{1 - y^2 - z^2}, y) \quad \text{for all } (y, z) \in \phi_1(U_{16}),$$

so $\phi_6 \circ \phi_1^{-1}$ is C^∞ on $\phi_1(U_{16})$.

Problem 5.4. Since M is a manifold, there exists a chart $(U_{\alpha_1}, \phi_{\alpha_1})$ about p. Then

$$\phi_{\alpha_1}|_{U_{\alpha_1} \cap U} \colon V_\alpha \cap U \to \phi_{\alpha_1}(V_{\alpha_1} \cap U)$$

is a homeomorphism with $\phi_{\alpha_1}(V_{\alpha_1} \cap U) \subset \mathbb{R}^n$ open, and it is compatible with all the charts in the maximal atlas. By maximality, it must belong to this atlas, so $U_{\alpha_1} \cap U = U_{\alpha_2}$ for some α_2. Then U_{α_2} is a coordinate open set such that $p \in U_{\alpha_2} \subset U$.

Problem 5.5. Each $\phi_\alpha \times \psi_i$ is a homeomorphism with image $\phi_\alpha \times \psi_i(U_\alpha \times V_i) = \phi_\alpha(U_\alpha) \times \psi_i(V_i)$ an open subset of $\mathbb{R}^m \times \mathbb{R}^n$. Furthermore, each transition map

$$(\phi_{\alpha_1} \times \psi_{i_1}) \circ (\phi_{\alpha_2} \times \psi_{i_2})^{-1} = (\phi_{\alpha_1} \circ \psi_{\alpha_2}^{-1}) \times (\psi_{i_1} \circ \psi_{i_2}^{-1})$$

is C^∞. Hence the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i \mid U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

is an atlas on $M \times N$.

6 Smooth Maps on a Manifold

Problem 6.1. (a) Since $\psi \circ \mathrm{id}^{-1} = \psi\colon \mathbb{R} \to \mathbb{R}$ is not C^∞, the chart $(\mathbb{R}, \psi)$ is not compatible with the chart $(\mathbb{R}, \mathrm{id})$, so the two differentiable structures are distinct.

(b) Let $F\colon \mathbb{R} \to \mathbb{R}'$ be defined by $F(x) = x^3$. Then F is a bijection and

$$\psi \circ F \circ \mathrm{id}^{-1}(x) = \psi \circ F(x) = \psi(x^3) = x$$

for all $x \in \mathbb{R}$, so F is C^∞. Similarly,

$$\mathrm{id} \circ F^{-1} \circ \psi(x) = \mathrm{id} \circ F^{-1}(x^{1/3}) = \mathrm{id}(x) = x$$

for all $x \in \mathbb{R}$, so F^{-1} is C^∞. Hence, F is a diffeomorphism from $\mathbb{R}$ onto $\mathbb{R}'$.

Problem 6.2. Let $p \in M$ and let (U, ϕ) be a chart about $p \in M$. Let (V, ψ) a chart about $q_0 \in N$. Then $(U \times V, \phi \times \psi)$ is a chart about $(p, q_0) \in M \times N$, and

$$(\phi \times \psi) \circ i_{q_0} \circ \phi^{-1}(p') = (\phi \circ \phi^{-1}(p'), \psi(q_0)) = (p', \psi(q_0))$$

for all $p' \in U \times V$, which is C^∞. It follows that the inclusion $i_{q_0}\colon M \to M \times N$ is C^∞.

Problem 6.3. We are $(\mathrm{GL}(V), \phi_e)$ and $(\mathrm{GL}(V), \phi_u)$. We want to compute $\phi_u \circ \phi_e^{-1}$ and $\phi_e \circ \phi_u^{-1}$. Given $[a^i_j] \in \mathrm{GL}(n, \mathbb{R})$, the automorphism $\phi_e^{-1}([a^i_j]) = L$ of V is such that $L(e_j) = \sum_i a^i_j e_i$ for all $j = 1, \dots, n$. Then $\phi_u(L) = [b^k_\ell]$ is such that $L(u_\ell) = \sum_k b^k_\ell u_k$ for all $\ell = 1, \dots, n$.

