Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Space Plasma Physics

Physical Processes

8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas

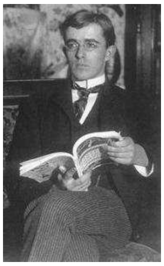
Used Material

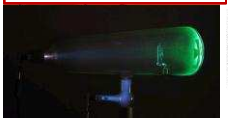
- Lecture notes from Eckart Marsch 2007
- Baumjohann&Treumann: Basic Space Plasma Physics
- Schindler: Physics of space plasma activity
- Priest: Solar MHD
- Kulsrud: Plasma Physics for Astrophysics
- Krall & Trivelpiece: Principles of Plasma Physics
- Chen: Introduction to plasma physics and controlled fusion
- Balescu: Plasma Transport (3 volumes)
- Spatschek: Theoretische Plasmaphysik

- Wikipedia, Google and YouTube

What is plasma?

In plasma physics we study ionized gases under the influence of electro-magnetic fields.



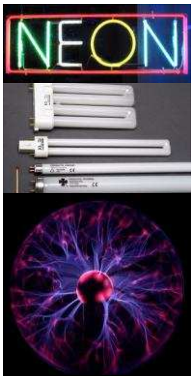


Levi Tonks (1897-1971) and Irving Langmuir (1881-1957, photo) first used the term „**plasma**“ for a collection of charged particles (Phys. Rev. 1929)

William Crookes (1832-1919) called ionized matter in a gas discharge (Crookes-tube, photo) „**4th state of matter**“ (Phil. Trans 1879)

Source: Wikipedia

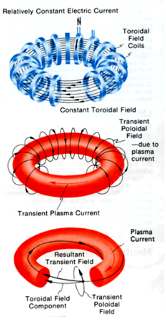
Industrial Plasmas



Neon-lights, Fluorescent lamps,Plasma globes

Electric Arcs

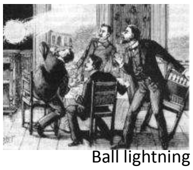
Semi conducter device fabrication



Nuclear Fusion, here a Tokamak

Natural Plasmas on Earth

St. Elmo's Fire



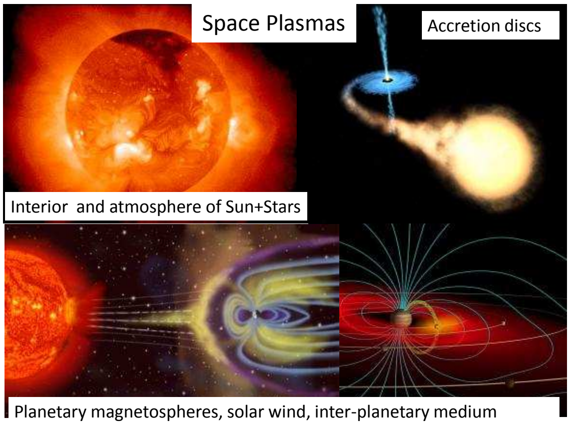
Ball lightning

Lightning

Aurorae

Source: Wikipedia

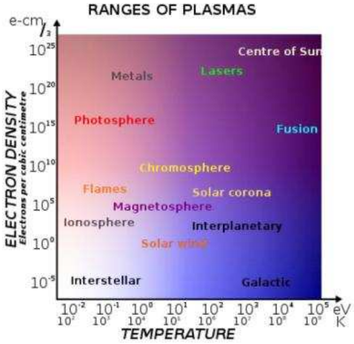
Space Plasmas



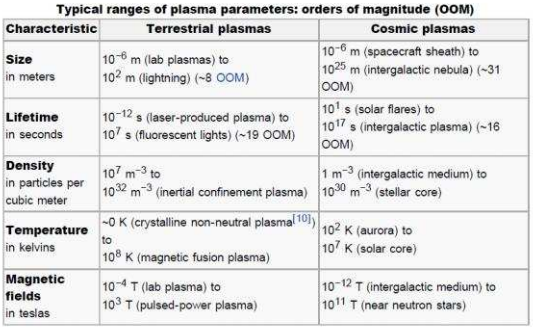
Accretion discs

Interior and atmosphere of Sun+Stars

Planetary magnetospheres, solar wind, inter-planetary medium



Source: Wikipedia



Typical ranges of plasma parameters: orders of magnitude (OOM)

Characteristic	Terrestrial plasmas	Cosmic plasmas
Size in meters	10^{-6} m (lab plasmas) to 10^{2} m (lightning) (~8 OOM)	10^{-6} m (spacecraft sheath) to 10^{25} m (intergalactic nebula) (~31 OOM)
Lifetime in seconds	10^{-12} s (laser-produced plasma) to 10^{7} s (fluorescent lights) (~19 OOM)	10^{1} s (solar flares) to 10^{17} s (intergalactic plasma) (~16 OOM)
Density in particles per cubic meter	10^{7} m^{-3} to 10^{32} m^{-3} (inertial confinement plasma)	1 m^{-3} (intergalactic medium) to 10^{30} m^{-3} (stellar core)
Temperature in kelvins	~0 K (crystalline non-neutral plasma[10]) to 10^{8} K (magnetic fusion plasma)	10^{2} K (aurora) to 10^{7} K (solar core)
Magnetic fields in teslas	10^{-4} T (lab plasma) to 10^{3} T (pulsed-power plasma)	10^{-12} T (intergalactic medium) to 10^{11} T (near neutron stars)

Source: Wikipedia

Source: Wikipedia

Comparison: Gas and Plasma

Property	Gas	Plasma
Electrical conductivity	**Very low**: Air is an excellent insulator until it breaks down into plasma at electric field strengths above 30 kilovolts per centimeter.[14]	**Usually very high**: For many purposes, the conductivity of a plasma may be treated as infinite.
Independently acting species	**One**: All gas particles behave in a similar way, influenced by gravity and by collisions with one another.	**Two or three**: Electrons, ions, protons and neutrons can be distinguished by the sign and value of their charge so that they behave independently in many circumstances, with different bulk velocities and temperatures, allowing phenomena such as new types of waves and instabilities.
Velocity distribution	**Maxwellian**: Collisions usually lead to a Maxwellian velocity distribution of all gas particles, with very few relatively fast particles.	**Often non-Maxwellian**: Collisional interactions are often weak in hot plasmas and external forcing can drive the plasma far from local equilibrium and lead to a significant population of unusually fast particles.
Interactions	**Binary**: Two-particle collisions are the rule, three-body collisions extremely rare.	**Collective**: Waves, or organized motion of plasma, are very important because the particles can interact at long ranges through the electric and magnetic forces.

Plasmas studied in this lecture

- Non-relativistic particle velocities v<<c
- Spatial and temporal scales are large compared to Planck length (1.6 10^{-35} m) and time (5.4 10^{-44} s) More precise: Any action variables like (momentum x spatial dimension, Energy x time) are large compared to Planck constant (h= 6.6 10^{-34} Js) Classic plasma, no quantum-mechanic effects.
- Plasmas violating these conditions (Quark-Gluon Plasma, relativistic plasma) are active areas of research, but outside the scope of this introductory course.

What is a plasma?

- A fully or partly ionized gas.
- Collective interaction of charged particles is more important than particle-particle collisions.
- Charged particles move under the influence of electro-magnetic fields (Lorentz-force)
- Charged particles cause electric fields, moving charged particles cause electric currents and thereby magnetic fields (Maxwell equations).

What is a plasma?

- In principle we can study a plasma by solving the Lorentz-force and Maxwell-equations selfconsistently.
- With typical 10^20-10^50 particles in space plasmas this is not possible.
  (Using some 10^9 or more particles with this approach is possible on modern computers)

=> Plasma models

Plasma models

- **Test particles:**
  Study motion of individual charged particles under the influence of external electro-magnetic (EM) fields
- **Kinetic models:**
  Statistic description of location and velocity of particles and their interaction + EM-fields. (Vlasov-equation, Fokker-Planck eq.)
- **Fluid models:**
  Study macroscopic quantities like density, pressure, flow-velocity etc. + EM-fields (MHD + multifluid models)
- **Hybrid Models:** Combine kinetic + fluid models

Maxwell equations for electro-magnetic fields

The motion of charged particles in space is strongly influenced by the self-generated electromagnetic fields, which evolve according to ***Ampere's and Faraday's*** (induction) laws (in this lecture we use the SI-system):

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

where [redacted] and [redacted] are the vacuum dielectric constant and free-space magnetic permeability, respectively. The charge density is [redacted] and the current density **j**. The electric field obeys ***Gauss*** law and the magnetic field is always free of divergence, i.e. we have:

$$\boxed{\nabla \cdot \mathbf{E} = \rho / \epsilon_0} \qquad \boxed{\nabla \cdot \mathbf{B} = 0}$$

Electromagnetic forces

The motion of charged particles in space is determined by the electrostatic ***Coulomb*** force and magnetic ***Lorentz*** force:

$$\boxed{\mathbf{F}_C = q\mathbf{E}} \qquad \boxed{\mathbf{F}_L = q(\mathbf{v} \times \mathbf{B})}$$

where q is the charge and **v** the velocity of any charged particle. If we deal with electrons and various ionic species (index, s), the ***charge and current densities*** are obtained, respectively, by summation over all kinds of species as follows:

$$\boxed{\rho = \sum_s q_s n_s} \qquad \boxed{\mathbf{j} = \sum_s q_s n_s \mathbf{v}_s}$$

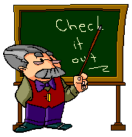
which together obey the ***continuity equation***, because the number of charges is conserved, i.e. we have:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0}$$

Electro-static effects

A plasma is quasi-neutral

- On large scales the positive (ions) and negative charges cancel each other.
- On small scales charge separations occur.
  Can we estimate these scales?
  => Do calculations on blackboard.



Debye-length

$$\Lambda_D = \sqrt{\frac{3\epsilon_0 k_B T}{2n^2}} \propto \boxed{\sqrt{\frac{T}{n}}}$$

Remark: Some text-books drop the factor sqrt(3/2)~1.2

Debye-length

- Electric potential for a charge in vacuum:

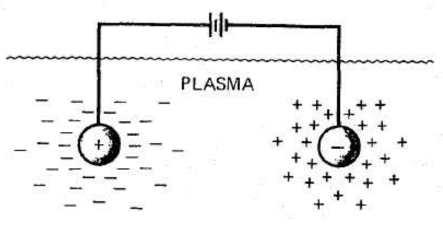
$$\Phi_0 = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

- Electric potential in a plasma:

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\exp\left(-\frac{r}{\Lambda_D}\right)$$

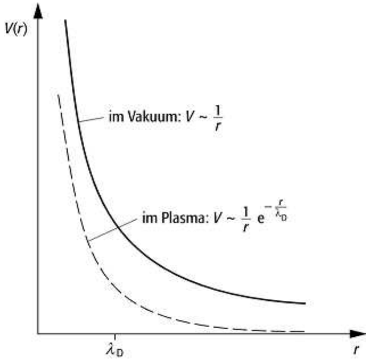
=> Debye shielding

Debye shielding



Source: Chen, Wikipedia

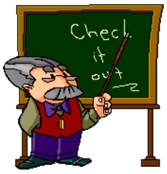
Debye shielding



Source: Technik-Lexikon

Plasma Oscillations

- How does a charged particle (say an electron) move in a non-magneticed plasma?
- Solve electrostatic Maxwell equation selfconsistently with equation of motion, here for the Coulomb force
- => Do calculations on blackboard.



