Solutions to
Basic Category Theory

Chapter 5 - Limits
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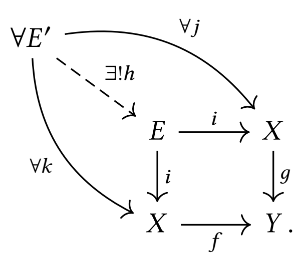
Contents

5 Limits

5.1 Limits: definition and examples

Exercise 5.1.33. This was done in Exercise 0.14(a).

Exercise 5.1.34. We will show that if E is an equaliser then it is not necessarily a pullback. If the above square is a pullback then it has the following universal property:



If the maps k and j are equal, then h indeed exists and is unique since E is a coequaliser. However, we do not expect this to be true if $k \neq j$ in general. Indeed, consider for instance the category

Set, $X = \{1, 2\}$, $Y = \{1\}$ and f, g the unique possible maps. Then the pullback is $(X \times X, \mathrm{pr}_1, \mathrm{pr}_2)$, and the equaliser is $(X, 1_X)$. Since $X \times X$ is not isomorphic to X, these are not equal.

On the other hand, the converse does hold: if the given square is a pullback then (E, i) is the equaliser of f and g. To show this, consider the diagram above which illustrates the universal property of the pullback. By taking k and j to be equal, we see that for any E' and a map $i' \colon E' \longrightarrow X$ such that $fi' = gi'$, there is a unique map $h \colon E' \longrightarrow E$ such that $ih = i'$. Hence (E, i) is the equaliser of f and g.

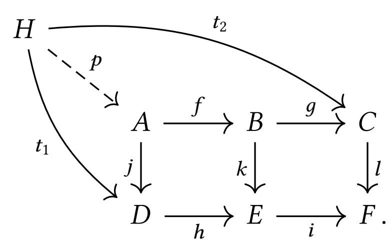
Exercise 5.1.35. (This is also Mac Lane's Exercise III.4.8.)

Assume we have a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ {\scriptstyle j}\downarrow & & {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} \\ D & \xrightarrow[h]{} & E & \xrightarrow[i]{} & F \end{array}$$

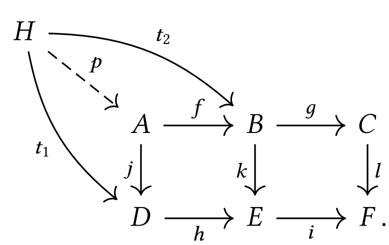
in some category $\mathscr{C}$ such that the right-hand square is a pullback.

First assume that the left-hand square is also a pullback. Let H be an object together with maps $t_1 \colon H \longrightarrow H$, $t_2 \colon H \longrightarrow C$ such that $iht_1 = lt_2$. We want to find a unique $p \colon H \longrightarrow A$ fitting in a commutative diagram as below:



Since the right-hand square is a pullback there is a unique $p' \colon H \longrightarrow B$ such that $kp' = ht_1$ and $gp' = t_2$. Then, since the left-hand square is a pullback there is a unique $p \colon H \longrightarrow A$ such that $jp = t_1$ and $p' = fp$. Then p satisfies $jp = t_1$ and $gfp = t_2$, and is unique as such. We deduce that the outer rectangle is a pullback.

Now assume that the outer rectangle is a pullback. Let H be an object of $\mathscr{C}$ and $t_1 \colon H \longrightarrow D$, $t_2 \colon H \longrightarrow B$ be such that $kt_2 = ht_1$. We want to find a unique $p \colon H \longrightarrow A$ filling the diagram



Since $lgt_2 = ikt_2 = iht_1$ and the right-hand square is a pullback, there exists a unique $p' \colon H \to B$ such that $gp' = gt_2$ and $kp' = ht_1$. As t_2 satisfies these equations, we have that $p' = t_2$. Now, since the outer rectangle is a pullback there exists a unique $p \colon H \to A$ such that $gfp = gt_2$ and $jp = t_1$. Then $gfp = gt_2$ and $kfp = hjp = ht_1$, i.e. fp satisfies the equations for p', and hence $fp = t_2$ by uniqueness. It follows that the left-hand square is a pullback.

Exercise 5.1.36. (a) Note that $(A \xrightarrow{p_I \circ h} D(I))_{I \in \mathbf{I}}$ is a cone on D. Indeed, if $I \xrightarrow{u} J$ in $\mathbf{I}$ then $Du \circ p_I \circ h = p_J \circ h$ since L is a cone. By the universal property of the limit, there is precisely one map $\widetilde{h} \colon A \to L$ such that $p_I \circ \widetilde{h} = p_I \circ h$ for all $I \in \mathbf{I}$. Since both h and h' satisfy this condition, it follows that $h = h'$.

(b) A diagram $D \colon \mathbf{I} \to \mathbf{Set}$ is a pair of sets X, Y. Its limit is the product $X \times Y$ together with the projections $\mathrm{pr}_X, \mathrm{pr}_Y$. A map $1 \to X \times Y$ is an element $(x, y) \in X \times Y$. Thus (a) in this case means that given $(x, y), (x', y') \in X \times Y$, if $x = x'$ and $y = y'$ then $(x, y) = (x', y')$.