Let $c^j_\ell, d^k_i \in \mathbb{R}$ be the scalars such that

$$u_\ell = \sum_j c^j_\ell e_j \;\text{ for all }\; \ell = 1, \dots, n \quad\text{ and }\quad e_i = \sum_k d^k_i u_k \;\text{ for all }\; i = 1, \dots, n,$$

Then

$$L(u_\ell) = \sum_j c^j_\ell L(e_j) = \sum_j c^j_\ell \left(\sum_i a^i_j \left(\sum_k d^k_i u_k \right) \right) = \sum_{i,j,k} c^j_\ell a^i_j d^k_i u_k \;\text{ for all }\; \ell = 1, \dots, n.$$

Therefore

$$\phi_u \circ \phi_e^{-1}([a^i_j]_{i,j}) = \left[\sum_{i,j} c^j_\ell a^i_j d^k_i \right]_{k,\ell},$$

so $\phi_u \circ \phi_e^{-1}$ is C^∞. Similarly, it follows that $\phi_e \circ \phi_u^{-1}$ is C^∞, so $(\mathrm{GL}(V), \phi_e)$ and $(\mathrm{GL}(V), \phi_u)$ are compatible. Thus $(\mathrm{GL}(V), \phi_u)$ and $(\mathrm{GL}(V), \phi_e)$ belong to the same maximal atlas, so they determine the same differentiable structure on $\mathrm{GL}(V)$.

Problem 6.4. Define $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $F(x, y, z) = (x, x^2 + y^2 + z^2 - 1, z)$. Then F can serve as a local coordinate system about $p = (x, y, z) \in \mathbb{R}^3$ if and only if it is a local diffeomorphism at p. By the Inverse Function Theorem this is equivalent to the condition

$$\frac{\partial(F^1, F^2, F^3)}{\partial(x, y, z)} = \det \begin{bmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{bmatrix} = 2y \neq 0.$$

It follows that F can serve as a coordinate system precisely at the points not on the y-axis.

7 Quotients

Problem 7.1. If $x \in f(f^{-1}(B))$, then $x = f(y)$ for some $y \in f^{-1}(B)$, so that $x = f(y) \in B \cap f(X)$. Conversely, if $x \in B \cap f(X)$ and $x = f(y)$ for $y \in X$, then $y \in f^{-1}(B)$, so $x = f(y) \in f(f^{-1}(B))$. If f is surjective, we have $f(X) = Y$ and thus $f(f^{-1}(B)) = B \cap f(X) = B \cap Y = B$.

Problem 7.2. We follow the hint. The function f is continuous by Proposition 7.1, and is clearly bijective. Since H^2 is compact, so is $H^2/\!\sim$. Finally, $S^2/\!\sim$ is Hausdorff by Problem 7.4. Thus, f is a continuous bijection from the compact space $H^2/\!\sim$ onto the Hausdorff space $S^2/\!\sim$, so it is a homeomorphism by Corollary A.36.

Problem 7.3. We shall prove that if $\sim$ is an open equivalence relation on a topological space S, then $S/\!\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

We follow the hint. First assume that $S/\!\sim$ is Hausdorff. Then the diagonal Δ in $(S/\!\sim) \times (S/\!\sim)$ is closed by Corollary 7.8. Let $\pi\colon S \longrightarrow S/\!\sim$ be the projection map. As π is continuous, so is $\pi \times \pi\colon S \times S \longrightarrow (S/\!\sim) \times (S/\!\sim)$. Thus $R = (\pi \times \pi)^{-1}(\Delta)$ is closed in $S \times S$. Conversely, assume that R is closed in $S \times S$. Let $[x] \neq [y] \in S/\!\sim$. Then $(x, y) \notin R$. As $(S \times S) \setminus R$ is open, there exist open neighbourhoods U of x and V of y respectively such that $(x, y) \in U \times V \subset (S \times S) \setminus R$. As π is open, $\pi(U)$ and $\pi(V)$ are open neighbourhoods of $[x]$ and $[y]$ respectively in $S/\!\sim$. If $[z] \in \pi(U) \cap \pi(V)$, then $[z] = [x'] = [y']$ for some $x' \in U$ and $y' \in V$, which is impossible as $U \times V \subset (S \times S) \setminus R$. Thus $\pi(U)$ and $\pi(V)$ are disjoint. We deduce that $S/\!\sim$ is Hausdorff.