Plasma Oscillations

- Plasma frequency

$$w_p = \sqrt{\frac{ne^2}{m\epsilon_0}} \propto \sqrt{n}$$

- Electron plasma frequency is often used to specify the electron density of a plasma.
  How? => Dispersion relation of EM-waves
  See lecture of plasma waves.

$$w_{pe} = \sqrt{\frac{n_e e^2}{m_e\epsilon_0}} \propto \sqrt{n_e}$$

Plasma parameter

- The plasma paramter g indicates the number of particles in a Debye sphere.

$$g = \frac{1}{n\,\Lambda_D^3} \propto \frac{n^{1/2}}{T^{3/2}}$$

- Plasma approximation: g << 1.
- For effective Debye shielding and statistical significants the number of particles in a Debye sphere must be high.

Plasma parameter g

- g gives a measure how collective effects dominate over single particle effects.
- Contrast between neutral gas and plasma:
  -Interaction region of a neutral atom is the atomic radius R and $n\ R^3 \ll 1$

  -Interaction region in a plasma is the Debye sphere and $1/n\,\Lambda_D^3 \ll 1$
- Plasma state can be derived from expansion of exact many body equations with g.

Collisions: Mean free path

- The mean free path is the distance a particle moves in average before it suffers a collision.
- Cross section σ for interaction of particles during collisions in a plasma is approximated with the Debye-length.
- Mean free path: $l_{\mathrm{mfp}} = \dfrac{1}{n\ \sigma}$
- Collision frequency: $\nu_c = n\ \sigma\ v$

$$\nu_c = n\ < \sigma(v)\ v >$$

Magnetized Plasmas

In order to study the transport of plasma and magnetic field lines quantitatively, let us combine **Maxwell's** equations with the simple phenomenological ***Ohm's law,*** relating the electric field in the plasma frame with its current:

$$\boxed{\mathbf{j} = \sigma_0(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

(Later we will derive Ohm's law more systematically from kinetic plasma theory)

Induction equation

Using ***Maxwell equations*** for slow time variations, without the displacement current yields the induction equation (with constant conductivity [redacted]:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma_0} \nabla^2 \mathbf{B}}$$

Convection

Diffusion

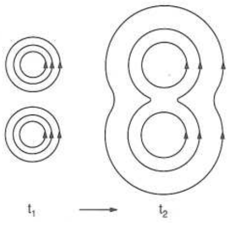
Exercise: Derive this equation for a spatial constant conductivity. How does the equation look if the conductivity varies spatially?

Magnetic diffusion

Assuming the plasma be at rest, the induction equation becomes a pure diffusion equation:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B}}$$

with the magnetic ***diffusion coefficient*** D_m = ([redacted]$)^{-1}$.

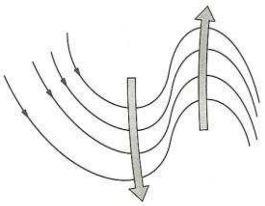


Under the influence of finite resistivity the magnetic field diffuses across the plasma, and field inhomogenities are smoothed out at time scale, [redacted]= [redacted]L_B^2, with scale length L_B.

Hydromagnetic theorem

In an ideal collisionless plasma in motion with infinite conductivity the induction equation becomes:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})}$$



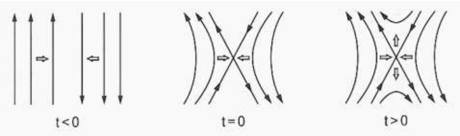
The field lines are constrained to move with the plasma -> ***frozen-in field***. If plasma patches on different sections of a bundle of field lines move oppositely, then the lines will be deformed accordingly. Electric field in plasma frame, **E**' = **0, ->** voltage drop around closed loop is zero**.**

Magnetic reconnection

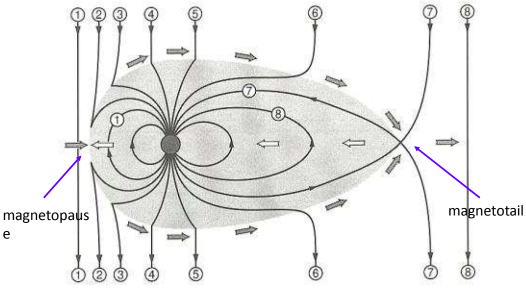
Assuming the plasma streams at bulk speed *V*, then the induction equation can be written in simple dimensional form as:

$$\frac{B}{\tau} = \frac{VB}{L_B} + \frac{B}{\tau_d}$$

The ratio of the first to second term gives the so-called ***magnetic Reynolds number***, R_m = [redacted]$_B V$, which is useful to decide whether a plasma is diffusion or convection dominated. Current sheet with converging flows -> magnetic merging at points where R_m [redacted]1. Field lines form *X-point* and *separatrix.*



Field line merging and reconnection in the Earth's magnetosphere



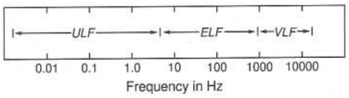
Waves in plasmas

In a plasma there are many reasons for spatio-temporal variations (waves or more generally ***fluctuations***): High temperature required for ***ionization*** ([redacted] 13.6 eV [redacted] 58000 K) implies fast *thermal particle motion*. As a consequence

-> microscopic fluctuating charge separations and currents
-> fluctuating electromagnetic fields.

There are also ***externally imposed disturbances*** which may propagate through the plasma and spread their energy in the whole plasma volume. The relevant frequency ranges are:

Ultra-low, extremely-low, and very-low frequency waves



Plasma waves are not generated at random. To exist they must satisfy two conditions:

-> their amplitude must exceed the thermal noise level
-> they must obey appropriate dynamic plasma equations

There is a large variety of ***wave modes*** which can be excited in a plasma. The mode structure depends on the composition, boundary conditions and theoretical description of the plasma.

We may represent any wave disturbance, ***A***(**x**,*t*), by its Fourier components (with amplitude, ***A***(**k**,[redacted] wave vector **k**, and frequency,[redacted]

$$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{k},\omega)\exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

Phase velocity (wave front propagation)

$$\mathbf{v}_{ph} = \omega\mathbf{k}/k^2$$

Group velocity (energy flow)

$$\mathbf{v}_{gr} = \partial\omega/\partial\mathbf{k}$$

Wave-particle interactions

Plasma waves in a warm plasma interact with particles through:

- *Cyclotron resonance:* [redacted] ***k*·v** = ± [redacted]$_e$
- *Landau resonance:* [redacted] ***k*·v** = 0
- *Nonlinear particle trapping in large-amplitude waves*
- *Quasilinear particle (pitch-angle) diffusion*
- *Particle acceleration in turbulent wave fields*

There is a large variety of ***wave-particle interactions***. They may occur in connection with linear plasma instabilities, leading to ***wave growth and damping***, or take place in coherent or turbulent wave fields, leading to particle acceleration and heating.

Summary

- Plasma is a quasi-neutral ionized gas moving under the influence of EM-fields.
- Thermal energy of particles is much larger as potential energy (free particles).
- Quasi-neutrality can be violated in Debye-sphere.
- To qualify as plasma, spatial dimensions must be much larger as the Debye-sphere and many particles are in the sphere for effective shielding.
- Collision frequency must be much smaller as the plasma frequency.

Exercises for Space Plasma Physics:
I. Basic Plasma Physics concepts

1. What are the main criteria that an ionized gas qualifies for being a plasma?

2. What is the Debye length? How does the Debye-length change with density and temperature? Please try to give a physical explanation for this behavior.

3. Plasma-parameter: What are the main differences of plasmas with few and many particles in a Debye sphere? In which category are typical space plasmas?

4. Can a quasineutral plasma create large scale (larger than Debye length) electric currents? If no, why not? If yes, how?

5. Derive the induction equation from Maxwell equations (without displacement current) and Ohm's law for
   (a) Spatial constant conductivity
   (b) Spatial varying conductivity.
   (c) Is a constant or a varying conductivity more likely in space plasmas?

6. Are electric or magnetic fields more important in space plasmas? Why?

7. Can magnetic reconnection happen in an ideal plasma?

8. Can one observe magnetic reconnection in numerical simulations of ideal plasmas? (Can the answer to this question be different from the previous answer?)

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Solar system space plasmas

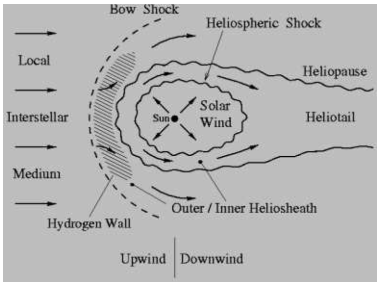
Plasmas differ by their chemical composition and the ionization degree of the ions or molecules (from different sources). Space Plasmas are mostly magnetized (internal and external magnetic fields).

- Solar interior and atmosphere
- Solar corona and wind (heliosphere)
- Planetary magnetospheres (plasma from solar wind)
- Planetary ionospheres (plasma from atmosphere)
- Coma and tail of a comet
- Dusty plasmas in planetary rings

Space plasma

- Space plasma particles are mostly free in the sense that their kinetic exceeds their potential energy, i.e., they are normally hot, T > 1000 K.
- Space plasmas have typically vast dimensions, such that the mean free paths of thermal particles are larger than the typical spatial scales --> they are collisionless.

Structure of the heliosphere



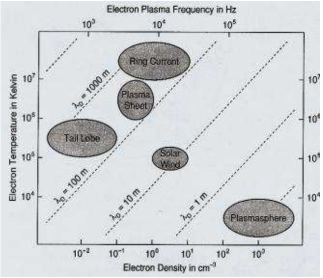
- **Basic plasma motions in the restframe of the Sun**
- **Principal surfaces (wavy lines indicate disturbances)**

Different plasma states

Plasmas differ by the charge, e_j, mass, m_j, temperature, T_j, density, n_j, bulk speed U_j and thermal speed, $V_j=(k_BT_j/m_j)^{1/2}$ of the particles (of species j) by which they are composed.

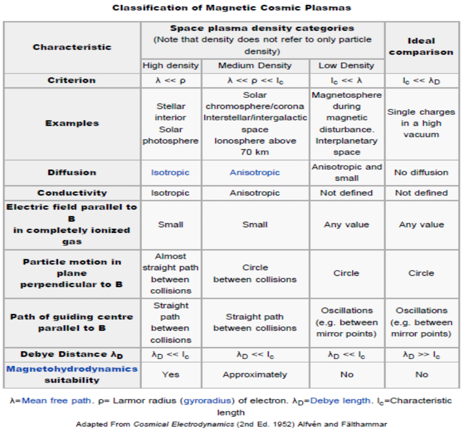
- Long-range (shielded) Coulomb potential
- Collective behaviour of particles
- Self-consistent electromagnetic fields
- Energy-dependent (often weak) collisions
- Reaction kinetics (ionization, recombination)
- Variable sources (pick-up)

Ranges of electron density and temperature for geophysical plasmas



Some plasmas, like the Sun's chromosphere or Earth's ionosphere are not fully ionized. Collisions between neutrals and charged particles couple the particles together, with a typical collision time, [redacted], say. Behaviour of a gas or fluid as a plasma requires that:

[redacted] >> 1

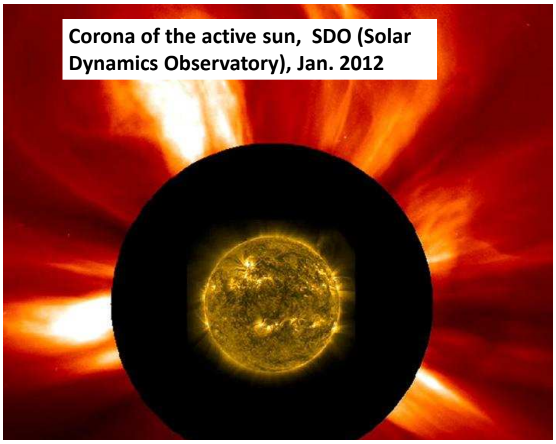


Classification of Magnetic Cosmic Plasmas

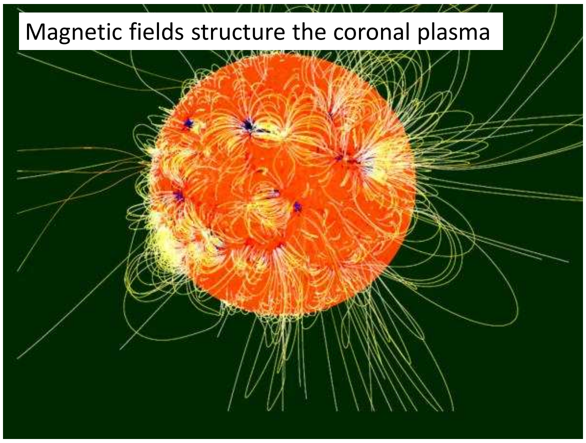
Characteristic	**Space plasma density categories** (Note that density does not refer to only particle density)			**Ideal comparison**
	High density	Medium Density	Low Density	
Criterion	$\lambda << \rho$	$\lambda << \rho << l_c$	$l_c << \lambda$	$l_c << \lambda_D$
Examples	Stellar interior Solar photosphere	Solar chromosphere/corona Interstellar/intergalactic space Ionosphere above 70 km	Magnetosphere during magnetic disturbance. Interplanetary space	Single charges in a high vacuum
Diffusion	Isotropic	Anisotropic	Anisotropic and small	No diffusion
Conductivity	Isotropic	Anisotropic	Not defined	Not defined
Electric field parallel to B in completely ionized gas	Small	Small	Any value	Any value
Particle motion in plane perpendicular to B	Almost straight path between collisions	Circle between collisions	Circle	Circle
Path of guiding centre parallel to B	Straight path between collisions	Straight path between collisions	Oscillations (e.g. between mirror points)	Oscillations (e.g. between mirror points)
Debye Distance λ_D	$\lambda_D << l_c$	$\lambda_D << l_c$	$\lambda_D << l_c$	$\lambda_D >> l_c$
Magnetohydrodynamics suitability	Yes	Approximately	No	No

λ=Mean free path. ρ= Larmor radius (gyroradius) of electron. λ_D=Debye length. l_c=Characteristic length

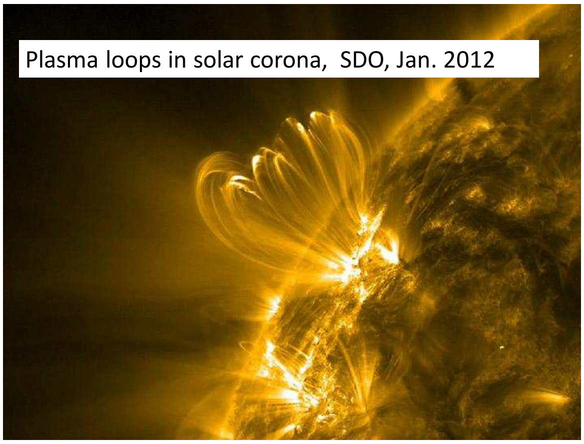
Adapted From *Cosmical Electrodynamics* (2nd Ed. 1952) Alfvén and Fälthammar



Corona of the active sun, SDO (Solar Dynamics Observatory), Jan. 2012



Magnetic fields structure the coronal plasma



Plasma loops in solar corona, SDO, Jan. 2012

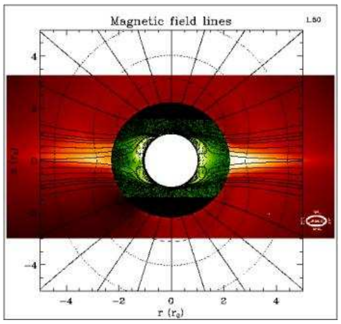
Solar coronal plasma can become unstable
Giant Prominence Erupts - April 16, 2012,
observed with SDO (Solar Dynamics Observatory)

Solar Eruptions

- The solar coronal plasma is frozen into the coronal magnetic field and plasma outlines the magnetic field lines.
- Coronal configurations are most of the time quasistatic and change only slowly.
- Occasionally the configurations become unstable and develop dynamically fast in time, e.g., in coronal mass ejections and flares.

Coronal magnetic field and density

Dipolar, quadrupolar, current sheet contributions



Polar field: B = 12 G

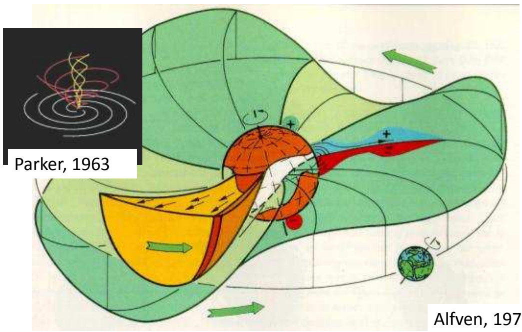
Current sheet is a symmetric disc anchored at high latitudes !

LASCO C1/C2 images (SOHO)

Banaszkiewicz et al., 1998;

Schwenn et al., 1997

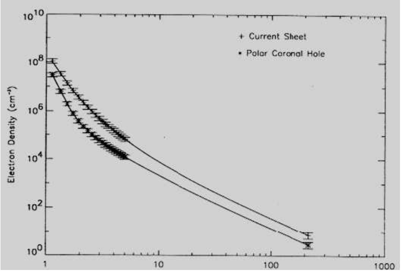
Solar wind stream structure and heliospheric current sheet



Parker, 1963

Alfven, 1977

Electron density in the corona



Heliocentric distance / R_s

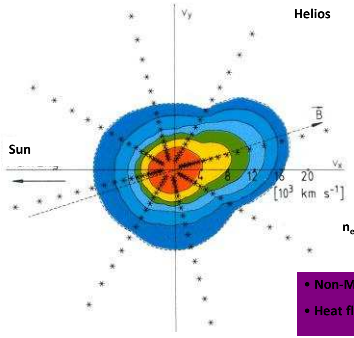
+ Current sheet and streamer belt, closed

• Polar coronal hole, open magnetically

Guhathakurta and Sittler, 1999, Ap.J., **523**, 812

Skylab coronagraph/Ulysses in-situ

Measured solar wind electrons



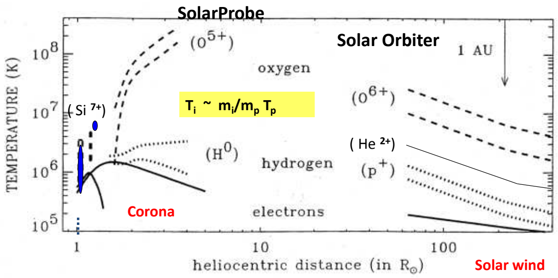
Helios

n_e = 3 -10 cm^{-3}

- **Non-Maxwellian**
- **Heat flux tail**

Pilipp et al., JGR, **92**, 1075, 1987

Temperatures in the corona and fast solar wind



SolarProbe

Solar Orbiter

(Si $^{7+}$)

(He $^{2+}$)

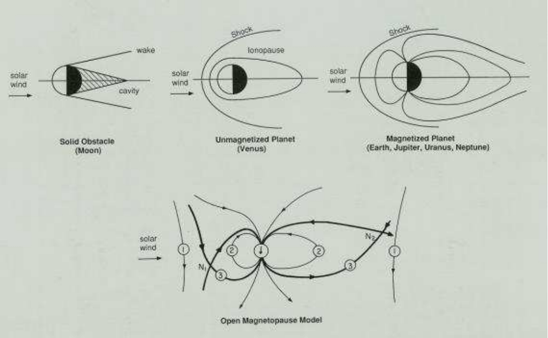
T_i ~ m_i/m_p T_p

Corona

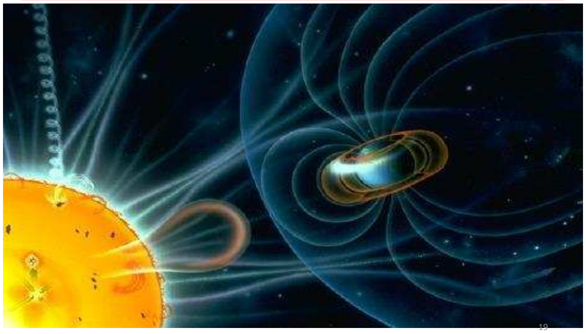
Solar wind

Cranmer et al., Ap.J., 2000;
Marsch, 1991

Boundaries between solar wind and obstacles



Space weather: Instabilities in the solar corona lead to huge eruptions, which can influence the Earth.

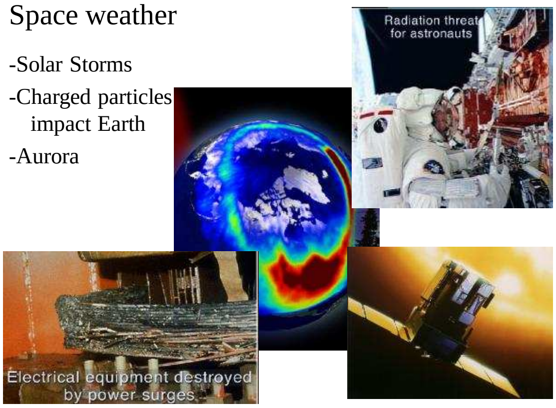


Space weather

-Solar Storms

-Charged particles impact Earth

-Aurora



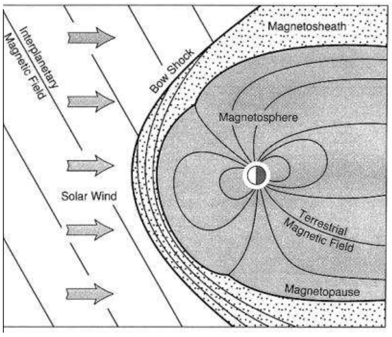
Space weather

- Solar wind and solar eruptions influence Earth and cause magnetic storms:
- Aurora
- Power cutoffs
- Destroyed satellites
- Harm for astronauts

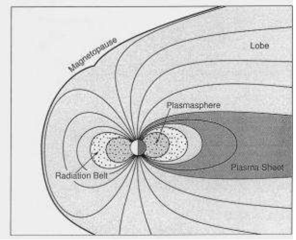
Movie: Solar wind's effects on Earth

http://www.youtube.com/watch?v=XuD82q4Fxgk

Schematic topography of solar-terrestrial environment



solar wind -> magnetosphere -> iononosphere

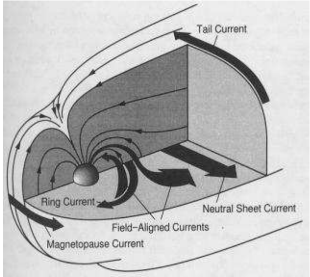


Magnetospheric plasma environment

The boundary separating the subsonic (after bow shock) solar wind from the cavity generated by the Earth's magnetic field, the magnetosphere, is called the magnetopause.

- The solar wind compresses the field on the dayside and stretches it into the magnetotail (far beyond lunar orbit) on the nightside.
- The magnetotail is concentrated in the 10 RE thick plasma sheet.
- The plasmasphere inside 4 RE contains cool but dense plasma of ionospheric origin.
- The radiation belt lies on dipolar field lines between 2 to 6 RE.

Magnetospheric current system



The distortion of the Earth's dipole field is accompanied by a current system.

- The currents can be guided by the strong background field, so-called field-aligned currents (like in a wire), which connect the polar cap with the magnetotail regions.
- A tail current flows on the tail surface and as a neutral sheet current in the interior.
- The ring current is carried by radiation belt particles flowing around the Earth in east-west direction.

