SOME THOUGHTS ON TRIGONOMETRIC IDENTITIES

Courses on advanced trigonometry often include a unit on proving trigonometric identities.

Example 1: Establish that $1-\sin^4(\theta)+\cos^4(\theta)=2\cos^2(\theta)$.

Example 2: Establish that $\dfrac{\csc(x)}{\tan(x)+\cot(x)}=\cos(x)$ for all values of x that make the expression on the left well defined.

One recalls that

$$\tan(x)=\frac{\sin(x)}{\cos(x)} \qquad \cot(x)=\frac{\cos(x)}{\sin(x)} \qquad \sec(x)=\frac{1}{\cos(x)} \qquad \csc(x)=\frac{1}{\sin(x)},$$

$$\sin(2x)=2\sin(x)\cos(x) \qquad \cos(2x)=\cos^2(x)-\sin^2(x)$$

and so on

Each identity can be established be writing each term solely in terms of sine and cosine of the same angle, clearing "fractions within fractions," performing basic algebraic operations, and applying the **same one and only one** trigonometric identity over and over again

$$\sin^2(x)+\cos^2(x)=1$$

to show that the expression on the left side of the equality is equivalent to the expression on the right.

This is not to trivialise the cleverness sometimes needed to conduct this task. For example, an insight is needed to think to rewrite $\cos^4(\theta)$ as $\left(\cos^2(\theta)\right)^2=\left(1-\sin^2(\theta)\right)^2$ in example 1.

One might also wish to perform some general algebraic operations on the equality given to you to produce an equivalent equality to establish instead. For instance, progress on example 2 might begin.

Example 2: Establish that $\dfrac{\csc(x)}{\tan(x)+\cot(x)} = \cos(x)$ for all values of x that make the expression on the left well defined.

Answer: We shall establish that, equivalently,

$$\csc(x) = \cos(x)\left(\tan(x)+\cot(x)\right).$$

Now the left side is $\dfrac{1}{\sin(x)}$.

The right side is …

The point is

> **Proving a trigonometric identity is an exercise in manipulating algebraic expressions in two unknowns S and C with the extra relation $S^2 + C^2 = 1$ to apply at any time desired.**

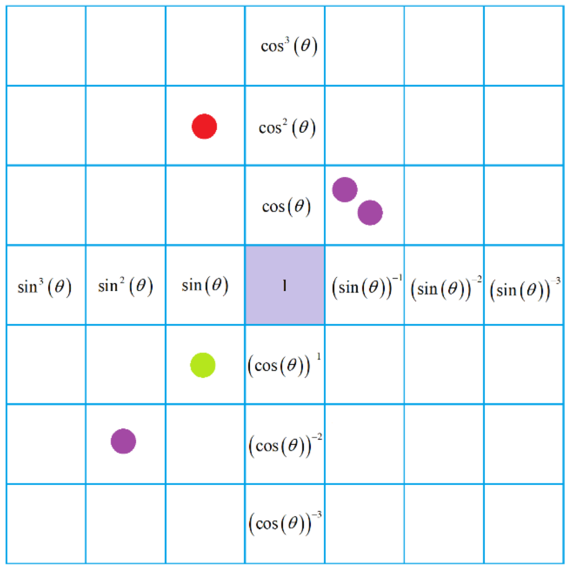
Global Math Project Ambassador Kiran Bacche points out that this can often be done purely visually using a system similar to Napier's Checkerboard in the story of Exploding Dots.

Start with a two-dimensional board. Label a central cell "1" and each cell on a horizonal axis by the powers of sine and each cell on a vertical axis by the powers of cosine. These are the "values" of these cells.

			$\cos^3(\theta)$			
			$\cos^2(\theta)$			
			$\cos(\theta)$			
$\sin^3(\theta)$	$\sin^2(\theta)$	$\sin(\theta)$	1	$(\sin(\theta))^{-1}$	$(\sin(\theta))^{-2}$	$(\sin(\theta))^{-3}$
			$(\cos(\theta))^{-1}$			
			$(\cos(\theta))^{-2}$			
			$(\cos(\theta))^{3}$			

Every other cell has value given by the product of the values of its matching horizontal axis and vertical axis cells. For example,