Problem 7.4. (a) For proving a map is open, it suffices to check that the images of basic open sets are open, if we have fixed a basis for the topology of the domain. Let $\pi\colon S^n \longrightarrow S^n/\!\sim$ be the projection map. Consider the basis for S^n consisting of balls $S^n \cap B(x, \varepsilon)$ centred at points $x \in S^n$ with radii $\varepsilon < 1/2$. If $U = S^n \cap B(x, \varepsilon)$ is one of these balls, then $\pi^{-1}(\pi(U)) = S^n \cap (B(x, \varepsilon) \cup B(-x, \varepsilon))$ is open in S^2, hence $\pi(U)$ is open in $S^2/\!\sim$. It follows that $\sim$ is an open relation.

(b) For a subset $A \subset S^n$, write $-A = \{a \in S^n \mid -a \in A\}$. Let R denote the graph of $\sim$ and let $(x, y) \in (S \times S) \setminus R$. Then $[x] \neq [y]$. Since S^n is Hausdorff, there exist neighbourhoods U of x and V of y respectively such that $U, -U, V$ and $-V$ are pairwise disjoint. Then $(x, y) \in U \times V \subset (S \times S) \setminus R$. Thus R is closed, and it follows from Theorem 7.7 that $S/\!\sim$ is Hausdorff.

Problem 7.5. Given $g \in G$, note that the map $r_g\colon S \longrightarrow S$ given by $s \mapsto sg$ is a homeomorphism with inverse $r_{g^{-1}}$. Let U be open in S. Then, for $g \in G$, the set $r_g(U)$ is open in G. Thus

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} r_g(U)$$

is open in G. By definition of quotient topology, it follows that $\pi(U)$ is open in S/G. We conclude that π is an open map.

Problem 7.6. Let $p\colon \mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$ denote the projection. Consider the subsets

$$V_1 = \{[t] \in \mathbb{R}/2\pi\mathbb{Z} \mid -\pi < t < \pi\} \quad \text{and} \quad V_1 = \{[t] \in \mathbb{R}/2\pi\mathbb{Z} \mid 0 < t < 2\pi\}$$

of $\mathbb{R}/2\pi\mathbb{Z}$. Note that
$$p^{-1}(V_1) = \bigcup_{n\in\mathbb{Z}}]2\pi n - \pi, 2\pi(n+1) - \pi[$$
is open in $\mathbb{R}$, so V_1 is open in $\mathbb{R}/2\pi\mathbb{Z}$. Similarly, V_2 is open. Now, note that each restriction $p_i = p|_{p^{-1}(V_i)} : p^{-1}(V_i) \longrightarrow V_i$ of p is again a quotient map. Consider the map
$$\psi_1' : p^{-1}(V_i) \longrightarrow \mathbb{R}, \qquad t \mapsto t - 2\pi n \ \text{ if } \ t \in]2\pi n - \pi, 2\pi(n+1) - \pi[.$$
Then ψ_1' is constant on the fibres of p_1, so by Proposition 7.1 it induces a well-defined continuous map $\psi_1 : V_i \longrightarrow \mathbb{R}$ given by $\psi_i([t]) = t$, where $-\pi < t < \pi$. The map ψ_1 is a homeomorphism onto its image $\psi_1(V_1) =]-\pi, \pi[$ with inverse given by $t \mapsto [t]$. Similarly, we have a well-defined continuous map $\psi_2 : V_2 \longrightarrow \mathbb{R}$ given by $\psi_2([t]) = t$, where $0 < t < 2\pi$, and ψ_2 is a homeomorphism onto its image $]0, 2\pi[$. Thus (V_1, ψ_1) and (V_2, ψ_2) are charts on $\mathbb{R}/2\pi\mathbb{Z}$. Moreover, $\psi_1(V_1 \cap V_2) =]0, \pi[= \psi_2(V_1 \cap V_2)$ and
$$\psi_2 \circ \psi_1^{-1} : \ ]0, \pi[\longrightarrow]0, \pi[$$
is the identity, and similarly for $\psi_1 \circ \psi_2^{-1}$. Thus $\{(V_1, \psi_1), (V_2, \psi_2)\}$ is a C^∞ atlas on $\mathbb{R}/2\pi\mathbb{Z}$.

