Solutions to exercises from Chapter 2 of Lawrence C. Evans' book 'Partial Differential Equations'

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1

Write down an explicit formula for a function u solving the initial value problem

$$\left.\begin{aligned} u_t + b \cdot Du + cu = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g \text{ on } \mathbb{R}^n \times \{t=0\} \end{aligned}\right\}$$

Solution. We use the method of characteristics; consider a solution to the PDE along the direction of the vector $(b,1)$: $z(s) = u(x+bs, t+s)$. We have $\dot{z}(s) = u_t(x+bs,t+s)+b\cdot Du(x+bs,t+s) = -cu(x+bs,t+s) = -cz(s)$, thus the PDE reduces to an ODE. The characteristic curves can be parametrized by (x_0+bs, t_0+s); and all the characteristic curves are parallel to each other. Given any (x_0,t_0) in $\mathbb{R}^n \times (0,\infty)$, we have $u(x_0,t_0) = u(x_0-bt_0,0)e^{-ct_0} = g(x_0-bt_0)e^{-ct_0}$; and this is an explicit formula for the solutions to the PDE.

2

Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad (x \in \mathbb{R}^n),$$

then $\Delta v = 0$.

Proof. By chain rule we have

$$D_{x_i} v(x) = \Sigma_{k=1}^n D_{x_k} u(Ox) o_{ik},$$

then

$$D_{x_i x_j} v(x) = \Sigma_{l=1}^n \Sigma_{k=1}^n D_{x_k x_l} u(Ox) o_{ik} o_{jl}.$$

Since O is orthogonal, $OO^T = I$, that is,

$$o_{ik} o_{il} = \begin{cases} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l. \end{cases}$$

Thus

$$\Delta v = \Sigma_{i=1}^n \Sigma_{k=1}^n \Sigma_{l=1}^n D_{x_k x_l} u(Ox) o_{ik} o_{il} = \Delta u = 0.$$

3

Modify the proof of the mean value formulas to show for $n \geq 3$ that

$$u(0) = \fint_{\partial B(0,r)} g \mathrm{d}S + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f \mathrm{d}x,$$

provided

$$\left.\begin{aligned} -\Delta u = f \text{ in } B^0(0,r) \\ u = g \text{ on } \partial B(0,r) \end{aligned}\right\}$$

Proof. Define

$$\phi(s) := \fint_{\partial B(x,s)} u(y) \mathrm{d}S(y) = \fint_{\partial B(0,1)} u(x + sz) \mathrm{d}S(z)$$

we shall have

$$\phi'(s) = ⨍_{\partial B(0,1)} Du(x+sz)\cdot z\mathrm{d}S(z) = ⨍_{\partial B(x,s)} D(y)\frac{y-z}{s}\mathrm{d}S(y)$$
$$= ⨍_{\partial B(x,s)} \frac{\partial u}{\partial \nu}\mathrm{d}S(y) = \frac{s}{n}⨍_{B(x,s)} \nabla^2 u(y)\mathrm{d}y = \frac{s}{n}⨍_{B(x,s)} \Delta u(y)\mathrm{d}y$$

We have $\phi(r) - \phi(\epsilon) =$

$$\int_\epsilon^r \phi'(s)\mathrm{d}s = \int_\epsilon^r \Big(\frac{s}{n}⨍_{B(0,s)} \Delta u(y)\mathrm{d}y\Big)\mathrm{d}s$$
$$= \int_\epsilon^r \Big(\frac{s}{n}⨍_{B(0,s)} f(y)\mathrm{d}y\Big)\mathrm{d}s = \int_\epsilon^r \Big(\frac{1}{n\alpha(n)s^{n-1}}\int_{B(0,s)} f(y)\mathrm{d}y\Big)\mathrm{d}s$$
$$= \frac{1}{n(2-n)\alpha(n)}\Big(\big(\frac{1}{s^{n-2}}\int_{B(0,s)} f(y)\mathrm{d}y\big)\Big|_\epsilon^r - \int_\epsilon^r \big(\frac{1}{s^{n-2}}\int_{\partial B(0,s)} f(y)\mathrm{d}y\big)\mathrm{d}s\Big)$$
$$= \frac{1}{n(n-2)\alpha(n)}\Big(\big(-\frac{1}{r^{n-2}}\int_{B(0,r)} f(y)\mathrm{d}y + \frac{1}{\epsilon^{n-2}}\int_{B(0,\epsilon)} f(y)\mathrm{d}y\big)$$
$$+ \int_\epsilon^r \big(\frac{1}{s^{n-2}}\int_{\partial B(0,s)} f(y)\mathrm{d}y\big)\mathrm{d}s\Big)$$

Now notice that

$$\frac{1}{\epsilon^{n-2}}\int_{B(0,\epsilon)} f(y)\mathrm{d}y \le C\epsilon^2;$$

and

$$\int_0^r \big(\frac{1}{s^{n-2}}\int_{\partial B(0,s)} f(y)\mathrm{d}y\big)\mathrm{d}s = \int_0^r\int_{\partial B(0,s)} \frac{f(y)}{s^{n-2}}\mathrm{d}y\mathrm{d}s$$
$$= \int_{B(0,r)} \frac{f(x)}{|x|^{n-2}}\mathrm{d}x$$

As $\epsilon \to 0$ we have

$$\frac{1}{\epsilon^{n-2}}\int_{B(0,\epsilon)} f(y)\mathrm{d}y + \int_\epsilon^r \big(\frac{1}{s^{n-2}}\int_{\partial B(0,s)} f(y)\mathrm{d}y\big)\mathrm{d}s \to \int_{B(0,r)} \frac{f(x)}{|x|^{n-2}}\mathrm{d}x$$

and $\phi(\epsilon) \to u(0)$. We have thus demonstrated

$$u(0) = \fint_{\partial B(0,r)} g \mathrm{d}S + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} (\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}) f \mathrm{d}x.$$

4

Give a direct proof that if $u \in C^2(U) \cap C^(\bar{U})$ is harmonic within a bounded open set U, then $\max_{\bar{U}} u = \max_{\partial U} u$.

