Common Core Algebra 2

<u>Chapter 3</u>:

Quadratic Equations & Complex Numbers

Chapter Summary:

- The strategies presented for solving quadratic equations in this chapter were introduced at the end of Algebra. The difference now is that solutions are not restricted to real numbers.

- In Section 3.2, complex numbers are defined and operations on complex numbers will be presented. This if followed by the technique of completing the square so that the Quadratic formula can be derived.

- In total, we will use five strategies for solving quadratic equations: graphing, square rooting, factoring, completing the square, and using the Quadratic Formula.

- The last two sections extend work with solving quadratic equations to solving nonlinear systems and solving quadratic inequalities. Each of these topics requires recall of connected skills from work with linear equations. Nonlinear systems are solved by methods of graphing, substitution, and elimination.

Section 3.1: Solving Quadratic Equations

Essential Question: How can you use the graph of a quadratic equation to determine the number of real solutions of equation?

What You Will Learn

- Solve quadratic equations by graphing.
- Solve quadratic equations algebraically.
- Solve real-life problems.

Solving Quadratic Equations by Graphing

A **quadratic equation in one variable** is an equation that can be written in the standard form:

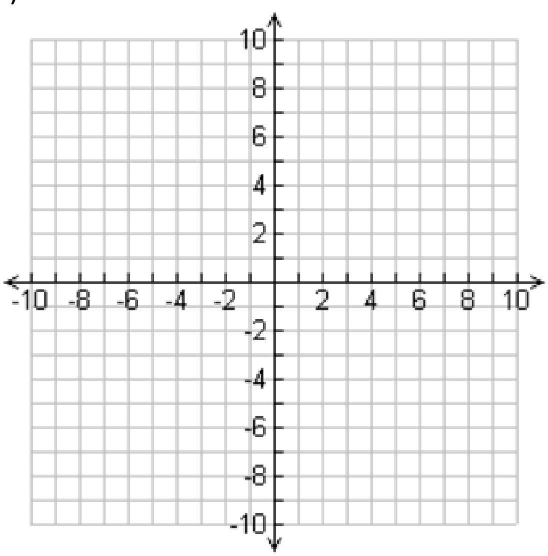
$ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$.

A **root of an equation** is a solution of the equation. We can use various methods to solve quadratic equations.

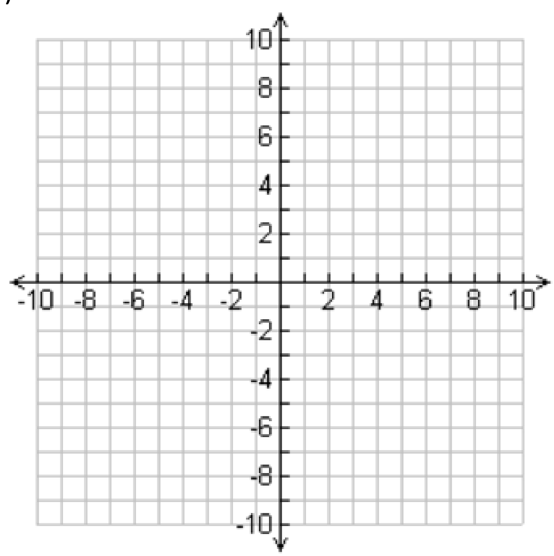
When solving a quadratic equation we are looking for all the possible values of x that make the equation true.

Example 1: Solve each equation by graphing.

a) $x^2 - x - 6 = 0$



b) $-2x^2 - 2 = 4x$



Solving Quadratic Equations Algebraically using Square Roots

When solving quadratic equations using square roots, you can use properties of square roots to write your solutions in different forms.

<u>Example 2</u>: Solve each equation using square roots.

a) $3x^2 - 9 = 0$ b) $4x^2 - 31 = 49$ c) $\frac{5}{2}(x+3)^2 = 5$

Solving a Quadratic Equations Algebraically by Factoring

When the left side of $ax^2 + bx + c = 0$ is factorable, we can solve the equation using the:
<u>Zero-Product Property</u>.

Core Concept

Zero-Product Property

Words If the product of two expressions is zero, then one or both of the expressions equal zero.

Algebra If A and B are expressions and $AB = 0$, then $A = 0$ or $B = 0$.

<u>Example 3</u>: Solve the following equations by factoring.

a) $x^2 - 4x = 45$ b) $2x^2 - 11x + 12 = 0$ c) $x^2 = 8x$

Zeros of a Quadratic Function

We know the x-intercepts of the graph of $f(x) = a(x-p)(x-q)$ are _____ and _____. The value of the function is equal to ___________ when $x = p$ and $x = q$, therefore the numbers p and q are called $zeros$ of the function.

A **zero of a function** f is an x-value for which $f(x) = 0$.