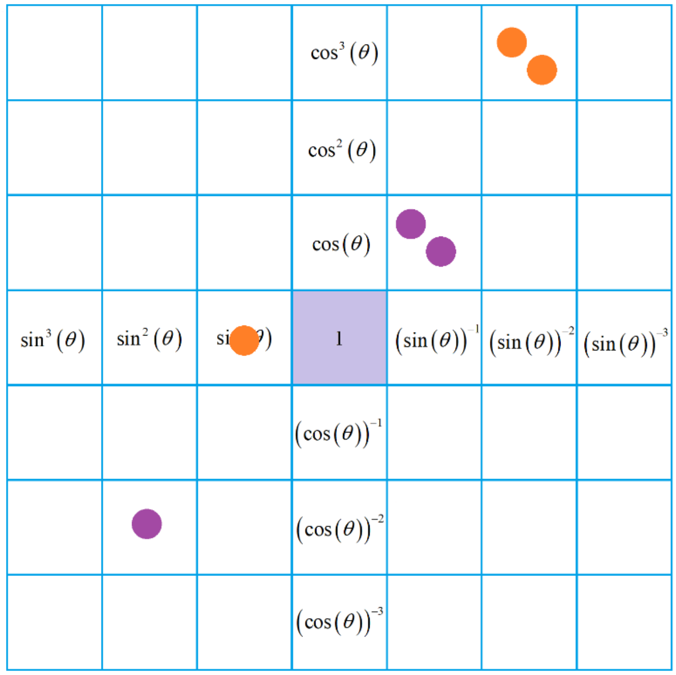
> The red dot has value $\sin(\theta)\cos^2(\theta)$.
>
> The green dot has value $\tan(\theta)$.
>
> The three purple dots collectively have value $\tan^2(\theta)+2\cot(\theta)$.



If we shifted each of the three dots upward two positions, they would then represent

$$\sin^2(\theta)+2\frac{\cos^3(\theta)}{\sin(\theta)},$$

which is the quantity we had before multiplied by $\cos^2(\theta)$. Next, shifting those three dots one place to the right has the effect of multiplying the quantity by $\sin^{-1}(\theta)$.

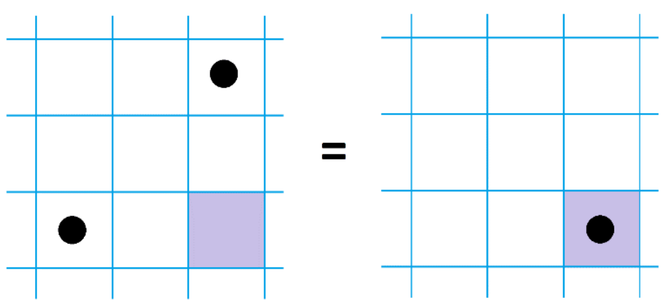


The orange dots have collective value $\cos^2(\theta)\sin^{-1}(\theta)$ *times the collective value of the purple dots.*

We have

Any two identical configurations of dots at different locations on the board have values that are multiples of each other.

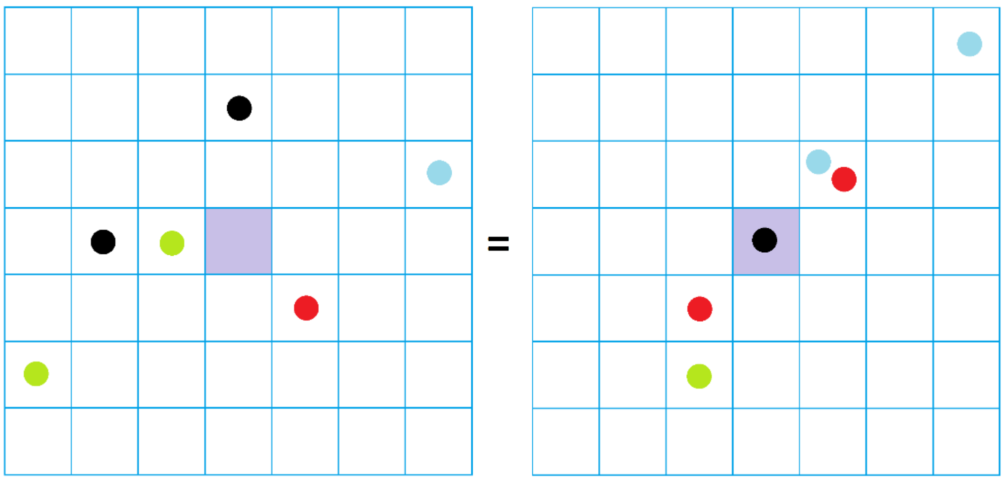
We also have our fundamental identity $\sin^2(\theta)+\cos^2(\theta)=1$.