Problem 7.7. (a) Recall from Example 5.7 that
$$U_1 = \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\}, \qquad U_2 = \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\},$$
and $\phi_\alpha : U_\alpha \longrightarrow \mathbb{R}$ for $\alpha = 1, 2$ are given by the formula $\psi_1(e^{it}) = t$ in their respective domains. Recall also that
$$A = \{e^{it} \mid -\pi < t < 0\} \quad \text{ and } \quad B = \{e^{it} \mid 0 < t < \pi\}.$$

Consider the C^∞ atlas $\{(V_1, \psi_1), (V_2, \psi_2)\}$ on $\mathbb{R}/2\pi\mathbb{Z}$ given in the solution of Problem 7.6, and let $p : \mathbb{R} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ denote the projection. Note that $\overline{\phi} : S^1 \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ is given by $e^{it} \mapsto [t]$. Consider the charts (U_1, ϕ_1) and (V_1, ψ_1). Then $\phi_1((\overline{\phi})^{-1}(V_1) \cap U_1)) = \phi_1(U_1) =]-\pi, \pi[$, and $\psi_1 \circ \overline{\phi} \circ \phi_1^{-1}$ is given by $t \mapsto t$ in this domain, so it is C^∞. Similarly, $\psi_2 \circ \overline{\phi} \circ \phi_2^{-1}$ is the C^∞ map $]0, 2\pi[\longrightarrow \mathbb{R}$, $t \mapsto t$. Now consider the charts (U_1, ϕ_1) and (V_2, ψ_2). Then $\phi_1((\overline{\phi})^{-1}(V_2) \cap U_1)) = \phi_1(B) =]0, \pi[$, and $\psi_1 \circ \overline{\phi} \circ \phi_2^{-1}$ is given by $t \mapsto t$ in this domain, so it is C^∞. Similarly for the charts (U_2, ϕ_1) and (V_1, ψ_1). We deduce that $\overline{\phi}$ is C^∞.

(b) This is analogous to part (a). Considering the atlas $\{(V_1, \psi_1), (V_2, \psi_2)\}$ for $\mathbb{R}/2\pi\mathbb{Z}$ and the atlas $\{(U_1, \phi_1), (U_2, \phi_2)\}$ for S^1, all composites $\phi_i \circ F \circ \psi_j^{-1}$, for $i, j = 1, 2$, are the identities in their respective domains.

(c) Note that the maps $\overline{\phi} : S^1 \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$ and $F : \mathbb{R}/2\pi\mathbb{Z} \longrightarrow S^1$ of parts (a) and (b) are inverses to each other and C^∞. Therefore F is a diffeomorphism.

Problem 7.8. (a) We follow the hint: that $\sim$ is an open equivalence relations follows from Problem 7.5.

(b) We follow the hint: since $F(k,n)$ is second countable (being a subspace of $\mathbb{R}^{n\times k}$) and $\sim$ is open by part (a), it follows that $G(k,n) = F(k,n)/{\sim}$ is second countable by Corollary 7.10.

(c) We follow the hint. Two matrices $A = \begin{bmatrix} a_1 & \cdots & a_k \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & \cdots & b_k \end{bmatrix}$ in $F(k,n)$ are equivalent if and only if all $(k+1)\times(k+1)$ minors of $\begin{bmatrix} A & B \end{bmatrix}$ are zero (recall Problem B.1). Let $(A,B) \in (S\times S)\setminus R$. Then there is some $(k+1)\times(k+1)$ minor of $\begin{bmatrix} A & B \end{bmatrix}$ which is non-zero, say corresponding to the rows $i_1,\dots,i_{k+1}$ and columns $j_1,\dots,j_{k+1}$. Since the function $F(k,n)\times F(k,n) \to \mathbb{R}$ taking (C,D) to the $(k+1)\times(k+1)$ minor of $\begin{bmatrix} C & D \end{bmatrix}$ corresponding to the rows $i_1,\dots,i_{k+1}$ and columns $j_1,\dots,j_{k+1}$ is continuous, it follows that in some neighbourhood U of (A,B) in $F(k,n)\times F(k,n)$, for all pairs $(C,D)\in U$ the $(k+1)\times(k+1)$ minor of $\begin{bmatrix} C & D \end{bmatrix}$ corresponding to the the rows $i_1,\dots,i_{k+1}$ and columns $j_1,\dots,j_{k+1}$ is non-zero, so that $U \subset (S\times S)\setminus R$. It follows that R is closed in $S\times S$.