Example 4: Find the zero(s) of each of the functions.

a) $f(x) = x^2 + 12x + 35$

b) $f(x) = 3x^2 + 5x$

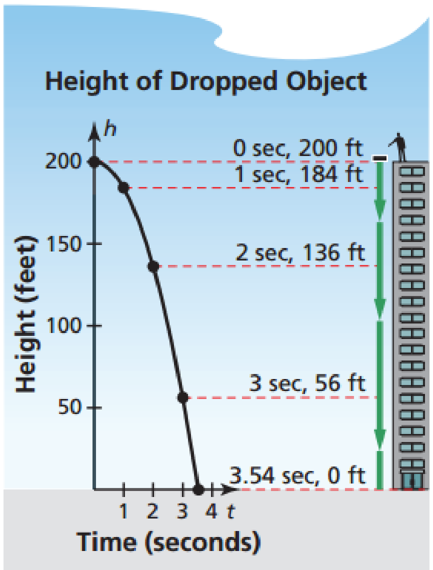
c) $f(x) = -x^2 + 2x + 3$

d) $f(x) = 4x^2 + 28x + 49$

e) $f(x) = 3x^2 + 9$

Solving Real-Life Problems

When an object is dropped, its height h (in feet) above the ground after t seconds can be modeled by the functions $h = -16t^2 + h_0$, where h_0 is the initial height (in feet) of the object.



The graph of $h = -16t^2 + 200$, representing the height of an object dropped from an initial height of 200 feet, is shown to the left.

How long does it take for this object to hit the ground?

The model $h = -16t^2 + h_0$ assumes that the force of air resistance on the object is negligible. Also, this model applies only to objects dropped on Earth. For planets with stronger or weaker gravitational forces, different models are used.

<u>Example 5</u>: Modeling a Dropped Object

For a science competition, students must design a container that prevents an egg from breaking when dropped from a height of 50 feet.

a) Write a function that gives the height h (in feet) of the container after t seconds.

b) How long does it take for the container to hit the ground?

c) Find and interpret $h(1) - h(1.5)$.

Section 3.2: Complex Numbers

Essential Question: What are the subsets of the set of complex numbers?

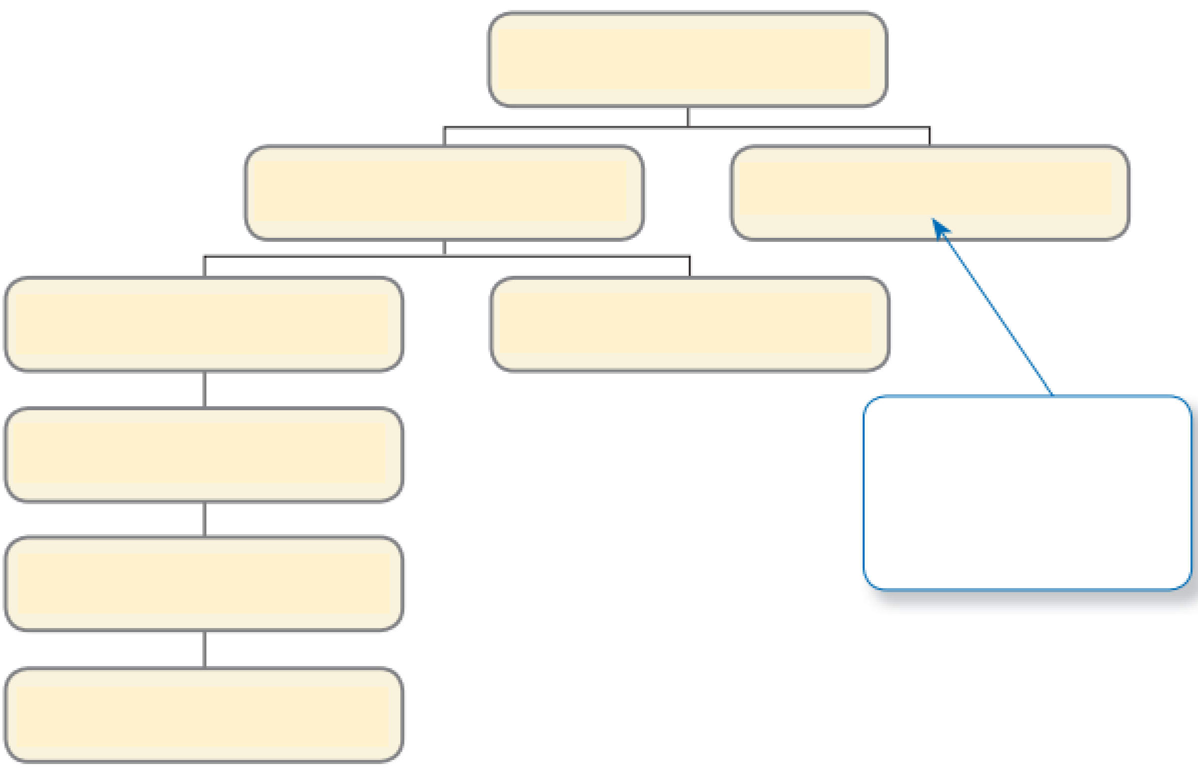
In your study of mathematics, you have probably worked with only *real numbers*, which can be represented graphically on the real number line.

In this lesson, the system of numbers is expanded to include *imaginary numbers*. The set of real numbers and imaginary numbers make up the set of *complex numbers*.

What You Will Learn

- Define and use the imaginary unit i
- Add, subtract, and multiply complex numbers.
- Find complex solutions and zeros.