But this equivalency of pattern holds *anywhere* on the board!

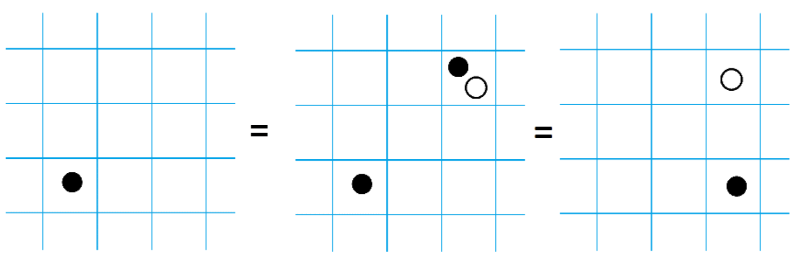
Practice: *Can you see that the picture on the left is equivalent to the one on the right?*

For instance, on the left, the two green dots represent $\sin^3(\theta)\cos^{-2}(\theta)+\sin(\theta)$ and the green dot in the right picture represents $\sin(\theta)\cos^{-2}(\theta)$, and this is basic identity multiplied by $\sin(\theta)\cos^{-2}(\theta)$.



Check that the red dots represent the basic identity multiplied through by $\sin^{-1}(\theta)\cos^{-1}(\theta)$ *and the blue dots the identity multiplied through by* $\sin^{-3}(\theta)\cos(\theta)$.

By using dots and anti-dot pairs we can also employ variations of our basic identity. For instance, we have the following.

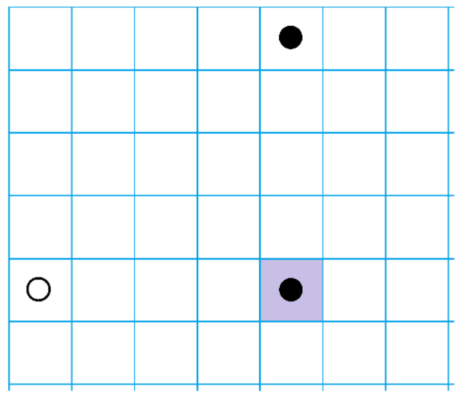


This shows that any multiple of $\sin^2(\theta)$ is can be replaced with a multiple of $1-\cos^2(\theta)$.

Let's now establish the two opening trigonometric identities.

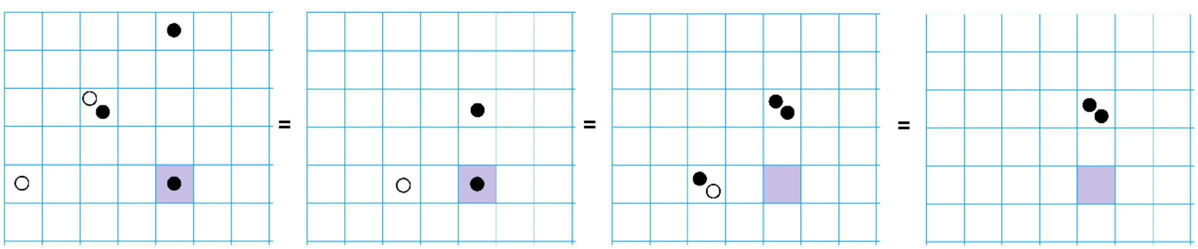
Example 1: Establish that $1-\sin^4(\theta)+\cos^4(\theta)=2\cos^2(\theta)$.

Answer: The left side is



Let's just play!

This is equivalent to



and this is $2\cos^2(\theta)$, as hoped!

Example 2: Establish that $\dfrac{\csc(x)}{\tan(x)+\cot(x)} = \cos(x)$ for all values of x that make the expression on the left well defined.