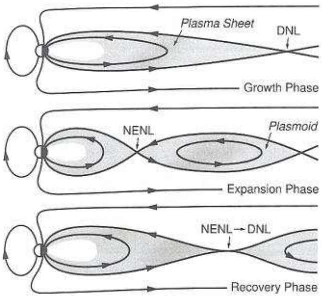
Magnetospheric substorm

Substorm phases:

- Growth
- Onset and expansion
- Recovery

***Magnetic reconnection**:*

- Southward solar wind magnetic field
- Perturbations in solar wind flow (streams, waves, CMEs)



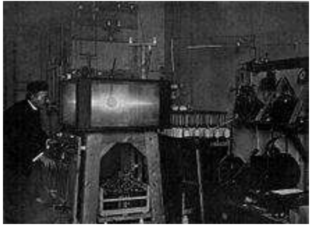
Magnetospheric substorm

- Growth phase: Can be well understood by a sequence of static plasma equilibria and analytic magneto-static models. Plasma is ideal (no resistivity)
- Onset and expansion: The equilibrium becomes unstable and free magnetic energy is set free. Studied with (resistive) MHD-simulations. Cause for resistivity are micro-instabilities (often used ad hoc resistivity models in MHD)
- Recovery phase: Not well studied

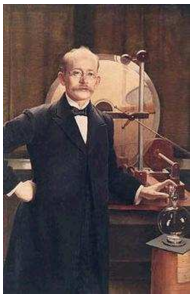
Aurora

Source: Wikipedia

Laboratory experiments, magnetic ball (terella)

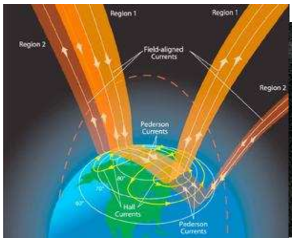


- Path of charged particles made visible in terella, glow in regions around pole.
- Cannot explain why actually in Earth aurora not occur at poles.
- Terellas replace by Computer simulations.

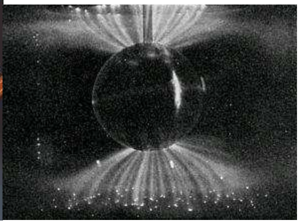


Kristian Birkeland, 1869-1917 first described substorms and investigated Aurora in laboratory.
Source: Wikipedia

Birkeland Currents



Aurora-like Birkeland currents in terella (source:Wikipedia)



- Moving charged particles cause electric currents parallel to the magnetic field lines connecting magnetosphere and ionosphere.
- 100.000 A (quiet times) to 1 million Ampere in disturbed times. => Joule heating of upper atmosphere.

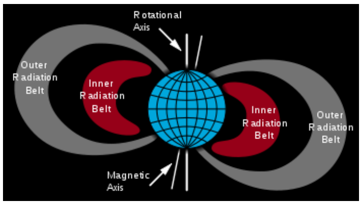
Aurora mechanism

- Atoms become ionized or excited in the Earth's upper atmosphere (above about 80 km) by collision with solar wind and magnetospheric particles accelerated along the Earth's magnetic field lines.
- Returning from ionized or excited states to ground state leads to emissions of photons.
- Ionized nitrogen atoms regaining an electron (blue light) or return to ground based from exited state (red light)
- Oxygen returning from excited state to ground state (red-brownish or green light, depending on absorbed energy in excited states)
- Returning to ground state can also occur by collisions without photon emission => Height dependence of emissions, different colours with height.

Magnetic Storms

- Largest magnetic storm ever measured was in 1859.
- Carrington noticed relation between a white light solar flare and geomagnetic disturbance.
- In 1859 the storm disrupted telegraph communication.
- Such a large storm today could initiate a cascade of destroyed transformators (by induced electric fields) and economic damage of over: 1000 billion Dollar. (source: Moldwin, the coronal current 2010)
- 2005 Hurricane Katrina in USA : 120 billion Dollar. 2011 Earthquake/Tsunami in Japan: 300 billion Dollar.
- Prediction of such storms would help to reduce the damage, e.g., by switching of electric power.

Van Allen radiation Belt



James van Allen
1914-2006
source of pictures: wikipedia

- Contains energetic particles originating from solar wind and cosmic rays.
- Inner belt: protons + electrons
- Outer belt: electrons
- Particles trapped in magnetic field (single particle model sufficient)

Planetary magnetospheres

- Rotation, size, mass, ...
- Magnetic field (moment) of planet and its inclination
- Inner/outer plasma sources (atmosphere, moons, rings)
- Boundary layer of planet and its conductivity
- Solar wind ram pressure (variable)

Dynamic equilibrium if ram pressure at magnetopause equals field pressure:

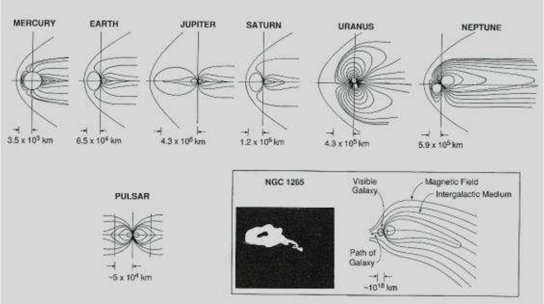
[redacted] $V_{sw}^2 = B^2/2$ [redacted] $= B_p^2 (R_p/R_m)^6 /2$ [redacted]

Stand-off distances: R_m/R_p = 1.6, 11, 50, 40 for M, E, J, S.

Planetary parameters and magnetic fields

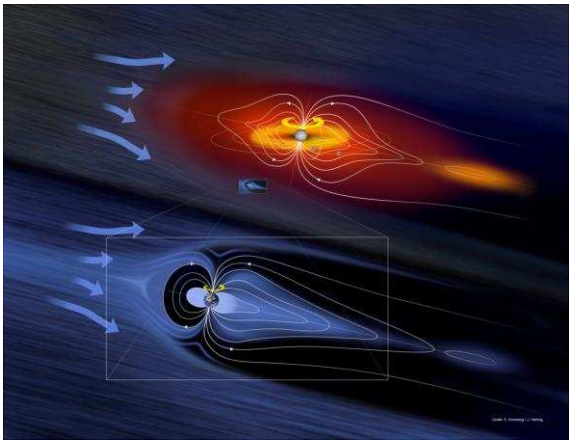
Parameter	Mercury	Earth	Jupiter	Saturn	Sun
Radius [km] (equator)	2425	6378	71492	60268	696000
Rotation period [h]	58.7 d	23.93	9.93	10.66	25-26 d
Dipole field [G] (equator)	340 nT	0.31	4.28	0.22	3-5
Inclination of equator [Degrees]	3	23.45	3.08	26.73	7.12

Magnetospheric configurations



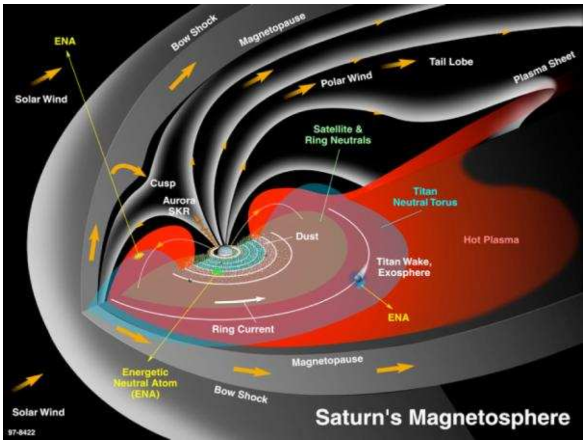
Jovian Magnetosphere

- Jupiter: fast rotation 10 h, mass-loading 1000 kg/s
- Dynamics driven largely by internal sources.
- Planetary rotation coupled with internal plasma loading from the moon Io may lead to additional currents, departure from equilibrium, magnetospheric instabilities and substorm-like processes.
- Regular (periodicity (2.5--4) days) release of mass from the Jovian magnetosphere and changes of the magnetic topology (Kronberg 2007).
- Jovian magnetospheric system is entirely internally driven and impervious to the solar wind (McComas 2007). [Debates are ongoing]



Saturn‘s Magnetosphere

- Saturn's moon Enceladus may be a more significant source of plasma for the Saturn's magnetosphere than Io is for Jovian magnetosphere (Rymer 2010).
- It is important to scale plasma sources, relative to the size of the magnetosphere, to better understand the importance of the internal sources (Vasyliunas 2008).
- At Saturn, auroral features and substorm onset have both been associated with solar wind conditions (Bunce 2005) confirming that both internal loading and the solar wind influence magnetospheric dynamics.



Key phenomena in space plasmas

- Dynamic and structured magnetic fields
- Plasma confinement and flows (solar wind)
- Formation of magnetospheres
- Shocks and turbulence
- Multitude of plasma waves
- Particle heating and acceleration
- Velocity distributions far from thermal equlibrium

Tools needed for space plasma research

- Investigate the motion of charged particles e.g. in radiation belt => Single particle model
- Tools to describe quiet states, where the plasma is in equilibrium (growth phase of magnetic substorms, energy built-up in solar coronal active regions) => Magnetostatics
- Tools to investigate activity (dynamic phase of substorms, coronal eruptions, waves) => MHD
- Cause for change from quiet to active states and tools to investigate energy conversion, reconnection => MHD + kinetic theory

Exercises for Space Plasma Physics:
II. Solar System Plasmas

1. How is a magnetosphere created?

2. What are magnetic storms and substorms?

3. Describe the physical mechanism, how a magnetic storm can destroy transformators. Would a large magnetic storm cause more harm in USA or in Europe? Are large magnetic storms now (year 2012) more or less likely than some five years ago?

4. In auroras we often see red light in high attitudes and green light lower down. Why? Hint: For oxygen it takes less than a second to emit green light, but stays up to about two minutes in excited state before it emits red.

5. Auroras occur in the so called Auroral oval. Why not over the poles? And why do auroral like emissions occur at the poles in laboratory experiments with a terella?

6. Are typical solar-system plasmas like magnetospheres and the solar corona thermodynamic equilibrium? Are they in force-equilibrium?

7. Show that Maxwell's $\nabla \cdot \vec{B}$ and $\nabla \cdot \vec{E}$ equations can be seen (used) as initial condition. If the divergence equations are fulfilled at an initial time, the other two (evolutionary) Maxwell equations ensure that these conditions are fulfilled for all times.

8. Use the electromagnetic potentials

$$\begin{aligned} \vec{B} &= \nabla \times \vec{A} \\ \vec{E} &= -\nabla \vec{\Phi} - \frac{\partial \vec{A}}{\partial t} \\ \nabla \cdot \vec{A} &= -\frac{1}{c^2} \frac{\partial \boldsymbol{\Phi}}{\partial t} \quad \text{(Lorenz Gauge)} \end{aligned}$$

   to derive wave equations for the potentials from Maxwell equations. How can one obtain the charge density and electric current density in a plasma?

Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Plasma models

- **Test particles:**
  Study motion of individual charged particles under the influence of external electro-magnetic (EM) fields
- **Kinetic models:**
  Statistic description of location and velocity of particles and their interaction + EM-fields. (Vlasov-equation, Fokker-Planck eq.)
- **Fluid models:**
  Study macroscopic quantities like density, pressure, flow-velocity etc. + EM-fields (MHD + multifluid models)
- **Hybrid Models:** Combine kinetic + fluid models

Single particle motion, Test particle model

- What is a test particle?
- Charged particles homogenous magnetic fields => Gyration
- Charged particles in inhomogenous fields => Drifts
- Adiabatic invariants
- Magnetic mirror and radiation belts

Test particles:

In the test-particle approach the charged particles move under the influence of electric and magnetic fields.
Back-reaction of the particles is ignored => model is not self-consistent.