THE COMPLEX NUMBER SYSTEM



Classifying Numbers

Example 1: Determine which subsets of the set of complex numbers.

a) $\sqrt{9}$ b) $\sqrt{0}$ c) $-\sqrt{4}$ d) $\sqrt{\frac{4}{9}}$ e) $\sqrt{-1}$

The Imaginary Unit i

Not all quadratic equations have real-number solutions. For example, $x^2 = -3$ has no real-number solutions since the square of any real number is never a negative number.

To overcome this problem, mathematicians created an expanded system of numbers using the imaginary unit i, defined as $i = \sqrt{-1}$. The imaginary unit i can be used to write the square root of any negative number.

Remember:

$i = \sqrt{-1}$

$i^2 =$ _____

Core Concept

The Square Root of a Negative Number

Property	Example
1. If r is a positive real number, then $\sqrt{-r} = i\sqrt{r}$.	$\sqrt{-3} = i\sqrt{3}$
2. By the first property, it follows that $(i\sqrt{r})^2 = -r$.	$(i\sqrt{3})^2 = i^2 \cdot 3 = -3$

Finding Square Roots of Negative Numbers

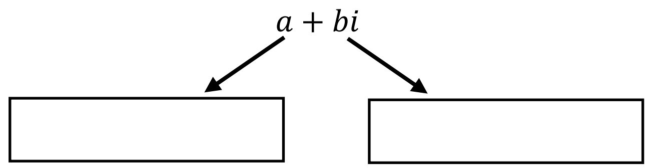
Example 2: Find the square root of each number.

a) $\sqrt{-25}$ b) $-5\sqrt{-9}$ c) $\sqrt{-72}$ d) $2\sqrt{-54}$

e) $\sqrt{-32}$ f) Simplify: $\sqrt{-25} + \sqrt{-49} + \sqrt{-48} - \sqrt{-75}$

Complex Numbers

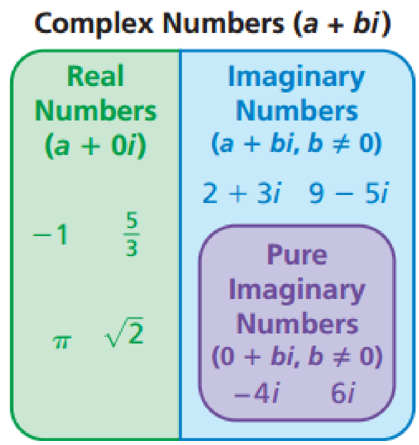
A complex number written in $standard\ form$ is a number $a + bi$ where a and b are real numbers.



If $b \neq 0$, then $a + bi$ is an imaginary number.

If $a = 0$ and $b \neq 0$, then $a + bi$ is a pure imaginary number.

The diagram to the right shows how different types of complex numbers are related.



Equality of Two Complex Numbers

Example 3: Find the values of x and y that satisfy each of the equations.

a) $2x - 7i = 10 + yi$ b) $x + 3i = 9 - yi$ c) $9 - 4yi = -2x + 3i$

Operations with Complex Numbers

Example 4: Add or subtract. Write the answer in standard form.

a) $(8 - i) + (5 + 4i)$ b) $(7 - 6i) - (3 - 6i)$ c) $13 - (2 + 7i) + 5i$

Example 5: Multiply. Write the answer in standard form.

a) $4i(-6+i)$

b) $(9-2i)(-4+7i)$

c) $(-3+2i)(-3+2i)$

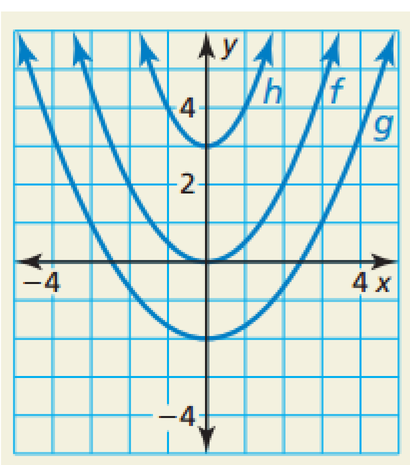
Complex Solutions and Zeros

Example 6: Solve the following quadratic equations.

a) $x^2+4=0$

b) $2x^2-11=-47$

c) HOW DO YOU SEE IT? The graphs of three functions are shown. Which function(s) has real zeros? Which function(s) have imaginary roots? Explain your reasoning.

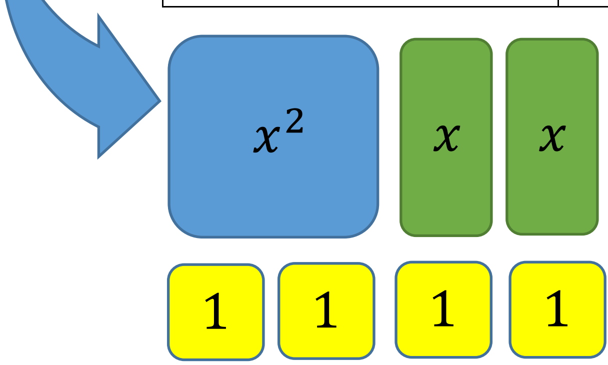


Section 3.3: Completing the Square

Essential Question: How can you complete the square for a quadratic expression?

Completing the Square → A process used by adding a term c to an expression of the form $x^2 + bx$ such that $x^2 + bx + c$ is a perfect square trinomial.