Proof. Define $u_\epsilon := u + \epsilon|x|^2$. Suppose u_ϵ is achieveing maximum in $\bar{U}$ at an interior point x_0, then $D(u_\epsilon(x_0)) = 0$ and $H = D_{ij}(u_\epsilon(x_0))$ is negative definite. Yet, $\Delta(u_\epsilon) = 2\epsilon \geq 0$, a contradiction, as the Laplacian is the trace of the Hessian. Letting $\epsilon \to 0$, we can conclude that u cannot attain an interior maximum.

5

We say $v \in C^2(\bar{U})$ is subharmonic if

$$-\Delta v \leq 0 \textbf{ in } U.$$

(a) Prove for subharmonic

$$v(x) \leq \fint_{B(x,r)} v \mathrm{d}y \textbf{ for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.
(c) Let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.
(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Solution. (a) We set

$$f(r) := \fint_{\partial B(x,r)} u(y) \mathrm{d}S(y) = \fint_{\partial B(0,1)} u(x + rz) \mathrm{d}S(z).$$

Taking derivative we have

$$f'(r) = \fint_{\partial B(0,1)} Du(x + rz) \cdot z \mathrm{d}S(z)$$

and consequently using Green's formula we have

$$\begin{aligned} f'(r) &= \fint_{\partial B(0,1)} Du(y) \cdot \frac{y-x}{r} \mathrm{d}S(z) \\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} \mathrm{d}S(y) \\ &= \frac{r}{n} \fint_{\partial B(x,r)} \Delta u(y) \mathrm{d}y \geq 0 \end{aligned}$$

This means that $f(r)$ is non-decreasing, therefore given $r > 0$

$$\begin{aligned} u(x) = \lim_{t \to 0} f(t) &= \lim_{t \to 0} \fint_{\partial B(x,t)} u(y) \mathrm{d}S(y) \\ &\leq \fint_{\partial B(x,r)} u(y) \mathrm{d}S(y) \end{aligned}$$

And this implies

$$\begin{aligned} \alpha(n) r^n u(x) &\leq \int_0^r \big(\int_{\partial B(x,s)} u(y) \mathrm{d}S(y) \big) \mathrm{d}s \\ u(x) &\leq \fint_{B(x,r)} u(y) \mathrm{d}y \end{aligned}$$

(b) Assume the subharmonic function v attains maximum at $x_0 \in U^0$, for any ball $B(x_0, r) \subset U_0$ we have by (a) $v(x_0) \leq \fint_{B(x_0,r)} v(y) \mathrm{d}y$. Yet, $v(x_0) \geq v(y)$ for any $y \in B(x_0, r)$, thus $v(x_0) = \fint_{B(x_0,r)} v(y) \mathrm{d}y$, and $v(x_0) = v(y)$ for any $y \in B(x_0, r)$. We can choose r so that $\partial \bar{U} \cap \partial B(x_0, r) = \{x_1\}$. We have thus shown that $\max\limits_{x \in \bar{U}} v = \max\limits_{x \in \partial U} v$.

(c) As ϕ is convex and smooth, $\phi'' \geq 0$; and $\Delta u = 0$. Hence

$$-\Delta \phi(u) = -(\sum_{i=1}^{n} \phi'' \cdot (u_{x_i})^2 + \phi' \cdot u_{x_i x_i}) \leq 0,$$

we can conclude that $\phi(u)$ is subharmonic.

(d) If u is harmonic then $\Delta u = 0$, thus $\Delta u_{x_i} = (\Delta u)_{x_i} = 0$, hence Du is harmonic. Since $|Du|^2$ is convex with respect to Du, the result follows from (c).

6

Let U be a bounded open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \le C(\max_{\partial U} |g| + \max_{\bar{U}} |f|)$$

where u is a smooth solution of

$$\begin{cases} -\Delta u = f \text{ in } U \\ \quad u = g \text{ on } \partial U. \end{cases}$$

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \le 0$, for $\lambda := \max_{\bar{U}} |f|$.)

Solution. We consider the function $v(x) := u(x) + \frac{|x|^2}{2n}\lambda$, where $\lambda = \max_{\bar{U}} |f|$. Indeed v is a subharmonic function since

$$-\Delta v = -\Delta\big(u + \frac{|x|^2}{2n}\lambda\big) \le (f - \max_{\bar{U}} |f|) \le 0.$$

The weak maximum principle holds for subharmonic functions-

$$\max_{\bar{U}} u \le \max_{\bar{U}} v = \max_{\partial U} v$$
$$= \max_{\partial U} \big(u + \frac{|x|^2}{2n}\lambda\big) \le \max_{\partial U} u + \max_{\partial U} \frac{|x|^2}{2n}\lambda = \max_{\partial U} g + \mathtt{C} \max_{\bar{U}} |f|$$

where $\mathtt{C}$ is some constant depending on U.

Replacing u by $-u$, we can conclude that there exists some constant C depending on U, such that

$$\max_{\bar{U}} |u| \le C\big(\max_{\partial U} |g| + \max_{\bar{U}} |f|\big).$$

7

Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r - |x|}{(r + |x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r + |x|}{(r - |x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Solution. We recall Poisson's formula for a ball

$$\begin{aligned} u(x) &= \frac{r^2-|x|^2}{n\omega(n)r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y) \\ &= r^{n-2}(r^2-|x|^2)⨍_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S(y), \end{aligned}$$

therefore,

$$\begin{aligned} r^{n-2}(r^2-|x|^2)\frac{u(0)}{(r+|x|)^n} \le u(x) \le r^{n-2}(r^2-|x|^2)\frac{u(0)}{(r-|x|)^n}, \\ r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \le u(x) \le r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0). \end{aligned}$$

8

Prove Theorem 15 in section 2.2.4. (Hint: Since $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies $\int_{\partial B(0,1)} K(x,y)\mathrm{d}S(y) = 1$ for each $x \in B^0(0,1)$.)