Exercise 5.1.37. Let L denote the given set

$$\{(x_I)_{I \in \mathbf{I}} \mid x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\},$$

and $p_J \colon L \to D(J)$ be given by $p_J((x_I)_{I \in \mathbf{I}}) = x_J$ for all $J \in \mathbf{I}$. First note that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a cone on D by definition. Now let $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ be any cone on D. We need to show that there is a unique map $h \colon A \to L$ such that $f_J = p_J \circ h$ for all $J \in \mathbf{I}$. Suppose such h exists. Given $a \in A$ write $h(a) = (h_I(a))_{I \in \mathbf{I}}$. Then, for all $J \in \mathbf{I}$, $f_J(a) = p_J \circ h(a) = h_J(a)$. This shows that if h exists then it is unique. Moreover, define $h \colon A \to L$ by $h(a) = (f_I(a))_{I \in \mathbf{I}}$. Note that $f_I(a) \in D(I)$ for each I, and if $I \xrightarrow{u} J$ is an arrow in $\mathbf{I}$ we have $(Du)(f_I(a)) = f_J(a)$ since h is a cone on D. Thus h well-defined, and moreover satisfies $f_J = p_J \circ h$ for all $J \in \mathbf{I}$. It follows that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone.

Exercise 5.1.38. (a) First note that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a cone on D. Indeed, if $I \xrightarrow{u} J$ is an arrow in $\mathbf{I}$ we have $p_J = \mathrm{pr}_J \circ p = Du \circ \mathrm{pr}_I \circ p = Du \circ p_I$ as (L, p) is the equaliser of s and t. Now suppose that $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ is an arbitrary cone on D. We shall find a unique map $h \colon A \to L$ such that $f_I = p_I \circ h$ for all $I \in \mathbf{I}$. The family $(f_I)_{I \in \mathbf{I}}$ induces a unique map $f \colon A \to \prod_{I \in \mathbf{I}} D(I)$ such that $\mathrm{pr}_I \circ f = f_I$ for all $I \in \mathbf{I}$. Given $I \xrightarrow{u} J$ an arrow in $\mathbf{I}$ then $\mathrm{pr}_J \circ f = f_J = Du \circ f_I = Du \circ \mathrm{pr}_I \circ f$, so f is a map such that $s \circ f = t \circ f$. Since (L, p) is an equaliser there is a unique map $h \colon A \to L$ such that $p \circ h = f$, and this last equation is equivalent to satisfying $p_I \circ h = f_I$ for all $I \in \mathbf{I}$. We conclude that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone on D.

(b) Existence of a terminal object is equivalent to existence of the empty product. Assuming binary products, we have all finite products by iteration. Hence the same proof above as in (a) applies, assuming the category $\mathbf{I}$ to be finite.

Exercise 5.1.39. (This is also Mac Lane's Exercise III.4.10.)

Denote by $*$ the terminal object of the category $\mathscr{C}$. Given two objects X, Y of $\mathscr{C}$, consider the pullback square

$$\begin{array}{ccc} E & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & & \downarrow \\ Y & \longrightarrow & * . \end{array}$$

We claim that (E, p_X, p_Y) is the product of X and Y. Indeed, $A \in \mathscr{C}$ and a pair of maps $f_X : A \to X$ and $f_Y : A \to Y$, then the respective compositions with $X \to *$ and $Y \to *$ are equal since $*$ is terminal, so there is a unique map $f : A \to E$ such that $p_X \circ f = f_X$ and $p_Y \circ f = f_Y$. That is, (E, p_X, p_Y) satisfies the universal property of the product.

Now, given parallel arrows $f, g : X \to Y$, consider the following pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{e} & X \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle (1_X, f)} \\ X & \xrightarrow[(1_X, g)]{} & X \times Y . \end{array}$$

Then $e = e'$ and $e : E \to X$ is the equaliser of f and g (see Exercise 5.1.34.)

Thus our category has all binary products, equalisers and a terminal object, so it has all finite limits by Proposition 5.1.26(b) (proved in Exercise 5.1.38(b)).

Exercise 5.1.40. (a) Monics in **Set** are precisely the injective maps (Example 5.1.30). Now $m : X \to A$ and $m' : X' \to A$ are isomorphic in $\mathbf{Monic}(A)$ if and only if there exists maps $f : X \to X'$ and $g : X' \to X$ such that $fg = 1_Y$, $gf = 1_X$ and $m'f = m$, $mg = m'$. These conditions are equivalent to existence of a bijection $f : X \to X'$ such that $m'f = m$. If such f exists then m and m' have the same image since f is a bijection. Conversely, if m and m' have the same image, then they are bijections onto the subset $m(X) = m(X')$ of A. Then we can define $f : X \to X'$ by $f(x) = m^{-1}(m(x))$, and this is a bijection such that $m'f = m$.

It follows that an isomorphism class of objects in $\mathbf{Monic}(A)$ corresponds precisely to a subset of A, namely the image of any representative of the class.

(b) Monics in each of these categories are precisely the injective morphisms, and the same proof as before applies, since the map $f(x) = m^{-1}(m(x))$ defined at the end is indeed a morphism in the respective category. Therefore subobjects of **Grp**, **Ring** and $\mathbf{Vect}_k$ are subgroups, subrings and subspaces respectively.

(c) As in the category of sets, a map f in **Top** is monic if and only if it is injective. Now, for $A \in \mathbf{Top}$ and monics $m : X \to A$, $m' : X' \to A$ representing objects of $\mathbf{Monic}(A)$, a map from m to m' is a homeomorphism $f : X \to X'$ such that $m'f = m$. In this case $f : X \to X'$ given by $f(x) = m^{-1}(m(x))$ may not be continuous. Hence subobjects of A are not its subspaces. They are subsets $U \subset A$ equipped with a topology finer than the subspace topology inherited from A.