Expression	Value of c needed to "complete the square"	Expression written as a binomial squared
1) $x^2 + 2x + c$		
2) $x^2 + 4x + c$		
3) $x^2 + 8x + c$		
4) $x^2 + 10x + c$		



Look for a pattern in the middle column of the table.

How are b and c related?

Rule:

Look for patterns in the last column of the table. Consider the general statement $x^2 + bx + c = (x + d)^2$.

How are b and d related?

Rule:

What You Will Learn

- Solve quadratic equations using square roots.
- Solve quadratic equations by completing the square.
- Write quadratic functions in vertex form.

Solving Quadratic Equations

Example 1: Solve the following quadratic equations.

a) $(x-8)^2 = 100$

b) $(x+4)^2 = 25$

c) $2(x-7)^2 = 40$

Solving Quadratic Equations by Completing the Square when [$\boldsymbol{a = 1}$]

Example 2: Solving the following quadratic equations by completing the square. Then identify the vertex.

a) $x^2 - 10x + 7 = 0$

b) $x^2 + 8x - 4 = 0$

c) $x^2 - 5x + 1 = 0$

d) $x^2 - 3x + 11 = 0$

Solving Quadratic Equations by Completing the Square [when $\boldsymbol{a \neq 1}$]

Example 3: Solve the following quadratic equation by completing the square. Then identify the vertex.

a) $3x^2 + 12x + 15 = 0$

b) $4x^2 + 24x - 11 = 0$

Modeling with Mathematics

Example 4: The height y (in feet) of a baseball t seconds after David Wright hits the ball can be modeled by the function:

$$y = -16t^2 + 96t + 3$$

Find the maximum height of the baseball.

METHOD #1

METHOD #2

How long does the ball take to hit the ground?

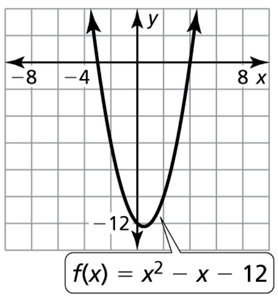
Name ____ Date _________

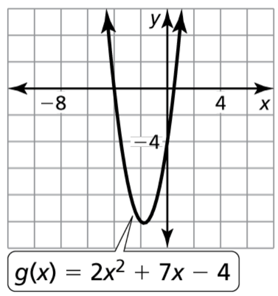
Chapter 3 Quiz

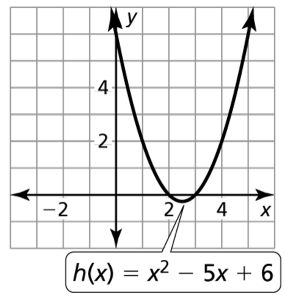
For use after Section 3.3

Solve the equation by using the graph. Check your solution(s).

1. $x^2 - x - 12 = 0$ **2.** $2x^2 - 4 = -7x$ **3.** $x^2 = 5x - 6$







Solve the equation using square roots or by factoring.

4. $x^2 = 3x - 1$ **5.** $x(x - 1) = 3$ **6.** $2(x^2 - 2x) = 5$

7. Find the values of x and y that satisfy the equation $5x + 8i = 30 + yi$.

Perform the operation. Write your answer in standard form.

8. $(3 + 4i) + (-6 + 2i)$ **9.** $(7 + 6i) - (4 - 3i)$

10. Find the zeros of the function $f(x) = 5x^2 + 2$.

Solve the equation by completing the square.

11. $x^2 + 16x - 22 = 0$ **12.** $x^2 - 12x + 26 = 0$

13. Write $y = x^2 + 4x - 5$ in vertex form. Then identify the vertex.

14. A water balloon is tossed into the air so that its height h (in feet) after t seconds can be modeled by the function $h(t) = -16t^2 + 80t + 5$.

a. What is the height of the balloon after 1 second?

b. For how long is the balloon more than 30 feet high?

c. What is the maximum height of the balloon?

15. A rectangular lawn measuring 24 feet by 16 feet is surrounded by a flower bed of uniform width. The combined area of the lawn and the flower bed is 660 square feet. What is the width of the flower bed?

Answers

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

7. ____________

8. ____________

9. ____________

10. ____________

11. ____________

12. ____________

13. ____________

____

14. a. ____________

b. ____________

c. ____________

15. ____________

<u>Section 3.4</u> – Using the Quadratic Formula

<u>Essential Question</u>: How can you derive a general formula for solving a quadratic equation?

What You Will Learn

- Solve quadratic equations using the Quadratic Formula.
- Analyze the discriminant to determine the number and type of solutions.
- Solve real-life problems.

DERIVATION OF THE QUADRATIC FORMULA

Let's begin with a quadratic equation in standard form: ____

Solving Equations Using the Quadratic Formula

Previously, you solved quadratic equations by completing the square. We developed a formula that gives the solutions of any quadratic equation by completing the square for the general equation $ax^2 + bx + c = 0$.

The formula for the solutions is called the **QUADRATIC FORMULA.**

Core Concept

The Quadratic Formula

Let a, b, and c be real numbers such that $a \neq 0$. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Solving an Equation with Two Real Solutions

Example 1: Solve $x^2 + 3x = 5$ using the Quadratic Formula. **SKETCH**

Example 2: Solve the equation using the Quadratic Formula.

