SOLVING TRIGONOMETRIC EQUATIONS – CONCEPT & METHODS

1. (by Nghi H. Nguyen, Udated 06/03/2020)

DEFINITION.

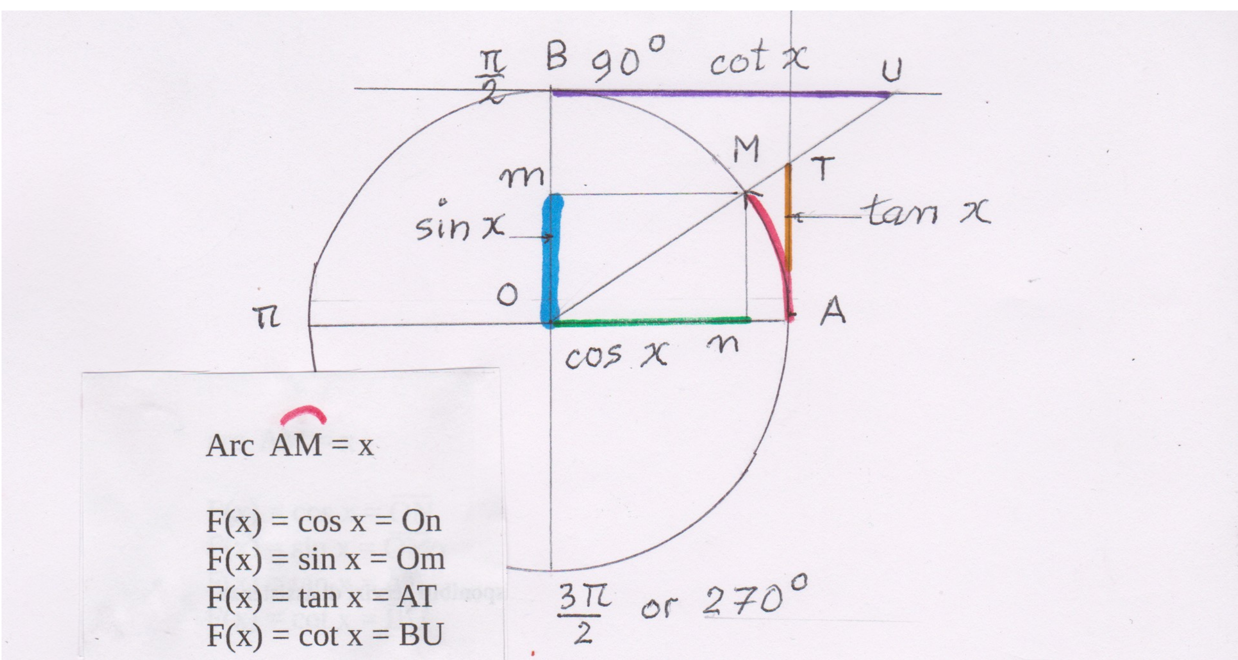
A trig equation is an equation containing one or many trig functions of the **variable arc x** that varies and rotates counter clockwise on the trig unit circle. Solving for **x** means finding the values of the trig arcs **x** whose trig functions make the trig equation true.

Example of trig equations:

tan (x + Л/3) = 1.5	sin (2x + Л/4) = 0.5	sin x + sin 2x = 0.75
cos x + sin 2x = 1	tan x + cot x = 1.732	2sin 2x + cos x = 1

Answers, or values of the solution arcs, are expressed in degrees or radians.

Examples:	x = 30°;	x = - 43°72	x = 360°
	x = Л/3	x = -2Л/3	x = 2Л



THE TRIG UNIT CIRCLE.

It is a circle with radius R = 1, and with an origin O. The unit circle defines the main trig functions of the variable arc x that varies and rotates counterclockwise on it. On the unit circle, the value of the arc x is exactly equal to the corresponding angle x.

When the variable arc AM = x (in radians or degree) varies on the trig unit circle:

- The horizontal axis OAx defines the function **f(x) = cos x.** When the arc x varies from 0 to 2Л, the function f(x) = cos x varies from 1 to (-1), then back to 1.

- The vertical axis OBy defines the function **f(x) = sin x**. When the arc x varies from 0 to 2Л, the function f(x) = sin x varies from 0 to 1, then to -1, then back to 0.

- The vertical axis AT defines the function **f(x) = tan x.** When x varies from -Л/2 to Л/2, the function f(x) varies from - ∞ to +∞.

- The horizontal axis BU defines the function f(x) = cot x. When x varies from 0 to Л, the function f(x) = cot x varies from +∞ to -∞.

The trig unit circle will be used as proof for solving trig equations & trig inequalities.

THE PERIODIC PROPERTY OF TRIG FUNCTIONS.

