Rational equations:

- equations containing one or more rational expressions
  - remember from Lesson 7 that rational expressions are simply fractions containing polynomials, so each equation we work with today will contain fractions
  - Example: $\frac{x+4}{5x-3} - \frac{2x-5}{10x+7} = 0$
- since we are working with equations now, rather than expressions, we do **NOT** have to work with the fractions; instead we can eliminate them using multiplication
  - in the case of $\frac{x+4}{5x-3} - \frac{2x-5}{10x+7} = 0$ we could multiply both sides of the equation by $5x - 3$ and $10x + 7$ to eliminate the fractions
  - if you prefer to get a common denominator and work with the fractions, you may, but I do not recommend it
- keep in mind that even if we eliminate the fractions, we still cannot have any answers that result in a denominator of zero
  - in the case of $\frac{x+4}{5x-3} - \frac{2x-5}{10x+7} = 0$, $x \neq \frac{3}{5}$ or $x \neq -\frac{7}{10}$
    - $5x - 3 \neq 0$

      $5x \neq 3$

      $x \neq \frac{3}{5}$
    - $10x + 7 \neq 0$

      $10x \neq -7$

      $x \neq -\frac{7}{10}$
  - the answer to $\frac{x+4}{5x-3} - \frac{2x-5}{10x+7} = 0$ is $x = -\frac{1}{6}$, and since $-\frac{1}{6}$ does not result in a denominator or zero, it is a valid answer
- **ALWAYS** exclude any values that make the denominator zero
- **ALWAYS** make sure that your solution(s) is/are not excluded

Steps for Solving Rational Equations:

1. factor all denominators
2. multiply both sides of the equation by the least common multiple of all the denominators to eliminate the fractions
3. distribute and combine like terms
4. isolate the variable
5. verify that your solution(s) is/are not restricted

Example 1: Solve the following equation. If there is no solution, write NO SOLUTION. If there are infinitely many solutions, list the restrictions using the notation $R - \{\quad\}$ (all real numbers except).

$$\frac{-6}{x-5} - \frac{3}{x^2-9x+20} = \frac{-5}{x-4}$$

$$(x-4)(x-5)\left(\frac{-6}{x-5} - \frac{3}{(x-4)(x-5)}\right) = \left(\frac{-5}{x-4}\right)(x-4)(x-5)$$

Example 2: Solve the following equation. If there is no solution, write NO SOLUTION. If there are infinitely many solutions, list the restrictions using the notation $R - \{\quad\}$ (all real numbers except).

$$\frac{4}{x-3} + \frac{6}{x+3} = \frac{10x-6}{x^2-9}$$

$$(x-3)(x+3)\left(\frac{4}{x-3} + \frac{6}{x+3}\right) = \left(\frac{10x-6}{(x-3)(x+3)}\right)(x-3)(x+3)$$

$$\frac{4(x-3)(x+3)}{x-3} + \frac{6(x-3)(x+3)}{x+3} = \frac{(10x-6)(x-3)(x+3)}{(x-3)(x+3)}$$

$$\frac{4\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} + \frac{6(x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = \frac{(10x-6)\cancel{(x-3)}\cancel{(x+3)}}{\cancel{(x-3)}\cancel{(x+3)}}$$

$$4(x+3) + 6(x-3) = 10x - 6$$

A true statement, such as $10x - 6 = 10x - 6$, is called an identity. An identity is an indication that an equation has infinitely many solutions. Keep in mind that this does **NOT** mean that every single number can take the place of x and result in a valid solution. We still must exclude values that result in a denominator of zero, such as -3 and 3 in the case of Example 2.

Example 3: Solve the following equation. If there is no solution, write NO SOLUTION. If there are infinitely many solutions, list the restrictions using the notation $R - \{\quad\}$ (all real numbers except).

$$\frac{3x-6}{x^2-4} - \frac{3}{x+2} = \frac{9}{x-2}$$

$$(x-2)(x+2)\left(\frac{3x-6}{(x-2)(x+2)} - \frac{3}{x+2}\right) = \left(\frac{9}{x-2}\right)(x-2)(x+2)$$

$$\frac{(3x-6)(x-2)(x+2)}{(x-2)(x+2)} - \frac{3(x-2)(x+2)}{x+2} = \frac{9(x-2)(x+2)}{x-2}$$

$$\frac{(3x-6)\cancel{(x-2)(x+2)}}{\cancel{(x-2)(x+2)}} - \frac{3(x-2)\cancel{(x+2)}}{\cancel{(x+2)}} = \frac{9\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$3x-6-3(x-2) = 9(x+2)$$

Be sure to **ALWAYS** check your answers in the original equation to be sure they do not result in a denominator of zero. If they do, they are not valid answers. If you only have one solution, and it is not valid, then you now have **NO SOLUTION**. No solution does not mean we didn't come up with any answers (on Example 3 we did, $x = -2$); it means the answer we came up with is invalid because it makes a denominator equal to zero.