(a) $x^2 - 6x + 4 = 0$

(b) $2x^2 + 4 = -7x$

(c) $5x^2 = x + 8$

Solving an Equation with One Real Solution

Example 3: Solve $25x^2 - 8x = 12x - 4$ using the Quadratic Formula. **SKETCH**

Solving an Equation with Imaginary Solutions

Example 4: Solve $-x^2 + 4x = 13$ using the Quadratic Formula. **SKETCH**

Example 5: Solve the following quadratic equations.

(a) $x^2 + 41 = -8x$

(b) $-9x^2 = 30x + 25$

(c) $5x - 7x^2 = 3x + 4$

ANALYZING THE DISCRIMINANT

In the Quadratic Formula, the expression:

$b^2 - 4ac$ is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can analyze the discriminant of a quadratic equation to determine the number and type of solutions of the equation.

Core Concept

Analyzing the Discriminant of $ax^2 + bx + c = 0$

Value of discriminant	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Number and type of solutions			
Graph of $y = ax^2 + bx + c$			

Example 6: Find the discriminant of the quadratic equation and describe the number and type of solutions of the equation.

(a) $4x^2 + 8x + 4 = 0$

(b) $\frac{1}{2}x^2 + x - 1 = 0$

(c) $5x^2 = 8x - 13$

(d) $7x^2 - 3x = 6$

(e) $4x^2 + 6x = -9$

(f) $-5x^2 + 1 = 6 - 10x$

Example 7: Writing an Equation

Find a possible pair of integer values of a and c so that the equation $ax^2 - 4x + c = 0$ has one real solution. The write the equation.

Find a possible pair of integer values for a and c so that the equation $ax^2 + 3x + c = 0$ has two real solutions. Then write the equation.

Concept Summary

Methods for Solving Quadratic Equations

Method	When to Use
Graphing	Use when approximate solutions are adequate.
Using square roots	Use when solving an equation that can be written in the form $u^2 = d$, where u is an algebraic expression.
Factoring	Use when a quadratic equation can be factored easily.
Completing the square	Can be used for *any* quadratic equation $ax^2 + bx + c = 0$ but is simplest to apply when $a = 1$ and b is an even number.
Quadratic Formula	Can be used for *any* quadratic equation.

Solving Real Life Problems

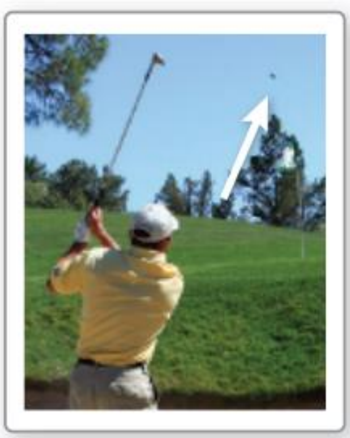
The function $h = -16t^2 + h_0$ is used to model the height of a *dropped* object. For an object that is *launched or thrown*, and extra term $v_0 t$ must be added to the model to account for the object's initial vertical velocity, v_0 (in feet per second).

Recall that h is the height (in feet), t is the time in motion (in seconds), and h_0 is the initial height (in feet).

$h = -16t^2 + h_0$ $\rightarrow$

$h = -16t^2 + v_0 t + h_0$ $\rightarrow$

As shown below, the value of v_0 can be ________________, ________________, or _____________ depending on whether the object is launched upward, downward, or parallel to the ground.



Example 8: A juggler tosses a ball into the air. The ball leaves the juggler's hand 4 feet above the ground and has an initial vertical velocity of 30 feet per second. The juggler catches the ball when it falls back to a height of 3 feet. How long is the ball in the air?

Example 9: A lacrosse player throws a ball in the air from an initial height of 7 feet. The ball has an initial vertical velocity of 90 feet per second. Another player catches the ball when it is 3 feet above the ground. How long is the ball in the air?

Example 10: In a volleyball game, a player on one team spikes the ball over the net when the ball is 10 feet above the court. The spike drives the ball downward with an initial vertical velocity of 55 feet per second. How much time does the opposing team have to return the ball before it touches the court?

<u>Section 3.5</u> – Solving Nonlinear Systems

<u>Essential Question</u>: How can you solve a nonlinear system of equations?

What You Will Learn

- Solve systems of nonlinear equations.
- Solve quadratic equations by graphing.

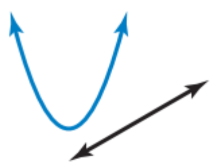
Systems of Nonlinear Equations

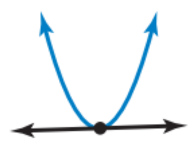
Previously, we've solved systems of *linear* equations by graphing, substitution, and elimination. You can also use these methods to solve a system of *nonlinear* equations.