All trig functions are periodic meaning they come back to the same values after the trig arc x rotates one period on the trig unit circle.

Examples:

The trig function f(x) = sin x has 2Л as period

The trig function f(x) = tan x has Л as period

The trig function f(x) = sin 2x has Л as period

The trig function f(x) = cos (x/2) has 4Л as period.

The trig function f(x) = cos (2x/3) has 3(2Л)/2 = 3Л as period

FIND THE ARC X WHOSE TRIG FUNCTIONS ARE KNOWN.

Before learning solving trig equations, you must know how to quickly find the solution arcs whose trig functions are known. Values of solution arcs (or angles), expressed in radians or degree, are given by **trig tables** or by calculators. Examples:

After solving get cos x = 0.732. Calculator gives the solution arc x1 = $42^{o}95$ degree. The trig unit circle will give another arc x2 = - $42^{o}95$ that has the same cos value (0.732).

After solving get sin x = 0.5. Trig table of special arcs gives the solution arc: x = Л/6. The unit circle gives another answer: x = 5Л/6. Since the arc x rotates many time on the unit circle, there are many extended answers:

x = Pi/6 +2KЛ (K is a real whole number) and x = 5Pi/6 + 2KЛ

CONCEPT IN SOLVING TRIG EQUATIONS.

To solve a trig equation, transform it into one or many basic trig equations. Solving trig equations finally results in solving 4 types of basic trig equations, or similar.

SOLVING BASIC TRIG EQUATIONS.

There are 4 types of common basic trig equations:

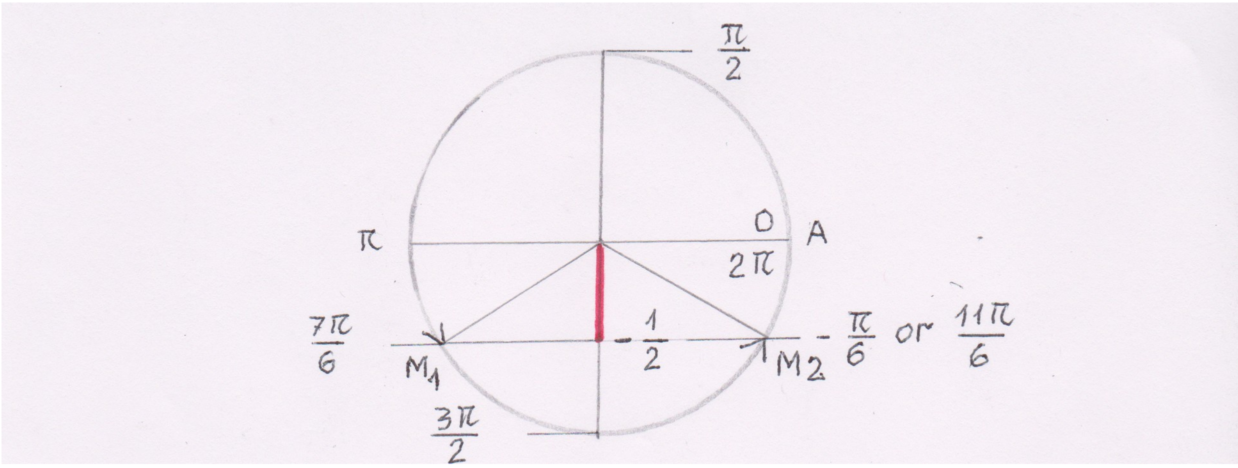
sin x = a	**cos x = a**	(**a** is a given number)
tan x = a	**cot x = a**	

Solving basic trig equations proceeds by considering the positions of the variable arc x that rotates on the trig unit circle, and by using calculator, or trig tables in trig books.

Example 1. Solve: sin x = - 1/2

Solution. Table of special arcs gives → -1/2 = tan (-Л/6) = tan (11Л/6) (co-terminal). The unit circle gives another solution arc (7Л/6) that has the same sine value.

x1 = 11Л/6	x2 = 7Л/6	Answers
x1 = 11Л/6 + 2kЛ	x2 = 7Л/6 + 2kЛ	Extended answers



<u>Example 2</u>. Solve: cos x = -1/2

Special trig Table and the unit circle give 2 solution arcs :

x = ± 2Л/3 Answers

x= ± 2Л/3 + 2k.Л Extended answers.

<u>Example 3</u>. Solve: tan (x – Л/4) = 0

Solution. Answer given by the unit circle :

x – Л/4 = 0 → x = Л/4 Answer

x = Л/4 + k.Л Extended answer

<u>Example 4</u>. Solve: cot 2x = 1.732

Solution. Answers given by unit circle, and calculator:

2x = 30^{o} + $k180^{o}$ Answer

x = 15^{o} + $k90^{o}$ Extended answers

<u>Example 5.</u> Solve: sin (x – 25^{o}) = 0.5

Solution. The trig table and unit circle gives:

a. $\sin (x - 25^\circ) = \sin 30^\circ$ | b. $\sin (x - 25^\circ) = \sin (180^\circ - 30^\circ)$.