Answer: We shall establish equivalently that

$$\csc(x) = \cos(x)\big(\tan(x)+\cot(x)\big)$$

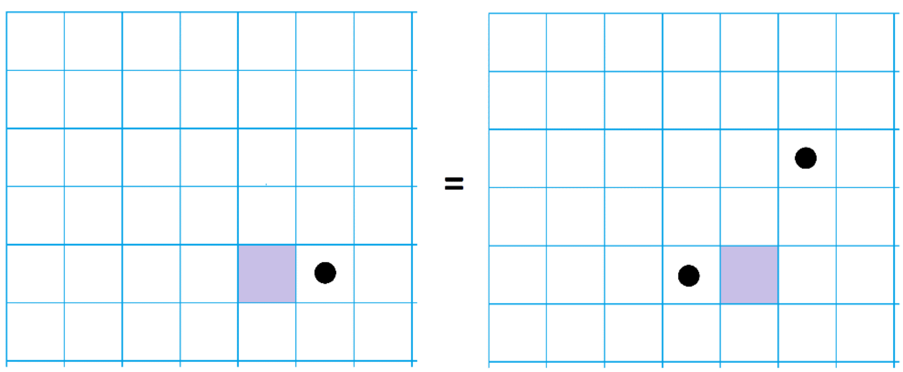
which is tantamount to showing

$$\csc(x) = \cos(x)\tan(x) + \cos(x)\cot(x)$$

or

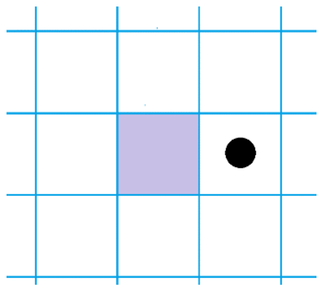
$$\frac{1}{\sin(x)} = \sin(x) + \frac{\cos^2(x)}{\sin(x)}.$$

This is our basic identity.

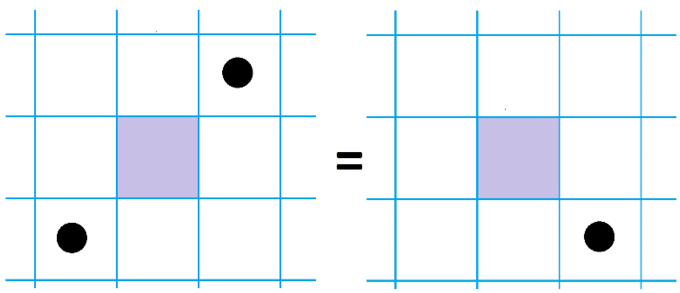


Alternative answer: Consider the expression $\dfrac{\csc(x)}{\tan(x)+\cot(x)}$.

The numerator looks like



The denominator looks like



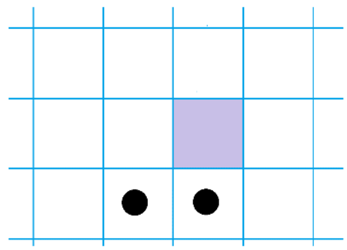
So the numerator is the denominator shifted up one unit. That is, the numerator is the denominator multiplied by $\cos(x)$.

SOME MORE PRACTICE

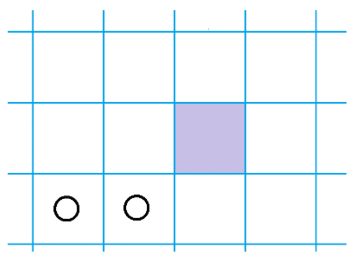
Example 3: Show that $(\sec(\theta)+\tan(\theta))(1-\sin(\theta))=\cos(\theta)$.

Answer: Let's do this by drawing a picture of $\sec(\theta)+\tan(\theta)$, then multiplying it by $-\sin(\theta)$ and adding the two pictures together!

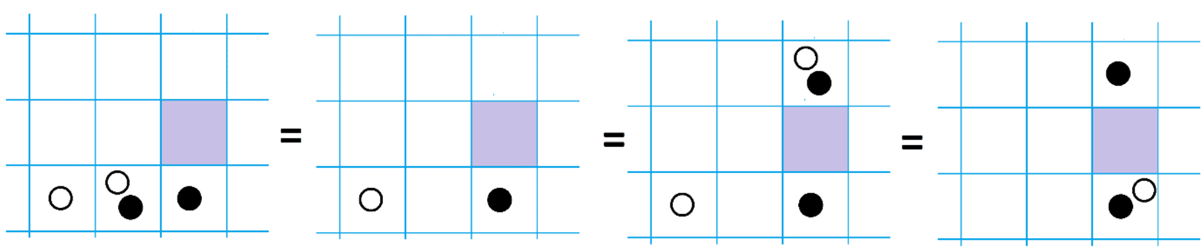
Here's $\sec(\theta)+\tan(\theta)$.