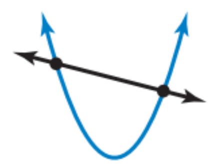
In a **<u>system of nonlinear equations</u>**, at least one of the equations in nonlinear. For instance, the nonlinear system shown below has a ________________________ equation and a ________________ equation.

$$y = x^2 + 2x - 4$$
$$y = 2x + 5$$

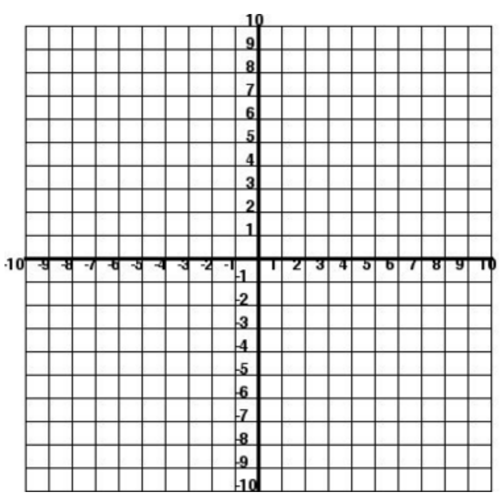
When the graphs of the equations in a system are a line and a parabola, the graphs can intersect in zero, one, or two points. So the system can have zero, one, or two solutions.


No solution


One solution

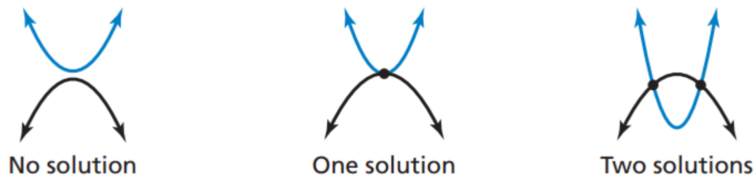

Two solutions

<u>Example 1</u>:
Solve the above system by graphing.



<u>Example 2</u>:
Solve the above system by substitution.

When the graphs of the equations in a system are a parabola that opens up and parabola that opens down, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions.



Example 3: Solving the following system by elimination.

$y = 2x^2 - 5x + 2$
$y = x^2 + 2x$

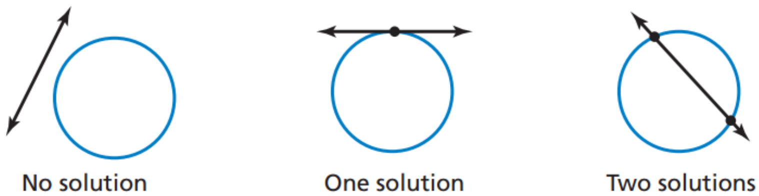
Example 4: Solve the following equations by any method.

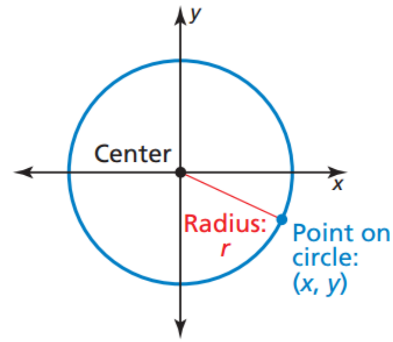
(a) $y = x^2 - 6x + 15$
$y = -(x-3)^2 + 6$

(c) $y = (x+4)(x-1)$
$y = -x^2 + 3x + 4$

Some nonlinear systems have equations of the form: $x^2 + y^2 = r^2$. This equation is the standard form of a circle with center $(0, 0)$ and radius r.

When the graphs of the equations in a system are a line and a circle, the graphs can intersect in zero, one, or two points. So, the system can have zero, one, or two solutions.





<u>Example 5</u>: Solve the following systems of equations.

(a) $x^2 + y^2 = 10$

$y = 3x + 10$

(b) $x^2 + y^2 = 1$

$y = \frac{1}{2}x + \frac{1}{2}$

<u>Section 3.6</u>: Quadratic Inequalities

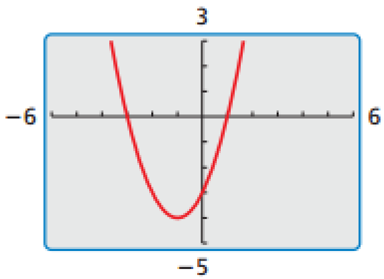
<u>Essential Question</u>: How can you solve a quadratic inequality?

What You Will Learn

- Graph quadratic inequalities in two variables.
- Solve quadratic inequalities in one variable.

<u>Exploration</u>: Solving a Quadratic Inequality

The graphing calculator screen shows the graph of: $f(x) = x^2 + 2x - 3$.



Explain how you can use the graph to solve the inequality:

$$x^2 + 2x - 3 \leq 0$$

Use the above graph to solve each of the following inequalities.

(a) $x^2 + 2x - 3 < 0$

(b) $x^2 + 2x - 3 \geq 0$

(c) $x^2 + 2x - 3 > 0$

Graphing Quadratic Inequalities in Two Variables

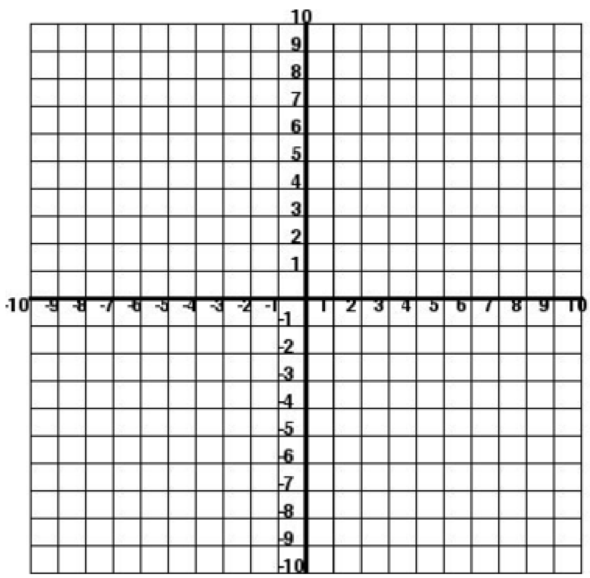
A **quadratic inequality in two variables** can be written in one of the following forms, where a, b, and c are real numbers and $a \neq 0$.