$x1 - 25^\circ = 30^\circ$ | $x2 - 25^\circ = 150^\circ$

$x1 = 55^\circ$ | $x2 = 175^\circ$ Answers

$x1 = 55^\circ + k.360^\circ$ | $x2 = 175^\circ + k.360^\circ$ (Extended answers)

TRANSFORMATIONS USED TO SOLVE TRIG EQUATIONS

To transform a complex trig equation into many basic trig equations (or similars), students can use common algebraic transformations (factoring, common factor, polynomials identities…), definitions and properties of trig functions, and **trig identities** (the most needed). There are about 31 trig identities, among them the last 14 identities, from # 19 to # 31, are called **transformation identities** since they are necessary tools to transform trig equations into basic ones.

<u>Example 6</u>: Transform the sum (sin a + cos a) into a product of 2 basic trig equations.

Solution

$\sin a + \cos a = \sin a + \sin (\pi/2 - a)$ Use Identity "Sum into Product" (# 28)

$= 2\sin \pi/4.\sin (a + \pi/4)$ Answer

<u>Example 7</u>. Transform the difference (sin 2a – sin a) into a product of 2 basic trig equations, using trig identity and common factor.

Solution. Use the Trig Identity: $\sin 2a = 2\sin a.\cos a$.

$\sin 2a - \sin a = 2\sin a.\cos a - \sin a = \sin a (2\cos a -1)$

THE COMMON PERIOD OF A TRIG EQUATION.

Unless specified, a trig equation $F(x) = 0$ must be solved covering one common period. This means you must find all the solution arcs x inside the common period of the equation.

The common period of a trig equation is equal to **the least multiple** of all periods of the trig functions presented in the trig equation. Examples:

- The equation $F(x) = \cos x - 2\tan x - 1$, has 2π as common period
- The equation $F(x) = \tan x + 3\cot x = 0$, has π as common period
- The equation $F(x) = \cos 2x + \sin x = 0$, has 2πas common period
- The equation $F(x) = \sin 2x + \cos x - \cos x/2 = 0$, has 4π as common period.
- The equation $F(x) = \tan 2x + \sin (x/3) = 0$ has 6π as period

METHODS TO SOLVE TRIG EQUATIONS

If the given trig equation contains only one trig function of x, solve it as a basic trig equation. If the given trig equation contains two or more trig functions of x, there are two main solving methods, depending on transformation possibilities.

1. METHOD 1 – Transform the given trig equation F(x) into a **product** of many basic trig equations, or similar:

$F(x) = f(x).g(x) = 0$ or $F(x) = f(x).g(x).h(x) = 0$

Example 8. Solve: $2\cos x + \sin 2x = 0$ $(0 < x < 2\pi)$

Solution. Replace sin 2x by using the trig identity "$\sin 2x = 2\sin x.\cos x$"

$2\cos x + 2\sin x.\cos x = 2\cos x (\sin x + 1)$

Next, solve the 2 basic trig equations: $\cos x = 0$ and $(\sin x + 1) = 0$

$\cos x = 0 \rightarrow x = \pi/2$ and $x = 3\pi/2$

$\sin x = - 1 \rightarrow x = 3\pi/2$

Example 9. Solve the trig equation: $\cos x + \cos 2x + \cos 3x = 0$ $(0 < x < 2\pi)$

Solution. Transform it into a product, using trig identity ($\cos a + \cos b$).

$\cos x + \cos 3x + \cos 2x = 2\cos 2x.\cos x + \cos 2x = \cos 2x (2\cos x + 1) = 0$

Next, solve the 2 basic trig equations: $\cos 2x = 0$ and $(2\cos x + 1) = 0$

Example 10. Solve: $\sin x - \sin 3x = \cos 2x$ $(0 < x < 2\pi)$

Solution. Using the trig identity (sin a – sin b), transform the equation into a product:

$\sin 3x - \sin x - \cos 2x = 2\cos 2x \sin x - \cos 2x = \cos 2x (2\sin x - 1) = 0$

Next, solve the 2 basic trig equations: $\cos 2x = 0$ and $(2\sin x - 1) = 0$

Example 11. Solve: $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

Solution. By using the "Sum into Product Identities", and then common factor, transform this trig equation into a product:

$\sin x + \sin 3x + \sin 2x = \cos x + \cos 3x + \cos 2x$
$2\sin 2x.\cos x + \sin 2x = 2\cos 2x.\cos x + \cos 2x$
$\sin 2x (2\cos x + 1) = \cos 2x (2\cos x + 1)$
$F(x) = (2\cos x + 1) (\sin 2x - \cos 2x) = 0$

Next, solve the 2 basic trig equations: $(2\cos x + 1) = 0$ and $(\sin 2x - \cos 2x) = 0$

METHOD 2 - If the given trig equation contains 2 or more trig functions, transform it into one trig equation that has **only one trig function variable**. The common trig functions to choose as variable are: $\sin x = t$; $\cos x = t$, $\cos 2x = t$, $\tan x = t$; $\tan x/2 = t$.