Exercise 5.1.41. (This is also (the dual of) Mac Lane's Exercise III.4.4.)

First assume that f is monic and consider the diagram

$$\begin{array}{ccc} X & \xrightarrow{1_X} & X \\ {\scriptstyle 1_X}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y\,. \end{array}$$

Suppose we are given arrows $g, g' \colon Z \longrightarrow X$, such that $f \circ g = f \circ g'$. We must show there is a unique $h \colon Z \longrightarrow X$ such that $1_X \circ h = g$ and $1_X \circ h = g'$. But f is monic, so $h = g = g'$ works.

Conversely, assume the diagram above is a pullback and let $g, g' \colon Z \longrightarrow X$ be such that $f \circ g = f \circ g'$. Then there is a unique $h \colon Z \longrightarrow X$ such that $1_X \circ h = g$ and $1_X \circ h = g'$, so $g = g'$.

Exercise 5.1.42. (This is also Mac Lane's Exercise III.4.5.)

Let $g, g' \colon Y \longrightarrow X'$ be arrows such that $m' \circ g = m' \circ g'$. Then $m \circ f' \circ g = f \circ m' \circ g = f \circ m' \circ g' = m \circ f' \circ g'$, so $f' \circ g = f' \circ g'$ as m is monic. Since the square is a pullback and $m \circ f' \circ g = f \circ m' \circ g$, there is a unique $h \colon Y \longrightarrow X'$ such that $m' \circ h = m' \circ g$ and $f' \circ h = f' \circ g$. Both g and g' satisfy the equations for h, so $g = g'$. It follows that m' is monic.

5.2 Colimits: definition and example

Exercise 5.2.21. First assume that $s = t$. Consider $X \xrightarrow{1_X} X$. Then $s1_X = t1_X$. Moreover, any map $Z \xrightarrow{e} X$ satisfies the condition $se = te$, and given any such map then e is the unique map $Z \longrightarrow X$ such that $1_X e = e$. Thus $(X, 1_X)$ is the equaliser of s and t. Similarly, $(Y, 1_Y)$ is the coequaliser of s and t.

Now suppose that the equaliser (Z, e) of s and t is an isomorphism. Then $se = te$ and $s = see^{-1} = tee^{-1} = t$. Similarly, if the coequaliser of s and t is an isomorphism then $s = t$.

Exercise 5.2.22. (a) By Example 5.2.9, the coequaliser of f and 1_X is the quotient $X/{\sim}$ where $\sim$ is the equivalence relation generated by $f(x) \sim x$ for all $x \in X$, together with the quotient map $X \longrightarrow X/{\sim}$. This is $X/{\sim}$ where $x \sim y$ if there exists $n \geq 1$ such that $f^n(x) = y$ or $f^n(y) = x$.

(b) In **Top**, the coequaliser is again the quotient $X/{\sim}$ as before, equipped with the quotient topology. Let $\theta \in [0, 2\pi]$ be irrational and consider the map $f \colon S^1 \longrightarrow S^1$ given by $e^{it} \mapsto e^{i(t+\theta)}$. Let $k \colon X \longrightarrow Y$ be the coequaliser of f and 1_X, where $Y = X/{\sim}$ is the quotient of X by the relation generated by $f(x) \sim x$ for all $x \in X$. Then Y is uncountable, for each equivalence class $[x]$ is countable. Let $U \subset Y$ be a non-empty open set, and $[x] \in U$. Since $k^{-1}(U)$ is open in X, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S^1 \subset k^{-1}(U)$. Then $B(z, \varepsilon) \cap S^1 \subset k^{-1}(U)$ for all $z \in X$ such that $[z] = [x]$. Now, let $z \in X$ be arbitrary. As θ is irrational, the set $A = \{f^n(x') \mid n \in \mathbb{Z}\}$ is dense in X, so there exists $w \in A \cap B(x, \varepsilon) \cap S^1$. Then $[w] = [z]$ and $w \in k^{-1}(U)$, so $k(z) = k(w) \in U$. Thus $k^{-1}(U) = X$, so $U = Y$. It follows that Y has the indiscrete topology.

Exercise 5.2.23. (a) Let i denote the inclusion $(\mathbb{N}, +, 0) \hookrightarrow (\mathbb{Z}, +, 0)$. Let $(M, +_M, 0_M)$ be a monoid and $f, g \colon (\mathbb{Z}, +, 0) \longrightarrow (M, +_M, 0_M)$ morphisms of monoids such that $fi = gi$. Then $f(n) = g(n)$

for all $n \geq 0$. Let $n < 0$. Then $0_M = f(0) = f(n-n) = f(n) +_M f(-n)$, and similarly $0_M = f(-n) +_M f(n)$, so $f(n)$ is the inverse of $f(-n)$ in M. Similarly $g(n)$ is the inverse of $g(-n)$ in M. Since inverses are unique and $f(-n) = g(-n)$, we have $f(n) = g(n)$. Thus $f(n) = g(n)$ for all $n < 0$ and therefore $f = g$.