Let's recall theorem 15 in section 2.2.4. (Poisson's formula for a ball) Assume $g \in C(\partial B(0,r))$ and define u by

$$u(x) = \frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n}\mathrm{d}S.$$

Then

(i) $u \in C^\infty(B^0(0,r))$

(ii) $\Delta u = 0$ in $B(0,r)$, and

(iii) $\lim_{\substack{x\to x^0 \\ x\in B^0(0,r)}} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

Proof. (1) For each fixed x, the mapping $y \mapsto G(x,y)$ is harmonic, except for $x=y$. As $G(x,y)=G(y,x)$, $x \mapsto G(x,y)$ is harmonic, except for $x=y$. Thus $x \mapsto \frac{\partial G}{\partial y_n}(x,y) = K(x,y)$ for $x, y \in B^0(0,r)$.

(2) Note that we have

$$1 = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{1}{|x-y|^n} \mathrm{d}S$$

for $x \in B^0(0,r)$. And as g is bounded, u is likewise bounded. Since $x \mapsto K(x,y)$ is smooth, for $x \neq y$, we can verify that $u \in C^\infty(B^0(0,r))$ with

$$\Delta u(x) = \int_{\partial B(0,r)} \Delta_x K(x,y) g(y) \mathrm{d}S = 0$$

.

(3) Now fix $x^0 \in \partial B(0,r), \epsilon > 0$. Choose $\delta > 0$ such that $|g(y) - g(x^0)| \leq \epsilon$ if $|y - x^0| < \delta, y \in \partial B(0,r)$. Then if $|x - x^0| < \frac{\delta}{2}, x \in B^0(0,r)$,

$$\begin{aligned} |u(x) - g(x^0)| &\leq \Big| \int_{\partial B(0,r)} K(x,y)[g(x) - g(x^0)] \mathrm{d}y \Big| \\ &\leq \Big| \int_{\partial B(0,r) \cap B(x^0,\delta)} K(x,y)[g(x) - g(x^0)] \mathrm{d}y \Big| + \\ &\Big| \int_{\partial B(0,r)/B(x^0,\delta)} K(x,y)[g(x) - g(x^0)] \mathrm{d}y \Big| := I + J \end{aligned}$$

We have $I \leq \epsilon \int_{\partial B(0,r)} K(x,y) \mathrm{d}y = \epsilon$. Furthermore if $|x - x^0| \leq \frac{\delta}{2}$ and $|y - x^0| \geq \delta$, we have $|y - x^0| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}|y - x^0|$. Hence

$$|y - x| \geq \frac{1}{2}|y - x^0|.$$

Thus

$$\begin{aligned} J &\leq 2\|g\|_{L^\infty} \int_{\partial B(0,r)/B(x^0,\delta)} K(x,y) \mathrm{d}y \\ &\leq \frac{2^{n+1}(r^2 - |x|^2)\|g\|_{L^\infty}}{n\alpha(n)r} \int_{\partial B(0,r)/B(x^0,\delta)} |y - x^0|^{-n} \mathrm{d}y \to 0 \end{aligned}$$

as $x \to x^{0+}$.

Combining this two estimates we can conclude that $\lim_{\substack{x \to x^0 \\ x \in B^0(0,r)}} u(x) = g(x^0)$ for each point $x^0 \in \partial B(0,r)$.

9

Let u be the solution of $\begin{cases} \Delta u = 0 \text{ in } \mathbb{R}^n_+ \\ u = g \text{ on } \partial\mathbb{R}^n_+ \end{cases}$ given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \leq 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$.)

Solution. We recall Poisson's formula for the half-space

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} \mathrm{d}S(y).$$

Notice that $u(0) = g(0) = 0$; following the hint we estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$:

$$\frac{1}{\lambda}\Big(u(\lambda e_n) - u(0)\Big) = \frac{2e_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{|y|}{\left(\sqrt{(\lambda^2 + |y|^2)}\right)^n} \mathrm{d}S(y).$$

We consider the integral

$$\begin{aligned} &\frac{2e_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap \{|y|\leq 1\}} \frac{|y|}{\left(\sqrt{(\lambda^2 + |y|^2)}\right)^n} \mathrm{d}S(y) \\ &= (n-1)\alpha(n-1) \int_0^1 \frac{r}{\left(\sqrt{\lambda^2 + r^2}\right)^n} r^{n-2} \mathrm{d}r \\ &= (n-1)\alpha(n-1) \int_0^1 \frac{1}{r\big((\lambda/r)^2 + 1\big)^{n/2}} \mathrm{d}r \end{aligned}$$

thus the integral diverges. Hence $\lim\limits_{\lambda\to 0} \frac{u(\lambda e_n)-u(0)}{\lambda}$ cannot be bounded, which implies $Du \cdot \vec{e}_n$ is unbounded at 0. Therefore Du is unbounded at 0.

10

(Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \| x| < 1, x_n > 0\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap x_n = 0$. Set

$$v(x) := \begin{cases} u(x) \textbf{ if } x_n \geq 0 \\ -u(x_1, ..., x_{n-1}, -x_n) \textbf{ if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. **Prove** $v \in C^2(U)$ **and thus** v **is harmonic within** U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U})$. **Show that** v **is harmonic within** U. **(Hint: Use Poisson's formula for the ball.)**

Solution. (a) Apparently $v(x)$ is of class C^2 in each of $\overline{U^+}$ and $U^- := B(0,1)/\overline{U^+}$. We can check that for $i \neq n$

$$\lim_{x_n \to 0^-} \partial_{x_i x_n} v(x_1, x_2, ..., x_n) = \partial_{x_i x_n} v(x_1, x_2, ..., 0)$$
$$= \partial_{x_i x_n} u(x_1, x_2, ..., 0) = \lim_{x_n \to 0^+} \partial_{x_i x_n} v(x_1, x_2, ..., x_n);$$

while $\partial_{x_n x_n} v(x_1, x_2, ..., 0) = 0$, and all other derivatives which do not involve x_n vanishes. So we can assure that $D^2 v$ exists and is continuous in U.