Equation of Motion, Lorentz-force:

$$\mathbf{v}(\mathbf{t}) = \frac{\partial \mathbf{r}(\mathbf{t})}{dt}$$

$$m\frac{\partial \mathbf{v}(\mathbf{t})}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Test particles, special cases

(calculate on blackboard + exercises)

- Static homogenous electric field, no B-field
- Static homgeneous magnetic field => Gyration, magnetic moment
- Static, hom. electromagnetic fields (exercise)
- Static inhomogeneous B-field.
- Homogeneous, time-varying electromagnetic fields (exercise).
- Generic cases of time-varying inhomogenous EM-fields => Treat numerically.

Particles in magnetic field

Kinetic energy is a constant of motion in B-Fields

$$\frac{d}{dt}\left(\frac{m}{2}v^2\right) = q\mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}) = 0$$

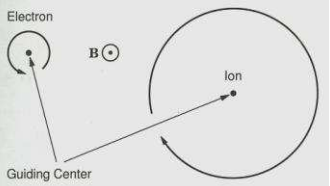
In B-Fields particles (Electrons, Protons) gyrate

Larmor (or gyro) frequency $w_g = \frac{|q|B}{m}$

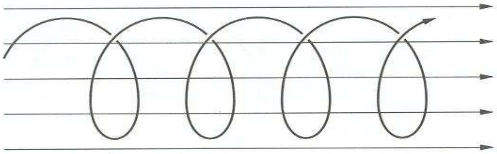
Larmor radius $r_g = \frac{v_\perp m}{B|q|}$

Gyration of ions and electrons

$$
\begin{aligned}
x - x_0 &= r_g \sin \omega_g t \\
y - y_0 &= r_g \cos \omega_g t
\end{aligned}
$$



The equation describes a circular orbit around the field with gyroradius, r_g, and gyrofrequency, ω_g. The orbit's center (x_0,y_0) is called the *guiding center*. The gyration represents a microcurrent, which creates a field opposite to the background one. This behaviour is called *diamagnetic effect*.

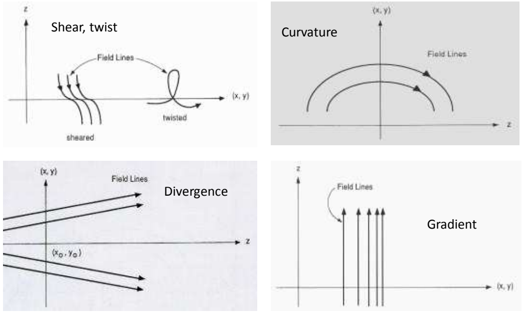


$$r_g = \frac{v_\perp}{|\,\omega_g\,|} = \frac{mv_\perp}{|\,q\,|\,B}$$

$$\alpha = \tan^{-1}\left(\frac{v_\perp}{v_\parallel}\right)$$

If one includes a constant speed parallel to the field, the particle motion is three-dimensional and looks like a *helix*. The *pitch angle* of the helix or particle velocity with respect to the field depends on the ratio of perpendicular to parallel velocity components.

Nonuniform magnetic fields in space



Particles in magnetic field

- Concept of gyrating particles remains useful for inhomogeneous and time-dependent magnetic fields => Drifts, Valid if:
- Spatial scales for B-field changes are large compared to Larmor radius.
- Electric fields, gravity, magnetic field curvature etc. also cause drifts
(some we will study in excercises)

Inhomogeneous B-fields

- Spatial scales for B-field changes are large compared to Gyro radius.

$$\epsilon = \left|\frac{\mathbf{v}\cdot\nabla\mathbf{B}}{w_g\mathbf{B}}\right| \ll 1$$

- Magnetic moment remains constant in weakly inhomogenous B fields
=> Adiabatic Invariant

$$\mu = \frac{mv_\perp^2}{2B}$$

Adiabatic invariants of motion

- We have a quantity which is small, like

$$\epsilon = \left|\frac{\mathbf{v}\cdot\nabla\mathbf{B}}{w_g\mathbf{B}}\right| \ll 1$$

- An adiabatic invariant stays constant over a period of the order $1/\epsilon$, while the electro-magnetic field changes of the order O(1) in the same period.
- What happens if $\epsilon \to 0$?
- Adiabatic invariant becomes constant of motion.

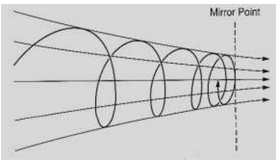
Magnetic mirror

Let us follow the guiding center of a particle moving along an inhomogeneous magnetic field by considering the magnetic moment:

$$\mu = \frac{mv^2 \sin^2 \alpha}{2B}$$

where we used the *pitch angle* α. Apparently, pitch angles at different locations are related by the corresponding magnetic field strengths:

$$\frac{\sin^2 \alpha_2}{\sin^2 \alpha_1} = \frac{B_2}{B_1}$$



The point where the angle reaches 90º is called the *mirror point*.

Adiabatic invariants of motion

In classical Hamiltonian mechanics the action integral

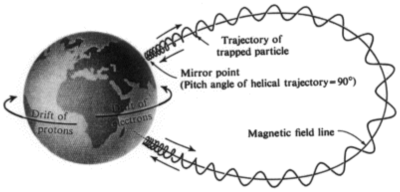
$$J_i = \oint p_i dq_i$$

is an invariant of motion for any change that is slow as compared to the oscillation frequency associated with that motion.

- *Bounce motion* between mirror points $J = \oint m v_\| ds$
- *Drift motion* in azimuthal direction, with planetary magnetic moment *M* $\Phi = \frac{2\pi m}{q^2} M = \text{const}$

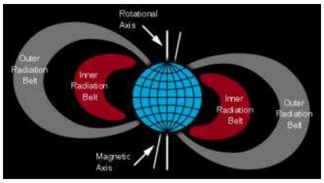
Magnetic flux, $\Phi_\mu = B \pi r_g^2$, through surface encircled by the gyro orbit is constant.

Application: Van Allen Belt

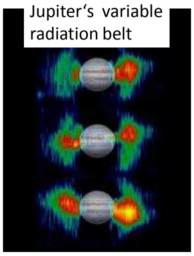


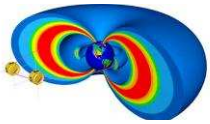
A *dipole magnetic field* has a field strength minumum at the equator and converging field lines at the polar regions (mirrors). Particles can be *trapped* in such a field. *They perform gyro, bounce and drift motions*.

Application: Van Allen Belt, Planetary radiation belts



Jupiter‘s variable radiation belt





Radiation Belt Storm Probes (RBSP), launch 2012(?)

Motion of particles in radiation belts can be modelled with Test-particle approach.

Magnetic drifts

Inhomogeneity will lead to a drift. A typical magnetic field in space will have gradients, and thus field lines will be curved. We Taylor expand the field:

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B}_0$$

where B_0 is measured at the guiding center and r is the distance from it. Modified equation of motion:

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v}\times\mathbf{B}_0) + q\left[\mathbf{v}\times(\mathbf{r}\cdot\nabla)\mathbf{B}_0\right]$$

Expanding the velocity in the small drift plus gyromotion, $v = v_g + v_\nabla$, then we find the stationary drift:

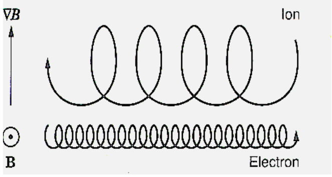
$$\mathbf{v}_\nabla = \frac{1}{B_0^2}\langle(\mathbf{v}_g \times (\mathbf{r}\cdot\nabla)\mathbf{B}_0)\times\mathbf{B}_0\rangle$$

Magnetic drifts

We time average over a gyroperiod and obtain:

$$\mathbf{v}_\nabla = \frac{mv_\perp^2}{2qB^3}(\mathbf{B}\times\nabla B)$$

The non-uniform magnetic field B leads to a *gradient drift* perpendicular to both, the field and its gradient:



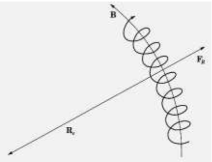
General force drifts

By replacing the electric field *E* in the drift formula by any field exerting a force *F/q*, we obtain the general *guiding-center drift*:

$$\mathbf{v}_F = \frac{1}{\omega_g}\left(\frac{\mathbf{F}}{m} \times \frac{\mathbf{B}}{B}\right)$$

In particular when the field lines are curved, the centrifugal force is

$$\mathbf{F}_R = m v_\parallel^2 \frac{\mathbf{R}_c}{R_c^2}$$



where R_c is the local radius of curvature.

Summary of guiding center drifts

E×B Drift:

Try to calculate these Drifts as excercise

Polarization Drift:

Gradient Drift: $\mathbf{v}_\nabla = \frac{m v_\perp^2}{2q B^3}(\mathbf{B} \times \nabla B)$ $\quad \mathbf{j}_\nabla = \frac{n_e(\mu_i + \mu_e)}{B^2}(\mathbf{B} \times \nabla B)$

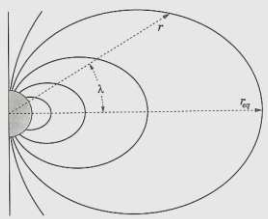
Curvature Drift: $\mathbf{v}_R = \frac{m v_\parallel^2}{q R_c^2 B^2}(\mathbf{R}_c \times \mathbf{B})$ $\quad \mathbf{j}_R = \frac{2n_e(W_{i\parallel} + W_{e\parallel})}{R_c^2 B^2}(\mathbf{R}_c \times \mathbf{B})$

Associated drifts are corresponding drift currents.

Gyrokinetic approach

- We can distinguish the motion of charged particles into gyration of the particle and motion of the gyration center.
- An exact mathematical treatment is possible within Hamilton mechanics by using non-canonical transformation.
- => Guiding-center approximation.
[Outside scope of this Lecture, see Balescue, Transport-Processes in Plasma, Vol. 1, 1988]
- Guiding center approach remains useful concept for selfconsistent kinetic plasma models.

Magnetic dipole field



At distances not too far from the surface the Earth's magnetic field can be approximated by a *dipole field* with a moment:

M_E = 8.05 10^{22} Am^2

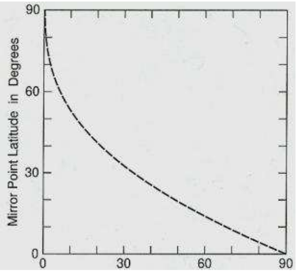
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{M_E}{r^3}(-2\sin\lambda\hat{\mathbf{e}}_r + \cos\lambda\hat{\mathbf{e}}_\lambda)$$

Measuring the distance in units of the Earth's radius, R_E, and using the equatorial surface field, B_E (= 0.31 G), yields with the so-called *L-shell* parameter ($L=r_{eq}/R_E$) the field strength as a function of latitude, λ, and of *L* as:

$$B(\lambda, L) = \frac{B_E}{L^3}\frac{(1+3\sin^2\lambda)^{1/2}}{\cos^6\lambda}$$

Dipole latitudes of mirror points

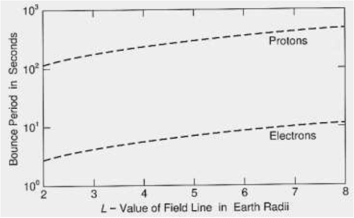
$$\sin^2\alpha_{eq} = \frac{B_{eq}}{B_m} = \frac{\cos^6\lambda_m}{(1+3\sin^2\lambda_m)^{1/2}}$$



Equatorial pitch angle in degrees

Latitude of mirror point depends only on pitch angle but not on *L* shell value.