(b) (This is also Mac Lane's Exercise I.5.4.) Let i denote the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. Let R be a ring and $\varphi, \psi \colon \mathbb{Q} \to R$ be morphisms such that $\varphi i = \psi i$. Then $\varphi = \psi$ on $\mathbb{Z}$. If $q \in \mathbb{Z} \setminus \{0\}$ then $1 = \varphi(1) = \varphi\left(\frac{q}{q}\right) = \varphi(q)\varphi\left(\frac{1}{q}\right)$, so $\varphi\left(\frac{1}{q}\right) = \varphi(q)^{-1} = \psi(q)^{-1} = \psi\left(\frac{1}{q}\right)$. Thus $\varphi\left(\frac{p}{q}\right) = \varphi(p)\varphi(q)^{-1} = \psi(p)\psi(q)^{-1} = \psi\left(\frac{p}{q}\right)$ for all $\frac{p}{q} \in \mathbb{Q}$, so $\varphi = \psi$. It follows that i is epic (and not surjective!).

Exercise 5.2.24. (a) Epics in **Set** are precisely the surjective maps (Example 5.2.18). In general, a surjective function $f \colon X \to Y$ induces an equivalence relation on X given by $x \sim x'$ if $f(x) = f(x')$. Given epics $e \colon A \to X$, $e' \colon A \to X'$, an isomorphism from e to e' is a map $f \colon X \to X'$ such that $fe = e'$ and there is a map $g \colon X' \to X$ with $ge' = e$ and $gf = 1$, $fg = 1$. This amounts to a bijection $f \colon X \to X'$ such that $fe = e'$. So assume there is such bijection. Let $a, a' \in A$. Then $e(a) = e(a') \implies fe(a) = fe(a') \implies e'(a) = e'(a')$, and similarly (using f^{-1}) $e'(a) = e'(a') \implies e(a) = e(a')$. Thus the equivalence relations induced on A by e and e' coincide. Conversely, assume that $e(a) = e(a') \iff e'(a) = e'(a')$ for all $a, a' \in A$. Define $f \colon X \to X'$ by $f(x) = e'(a)$ where a is any preimage of x under e. This is well-defined by assumption, and is a bijection such that $fe = e'$. Thus e and e' are isomorphic in $\mathbf{Epic}(A)$.

We deduce that the quotient objects of A are in canonical one-to-one correspondence with the equivalence relations of A, namely a quotient object represented by some $e \colon A \to X$ corresponds to the equivalence relation on A such that $a \sim a'$ if and only if $e(a) = e(a')$.

(b) (The proof that epic $\implies$ surjective on **Grp** can be found in Mac Lane's Exercise I.5.5.) An epimorphism $G \xrightarrow{\varphi} H$ induces an isomorphism $\widetilde{\varphi} \colon G/\ker\varphi \to H$. Two epimorphisms $G \xrightarrow{\varphi} H$, $G \xrightarrow{\psi} H'$ are isomorphic in $\mathbf{Epic}(G)$ if and only if there is an isomorphism $\theta \colon H \to H'$ such that $\theta\varphi = \psi$. If there is such θ then $\ker\varphi = \ker\psi$. Conversely, if $\ker\varphi = \ker\psi$, let $\theta = (\widetilde{\psi})^{-1}\widetilde{\varphi} \colon H \to H'$. Then θ is an isomorphism, and is given by $\varphi(g) \mapsto \psi(g)$ for $g \in G$, so $\theta\varphi = \psi$. Thus $\varphi, \psi \in \mathbf{Epic}(G)$ are isomorphic if and only if $\ker\varphi = \ker\psi$. Hence the correspondence $\varphi \leftrightarrow \ker\varphi$ is one-to-one between quotient objects of G and normal subgroups of G. (Any kernel is normal, and any normal subgroup N of G is the kernel of $G \to G/N$.)

Exercise 5.2.25. (a) Let $m \colon A \to B$ be a morphism.

First suppose that m is split monic and let $e \colon B \to A$ be such that $em = 1_A$. Consider the maps $me, 1_B \colon B \to B$. Then $mem = m1_A = 1_B m$. Moreover, if $h \colon C \to B$ is a map such that $meh = 1_B h = h$, then $eh \colon C \to A$ satisfies $m(eh) = h$, and if $h' \colon C \to A$ satisfies $mh' = h$ then $h' = emh' = eh$. Thus m is the equaliser of the maps $me \colon B \to B$ and $1_B \colon B \to B$, so it is regular monic.

Now assume that m is regular monic, and let C be an object and $f, g \colon B \to C$ maps of which m is an equaliser. Suppose that $h, h' \colon X \to A$ are maps such that $mh = mh'$. Since m is an equaliser, given any map $k \colon X \to B$ such that $fk = gk$ then there exists a unique $\widetilde{h} \colon X \to A$ such that

$m\widetilde{h} = k$. Taking $k = mh = mh'$ then both h and h' satisfy the condition for $\widetilde{h}$, so $h = h'$ by uniqueness. Thus m is monic.

(b) Let $\varphi\colon A \to B$ be monic in **Ab**, that is, φ is a monomorphism. Consider $C = B/\operatorname{im}\varphi$, and the maps $\pi\colon B \to C$, $0\colon B \to C$, where π is the projection. Then $\pi\varphi = 0\varphi = 0$, and is universal as such. Indeed, if $\psi\colon X \to B$ is a homomorphism such that $\pi\psi = 0$, then $\psi(X) \subset \varphi(A)$, and we can define a map $\theta\colon X \to A$ by $\theta(x) = a$ if $\psi(x) = \varphi(a)$. This is well-defined since φ is injective. Moreover, it is the unique map with the property $\varphi\theta = \psi$. Thus $\varphi\colon A \to B$ is the equaliser of $\pi\colon B \to \operatorname{im}\varphi$ and $0\colon B \to C$. It follows that all monics in **Ab** are regular monics.