Since $v(x) = u(x)$ in U^+ and $v(x) = -u(x_1, x_2, ..., -x_n)$ if $x_n < 0$. And we have already known that u is harmonic in U^+. Given any x with $x_n < 0$, we have $v(x)$ satisfies a local version of mean value theorem-

$$v(x) = \fint_{\partial B(x,\delta)} v(y) \mathrm{d}y$$

for small enough $\delta > 0$. The local mean value property is equivalent to harmonicity. Finally, if $x_n = 0$, we also have

$$v(x) = \fint_{\partial B^+(x,\delta)} v(y) \mathrm{d}y + \fint_{\partial B^-(x,\delta)} v(y) \mathrm{d}y = 0$$

since $v(x) = -v(x_1, x_2, ..., -x_n)$. We therefore can conclude that $v(x)$ is a harmonic function in U.

(b) Now assume that u is merely continuous on the boundary, we would like to find a harmonic function w such that $w = v$ on $\partial B(0,1)$ by the Poisson's formula for the ball-

$$w(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B(0,1)} \frac{v(y)}{|x-y|^n} \mathrm{d}S(y),$$

we shall see that $w(x) = 0$ on $x_n = 0$. Therefore, $w = u$ on ∂U^+ and by the maximum principle we have $u - w = 0$ throughout U^+. So $w \equiv v$ in $\overline{U^+}$ and we can similarly show $w \equiv v$ in $\overline{U^-}$. Therefore, v is harmonic within U.

11

(Kelvin transform for Laplace's equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ **of a function** $u : \mathbb{R}^n \to \mathbb{R}$ **is**

$$\overline{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|^2)|x|^{2-n}, (x \neq 0),$$

where $\bar{x} = x/|x|^2$**. Show that if** u **is harmonic, then so is** $\bar{u}$**. (Hint: First show that** $D_x\bar{x}(D_x\bar{x})^T = |x|^4 I$**. The mapping** $x \to \bar{x}$ **is** *conformal*, **meaning angle preserving.)**

Proof. We denote $\psi(x) = \bar{x}$, and compute

$$\frac{\partial}{\partial x_i}\psi^j(x) = \frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4},$$

where δ_{ij} is Kronecker's delta symbol. We write $D_x\bar{x}$ in coordinate free expression:

$$D_x\bar{x} = |x|^{-2}(I - 2\frac{xx^T}{|x|^2}),$$

thus $D_x\bar{x}(D_x\bar{x})^T = |x|^{-4}(I - 4|x|^{-2}xx^T + 4|x|^{-4}xx^Txx^T) = |x|^{-4}I$.

We now calculate $\Delta\psi$; since careful computations shall show that

$$\psi^j_{x_i x_i} = -2|x|^{-4}(x_j + 2\delta_{ij}x_i) + 8|x|^{-6}x_i^2 x_j.$$

Thus, $\Delta\psi = n(2-n)\frac{x}{|x|^4}$.

Hence

$$\Delta\big(u(x/|x|^2)|x|^{2-n}\big) = \Delta\big(|x|^{2-n}\big)u(\bar{x}) + 2DuD\bar{x}\big(D|x|^{2-n}\big)^T$$
$$+|x|^{2-n}Du \cdot \Delta\bar{x} + |x|^{2-n}\mathrm{Tr}\big((D\bar{x})^T D^2 u D\bar{x}\big).$$

Note that the first term is 0 for $x \neq 0$; and the last term is also 0 as $\mathrm{Tr}\big((D\bar{x})^T D^2 u D\bar{x}\big) = |x|^{-4}\Delta u = 0$. While we have

$$2DuD\bar{x}\big(D|x|^{2-n}\big)^T = 2Du\big(|x|^{-2}(I - 2\frac{xx^T}{|x|^2})\big)(2-n)|x|^{-n}x$$
$$= 2(n-2)|x|^{-2-n}Du \cdot x = -|x|^{2-n}Du \cdot \Delta\bar{x}.$$

Therefore the Kelvin transform preserves harmonic functions.

12

Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) := x \cdot Du(x,t) + 2tu_t(x,t)$ solves the heat equation as well.

Proof. (a) We compute to show that the dilation of u also satisfies the heat equation, since

$$\frac{\partial}{\partial t} u_\lambda(x,t) = \frac{\partial}{\partial t} u(\lambda x, \lambda^2 t) = \lambda^2 u_t$$

and

$$\frac{\partial^2}{\partial x_i^2} u_\lambda(x,t) = \lambda^2 u_{x_i x_i}.$$

We see that u_λ solves the heat equation.

(b) Differentiating u_λ with respect to λ we shall have

$$\frac{\partial}{\partial \lambda} u_\lambda = x \cdot Du(\lambda x, \lambda^2 t) + 2t\lambda u_t(\lambda x, \lambda^2 t).$$

Now let $\lambda = 1$ we then see that

$$\frac{\partial}{\partial \lambda} u_\lambda = x \cdot Du(x,t) + 2tu_t(x,t) = v(x,t).$$

Since u is smooth, the mixed partials are equal under exchanges of orders of differentiation. And u_λ solves the heat equation, that is

$$(u_\lambda)_t - \Delta(u_\lambda) = 0.$$

We differentiate both sides with respect to λ and conclude that $v(x,t)$ is also a solution to the heat equation.

13

Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show $u_t = u_{xx}$ if and only if $v'' + \frac{z}{2} v' = 0$. Show that the general solution is

$$v(z) = c\int_0^z e^{-s^2/4}\mathrm{d}s + d.$$

(b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution. (Hint: What is the initial condition for u?)

Solution. (a) Differentiate v we have $u_t = \frac{1}{2}xt^{-\frac{3}{2}}v'$ and $u_{xx} = \frac{1}{t}v''$. By the heat equation we get $v'' + \frac{x}{2\sqrt{t}}v' = 0$. Vice versa, if v satisfies the ODE then u solves the heat equation. We solve the ODE, notice that $(e^{\frac{z^2}{4}}v(z)')' = 0$, then the solution shall be

$$v(z) = c\int_0^z e^{-s^2/4}\mathrm{d}s + d.$$

(b) The initial condition is the $\delta-$distribution. Differentiate $u(x,t) = v(z)$ with respect to x we have $u_x(x,t) = \frac{c}{\sqrt{t}}e^{-\frac{x^2}{4t}}$, which should also be a solution to the heat equation. Since the integral of $\delta-$function is 1,

$$1 = c\int_{-\infty}^{\infty} e^{-\frac{s^2}{4}}\mathrm{d}s = 2c\sqrt{\pi},$$

we must have $c = \frac{1}{2\sqrt{\pi}}$, thus $\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ is the fundamental solution for $n = 1$.