(d) Since the graph R in $F(k,n)\times F(k,n)$ of the equivalence relation $\sim$ is closed and $\sim$ is open by part (a), it follows from Theorem 7.7 that $G(k,n) = F(k,n)/{\sim}$ is Hausdorff.

(e) Suppose that $i = 1$ and $j = 2$. Let $A \in V_{12}$, so that A_{12} is non-singular by assumption. If $g \in \mathrm{GL}(2,\mathbb{R})$, then

$$Ag = \begin{bmatrix} A_{12}g \\ A_{13}g \end{bmatrix},$$

and $A_{12}g$ is non-singular. Thus $Ag \in V_{12}$ for all $g \in \mathrm{GL}(2,\mathbb{R})$. It is clear that a similar proof works in the case that i and j are not 1 and 2 respectively.

(f) Let $A, B \in V_{12}$ and suppose that $A \sim B$. Then there exists $g \in \mathrm{GL}(2,\mathbb{R})$ such that

$$Ag = \begin{bmatrix} A_{12}g \\ A_{13}g \end{bmatrix} = \begin{bmatrix} B_{12} \\ B_{13} \end{bmatrix},$$

and hence

$$\widetilde{\phi}_{12}(B) = B_{34}B_{12}^{-1} = (A_{34}g)(A_{12}g)^{-1} = A_{34}A_{12}^{-1} = \widetilde{\phi}_{12}(A).$$

Thus, by Proposition 7.1 we have a well-defined continuous map $\phi_{12}\colon U_{12} \to \mathbb{R}^{2\times 2}$ given by $\phi_{12}([A]) = A_{34}A_{12}^{-1}$. On the other hand, we have a map $\psi\colon \mathbb{R}^{2\times 2} \to U_{12}$ given by sending $g \in \mathbb{R}^{2\times 2}$ to the class of

$$\begin{bmatrix} I \\ g \end{bmatrix}.$$

It is clear that $\phi_{12}\circ\psi$ is the identity map of $\mathbb{R}^{2\times 2}$. Since every element of U_{12} has a representative of the form $\begin{bmatrix} I \\ g \end{bmatrix}$, it follows that $\psi\circ\phi_{12}$ is the identity on U_{12}. We deduce that ϕ_{12} is a homeomorphism.

(g) For all i, j, the homeomorphism $\phi_{ij}\colon U_{ij} \to \mathbb{R}^{2\times 2}$ is given by $\phi_{ij} = A_{k\ell}A_{ij}^{-1}$, where $\{k,\ell\} = \{1,2,3,4\}\setminus\{i,j\}$. By definition, we have

$$\phi_{12}\circ\phi_{23}^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \phi_{12}\left(\begin{bmatrix} 1 & 0 \\ a & b \\ c & d \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix}^{-1} = \frac{1}{b-a}\begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix}\begin{bmatrix} b & 0 \\ -a & 1 \end{bmatrix} = \frac{1}{b-a}\begin{bmatrix} cb-da & d \\ -a & 1 \end{bmatrix},$$

and this is defined precisely for all $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\det \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} = b - a \neq 0$. This is C^∞ being a polynomial on the entries of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(h) By definition, an element of $GL(2,4) = F(2,4)/\sim$ is a class of a 4×2 matrix of rank 2. For any such matrix A, we must have A_{ij} non-singular for some $i \neq j$ by Problem B.1. Thus $\{U_{ij} \mid 1 \leq i < j \leq 4\}$ is an open cover of $G(2,4)$ and the (U_{ij}, ϕ_{ij}) defined are charts on $G(2,4)$. They are all pairwise C^∞-compatible by a similar computation as that of (g), since any transition function is a polynomial in the entries.

Problem 7.9. We follow the hint. By Exercise 7.11, we have a homeomorphism $\mathbb{R}P^n \approx S^n/\sim$ where $S^n/\sim$ is the quotient of S^n which identifies antipodal points. Since S^n is compact, so is $S^n/\sim$. Therefore, $\mathbb{R}P^n$ is compact.

Please send comments, suggestions and corrections by e-mail, or at website.
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