To show that not all monics in **Ab** are split monics consider $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(1) = 2$. Then φ is a monomorphism, but there is no $\psi\colon \mathbb{Z} \to \mathbb{Z}$ such that $\psi\varphi = 1_{\mathbb{Z}}$. Indeed, any such ψ would send 2 to 1, which is not possible.

(c) We will show that the regular monics in **Top** are precisely the embeddings, i.e. the injective maps which are homeomorphisms onto their image. Let $h\colon X \to Y$ be regular monic in **Top**. In particular it is monic, i.e. it is injective. Let $\overline{h}\colon X \to h(X)$ be obtained from h by restricting its codomain. Let $f, g\colon Y \to Z$ be maps of which h is an equaliser. Consider the inclusion map $i\colon h(X) \hookrightarrow Y$. Then $fi = gi$, so there is a unique $k\colon h(X) \to X$ such that $i = hk = i\overline{h}k$. As i is monic, it follows that $\overline{h}k = 1_{h(X)}$. Thus, the inverse function $k\colon h(X) \to X$ of $\overline{h}$ is continuous, so h is an embedding. Conversely, assume that $h\colon X \to Y$ is an embedding. Consider the pushout

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & Z\,. \end{array}$$

By the construction of pushouts in **Top**, $Z = Y \amalg Y/\!\sim$ where an element $h(x) \in Y$ in the first summand is identified with the same element $h(x) \in Y$ in the second summand, and f, g are the inclusions. The equaliser of f, g is then the inclusion $Y' \to Y$ of the subspace Y' of Y where both f and g are equal, i.e. $h(X) \to Y$. Since h is an embedding this is the same as $h\colon X \to Y$.

Exercise 5.2.26. A regular epic is thus a map which is a coequaliser, and a split epic is a map with a right inverse. As in Exercise 5.2.25(a), split epic $\implies$ regular epic $\implies$ epic.

(a) Let $f\colon A \to B$ be map in a category. If f is an isomorphism, then the equations $f^{-1}f = 1$ and $ff^{-1} = 1$ imply that f is split epic and split monic, in particular monic and regular epic. Conversely, assume that f is both monic and regular epic. Let $g, h\colon C \to A$ be maps of which f is a coequaliser. Then $fg = fh$, so $g = h$ as f is monic. By Exercise 5.2.21, f is an isomorphism

(b) It suffices to prove that epic $\implies$ split epic in **Set**. Assume $f\colon A \to B$ is epic in **Set**, i.e. f is surjective. By the axiom of choice it follows that f has a section, that is, there exists $g\colon B \to A$ such that $fg = 1_B$. Thus f is split epic.

(c) In **Top** the epimorphisms are precisely the surjective (continuous) maps. Let $X \in$ **Top** and suppose that $X' \in$ **Top** is another space with the same underlying set as X, but whose topology is strictly finer. Then the identity $X' \to X$ is a map in **Top**, and is epic, but is is not split since the identity $X \to X'$ is not continuous. It follows from (b) that **Top** does not satisfy the axiom of choice.

In **Grp** the epics are precisely the surjections (c.f. Exercise 5.2.24 and Mac Lane's Exercise I.5.5.) Consider the quotient map $\pi\colon \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ of the free group $\mathbb{Z} * \mathbb{Z}$ on two generators onto its abelianisation $\mathbb{Z} \oplus \mathbb{Z}$. Then π is epic, but is not split. Indeed, no map $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ is as since $\mathbb{Z} \oplus \mathbb{Z}$ is countable and $\mathbb{Z} * \mathbb{Z}$ is uncountable. It follows from (b) that **Grp** does not satisfy the axiom of choice.

Exercise 5.2.27. First we analyse stability under pullbacks. By Exercise 5.1.42, monics are stable under pullbacks. Now consider the category generated by the graph

$$\begin{array}{ccccc} \bullet & \xrightarrow{a} & \bullet & & \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} & & \\ \bullet & \xrightarrow[e]{} & \bullet & \overset{f}{\underset{g}{\rightrightarrows}} & \bullet \end{array}$$

with the relations $fe = ge$, $eb = ca$ and $feb = geb$. Then e is the equaliser of f and g, the square is a pullback, but a is not an equaliser, as can be check. It follows that regular monics are not stable under pullbacks. Finally, note that split monics are not stable under pullbacks either: let X and Y be disjoint empty subsets of a set Z. Then

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & Z \end{array}$$

is a pullback on **Set**, where $X \hookrightarrow Z$ is split monic but $\emptyset \hookrightarrow Y$ is not.

Next, epics are not stable under pullbacks. Consider the category **Haus** of Hausdorff topological spaces. If $A \subset B$ is an inclusion of a dense subspace in **Haus**, it is epic, for if two maps into a Hausdorff space agree on a dense subset, they are equal. It follows that in the pullback diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow \\ \mathbb{R} \setminus \mathbb{Q} & \hookrightarrow & \mathbb{R} \end{array}$$

in **Haus** the map $\mathbb{Q} \hookleftarrow \mathbb{R}$ is epic, but $\emptyset \to \mathbb{R} \setminus \mathbb{Q}$ is clearly not. Now, an example similar to the one of regular monics above (where now f and g are maps into the left-bottom corner) shows that regular epics are not stable under pullbacks either.