Example 12. Solve $3\sin^2 x - 2\cos^2 x = 4\sin x + 7$ (1) $(0 < x < 2\pi)$

Solution. Replace in the equation ($\cos^2 x$) by ($1 - \sin^2 x$), then put the equation in standard form.

$3\sin^2 x - 2 + 2\sin^2 x - 4\sin x - 7 = 0$.

Call $\sin x = t$, we get: $5t^2 - 4t - 9 = 0$.

This is a quadratic equation in t, with 2 real roots: $t1 = -1$ and $t2 = 9/5$. The second real root t2 is rejected since sin x must < 1. Next, solve for $t = \sin x = -1 \rightarrow x = 3\pi/2$

Check the given equation (1) by replacing $\sin x = \sin 3\pi/2$:

$3 - 0 = -4 + 7$ The answer is correct

Example 13. Solve: $\sin^2 x + \sin^4 x - \cos^2 x = 0$ $(0, 2\pi)$

Solution. Choose $\cos x = t$ as function variable.

$$(1 - t^2)(1 + 1 - t^2) - t^2 = 0$$
$$t^4 - 4t^2 + 2 = 0$$

It is a bi-quadratic equation. There are 2 real roots. $D^2 = b^2 - 4ac = 16 - 8 = 8$

$t^2 = -b/2a + d/2a = 2 + 1.414$ (rejected because > 1) and

$t^2 = 2 - 1.414 = 0.586$ (accepted since < 1).

$\cos x = t = \pm 0.77$

Next solve the 2 basic trig equations: $\cos x = t = 0.77$ and $\cos x = t = -0.77$.

Example 14. Solve: $\cos x + 2\sin x = 1 + \tan x/2$ $(0 < x < 2\pi)$

Solution. Choose $t = \tan x/2$ as function variable. Replace $\sin x$ and $\cos x$ in terms of t

$1 - t^2 + 4t = (1 + t)(1 + t^2)$

$t^3 + 2t^2 - 3t = t(t^2 + 2t - 3) = 0$

The quadratic equation $(t^2 + 2t - 3 = 0)$ has 2 real roots: 1 and -3.

Next, solve 3 basic trig equations: $t = \tan x/2 = 0$; $t = \tan x/2 = -3$; and $t = \tan x/2 = 1$.

Example 15. Solve: $\tan x + 2\tan^2 x = \cot x + 2$ $(-\pi/2 < x < \pi/2)$

Solution. Choose $\tan x = t$ as function variable.

$t + 2t^2 = 1/t + 2$
$(2t + 1)(t^2 - 1) = 0$

Next, solve the 3 basic trig equations:

$(2\tan x + 1) = 0$; $(\tan x - 1) = 0$; and $(\tan x + 1) = 0$

SOLVING SPECIAL TYPES OF TRIG EQUATIONS.

There are a few special types of trig equations that require specific transformations.

Examples: a sin x + b cos x = c

a (sin x + cos x) + b cos x.sin x = c

a.sin^2 x + b.sin x.cos x + c.cos ^2 x = 0

CONCLUSION.

Solving trig equations is a tricky work that often leads to errors and mistakes. Therefore, the answers should be always carefully checked.

After solving, students may check the answers by using a graphing calculator to directly graph the given trig equation F(x) = 0.

Note. Graphing calculators give answers (real roots) in decimals. For example Л, or 180^o is given by the value 3.14.

REMARK

On the trig unit circle, the values of the arc x and the corresponding angle x are exactly equal.

- The **french concept** to select the **arc x** as variable, instead of the **angle x**, makes the whole trigonometric study more convenient, more concrete, and less absurd.

- The expressions such as “Arc tan 1/3”, and “Arc sin 3/4” make sense with this concept.

- About the trig functions **tan x** and **cot x**, when they go to infinities, the notion of infinity is understandable as two axis lines going parallel to each other. On the contrary, the infinitive notion of these functions of an angle x is fully absurd.

- Finally, we can have a second approach to solve complex trig inequalities by using the bi-unit circle, or triple unit circle.

(This article was written by Nghi H. Nguyen, author of the new Transforming Method to solve quadratic equations. Updated 6/3/2020)