Bounce period as function of *L* shell

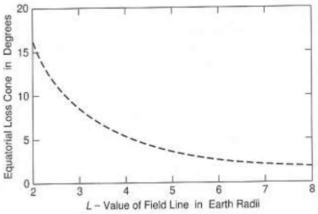


Energy, *W*, is here 1 keV and α_{eq} = 30°.

Bounce period, τ_b, is the time it takes a particle to move back and forth between the two mirror points (*s* is the path length along a given field line).

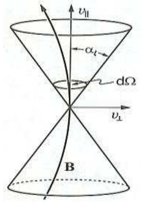
$$\tau_b = 4\int_0^{\lambda_m}\frac{ds}{v_\parallel} = 4\int_0^{\lambda_m}\frac{ds}{d\lambda}\frac{d\lambda}{v_\parallel}$$

Equatorial loss cone for different *L*-values



$$\sin^2 \alpha_\ell = (4L^6 - 3L^5)^{-1/2}$$

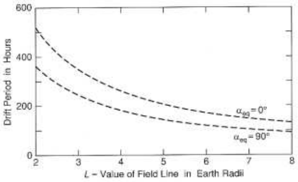
If the mirror point lies too deep in the atmosphere (below 100 km), particles will be absorbed by collisions with neutrals.



The loss-cone width depends only on L but not on the particle mass, charge or energy.

Period of azimuthal magnetic drift motion

Here the energy, *W*, is 1 keV and the pitch angle: α_{eq} = 30° and 90°.



$$\langle \tau_d \rangle \approx \frac{\pi q B_E R_E^2}{3LW}(0.35 + 0.15 \sin \alpha_{eq})^{-1}$$

Drift period is of order of several days. Since the magnetospheric field changes on smaller time scales, it is unlikely that particles complete an undisturbed drift orbit. Radiation belt particles will thus undergo radial (*L*-shell) diffusion!

Summary: Single particle motion

- Gyromotion of ions and electrons arround magnetic field lines.
- Inhomogeneous magnetic fields, electric fields and other forces lead to particle drifts and drift currents.
- Bouncing motion of trapped particles to model radiation belt.
- Constants of motion and adiabatic invariants.
- So far we studied particles in external EM-fields and ignored fields and currents created by the charged particles and collision between particles.

Exercises for Space Plasma Physics:
III. Single particle motion

1. Under which conditions is the test-particle approach a suitable approximation?

2. What are constants of motions? What are adiabatic invariants?

3. What conditions have to be fulfilled that the motion of charged particles can be described as Drift?

4. How do gyro-radius and frequency change with the magnetic field strength, particle mass, charge and temperature?

5. How do electrons (charge $-e$, mass m_e) and ions (charge $+e$, mass m_i) move under the influence of a constant homogeneous magnetic field $\vec{B} = B_0 \vec{e}_z$ and

   (a) A constant homogeneous electric field $\vec{E} = E_0 \, \vec{e}_x$

   (b) A homogenous, but slowly temporal varying electric field

   $$\vec{E} = E(t) \, \vec{e}_x$$

   Hint: In the moving frame (Lorentz transform of E) a free particle does not feel an electric field $\vec{E}' = \vec{E} + \vec{v} \times \vec{B} = 0$ and you can average over the gyroperiod assuming that the temporal changes of the electric field are much slower than the gyro-frequency.

   (c) Do these drift motions produce electric currents? If yes, calculate them.

Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Statistical description of a plasma

- The complete statistic description of a system with N particles is given by the distribution function

$$F(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N, t)$$

$$\int F\, dx_1, dx_2, \ldots, dx_N, dv_1, dv_2, \ldots, dv_N = 1$$

- Hyperspace for N particles has dimension 6 N +1
- N is typically very large (For Sun: N~ 10^{57})

=> No chance to compute or estimate F

Liouville Equation

after Joseph Liouville 1809-1889

The many body distribution

$$F(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N, t)$$

obeys the Liouville equation

$$\frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F}{\partial x_i} \cdot v_i + \frac{\partial F}{\partial v_i} \cdot a_i^T \right) = 0$$

a_i^T acceleration of particle i due to external and interparticle forces

We define the one-particle distribution function $f_\alpha^{(1)}(x_1, v_1, t)$ by integrating $F(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N, t)$ over coordinates and velocities of all but one particle of type α (say ions and electrons) and multiplying over number of particle N_α for each species

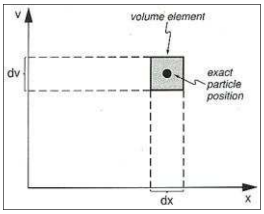
$$\overline{n}_\alpha f_\alpha^{(1)}(x_1, v_1, t) = N_\alpha \int F\, dx_2, \ldots, dx_N, dv_2, \ldots, dv_N$$

where $\overline{n}_\alpha = N_\alpha / V$ and V is the volume.

$\overline{n}_\alpha f_\alpha^{(1)}\, dx_1\, dy_1$ is the number of particles at x_1 with velocity v_1 in the range $dx_1\, dy_1$.

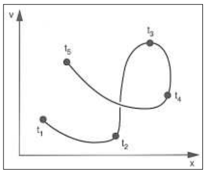
For plasmas in equilibrium $f_\alpha^{(1)}$ has a maxwellian distribution in velocity space. Space plasmas are, however, often far away from equilibrium and the distribution is non-maxwellian.

Phase space



Six-dimensional *phase space* with coordinates axes **x** and **v** and volume element d**x**d**v**

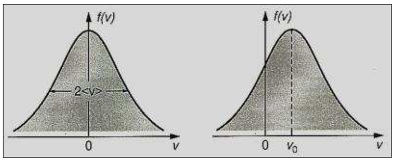
Many particles (i=1, N) having time-dependent position $\mathbf{x}_i$(t) and velocity $\mathbf{v}_i$(t). The particle path at subsequent times (t_1,.., t_5) is a curve in phase space



Maxwellian velocity distribution function

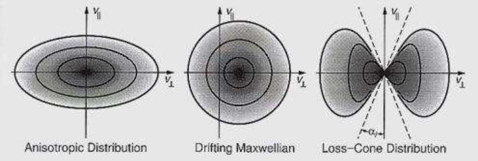
The general equilibrium VDF in a uniform thermal plasma is the *Maxwellian* (Gaussian) distribution.

The average velocity spread (variance) is, <v> = $(2k_BT/m)^{1/2}$, and the mean drift velocity, $\mathbf{v}_0$.



$$f(\mathbf{v}) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m(\mathbf{v} - \mathbf{v}_0)^2}{2k_B T} \right)$$

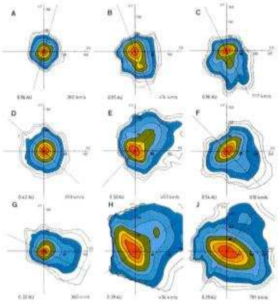
Anisotropic model velocity distributions



The most common anisotropic VDF in a uniform thermal plasma is the *bi-Maxwellian* distribution. Left figure shows a sketch of it, with $T_\perp > T_\parallel$

$$f(v_\perp, v_\parallel) = \frac{n}{T_\perp T_\parallel^{1/2}} \left(\frac{m}{2\pi k_B}\right)^{3/2} \exp\left(-\frac{mv_\perp^2}{2k_B T_\perp} - \frac{mv_\parallel^2}{2k_B T_\parallel}\right)$$

Measured solar wind proton velocity distributions



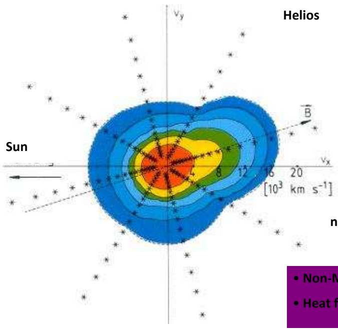
Helios

- **Temperature anisotropies**
- **Ion beams**
- **Plasma instabilities**
- **Interplanetary heating**

Plasma measurements made at 10 s resolution (> 0.29 AU from the Sun)

Marsch et al., JGR, **87**, 52, 1982

Measured solar wind electrons



n_e = 3 -10 cm^{-3}

- **Non-Maxwellian**
- **Heat flux tail**

Pilipp et al., JGR, **92**, 1075, 1987

Set of reduced distribution functions

We can define an entire set of reduced distribution functions from F, for example the two-particle distribution function $f^{(2)}_{\alpha,\gamma}(x_1, v_1, x_2, v_2, t)$ by integrating $F(x_1, x_2, \ldots, x_N, v_1, v_2, \ldots, v_N, t)$ over coordinates and velocities of all but two particles:

$$\overline{n}_\alpha \overline{n}_\gamma\, f^{(2)}_{\alpha,\gamma}(x_1, v_1, x_2, v_2, t) = N_\alpha N_\gamma \int F\, dx_3, \ldots, dx_N, dv_3, \ldots, dv_N$$

where α and γ can be the same type or different species. (say describing the interaction of 2 electrons, 2 ions, or 1 ion and 1 electron)

Similar we can define the three-particle distribution function and so on.

How to proceed?

$$\frac{\partial F}{\partial t} + \sum_i \left(\frac{\partial F}{\partial x_i} \cdot v_i + \frac{\partial F}{\partial v_i} \cdot a_i^T\right) = 0$$

- We integrate over the Liouville equation and get evolutionary equations for the reduced distribution functions.
- This reduces the number of dimensions from 6 N+1 to 7 for the one particle distribtion function, 13 for the two-particle distribution function etc.

=> Derivation on the blackboard

Set of reduced equations

After integrating the Lioville equation we get the evolutionary equation for the one particle distribution function.

closed form

$$\boxed{\frac{\partial f_\alpha^{(1)}}{\partial t} + v_1 \cdot \frac{\partial f_\alpha^{(1)}}{\partial x_1} + a_1^E \frac{\partial f_\alpha^{(1)}}{\partial v_1}} = \left.\frac{\partial f_\alpha^{(1)}}{\partial t}\right|_{\text{particle}}$$

where the term on the right side is due to particle interaction. We calculated (see blackboard) this term as :

$$\left.\frac{\partial f_\alpha^{(1)}}{\partial t}\right|_{\text{particle}} = -\sum_\beta \overline{n}_\beta \int a_{1,\beta} \frac{\partial}{\partial v_1} f^{(2)}_{\alpha,\beta}(x_1, v_1, x_\beta, v_\beta, t)\, dx_\beta\, dv_\beta$$

We do not know the two particle distribution function! How can we derive it?

- We reduced the high-dimensional (6N+1) Liouville equation to a set of equations for reduced distribution-functions.

Set of reduced equations

- Problem: Equation for the one particle distribution function contains the two particle distribution function on the right side.
- In principle we know, how to derive that one: Do the corresponding integration over the Liouville equation (we do not show that explicitely in this lecture)
- Problem: Equation for the two particle distribution function contains the 3-particle distribution function on right side, and so on.

BBGKY-Hierarchy
(Bogoliubov,Born,Green,Kirkwood,Yvon)

The full set of these equations is equivalent to the Liouville-equation.

- Problem: Equation for $f^{(n)}$ contains $f^{(n+1)}$.
- $\Rightarrow$ We cannot solve the full set of equations in the BBGKY-hierarchy. This is as complicated as solving the Liouville-equation directly.
- How to proceed?

Remark: This hierarchy of equations was first published in french by J. Yvon (1935), but this work was hardly recognized that time and got only attention after it was re-discovered in the end of the 1940th.

BBGKY-Hierarchy

- We must cut-off the hierarchy at some point.
- This means we make a suitable assumption for the term containing $f^{(n+1)}$, without computing it exactly.
- For a plasma we cut already after the first equation (for $f^{(1)}$) and make assumptions regarding $f^{(2)}$

$\Rightarrow$ **Kinetic Equations**.