Here's the same picture multiplied by $-\sin(\theta)$.



Adding the tow pictures gives the left side of our equality.



which simplifies to $\cos(\theta)$.

Practice 4: Show that $\left(\sin(A)+\cos(A)\right)\left(1-\sin(A)\cos(A)\right)=\sin^3(A)+\cos^3(A)$.

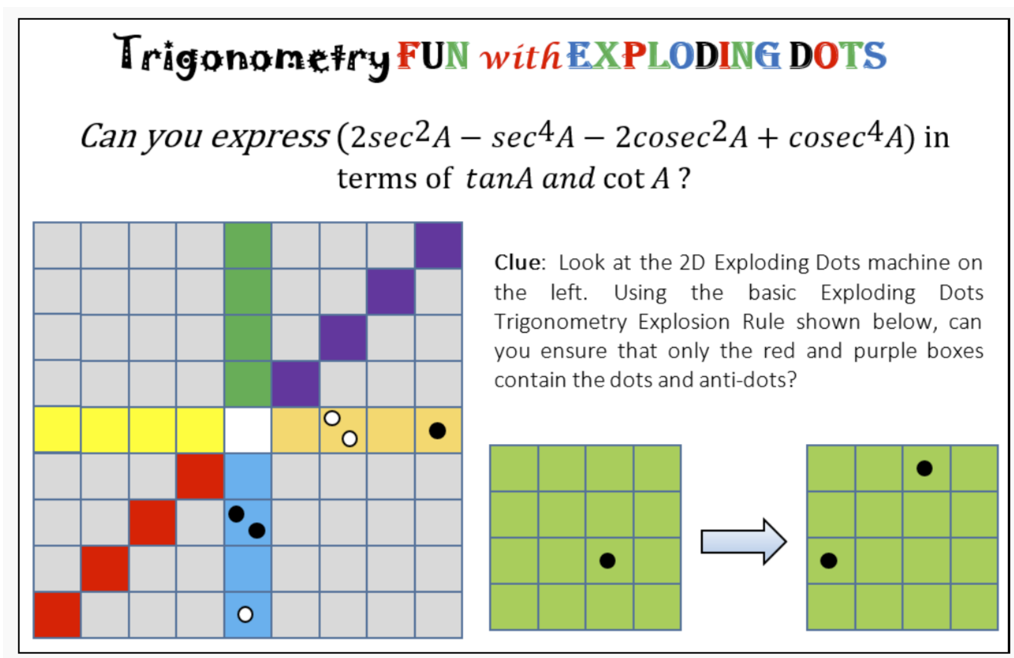
Practice 5: Show that $\dfrac{1}{\tan(x)+\cot(x)}=\sin(x)\cos(x)$.

Practice 6: Show that $\dfrac{1}{\sec(y)+\tan(y)}=\sec(y)-\tan(y)$.

Practice 7: Show that $\dfrac{\sin(x)-2\sin^3(x)}{2\cos^3(x)-\cos(x)}=\tan(x)$.

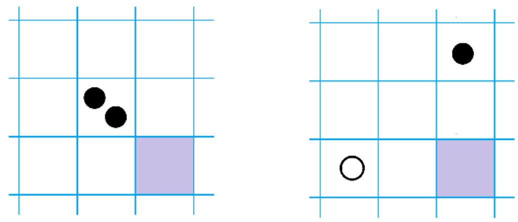
(Perhaps show pictorially that the numerator on the left is a multiple of the denominator on the left.)

Practice 8: This is directly from Kiran.



Practice 9: Show that $\dfrac{1+\sin(2A)-\cos(2A)}{1+\sin(2A)+\cos(2A)}=\tan(A)$.

<u>Hint</u>: $\sin(2A)$ and $\cos(2A)$ look like this:



sin(2A) cos(2A)

Practice 10: This is directly from Kiran.