$$y < ax^2 + bx + c \qquad\qquad y > ax^2 + bx + c$$

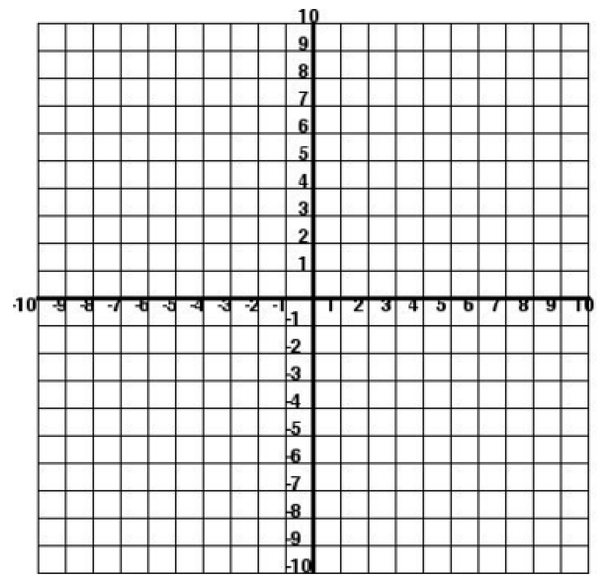
$$y \leq ax^2 + bx + c \qquad\qquad y \geq ax^2 + bx + c$$

<u>Example 1</u>: Graph the following inequalities.

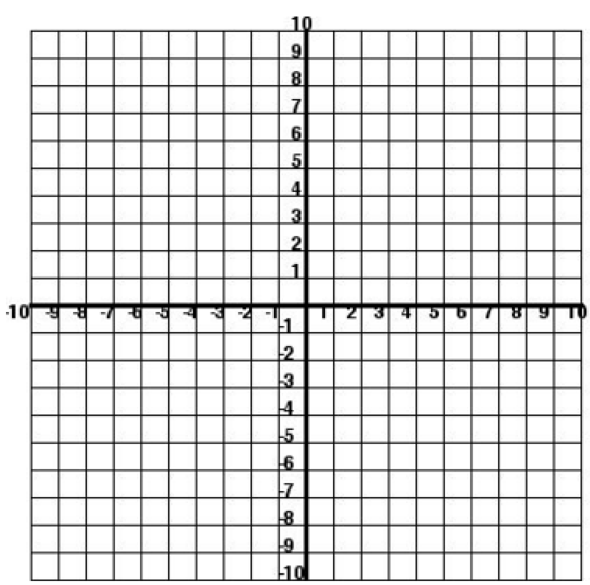
$y < -x^2 - 2x - 1$



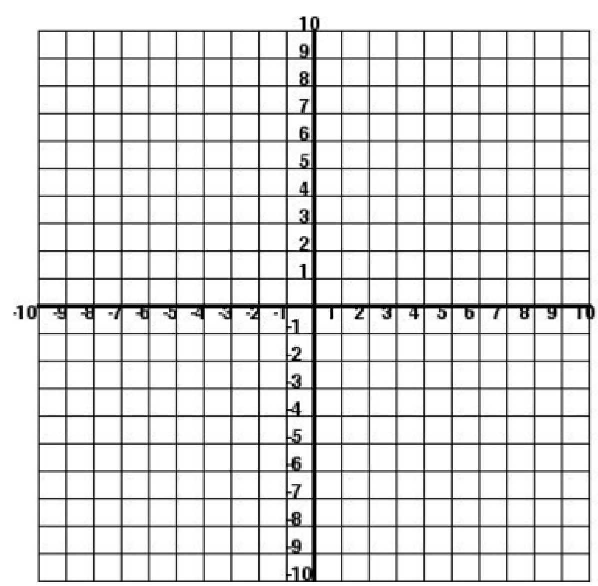
$y > x^2 - 4x + 3$



$y \leq x^2 - 4$

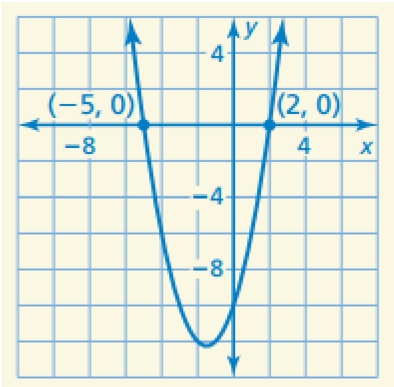


$y \geq -x^2 + 3x - 2$



Solving Inequalities by Graphing

Example 2: Using the graph of $f(x) = x^2 + 3x - 10$, solve each of the following inequalities.