14

Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f \text{ in } \mathbb{R}^n \times (0,\infty) \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$.

Solution. First we consider the homogeneous equation with initial boundary data

$$\begin{cases} v_t - \Delta v + cv = 0 \text{ in } \mathbb{R}^n \times (0,\infty) \\ v = h \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

We take the Fourier transform with respect to the spacial variable and the equation shall become an ODE

$$\begin{cases} \widehat{v}_t + |\xi|^2\widehat{v} + c\widehat{v} = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ \widehat{v} = \widehat{h} \text{ on } \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Solving the ODE- $\widehat{v} = e^{-(|\xi|^2 - c)t}\widehat{h(\xi)}$, then taking the inverse Fourier transform we have

$$v(x,t) = \frac{e^{-ct}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} h(y)\mathrm{d}y.$$

By Duhamel's principle we conclude that the solution to the non-homogeneous problem is

$$u(x,t) = \frac{e^{-ct}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} g(y)\mathrm{d}y$$
$$+ \int_0^t \frac{e^{-c(t-s)}}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4(t-s)}} f(y,s)\mathrm{d}y\mathrm{d}s.$$

15

Given $g : [0, \infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\mathrm{d}s$$

for a solution of the initial/boundary value problem

$$\begin{cases} u_t - u_{xx} = 0 \textbf{ in } \mathbb{R}_+ \times (0, \infty) \\ u = 0 \textbf{ on } \mathbb{R}^+ \times \{t = 0\} \\ u = g \textbf{ on } \{x = 0\} \times [0, \infty) \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x < 0\}$ by odd reflection.)

Solution. We follow the hint and extend our function $v(x,t) := u(x,t) - g(t)$ by odd extension to $\{x < 0\} : v(x,t) := g(t) - u(-x,t)$. Then our problem becomes

$$\begin{cases} v_t - v_{xx} = -g_t \text{ in } \mathbb{R}_+ \times (0,\infty) \\ v_t - v_{xx} = g_t \text{ in } \mathbb{R}_- \times (0,\infty) \\ v = 0 \text{ on } \mathbb{R} \times \{t = 0\} \end{cases}$$

Apply the formula for the solution to the heat equation we have

$$v(x,t) = \int_{-\infty}^{\infty} \Phi(x-y,t)0\mathrm{d}y$$
$$- \int_0^t \int_0^{\infty} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s$$
$$+ \int_0^t \int_{-\infty}^{0} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s$$
$$= - \int_0^t \int_0^{\infty} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s + \int_0^t \int_{-\infty}^{0} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s$$
$$= - \int_0^t \int_{-\infty}^{\infty} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s + 2\int_0^t \int_{-\infty}^{0} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s$$
$$= - \int_0^t g^{'}(s)\mathrm{d}s + 2\int_0^t \int_{-\infty}^{0} \Phi(x-y,t-s)g^{'}(s)\mathrm{d}y\mathrm{d}s$$

We denote $h(s) := \int_{-\infty}^{0} \Phi(x-y,t-s)\mathrm{d}y$ and $z := \frac{(x-y)}{\sqrt{4(t-s)}}$ and compute-

$$h(s) = \int_{-\infty}^{0} \frac{e^{-\frac{(x-y)^2}{4(t-s)}}}{(4\pi(t-s))^{1/2}}\mathrm{d}y = \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4(t-s)}}}^{\infty} e^{-z^2}\mathrm{d}z$$

$$h'(s) = -\frac{x}{4\sqrt{\pi}(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}};$$

and continue

$$v(x,t) = -g(t) + 2\big[h(s)g(s)\big]_0^t - 2\int_0^t g(s)h^{'}(s)\mathrm{d}s$$
$$= -g(t) + \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\mathrm{d}s.$$

Therefore we have

$$u(x,t) = v(x,t) + g(t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s)\mathrm{d}s.$$

16

Give a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$. (Hint: Define $u_\epsilon := u - \epsilon t$ for $\epsilon > 0$, and show u_ϵ cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

Proof. We denote $U_T := U \times (0,T]$ and $\Gamma_T := \bar{U} \times \{0\} \cup (\partial U \times [0,T])$. Claim: if $\Delta u - u_t \geq 0$ in U_T, then $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$.

First we consider the case where $\Delta u - u_t > 0$. For any $\epsilon \in (0,T)$ exsists a point $(x_0,t_0) \in \overline{U_{T-\epsilon}}$ such that $u(x_0,t_0) = \max_{\bar{U}_{T-\epsilon}} u(x,t)$ since u is continuous and $\overline{U}_{T-\epsilon}$ is compact. If $(x_0,t_0) \in U_{T-\epsilon}$ by derivative tests we have $\Delta u(x_0,t_0) \leq 0, u_t(x_0,t_0) = 0$ and $\nabla u(x_0,t_0) = 0$, a contradiction to $\Delta u - u_t > 0$. Hence we must have $(x_0,t_0) \in \Gamma_{T-\epsilon}$. Letting $\epsilon \to 0$ we have $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$.

Now we consider the case where $\Delta u - u_t \geq 0$. For $\epsilon > 0$, we define a function $u_\epsilon(x,t) := u(x,t) - \epsilon t$, notice that

$$\Delta u_\epsilon = \Delta u = u_t > (u_t - \epsilon) = (u_\epsilon)_t.$$

So we have by our previous reasoning that $\max_{\bar{U}_T} u_\epsilon = \max_{\Gamma_T} u_\epsilon$. Letting $\epsilon \to 0$, we have the desired conclusion.

17

We say $v \in C_1^2(U_T)$ is a *subsolution* of the heat equation if

$$v_t - \Delta v \leq 0$$

in U_T.