Finally, note that split epics are stable under pullbacks. Indeed, let

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a pullback square in some category, where f is split epic. Then there exists $s\colon Z \to X$ a right inverse of f. Then the maps $1_Y\colon Y \to Y$ and $sg\colon Y \to X$ satisfy $g1_Y = fsg$, so by the universal property of the pullback there is a map $s'\colon Y \to W$ such that $f's' = 1_Y$. Thus f' is split epic.

Now we analyse stability under composition. If f and g are monic and h_1, h_2 are maps such that $fgh_1 = fgh_2$, we have $gh_1 = gh_2$ since f is monic, and in turn $h_1 = h_2$ since g is monic. Thus monics are stable under composition. Dually, epics are stable under composition too.

Finally, regular monics are not stable under compositions. Consider the full subcategory **FHaus** of **Haus** spanned by the functionally Hausdorff spaces (or completely Hausdorff spaces): those spaces X such that for any $x, y \in X$, there exists a continuous map $f\colon [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. Let $A = \{1/n \mid n \in \mathbb{Z}_+\}$, $B = A \cup \{0\}$, and C the subspace whose underlying set is $\mathbb{R}$ and having basis $\mathscr{T} \cup \{\mathbb{R} \setminus A\}$ for its topology, where $\mathscr{T}$ is a basis for the standard topology on $\mathbb{R}$. Then it can be prove that the inclusions $A \subset B$ and $B \subset C$ are regular monics in **FHaus**, but their composition is not. Details would take us too far into analysis and hence are omitted. Since regular epics in a category $\mathscr{C}$ are precisely regular monics in $\mathscr{C}^{\text{op}}$, it follows that regular epics are not stable under composition either.

5.3 Interactions between functors and limits

Exercise 5.3.8. Let $F\colon \mathscr{A} \times \mathscr{A}$ be given on objects by $F(X,Y) = X \times Y$ (for our previously made choice). For a morphism $(f,g)\colon (X,Y) \to (X',Y')$ in $\mathscr{A} \times \mathscr{A}$ define $F(f,g)\colon X \times Y \to X' \times Y'$ to be the unique map $X \times Y \to X' \times Y'$ induced by the maps $f \circ p_1^{X,Y}\colon X \times Y \to X'$, $g \circ p_2^{X,Y}\colon X \times Y \to Y'$ using the universal property of the product $X' \times Y'$. Then clearly $F(1_X, 1_Y) = 1_{X,Y}$. Consider two maps $(f,g)\colon (X,Y) \to (X',Y')$ and $(h,k)\colon (X',Y') \to (X'',Y'')$ in $\mathscr{A} \times \mathscr{A}$. Then $F(hf, gk)$ is the unique map $X \times Y \to X'' \times Y''$ such that $p_1^{X'',Y''} \circ F(hf,kg) = h \circ f \circ p_1^{X,Y}$ and $p_2^{X'',Y''} \circ F(hf,kg) = k \circ g \circ p_2^{X,Y}$. Since $p_1^{X'',Y''} \circ F(h,k) = h \circ p_1^{X',Y'}$ we have

$$p_1^{X'',Y''} \circ F(h,k) \circ F(f,g) = h \circ p_1^{X',Y'} \circ F(f,g) = h \circ f \circ p_1^{X,Y}.$$

Similarly $p_2^{X'',Y''} \circ F(h,k) \circ F(f,g) = k \circ g \circ p_1^{X,Y}$. By uniqueness it follows that $F(hf,gk) = F(h,k) \circ F(f,g)$. Thus F is indeed a functor.

Exercise 5.3.9. Given $A, X, Y \in \mathscr{A}$ define

$$\begin{aligned} \varphi_{A,X,Y}\colon \mathscr{A}(A, X \times Y) &\to \mathscr{A}(A,X) \times \mathscr{A}(A,Y) \\ f &\mapsto (\text{pr}_X \circ f, \text{pr}_Y \circ f). \end{aligned}$$

Then $\varphi_{A,X,Y}$ is bijective with inverse given by $(g,h) \mapsto f$, where f is the unique map $A \to X \times Y$ induced by the universal property of the product. Given $g\colon A' \to A$, $h\colon X \to X'$ and $k\colon Y \to Y'$ consider the diagram

$$\begin{array}{ccc} \mathscr{A}(A, X \times Y) & \xrightarrow{\varphi_{A,X,Y}} & \mathscr{A}(A,X) \times \mathscr{A}(A,Y) \\ {\scriptstyle g^*(h \times k)_*}\downarrow & & \downarrow{\scriptstyle (g^*h_*, g^*k_*)} \\ \mathscr{A}(A', X' \times Y') & \xrightarrow[\varphi_{A',X',Y'}]{} & \mathscr{A}(A',X') \times \mathscr{A}(A',X')\,. \end{array}$$

For any $f\colon A \to X \times Y$ we have

$$(g^*h_*, g^*k_*) \circ \varphi_{A,X,Y}(f) = (h \circ \mathrm{pr}_X \circ f \circ g, k \circ \mathrm{pr}_Y \circ f \circ g)$$

and

$$\varphi_{A',X',Y'} \circ g^*(h \times k)_*(f) = (\mathrm{pr}_{X'} \circ (h \times k) \circ f \circ g, \mathrm{pr}_{Y'} \circ (h \times k) \circ f \circ g).$$

Thus, proving naturality of φ reduces to proving commutativity of the diagrams

$$\begin{array}{ccc} X \times Y & \xrightarrow{h \times k} & X' \times Y' \\ {\scriptstyle \mathrm{pr}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{X'}} \\ X & \xrightarrow[h]{} & X', \end{array} \qquad \begin{array}{ccc} X \times Y & \xrightarrow{h \times k} & X' \times Y' \\ {\scriptstyle \mathrm{pr}_Y}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{Y'}} \\ Y & \xrightarrow[k]{} & Y'. \end{array}$$

But $h \times k$ is by definition the unique map such that the above diagrams commute. It follows that $\varphi_{A,X,Y}$ is an isomorphism $\mathscr{A}(A, X \times Y) \cong \mathscr{A}(A, X) \times \mathscr{A}(A, Y)$ natural in $A, X, Y \in \mathscr{A}$.