Example 4: Solve the following equationS. If there is no solution, write NO SOLUTION. If there are infinitely many solutions, list the restrictions using the notation $R - \{\quad\}$ (all real numbers except).

a.

$$17 - \frac{3}{x} = -16$$

$$x\left(17 - \frac{3}{x}\right) = (-16)x$$

$$17x - 3 = -16x$$

$$33x = 3$$

$$x = \frac{3}{33}$$

$$\boldsymbol{x = \frac{1}{11}}$$

b.

$$\frac{3x+1}{6x-1} = \frac{2x+5}{4x-13}$$

$$(4x-13)(6x-1)\left(\frac{3x+1}{6x-1}\right) = \left(\frac{2x+5}{4x-13}\right)(4x-13)(6x-1)$$

$$(4x-13)(3x+1) = (2x+5)(6x-1)$$

$$12x^2 + 4x - 39x - 13 = 12x^2 - 2x + 30x - 5$$

$$12x^2 - 35x - 13 = 12x^2 + 28x - 5$$

$$-8 = 63x$$

$$\boldsymbol{-\frac{8}{63} = x}$$

c.

$$\frac{5}{2x+3} + \frac{4}{2x-3} = \frac{14x+3}{4x^2-9}$$

d.

$$\frac{6}{x+3} - \frac{5}{x-2} = \frac{-20}{x^2+x-6}$$

e. $\frac{1}{2x-1} = \frac{4}{8x-4}$

$\frac{1}{2x-1} = \frac{4}{4(2x-1)}$

$\frac{1}{2x-1} = \frac{1}{2x-1}$

Since it is true that $\frac{1}{2x-1}$ is equal to itself, this equation is an identity. That means we have infinitely many solutions. The only value that will not make this equation true is any number that results in a denominator of zero.

$2x - 1 \neq 0$

$2x \neq 1$

$x \neq \frac{1}{2}$

So x can be any real number except for $\frac{1}{2}$. We express this as $R - \left\{\frac{1}{2}\right\}$

$\mathbb{R} - \left\{\frac{1}{2}\right\}$

f. $\frac{-1}{x+9} - \frac{18}{(x+9)(x-9)} = \frac{-2}{x-9}$

$(x+9)(x-9)\left(\frac{-1}{x+9} - \frac{18}{(x+9)(x-9)}\right) = \left(\frac{-2}{x-9}\right)(x+9)(x-9)$

$\frac{-1(x+9)(x-9)}{x+9} - \frac{18(x+9)(x-9)}{(x+9)(x-9)} = \frac{-2(x+9)(x-9)}{x-9}$

$\frac{-1\cancel{(x+9)}(x-9)}{\cancel{x+9}} - \frac{18\cancel{(x+9)(x-9)}}{\cancel{(x+9)(x-9)}} = \frac{-2(x+9)\cancel{(x-9)}}{\cancel{x-9}}$

$-1(x-9) - 18 = -2(x+9)$

$-x + 9 - 18 = -2x - 18$

$-x - 9 = -2x - 18$

$x = -9$

The rational equation we started with produced one answer, $x = -9$. However that one answer is not valid because replacing x with -9 results in two of the rational expressions having denominators of zero $\left(\frac{-1}{x+9}\right.$ and $\left.\frac{18}{(x+9)(x-9)}\right)$. Therefore the one answer we had is invalid, so we now have no solution.

NO SOLUTION

Answers to Examples:

1. $x = -4$ *; 2.* $INFINITELY\ MANY\ SOLUTIONS, \mathbb{R} - \{-3, 3\}$ *;*

3. $NO\ SOLUTION$ *; 4a.* $\frac{1}{11}$ *; 4b.* $x = -\frac{8}{63}$ *;*

4c. $NO\ SOLUTION$ *; 4d.* $x = 7$ *;*

4e. $INFINITELY\ MANY\ SOLUTIONS, \mathbb{R} - \left\{\frac{1}{2}\right\}$ *;*

4f. $NO\ SOLUTION$ *;*