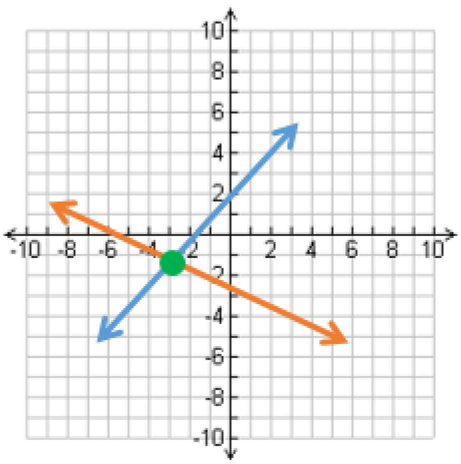
Solving Systems of Linear Equations by Substitution

Note: There are two Solving Systems of Linear Equations handouts, one by Substitution and another by Elimination.

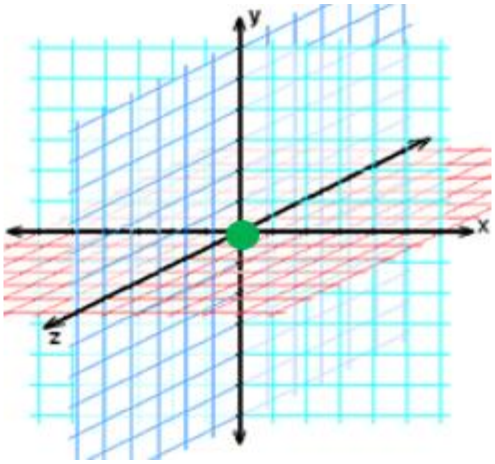
A **linear equation** is an equation for a **line**. A system of equations involves one or more equations working together. **This handout focuses on systems of equations with one solution for the system.** These systems are known as “consistent and independent” and have one point of intersection.

Case 1: Two variable linear equations in two-dimensional space.
Number of unknown variables: 2
Sample Diagram:



Solution: **(x, y) = (-3, -1)**

Case 2: Three variable linear equations in three-dimensional space.
Number of unknown variables: 3
Sample Diagram:



Solution: **(x, y, z) = (0, 0, 0)**

Note: It is possible for a system of linear equations to have no solution (i.e. "inconsistent") or infinitely many solutions (i.e. "consistent and dependent").

PART A – Two Equations with Two Unknowns

Whenever we have two equations with two unknown variables, we solve for the unknown variables using algebra and a process known as **Substitution**.
In general, solving "by substitution" works by solving for a variable in one of the equations and then ***substituting*** it into the other equation. Then you back-solve for the first variable.

Example 1:

Given the linear equations,

1) **3x + 4y = 20** and
2) **5y = 10**

solve for the values of **x and y** by **SUBSTITUTION**.

Step 1:

Try to define an expression for one of the unknown variables. Let's solve for y in equation 2.

5y = 10
y = 2

Step 2:

Since we now know the value of y we can *substitute* **y=2** into equation 1 to solve for the value of x.

3x + 4(2) = 20
3x + 8 = 20
3x = 12
x = 4

Step 3:

Solution (x, y) = 4, 2

Step 4:

Check!

3(4) = 4(2) = 20

20 = 20

L.H.S = R.H.S.

Example 2:

Given the linear equations,

1) **2x + 5y = 20** and
2) **6x + y = 10**

Solve for the values of **x and y** by **Substitution**.

Step 1:

Try to define an expression for one of the unknown variables. Let's solve for y in equation 2).

6x + y = 10

y = 10 – 6x

Step 2:

Substitute **y = 10 – 6x** into equation 1) and solve for the value of **x**.

2x + 5(10-6x) = 30

2x + 50 – 30x = 30

-28x = -20

$\mathbf{x = \frac{5}{7}}$

Step 3:

Since we now know the value of **x,** we can *substitute* $x = \frac{5}{7}$ into any of the equations to solve for the value of **y**.

$y = 10 - 6(\frac{5}{7})$

$\boldsymbol{y = \frac{40}{7}}$

Step 4:

Solution

$(x, y) = (\frac{5}{7}, \frac{40}{7})$

PART B – Three Equations with Three Unknowns

Whenever we have three unknown variables we need three equations in order to solve for a consistent solution.

Example 3:

Given the linear equations,

1) **x + 3y - 2z**
2) **y + 4z = 13**
3) **5z + 2x = 16**

solve for the values of **x, y and z** by **Substitution**.

Step 1:

Since we have three unknown variables we need to define two variables. Let's solve for **y** in equation 2) and **x** in equation 3).

For y,

y + 4z = 13

y = 13 – 4z

For x,

5z + 2x = 16

$$\mathbf{x} = \frac{(\mathbf{16} - \mathbf{5}z)}{\mathbf{2}}$$

Step 2:

Substitute **y = 13 – 4z** and $\mathbf{x} = \frac{(\mathbf{16}-\mathbf{5}z)}{\mathbf{2}}$ into equation 1) and solve for the value of **z.**

$$\frac{(\mathbf{16} - \mathbf{5}z)}{\mathbf{2}} + 3(\mathbf{13} - \mathbf{4z}) - 2z = 14$$

$8 - \frac{5}{2}z + 39 - 12z - 2z = 14$

$-\frac{5}{2}z - 14z = 14 - 8 - 39$

$-\frac{5}{2}z - \frac{28}{2}z = -33$

$-\frac{33}{2}z = -33$

z = 2

Step 3:

Substitute **z = 2** into the equations that define **x and y**.

For y;

y = 13 – 4(2)

y = 5

For x;

$$x = \frac{(16 - 5(2))}{2}$$

$x = 3$

Step 4:

Solution

(x, y, z) = (3, 5, 2)

As linear systems of equations become larger and larger, solving by substitution can become quite long. Solving linear systems by **Elimination** may help simplify some of those calculations.

Outlined here is a summary of steps needed to solve linear equations by **Substitution**.

Step 1:	To solve for a consistent system, check to see if the number of equations is equal to the number of unknown variables.
Step 2:	Define a value or expression for at least one of the unknown variables.
Step 3:	Substitute this defined value into at least one of the equations to help solve for the other unknown variable(s). (We may need to repeat these steps depending on how many unknowns are involved.)
Step 4:	Once we have the actual value of an unknown variable, we can substitute this value into one of the equations to get the value of the other unknown variable(s).
Step 5:	Check! Once you have found the values of the variables substitute them into one of the equations and simplify. If the left hand side of the equation is equal to the right hand side, then you are done.

Exercises:

Solve the following systems of linear equations.

1) $2x + 3x - 4 = 6$
   $x + 4y - 2 = 20$
2) $4x - 2y = 2$
   $5x + 3y = 36$
   (Hint: Simplify the first equation.)

3) $2x + y – 4z = -15$
$x + 2z = 14$
$3z – y = 13$

4) $4x – 3y + z = -1$
$x + 5y – z = 12$
$2y + z = 10$
(Hint: Define the two simplest equations in terms of a common unknown.)

Solutions:

1) $x = 2$, $y = 5$
2) $x = 3$, $y = 7$
3) $x = 2$, $y = 5$, $z = 6$
4) $x = 1$, $y = 3$, $z = 4$