Exercise 5.3.10. Let $F\colon \mathscr{A} \to \mathscr{B}$ create limits. Let $D\colon \mathbf{I} \to \mathscr{A}$ be a diagram. Then, for any limit cone $(B \xrightarrow{q_I} FD(I))_{I \in \mathbf{I}}$ on FD, there is a unique cone $(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ on D such that $F(A) = B$ and $F(p_I) = q_I$ for all $I \in \mathbf{I}$, and the cone $(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone. In particular, if $(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a cone on D such that $(F(A) \xrightarrow{F(p_I)} FD(I))_{I \in \mathbf{I}}$ is a limit cone on FD, then $(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is the unique cone given by the property of creating limits, and hence it is a limit cone. Therefore F reflects limits.

Exercise 5.3.11. (a) If $D\colon \mathbf{I} \to \mathbf{Set}$ is a diagram then

$$L = \varprojlim_{\mathbf{I}} D \cong \{(x_I)_{I \in \mathbf{I}} \mid x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}$$

is the limit cone on D with projections $L \xrightarrow{p_J} D(J)$ given by $p_J((x_I)_{I \in \mathbf{I}}) = x_J$ for all $J \in \mathbf{I}$. This formula was given in Example 5.1.22 and proven in Exercise 5.1.37.

Let $D\colon \mathbf{I} \to \mathbf{Grp}$ be a diagram in $\mathbf{Grp}$ and $(B \xrightarrow{q_I} UD(I))_{I \in \mathbf{I}}$ be a limit cone on UD in $\mathbf{Set}$. Then

$$B' = \{(x_I)_{I \in \mathbf{I}} \mid x_I \in UD(I) \text{ for all } I \in \mathbf{I} \text{ and } (UDu)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\} \in \mathbf{Set}$$

is also a limit cone on UD with projections $B' \xrightarrow{q'_J} UD(J)$ given by $q'_J((x_I)_{I \in \mathbf{I}}) = x_J$ for all $J \in \mathbf{I}$, so there is a unique isomorphism $h\colon B \to B'$ such that $q'_J h = q_J$ for all $J \in \mathbf{I}$.

Endow the set B' with a canonical group structure as a subgroup of $\prod_{i \in \mathbf{I}} D(I) \in \mathbf{Grp}$, call this group A', and let $p'_J\colon A' \to D(J)$, for $J \in \mathbf{I}$, denote the projection homomorphism whose underlying set function is q'_J. Now, the bijection $h\colon B \to B'$ endows B with a group structure, and call this group A, so that $h\colon A \to A'$ is an isomorphism of groups. For $J \in \mathbf{I}$, let $p_J\colon A \to D(J)$

be the projection, so that its underlying set function is q_J. Then $U(A) = B$ and $U(p'_I) = p_I$ for all $I \in \mathbf{I}$, and clearly $(A, (p'_I)_{I\in\mathbf{I}})$ is unique as such.

It remains to show that $(A \xrightarrow{p'_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D. So let $(C \xrightarrow{f_I} D(I))_{I\in\mathbf{I}}$ be a cone on D. Then $(U(C) \xrightarrow{U(f_I)} UD(I))_{I\in\mathbf{I}}$ is a cone on UD, so there is a unique map $g\colon U(C) \to B$ such that $p_I g = U(f_I)$ for all $I \in \mathbf{I}$. What we must prove is that g is a group homomorphism as a map $C \to A$. But composing with h gives a map $hg\colon U(C) \to B'$ which is the unique one given by the universal property. This hg is a group homomorphism when viewed as a map $C \to A'$. Since we defined the group structure on A declaring h to be an isomorphism of groups, it follows that $g\colon C \to A$ is indeed a homomorphism, and is of course the unique such that $p'_I g = f_I$ for all $I \in \mathbf{I}$. Thus $(A \xrightarrow{p'_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D. We conclude that $U\colon \mathbf{Grp} \to \mathbf{Set}$ creates limits.

(b) The above proof works when **Grp** is replaced by **Ab**, **Ring** or $\mathbf{Vect}_k$.

Exercise 5.3.12. Let $D\colon \mathbf{I} \to \mathscr{A}$ be a diagram. Then there exists a limit cone $(B \xrightarrow{q_I} FD(I))_{I\in\mathbf{I}}$ on FD. Since F creates limits there is a unique cone $(A \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ such that $F(A) = B$, $F(p_I) = q_I$ for all $I \in \mathbf{I}$, and $(A \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D. It follows $\mathscr{A}$ has limits of shape $\mathbf{I}$.