(a) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2}\mathrm{d}y\mathrm{d}s$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ **be smooth and convex. Assume** u **solves the heat equation and** $v := \phi(u)$. **Prove** v **is a subsolution.**

(d) Prove $v := |Du|^2 + u_t^2$ **is a subsolution, whenever** u **solves the heat equation.**

Proof. (a) Without loss of generality, we shift (x,t) to $(0,0)$, and upon mollifying if necessary, we assume that u is smooth. We write $E(r) = E(0,0,;r)$. We define the function

$$\phi(r) := \frac{1}{4r^n} \iint_{E(r)} u(y,s)\frac{y^2}{s^2}\mathrm{d}y\mathrm{d}s = \iint_{E(1)} u(ry, r^2 s)\frac{y^2}{s^2}\mathrm{d}y\mathrm{d}s.$$

We shall use an identity: $\iint_{E(1)} \frac{|y|^2}{|s|^2}\mathrm{d}y\mathrm{d}s = 4$. We calculate $\phi^{'}(r)$:

$$\begin{aligned}
\phi^{'}(r) &= \iint_{E(1)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2ru_s\frac{|y|^2}{s}\mathrm{d}y\mathrm{d}s \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2ru_s\frac{|y|^2}{s}\mathrm{d}y\mathrm{d}s \\
&=: A + B
\end{aligned}$$

Also, let us introduce a function $\psi := \frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r$ and since $\Phi(y,-s) = r^{-n}$ on $\partial E(r)$ we have $\psi = 0$ on $\partial E(r)$. We utilize the formula to write

$$\begin{aligned}
B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i}\mathrm{d}y\mathrm{d}s \\
&= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^{n} u_{sy_i} y_i \psi\mathrm{d}y\mathrm{d}s;
\end{aligned}$$

due to integral by parts and the fact that $\psi = 0$ on $\partial E(r)$. Integrating by parts with respect to s, we discover

$$B = \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \psi_s \mathrm{d}y\mathrm{d}s$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi + 4\sum_{i=1}^{n} u_{y_i} y_i \big(-\frac{n}{2s} - \frac{|y|^2}{4s^2}\big)\mathrm{d}y\mathrm{d}s$$

$$= \frac{1}{r^{n+1}} \iint_{E(r)} -4nu_s\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \mathrm{d}y\mathrm{d}s - A.$$

Consequently since $u_t - \Delta u \leq 0$,

$$\phi^{'}(r) = A + B$$

$$\geq \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta u\psi - \frac{2n}{s}\sum_{i=1}^{n} u_{y_i} y_i \mathrm{d}y\mathrm{d}s$$

$$= \sum_{i=1}^{n} \frac{1}{r^{n+1}} \iint_{E(r)} 4nu_{y_i}\psi_{y_i} - \frac{2n}{s} u_{y_i} y_i \mathrm{d}y\mathrm{d}s = 0$$

Hence $\phi(r)$ is non-decreasing; and therefore

$$\phi(r) = \frac{1}{4r^n} \iint_{E(r)} u(y,s)\frac{y^2}{s^2}\mathrm{d}y\mathrm{d}s \geq$$

$$\geq \lim_{t\to 0}\phi(t) = u(0,0)\big(\lim_{t\to 0}\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2}\mathrm{d}y\mathrm{d}s\big) = 4u(0,0).$$

As

$$\frac{1}{t^n}\iint_{E(t)} \frac{|y|^2}{s^2}\mathrm{d}y\mathrm{d}s = \iint_{E(1)} \frac{|y|^2}{s^2}\mathrm{d}y\mathrm{d}s.$$

(b) Suppose that there exists some point $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = M := \max_{\bar{U}_T} u$. Then for sufficiently small $r > 0$ we have $E(x_0, t_0; r) \subset U_T$; and we have

$$M = u(x_0, t_0) \leq \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s)\frac{|x_0 - y|^2}{(t_0 - s)^2}\mathrm{d}y\mathrm{d}s \leq M$$

since

$$\frac{1}{4r^n}\iint_{E(x_0,t_0;r)}\frac{|x_0-y|^2}{(t_0-s)^2}\mathrm{d}y\mathrm{d}s=1.$$

We shall have for all $(y,s)\in E(x_0,t_0;r)$, $u(y,s)=u(x_0,t_0)$.

Draw any line segment L in U_T connecting (x_0,t_0) with some other point $(y_0,s_0)\in U_T$ with $s_0<t_0$. Consider

$$r_0:=\min\{s\le s_0|u(x,t)=M \text{ for all points } (x,t)\in L, s\le t\le t_0\}.$$

Since u is continuous, the minimum is attained. Assume $r_0>s_0$. Then $u(z_0,r_0)=M$ for some point (z_0,r_0) on $L\cap U_T$ and so $u\equiv M$ on $E(z_0,r_0;r)$ for sufficiently small r. Since $E(z_0,r_0;r)$ contains $L\cap\{r_0-\sigma\le t\le r_0\}$ for some small $\sigma>0$, we have a contradiction. Thus $r_0=s_0$ and hence $u\equiv M$ on L.

(c) A direct computation shall show that

$$\partial_t(\phi(u))-\Delta(\phi(u))=\phi^{'}(u_t-\Delta u)-\phi^{''}|Du|^2.$$

As ϕ is convex, we have $\phi^{''}\ge 0$. Thus $\partial_t(\phi(u))-\Delta(\phi(u))\le 0$.

(d) If u solves the heat equation, then u_t and Du each solves the heat equation; by part (c), $(u_t)^2$ and $|Du|^2$ are two subsolutions. Since the heat equation is linear, we simply apply superposition principle to conclude.

18

(Stoke's rule) Assume u solves the initial value problem

$$\begin{cases} u_{tt}-\Delta u=0 \textbf{ in } \mathbb{R}^n\times(0,\infty)\\ u=0, u_t=h \textbf{ on } \mathbb{R}^n\times\{t=0\}\end{cases}$$

Show that $v:=u_t$ solves

$$\begin{cases} v_{tt}-\Delta v=0 \textbf{ in } \mathbb{R}^n\times(0,\infty)\\ v=h, v_t=0 \textbf{ on } \mathbb{R}^n\times\{t=0\}\end{cases}$$

This is *Stoke's rule.*

Solution. Suppose that u solves

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ u = 0, u_t = h \text{ on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

we differentiate the wave equation with respect to time and we have

$$u_{tt} - \Delta u_t = 0, \text{ that is, } v_{tt} - \Delta v = 0,$$

where $v := u_t$.