- We have to simplify the term

$$\sum_\beta \overline{n}_\beta \int a_{1,\beta} \frac{\partial}{\partial v_1} f^{(2)}_{\alpha,\beta}(x_1, v_1, x_\beta, v_\beta, t)\, dx_\beta\, dv_\beta$$

Kinetic Equations

- Particle interactions in a plasma are long range and we divide the particle interaction forces in:
  - Average force due to many distant particles.
  - Force due to nearest neighbours (Collisions).
- The average forces due to many distant particles do not depend on the exact position of these individual particles and we treat them together with the external forces.

$$\mathbf{a} = \mathbf{a}_{\text{ext}} + \langle \mathbf{a}_{\text{int}} \rangle$$

Kinetic Equations

$$\frac{\partial f^{(1)}_\alpha}{\partial t} + \mathbf{v_1} \cdot \frac{\partial f^{(1)}_\alpha}{\partial \mathbf{x_1}} + \frac{q_\alpha}{m_\alpha} \langle \mathbf{E} + \mathbf{v_1} \times \mathbf{B} \rangle \cdot \frac{\partial f^{(1)}_\alpha}{\partial \mathbf{v_1}} = \xcancel{\left. \frac{\partial f^{(1)}_\alpha}{\partial t} \right|_c}$$

~~where we get for the binary-collision rate~~

$$\xcancel{\left. \frac{\partial f^{(1)}_\alpha}{\partial t} \right|_c = -\sum_\beta \int \left(\mathbf{a}_{1,\beta} - \langle \mathbf{a}^{\text{int}}_{1,\beta} \rangle \right) \cdot \frac{\partial}{\partial \mathbf{v_1}} f^{(2)}_{\alpha,\beta}(x_1, v_1, x_\beta, v_\beta, t)\, dx_\beta\, dv_\beta}$$

Simplest approach: Neglect the binary-collision rate. Only the average forces created by the other particles are considered => Collisionless plasma => Vlasov equation

Two problems remain:

- How do we get the average fields $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$?
- The collision term $\left. \frac{\partial f^{(1)}_\alpha}{\partial t} \right|_c$ still contains $f^{(2)}$.

Vlasov Equation

$$\frac{\partial f^{(1)}_\alpha}{\partial t} + \mathbf{v_1} \cdot \frac{\partial f^{(1)}_\alpha}{\partial \mathbf{x_1}} + \frac{q_\alpha}{m_\alpha} \langle \mathbf{E} + \mathbf{v_1} \times \mathbf{B} \rangle \cdot \frac{\partial f^{(1)}_\alpha}{\partial \mathbf{v_1}} = 0$$

We get average fields $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ from Maxwell equations

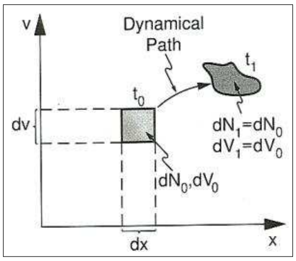
$$\nabla \cdot \langle \mathbf{E} \rangle = \frac{\langle \rho_q \rangle}{\epsilon_0}$$

$$\nabla \times \langle \mathbf{B} \rangle = \mu_0 \langle \mathbf{J} \rangle + \mu_0 \epsilon_0 \frac{\partial \langle \mathbf{E} \rangle}{\partial t} = 0$$

How do we get the average charge density $\langle \rho_q \rangle$ and the average electric current $\langle \mathbf{J} \rangle$?


Anatoly Vlasov
(1908-1975)

Vlasov equation



The Vlasov equation expresses phase space density conservation. A 6D-volume element evolves like in an incompressible fluid.

Vlasov equation is nonlinear via closure with Maxwell‘s equations.

Vlasov Equation

- We have actually to solve two Vlasov equations: for ions $f_i^{(1)}$ and electrons $f_e^{(1)}$.
- We write short f_i and f_e from now on and also $\mathbf{x}$ and $\mathbf{v}$ instead of $\mathbf{x_1}$ and $\mathbf{v_1}$.
- The fields $\langle \mathbf{E} \rangle$ and $\langle \mathbf{B} \rangle$ are of course unique and couple these two Vlasov equations.
- The average charge density $\langle \rho_q \rangle$ and the average electric current $\langle \mathbf{J} \rangle$ are only functions of the location and time, but not the velocity space.
- $\langle \rho_q \rangle = \langle \rho_{qi} \rangle + \langle \rho_{qe} \rangle$, $\langle \mathbf{J} \rangle = \langle \mathbf{J_i} \rangle + \langle \mathbf{J_e} \rangle$
- We have to relate the macroscopic quantities $\langle \rho_q(\mathbf{x}, t) \rangle$ and $\langle \mathbf{J}(\mathbf{x}, \mathbf{t}) \rangle$ to the distribution functions $f_i(\mathbf{x}, \mathbf{v}, t)$ and $f_e(\mathbf{x}, \mathbf{v}, t)$.

Macroscopic variables of a plasma

- We take moments of the distribution function f_α, where α stands for ions and electrons. Moment means that we integrate in velocity space over quantities like

$$\int \mathbf{v}^n f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

- Zero moment, densities (particle-, mass-, charge density):

$$n_\alpha(\mathbf{x}, t) = \overline{n}_\alpha \int f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

$$\rho_{m\,\alpha}(\mathbf{x}, t) = \overline{n}_\alpha m_\alpha \int f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

needed in Vlavov-Maxwell system

$$\boxed{\rho_{q\,\alpha}(\mathbf{x}, t) = \overline{n}_\alpha q_\alpha \int f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}}$$

Macroscopic variables of a plasma

- First moment (particle flux $\mathbf{\Gamma}$, macroscopic plasma flow $\mathbf{V}$, electric current density $\mathbf{J}$):

$$\begin{aligned} \Gamma_\alpha(\mathbf{x}, t) &= \overline{n}_\alpha \int \mathbf{v}\, f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \\ &= n_\alpha(\mathbf{x}, t)\, \mathbf{V}_\alpha \end{aligned}$$

$$\mathbf{V}_\alpha(\mathbf{x}, t) = \frac{\int \mathbf{v}\, f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}}{\int f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}}$$

$$\boxed{\begin{aligned} J_\alpha(\mathbf{x}, t) &= q_\alpha \overline{n}_\alpha \int \mathbf{v}\, f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \\ &= q_\alpha\, n_\alpha(\mathbf{x}, t)\, \mathbf{V}_\alpha(\mathbf{x}, t) \end{aligned}}$$

needed in Vlavov-Maxwell system

Macroscopic variables of a plasma

- Second moment (Pressure Tensor P, scalar pressure p):

$$P_\alpha(\mathbf{x}, t) = \overline{n}_\alpha m_\alpha \int (\mathbf{v} - \mathbf{V}_\alpha)(\mathbf{v} - \mathbf{V}_\alpha)\; f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

- For spherical symmetric velocity distributions the pressure tensor becomes diagonal

$$P_\alpha(\mathbf{x}, t) = \begin{pmatrix} p_\alpha & 0 & 0 \\ 0 & p_\alpha & 0 \\ 0 & 0 & p_\alpha \end{pmatrix}$$

with the scalar pressure

$$\begin{aligned} p_\alpha(\mathbf{x}, t) &= \frac{\overline{n}_\alpha m_\alpha}{3} \int (\mathbf{v} - \mathbf{V}_\alpha)^2\, f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v} \\ &= n_\alpha\, k_b\, T_\alpha \end{aligned}$$

The assumption of a scalar pressure is popular for it's simplicity, but not valid in some space plasmas like the solar wind. This leads to an anisotropic pressure tensor.

Macroscopic variables of a plasma

- Third moment (Heat flux $\mathbf{H}$):

$$\mathbf{H}_\alpha(\mathbf{x}, t) = \frac{\overline{n}_\alpha m_\alpha}{2} \int \mathbf{v}\,(\mathbf{v} \cdot \mathbf{v})\, f_\alpha(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{v}$$

- It is possible to compute additional moments, but we cannot necessarily relate these higher moments to physical quantities.
- First and second moments are sufficient to close the Vlasov-Maxwell system.
- We can derive also equations for the derived macroscopic quantities. This leads to a fluid description of the plasma (like MHD) and we do that soon.
- In the following we continue to study the Vlasov-Maxwell system, kinetic equations and make more sophisticated approaches for collisions.

Vlasov-Maxwell Equations

kinetic description of a collisionless plasma

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \sum_\alpha \overline{n}_\alpha q_\alpha \int f_\alpha \, d\mathbf{v} + \frac{\rho_{q\,\mathrm{ext}}}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_\alpha \overline{n}_\alpha q_\alpha \int \mathbf{v} \, f_\alpha \, d\mathbf{v} + \mu_0 \, \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}_{\mathrm{ext}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

α stands for ions and electrons.

Properties of Vlasov Equation

- Vlasov Equation conserves particles
- Distribution functions remains positive
- Vlasov equation has many equilibrium solutions

$$\mathbf{v} \cdot \frac{\partial f_{\alpha\,0}}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_{\alpha\,0}}{\partial \mathbf{v}} = 0$$

Index 0 stands for equilibrium. Any distribution function $f_{\alpha\,0}$ which depends only on constants of motions (say $a(x,v)$, $b(x,v), \dots$) of the particle trajectories solves the stationary Vlasov equation. (show proof on blackboard)

$$f_{\alpha\,0}(x,v) = f_{\alpha\,0}(a(x,v), b(x,v), \dots)$$

Stability of Vlasov equilibria

- $\frac{\partial f_{\alpha 0}}{\partial t} = 0$ does not guaranty stability of $f_{\alpha\,0}$.
- Assume a nearby disturbed case

$$f_\alpha(t) = f_{\alpha\,0} + \delta f_\alpha(t)$$

- How does it develop with the time-dependent Vlasov-equation?
  - If $\delta f_\alpha(t)$ decreases in time, the system will finally reach $f_{\alpha\,0} \Rightarrow$ Stable
  - If $\delta f_\alpha(t)$ increases, the system will get further away from $f_{\alpha\,0} \Rightarrow$ Unstable
  - If f decreases monotonically with v^2 then the equilibrium is stable (Proof on blackboard):

$$\frac{\partial f}{\partial v^2} < 0$$

Example: The well known Maxwell-Boltzmann-distribution

$$f(\mathbf{v}) \propto \exp\left(-\frac{m\,v^2}{2\,k_b\,T}\right)$$

James Clerk Maxwell, 1831-1871

is stable!

(Proof on blackboard)

Ludwig Boltzmann, 1844-1906

Further Examples

- Drifting Maxwellian ?

$$f(\mathbf{v}) \propto \exp\left(-\frac{m\,(\mathbf{v}-\mathbf{u_D})^2}{2\,k_b\,T}\right)$$

- Maxwellian with a non-thermal feature ?

$$f(v) =\propto \exp\left(-\frac{m\,v^2}{2\,k_b\,T}\right) + \epsilon \exp\left(-10 \cdot \frac{m\,(\mathbf{v}-\mathbf{u_D})^2}{2\,k_b\,T}\right)$$

Investigate stability as an excercise.

Kinetic equations of first order

$$\frac{\partial f_\alpha^{(1)}}{\partial t} + \mathbf{v_1} \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{x_1}} + \frac{q_\alpha}{m_\alpha}\langle \mathbf{E} + \mathbf{v_1} \times \mathbf{B} \rangle \cdot \frac{\partial f_\alpha^{(1)}}{\partial \mathbf{v_1}} = \left.\frac{\partial f_\alpha^{(1)}}{\partial t}\right|_c$$

where we get for the binary-collision rate

$$\left.\frac{\partial f_\alpha^{(1)}}{\partial t}\right|_c = -\sum_\beta \int \left(\mathbf{a}_{1,\beta} - \langle \mathbf{a}_{1,\beta}^{\mathrm{int}} \rangle\right) \cdot \frac{\partial}{\partial \mathbf{v_1}} f_{\alpha,\beta}^{(2)}(x_1, v_1, x_\beta, v_\beta, t)\, dx_\beta \, dv_\beta$$

- Vlasov approach: neglect 2 particle distribution function completely
- Now we make a more sophisticated approach, the Mayer cluster expansion.