Now suppose $(A \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D. There exists a limit cone $(B \xrightarrow{q_I} FD(I))_{I\in\mathbf{I}}$ on FD, and a unique cone $(A' \xrightarrow{p'_I} D(I))_{I\in\mathbf{I}}$ such that $F(A') = B$, $F(p'_I) = q_I$ for all $I \in \mathbf{I}$, and $(A' \xrightarrow{p'_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D. Thus there is a unique isomorphism $h\colon A' \to A$ such that $p_I h = p'_I$ for all $I \in \mathbf{I}$. Then $F(h)\colon B \to F(A)$ is an isomorphism. Consider the cone $(F(A) \xrightarrow{F(p_I)} FD(I))_{I\in\mathbf{I}}$. Given any cone $(C \xrightarrow{f_I} FD(I))_{I\in\mathbf{I}}$, there is a unique $f\colon C \to B$ such that $q_I f = f_I$ for all $I \in \mathbf{I}$. Then $F(h)f\colon C \to F(A)$ satisfies $F(p_I)F(h)f = F(p'_I)f = f_I$ for all $I \in \mathbf{I}$. Moreover, if $h'\colon C \to F(A)$ satisfies $F(p_I)h' = f_I$ for each I then $F(h)^{-1}h'$ satisfies $q_I F(h)^{-1}h' = F(p'_I h^{-1})h' = F(p_I)h' = f_I$, so we must have $F(h)^{-1}h' = f$ by uniqueness of f, i.e. $h' = F(h)f$ is unique. It follows that $(F(A) \xrightarrow{F(p_I)} FD(I))_{I\in\mathbf{I}}$ is a limit cone on FD. We conclude that F preserves limits.

Exercise 5.3.13. (a) Let $S \in \mathbf{Set}$ be arbitrary and let $f\colon B \to B'$ be epic in $\mathscr{B}$. Then $G(f)\colon G(B) \to G(B')$ is epic in **Set**. By Exercise 5.2.26(b), $G(f)$ has a right inverse, say $h\colon G(B') \to G(B)$. Then the induced map $G(f)_*\colon \mathbf{Set}(S, G(B)) \to \mathbf{Set}(S, G(B'))$ has a right inverse, namely $h_*\colon \mathbf{Set}(S, G(B')) \to \mathbf{Set}(S, G(B))$. In particular, $G(f)_*$ is surjective.

Since $G \dashv F$ there is an isomorphism $\varphi_{S,B}\colon \mathscr{B}(F(S), B) \to \mathscr{A}(S, G(B))$ natural in $S \in \mathbf{Set}$ and $G(B)$, so that we have a commutative square

$$\begin{array}{ccc} \mathscr{B}(F(S), B) & \xrightarrow{\varphi_{S,B}} & \mathbf{Set}(S, G(B)) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle G(f)_*} \\ \mathscr{B}(F(S), B') & \xrightarrow[\varphi_{S,B'}]{} & \mathbf{Set}(S, G(B')) . \end{array}$$

Since $\varphi_{S,B}, \varphi_{S,B'}$ are isomorphisms and $G(f)_*$ is surjective, it follows that $\mathscr{B}(F(S), f)$ is surjective, i.e. it is epic. Therefore $F(S)$ is projective. We conclude that $F(S)$ is projective for all sets S.

(b) Recall that a map in **Ab** is epic if and only if it is surjective (Example 5.2.19). Consider $\mathbb{Z}/2\mathbb{Z} \in \mathbf{Ab}$. Then the unique non-trivial map $f : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is epic in **Ab**, but $f_* : \mathbf{Ab}(\mathbb{Z}/2, \mathbb{Z}) \to \mathbf{Ab}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is not epic, for $\mathbf{Ab}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$ and $\mathbf{Ab}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Thus $\mathbb{Z}/2\mathbb{Z}$ is not projective in **Ab**.

(c) Let k be a vector space and $U \in \mathbf{Vect}_k$ arbitrary. Let $f^{\mathrm{op}} : W \to V$ be epic in $\mathbf{Vect}_k^{\mathrm{op}}$, that is, $f : V \to W$ is monic (i.e. injective) in $\mathbf{Vect}_k$. We shall prove that $f^* : \mathbf{Vect}_k(W, U) \to \mathbf{Vect}_k(V, U)$ is surjective. Given a linear map $L : V \to U$ we need $\widetilde{L} : W \to U$ such that the diagram

$$\begin{array}{ccc} V & & \\ {\scriptstyle f}\downarrow & \searrow^{L} & \\ W & \xrightarrow[\widetilde{L}]{} & U \end{array}$$

commutes. Taking a basis $\{v_i\}_{i\in J}$ for V then $\{f(v_i)\}_{i\in J}$ is linear independent in W, so extends to a basis for W. Then we may define $\widetilde{L}$ by sending each $f(v_i)$ to $L(v_i)$ and all other basis elements to 0. Thus U is injective. Since U was arbitrary, it follows that every k-vector space is injective.

Now consider $\mathbb{Z} \in \mathbf{Ab}$ and $f : \mathbb{Z} \to \mathbb{Q}$ the inclusion map, which is monic. Then $f^* : \mathbf{Ab}(\mathbb{Q}, \mathbb{Z}) \to \mathbf{Ab}(\mathbb{Z}, \mathbb{Z})$ is not epic, for $\mathbf{Ab}(\mathbb{Q}, \mathbb{Z}) = 0$ and $\mathbf{Ab}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$. It follows that $\mathbb{Z}$ is not injective in **Ab**.

Please send comments, suggestions and corrections by e-mail, or at website.
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