On $\mathbb{R}^n \times \{t = 0\}$ since $u = 0$ so $\Delta u = 0$, therefore $v_t = u_{tt} = \Delta u = 0$. Hence $v := u_t$ solves the initial value problem

$$\begin{cases} v_{tt} - \Delta v = 0 \text{ in } \mathbb{R}^n \times (0, \infty) \\ v = h, v_t = 0 \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

19

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x, y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x+t, \eta = x-t$, show $u_{tt}-\Delta u = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to derive D'Alembert's formula.

(d) Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

Solution. (a) It is obvious by integrating with respect to x and y.

(b) Since $u(x, t) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2})$, differentiate we get $u_{\xi\eta} = \frac{1}{4}(\Delta u - u_{tt})$. Hence $u_{\xi\eta} = 0$ iff u solves the wave equation.

(c) The general solution is given by $u = F(\xi)+G(\eta) = F(x+t)+G(x-t)$. From the initial condition we have $F(x) + G(x) = g(x)$ and $F^{'}(x) - G^{'}(x) = h(x)$.

Integrating the second the equation $F(x) - G(x) = \int_0^x h(y)\mathrm{d}y$, then $F(x) = \frac{1}{2}\big(g(x) + \int_0^x h(y)\mathrm{d}y\big)$ and $G(x) = \frac{1}{2}\big(g(x) - \int_0^x h(y)\mathrm{d}y\big)$. We then arrive at D'Alembert's formula

$$u(x, t) = \frac{1}{2}\big(g(x + t) + g(x - t)\big) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y.$$

(d) If $F(z) = g(z) + \int_0^z h(y)\mathrm{d}y = 0$, the solution is a right-moving wave; if $G(x) = g(z) - \int_0^z h(y)\mathrm{d}y = 0$, the wave is moving towards the left.

20

Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there exist for all profiles ϕ solutions of the wave equation in $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$ having the form

$$u(x,t) = \alpha(r)\phi(t - \beta(r)).$$

Here $r = |x|$ and we assume $\beta(0) = 0$. Show that this is possible only if $n = 1$ or 3, and compute the form of the functions α, β.

Proof. Setting $v(r,t) := u(x,t)$ we obtain the $n-$dimentional radially symmetric wave equation

$$v_{rr} + \frac{n-1}{r}v_r = v_{tt} \qquad (RW).$$

If distortionless radially symmetric wave propagation is possible, then given any reasonable ϕ the function $v(r,t) = \alpha(r)\phi(t - \beta(r))$ is a solution of (RW). Computing partial derivatives

$$v_{tt} = \alpha\phi^{''}$$
$$v_r = \alpha^{'}\phi - \alpha\phi^{'}\beta^{'}$$
$$v_{rr} = \alpha^{''}\phi - 2\alpha^{'}\phi^{'}\beta^{'} - \alpha\phi^{''}(\beta^{'})^2 - \alpha\phi^{'}\beta^{''}$$

Plugging these into (RW) we have

$$\alpha^{''}\phi - 2\alpha^{'}\phi^{'}\beta^{'} - \alpha\phi^{''}(\beta^{'})^2 - \alpha\phi^{'}\beta^{''} + \frac{n-1}{r}\left(\alpha^{'}\phi - \alpha\phi^{'}\beta^{'}\right) = \alpha\phi^{''}$$

The only possible way the above equation holds for all reasonable ϕ is that the coefficients of $\phi, \phi^{'}, \phi^{''}$ to equal to zero. Equating the coefficients gives $\beta^{'} = 1$, thus $\beta^{''} = 0$; plugging this into the equation gives

$$2\alpha^{'} + \frac{n-1}{r}\alpha = 0, \quad \alpha^{''} + \frac{n-1}{r}\alpha^{'} = 0 \qquad (*)$$

The solutions to $(*)$ are of the form Kr^p where K and p are constants. Plugging this ansatz for α into (1) and (2) gives

$$2p + n - 1 = 0, \quad p(p-1) + p(n-1) = 0 \qquad (**)$$

Equations $(**)$ have solutions only for $p = 1$ or 3. Moreover, when $p = 1$, we have $\alpha(r) = 1$.

21

(a) Assume $E = (E^1, E^2, E^3)$ and $B = (B^1, B^2, B^3)$ solve Maxwell's equations $\begin{cases} E_t = \mathbf{curl} B, B_t = -\mathbf{curl} E \\ \mathbf{div} B = \mathbf{div} E = 0. \end{cases}$

Show $E_{tt} - \Delta E = 0, B_{tt} - \Delta B = 0.$

(b) Assume that $u = (u^1, u^2, u^3)$ solves the evolution equations of linear elasticity $u_{tt} - \mu\Delta u - (\lambda + \mu)D(\mathbf{div} u) = 0$ in $\mathbb{R}^3 \times (0, \infty)$.

Show $w := \mathbf{div} u$ and $w := \mathbf{curl} u$ each solve wave euqations but with differing speeds of propagation.

Proof. (a) Taking curl on both sides of the equations and computing carefully we have the identity $\nabla \times E_t = \nabla(\nabla \cdot B) - \Delta B = -\Delta B$, and $\nabla \times B_t = \Delta E - \nabla(\nabla \cdot E) = \Delta E$ by the divergence free condition. Differentiating both sides of the equations with respect to time we have $E_{tt} = \nabla \times B_t$ and $B_{tt} = -\nabla \times E_t$. We have thus shown that $E_{tt} - \Delta E = 0$, and $B_{tt} - \Delta B = 0$.

(b) Taking divergence of the equation we have

$$\text{div} u_{tt} - \mu(\Delta \text{div} u) - (\lambda + \mu)\text{div}(D(\text{div} u)) = 0,$$

that is $w_{tt} - (2\mu + \lambda)\Delta w = 0$.