(a) $f(x) \leq 0$ (b) $f(x) \geq 0$ (c) $f(x) < 0$

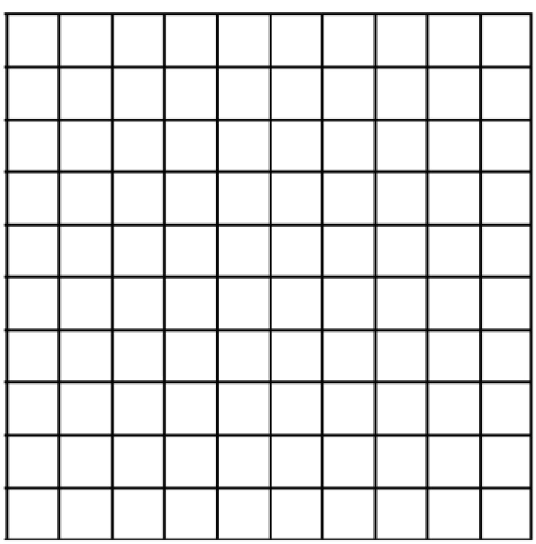
(d) $f(x) > 0$ (e) $f(x) \leq -6$ (f) $f(x) > -6$

Using a Quadratic Inequality in Real Life

Example 3: A manila rope used for rappelling down a cliff can safely support a weight W (in pounds) provided:

$$W \leq 1480d^2$$

where d is the diameter (in inches) of the rope. Graph the inequality and interpret the solution.



Modeling with Mathematics

Example 4:

A rectangular parking lot must have a perimeter of 440 feet and an area of at least 8000 square feet. Describe the possible lengths of the parking lot.

(a) Write an equation to represent the perimeter of the parking lot.

(b) Write an inequality to represent the area of the parking lot.

What can we do with the above equation and inequality to help us find all possible lengths?

Review for Chapter 3 Test (Part 2)

Solve the equation using any method. Provide a reason for your choice.

1. $0 = x^2 + 2x + 3$

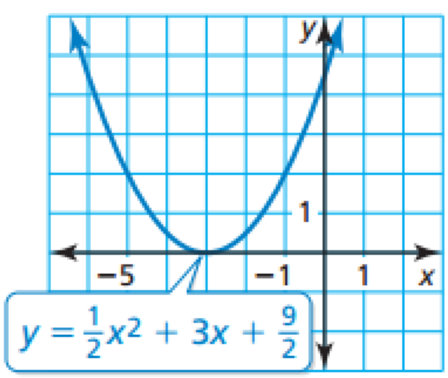
2. $6x = x^2 + 7$

3. $x^2 + 49 = 85$

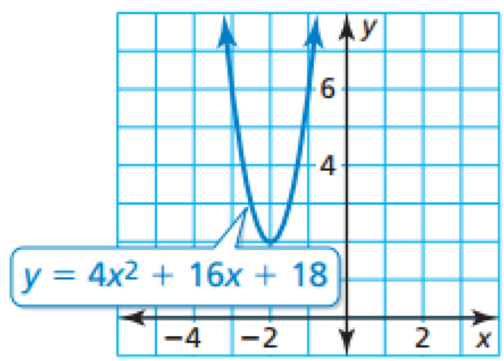
4. $(x + 4)(x - 1) = -x^2 + 3x + 4$

Example how to use the graph to find the number and type of solutions of the quadratic equation. Justify your answer by using the discriminant.

5.



6.



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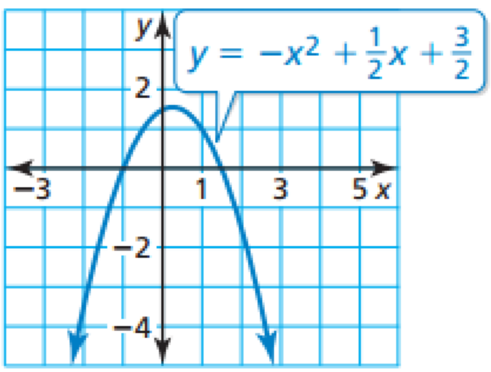


Solve the system of equations.

8. $x^2 + 66 = 16x - y$

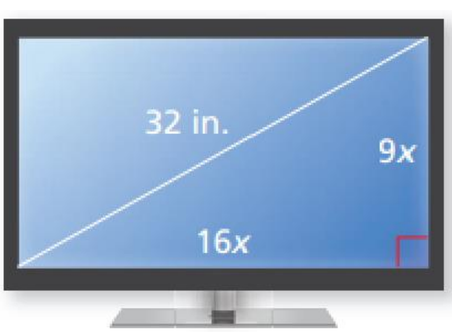
   $2x - y = 18$

9. $0 = x^2 + y^2 - 40$

   $y = x + 4$

10. Write $(3 + 4i)(4 - 6i)$ as a complex number in standard form.

11. The *aspect ratio* of a widescreen TV is the ratio of the screen's width to its height, or 16:9. What are the width and the height of a 32-inch widescreen TV? [Hint: Use Pythagorean Theorem]



12. The shape of the Gateway Arch in St. Louis, Missouri, can be modeled by $y = -0.0063x^2 + 4x$, where x is the distance (in feet) from the left foot of the arch and y is the height (in feet) of the arch above the ground. For what distances x is the arch more than 200 feet above the ground?