Taking curl of the equation we have

$$\text{curl} u_{tt} - \mu\Delta(\text{curl} u) - (\lambda + \mu)\text{curl}(D(\text{div} u)) = 0,$$

that is $v_{tt} - \mu\Delta v = 0$.

22

Let u denote the density of particles moving to the right with speed one alone the real line and let v denote the density of particles

moving to the left with speed one. If at rate $d > 0$ right-moving particles randomly become left-moving, and vice versa, we have the system of PDE $\begin{cases} u_t + u_x = d(v-u) \\ v_t - v_x = d(u-v). \end{cases}$

Show that both $w := u$ and $w := v$ solve the telegraph equation $w_{tt} + 2dw_t - w_{xx} = 0$.

Proof. Differentiating both sides of the equations with respect to t, and to x we have

$$u_{tt} + u_{xt} = d(v_t - u_t), v_{tt} - v_{xt} = d(u_t - v_t).$$

$$u_{tx} + u_{xx} = d(v_x - u_x), v_{tx} - v_{xx} = d(u_x - v_x).$$

Subtracting and using the equations we have

$$u_{tt} - u_{xx} = d(v_t - v_x + u_x - u_t) = d(u - v + v - u) - 2du_t = -2du_t.$$

Adding and using the equations we have

$$v_{tt} - v_{xx} = d(-v_t - v_x + u_x + u_t) = d(u - v + v - u) - 2dv_t = -2dv_t.$$

Hence both u and v solve the telegraph equation.

23

Let S denote the square lying in $\mathbb{R}\times(0,\infty)$ with corners at the points $(0,1),(1,2),(0,3),(-1,2)$. Define $f(x,t) := \begin{cases} -1 \textbf{ for } (x,t) \in S \cap \{t > x+2\} \\ 1 \textbf{ for } (x,t) \in S \cap \{t < x+2\} \\ 0 \textbf{ otherwise.} \end{cases}$

Assume u solves $\begin{cases} u_{tt} - u_{xx} = f \textbf{ in } \mathbb{R}\times(0,\infty) \\ u = 0, u_t = 0 \textbf{ on } \mathbb{R}\times\{t=0\}. \end{cases}$ **Describe the shape of u for $t > 3$.**

Solution. By Duhamel's principle, the solution to the nonhomogeneous wave equation is given by

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y, t-s)\mathrm{d}y\mathrm{d}s \qquad (*)$$

See Fig. 2 attached - at time T_1, $u(x,T_1)=0$ at any point which is not between P_1 and P_2. For example, the double integral $(*)$ vanishes when taken over triangles $\triangle Q_1A_1B_1$ and $\triangle Q_2A_2B_2$ because of the nature of this particular function $f(x,t)$. The force affects $u(x,t)$ only in the shaded comet shaped region; at times T_1 and T_2 the shape of the string is illustrated by the broken lines superimposed on Fig.2. The pointed pulse travels with unit speed in the positive x-direction. The solution is called a "one way wave".

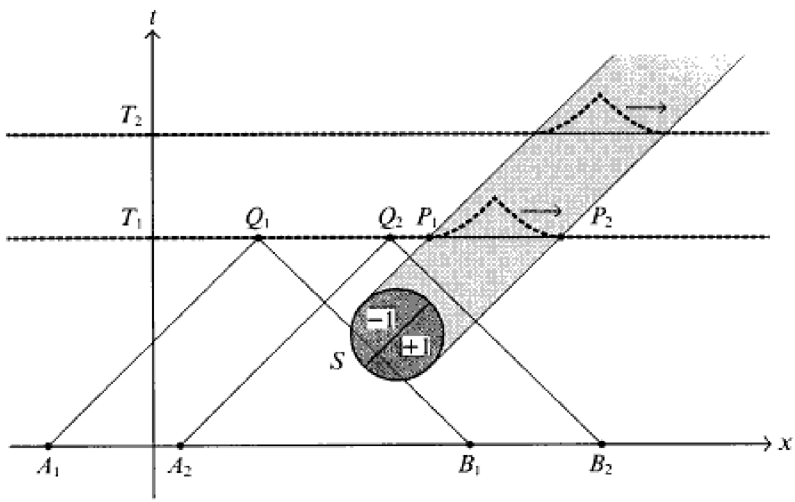

FIG. 2

24

(Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\left.\begin{aligned} u_{tt} - u_{xx} = 0 \text{ in } \mathbb{R}\times(0,\infty) \\ u = g,\ u_t = h \text{ on } \mathbb{R}\times\{t=0\} \end{aligned}\right\}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\mathrm{d}x$ and the potential energy is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\mathrm{d}x$. Prove

(a) $k(t)+p(t)$ is constant in t, (b) $k(t)=p(t)$ for all large enough times t.

Proof. Take derivative of $k(t)$ and$p(t)$, we have

$$k_t(t) = \frac{\partial}{\partial t}\frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} u_t \cdot u_{tt}\mathrm{d}x$$

$$p_t(t) = \frac{\partial}{\partial t}\frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\mathrm{d}x = \int_{-\infty}^{\infty} u_x \cdot u_{xt}\mathrm{d}x = -\int_{-\infty}^{\infty} \Delta u \cdot u_t\mathrm{d}x.$$

Hence, $k_t(t) + p_t(t) = \int_{-\infty}^{\infty} u_t \cdot (u_{tt} - u_{xx})\mathrm{d}x = 0$ which implies that $k(t) + p(t)$ is constant over t. D'Alembert's formula gives us the explicit solution $\frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\mathrm{d}y$ and thus $k(t) - p(t) = \frac{1}{8}\int_{-\infty}^{\infty}[(h(x+t) - h(x-t))^2 - (g^{'}(x+t) + g^{'}(x-t) + h(x+t) + h(x-t))^2]\mathrm{d}x$. Since both g and h are compactly supported, for large enough t the above integral vanishes. As $t \to \infty$, we have $k(t) - p(t) = 0$, that is $k(t) = p(t)$ for all large enough times t.