Mayer cluster expansion

$$f^{(2)}_{\alpha,\beta} = \underbrace{f^{(1)}_{\alpha}(x_1,v_1,t)\,f^{(1)}_{\beta}(x_2,v_2,t)}_{\text{Particles are independent from each other}} + \underbrace{g^{(2)}_{\alpha,\beta}(x_1,v_1,x_2,v_2,t)}_{\text{Correlation of the 2 particles}}$$

- In Vlasov theory we assumed $g^{(2)}_{\alpha,\beta} = 0$, which means that the single particle distribution functions are uncorrelated.
- We take now correlations into account, but assume
  $$g^{(2)}_{\alpha,\beta} \ll f^{(1)}_{\alpha}\, f^{(1)}_{\beta}$$
  This is a valid assumption, because in a small volume
  $$V \ll \Lambda_D^3$$
  The joint distribution function $f^{(2)}_{\alpha,\beta}$ is determined by the many particles outside V and not by the seperation of two particle from each other.

Collisions

Plasmas may be *collisional* (e.g., fusion plasma) or *collisionsless* (e.g., solar wind). Space plasmas are usually collisionless. => Vlasov equation

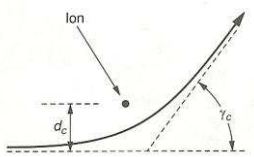
Ionization state of a plasma:

• *Partially ionized*: Earth's ionosphere or Sun's photosphere and chromosphere, dusty and cometary plasmas
=> Collisions with neutrals dominate

• *Fully ionized*: Sun's corona and solar wind and most of the planetary magnetospheres
=> Coulomb collisions or collisionless

Coulomb collisions

Charged particles interact via the Coulomb force over distances much larger than atomic radii, which enhances the cross section as compared to hard sphere collisions, but leads to a preference of small-angle deflections.



$$\nu_{ei} = n_e \sigma_c \langle v_e \rangle$$

Impact or collision parameter, d_c, and scattering angle, [redacted]

Models for the collision terms

Neutral-ion collisions are described by a Maxwell Boltzmann collision terms as in Gas-dynamics. (f_n : distribution function of neutrals, [redacted] collision frequency):

$$\left(\frac{\partial f}{\partial t}\right)_c = \nu_n (f_n - f)$$

Coulomb Collisions lead to the Fokker-Planck-Equation which can often be described as a diffusion process.

$$\left(\frac{\partial f}{\partial t}\right)_c = \nabla_{\mathbf{v}} \cdot (\mathbf{D} \cdot \nabla_{\mathbf{v}} f)$$

Further sophistications are to take into account that Coulomb collisions occur not in vaccum, but are Debye-shielded and to consider wave-particle interactions.

Kinetic Equations

- In Kinetic theory we have a statistic description in a 6D phase space (configuration and velocity space).
- Kinetic equations are a first order approximation of the BBGKY-Hierarchy.
- Equation for Particle distribution function is coupled with Maxwell-equations => Difficult to solve.
- Most space plasmas are collisionless => Vlasov equation
- Collission terms depend on nature of process:
  - collision with neutrals => Maxwell-Boltzmann
  - Coloumb collisions: Fokker-Planck
  - Coloumb c. with Debye-shielding => Lennard Balescu

How to proceed?

- Due to the nonlinear coupling with Maxwell-equations, the Kinetic equations are difficult to solve.
- In numerical simulations one has to resolve relevant plasma scales like Debye-length, gyro-radii of electrons and ions and also the corresponding temporal scales (gyro-frequencies, plasma frequencies), which are orders of magnitude smaller and faster as the macroscopic scales (size of magnetosphere or active region)
- We often not interested in details of the velocity distribution function.
- => Integration over the velocity space lead to fluid equations like MHD (3D instead of 6D space)

Exercises for Space Plasma Physics: IV. Kinetic equations

1. What is the difference between the Vlasov-, Boltzmann- and Fokker-Planck equation?

2. Why do physicists use the Fokker-Planck equation in fully ionized plasmas instead of the Boltzmann equation used for normal gases?

3. How does the entropy $S = -\sum_\alpha \int f_\alpha \, \ln f_\alpha \, d\mathbf{x} \, d\mathbf{v}$ evolve in a Vlasov Maxwell system?

4. In the lecture we showed the Mayer-Cluster-expansion for the two-particle distribution function $f^{(2)}_{\alpha,\beta}$. Can you imagine, how the corresponding expansion for a three particle-distribution function $f^{(3)}_{\alpha,\beta,\gamma}$ looks like? Are three-particle correlations assumed to be more or less important than two-particle correlations in space plasmas?

5. Check if the following distribution functions are stable or unstable:
   - Drifting Maxwellian (with Drift velocity $\mathbf{u_D}$):

     $$f(\mathbf{v}) \propto \exp\left(-\frac{m\,(\mathbf{v}-\mathbf{u_D})^2}{2\,k_b\,T}\right)$$

   - Maxwellian with a non-thermal feature like a drifting beam

     $$f(v) = \exp\left(-\frac{m\,v^2}{2\,k_b\,T}\right) + \epsilon\exp\left(-10\cdot\frac{m\,(\mathbf{v}-\mathbf{u_D})^2}{2\,k_b\,T}\right)$$

Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Multi-fluid theory

- Full plasma description in terms of particle distribution functions (VDFs), $f_s(\mathbf{v},\mathbf{x},t)$, for species, s.

- For slow large-scale variations, a description in terms of moments is usually sufficient -> **multi-fluid** (density, velocity and temperature) description.

- Fluid theory is looking for evolution equations for the basic macroscopic moments, i.e. number density, $n_s(\mathbf{x},t)$, velocity, $\mathbf{v}_s(\mathbf{x},t)$, pressure tensor, $\mathbf{P}_s(\mathbf{x},t)$, and kinetic temperature, $T_s(\mathbf{x},t)$. For a two fluid plasma consisting of electrons and ions, we have s=e,i.

Magneto-hydro-dynamics (MHD)

- Fluid equations reduce from the 6D space from kinetic theory to our usual 3D configuration space (and the time).
- In Fluid theory, we miss some important physics, however. With single fluid MHD even more than multi fluid (2 Fluid: electrons and ions).
- 3 fluid (+neutral particles) and 4 fluid models are popular in space plasma physics, too, also one can use different fluids for different Ion-species.
- Hybrid-model: Ions treated as particles and electrons as fluid.

Fluid models, Magneto-Hydro-Dynamics

- If we solved the kinetic equations (Vlasov or Fokker-Planck-equation) we derive macroscopic variables by taking velocity-moments of the distribution functions.

$$n_\alpha(\mathbf{x},t) = \overline{n}_\alpha \int f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}$$

$$\mathbf{V}_\alpha(\mathbf{x},t) = \frac{\int \mathbf{v}\, f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}}{\int f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}}$$

$$P_\alpha(\mathbf{x},t) = \overline{n}_\alpha\, m_\alpha \int (\mathbf{v}-\mathbf{V}_\alpha)(\mathbf{v}-\mathbf{V}_\alpha)\ f_\alpha(\mathbf{x},\mathbf{v},t)\, d\mathbf{v}$$

Fluid models, Magneto-Hydro-Dynamics

- Solving the kinetic equations is, however, very difficult, even numerically. One has to resolve all spatial and temporal plasma scales like Debey-length/frequency, Gyro-radius/frequency etc.
- In space plasmas these scales are often orders of magnitudes smaller as the macroscopic scales we are interested in (like size of solar active regions or planetary magnetospheres).
- Solution: We take velocity-moments over the kinetic equations (Vlasov-equation), instead of (generally unknown) solution of the kinetic equation.

Continuity equation

Evolution equations of moments are obtained by taking the corresponding moments of the Vlasov equation:

$$\boxed{\int \Big[\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\ \cdot\nabla_{\mathbf{v}} f_s\Big] d^3v = 0}$$

Taking the zeroth moment yields for the first term:

$$\boxed{\frac{\partial}{\partial t}\int f_s d^3v = \frac{\partial n_s}{\partial t}}$$

In the second term, the velocity integration and spatial differentiation can be interchanged which yields a divergence:

$$\boxed{\nabla_{\mathbf{x}}\cdot\int \mathbf{v} f_s d^3v = \nabla\cdot(n_s\mathbf{v}_s)}$$

In the force term, a partial integration leads to a term, which does not contribute.

$$\boxed{\int f_s \nabla_{\mathbf{v}}\cdot(\mathbf{E}+\mathbf{v}\times\mathbf{B}) d^3v}$$

Continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s) = 0$$

We have again (like in BBGKY-hierarchy) the problem, that the equation for the 0-moment (density) contains the first moment

(velocity **v** or momentum n**v**).

Momentum equation

The evolution equation for the flow velocity/momentum is obtained by taking the first moment of the Vlasov equation:

$$\int \mathbf{v}\Big[\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f_s + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f_s\Big] d^3v = 0$$

Since the phase space coordinate **v** does not depend on time, the first term yields the time derivative of the flux density:

$$\frac{\partial}{\partial t}\int \mathbf{v} f_s d^3v = \frac{\partial}{\partial t}(n_s \mathbf{v}_s)$$

In the second term, velocity integration and spatial differentiation can be exchanged, and **v**(**v**·∇ = ∇(**vv**) be used. With the definition of the pressure tensor we get:

$$\nabla_{\mathbf{x}} \cdot \int \mathbf{v}\mathbf{v} f_s d^3v = \nabla \cdot (n_s \mathbf{v}_s \mathbf{v}_s) + \frac{1}{m_s}\nabla \cdot \mathbf{P}_s$$

In the third term, a partial integration with respect to the velocity gradient operator ∇ gives the remaining integral:

$$\int f_s (\nabla_{\mathbf{v}} \mathbf{v}) \cdot (\mathbf{E} + \mathbf{v} \times \mathbf{B}) d^3v = n_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B})$$

We can now add up all terms and obtain the final result:

$$\frac{\partial (n_s \mathbf{v}_s)}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s \mathbf{v}_s) + \frac{1}{m_s}\nabla \cdot \mathbf{P}_s - \frac{q_s}{m_s} n_s (\mathrm{E} + \mathrm{v}_s \times \mathrm{B}) = 0$$

This momentum density conservation equation for species s corresponds to the Navier-Stokes equation for neutral fluids. In a plasma the equation becomes more complicated due to coupling with Maxwell-equations via the Lorentz-force.

Momentum equation or equation of motion

$$\frac{\partial (n_s \mathbf{v}_s)}{\partial t} + \nabla \cdot (n_s \mathbf{v}_s \mathbf{v}_s) + \frac{1}{m_s}\nabla \cdot \mathbf{P}_s - \frac{q_s}{m_s} n_s (\mathrm{E} + \mathrm{v}_s \times \mathrm{B}) = 0$$

This equation for the first-moment (velocity, momentum) contains the second moment (pressure tensor).

And so on .....

Equation for moment k, contains always the moment k+1 => **Hierarchy of moments**

Hierarchy of moments

- As in kinetics (BBGKY) we have to cut the hierarchy somewhere by making suitable approximations.
- Number of (scalar) variables increase by taking higher orders of moments v^n.
- MHD, 5 moments: density, 3 components of velocity and a scalar (isotropic) pressure (or temperature). These 5 moments are called plasmadynamic variables.
- Fluid equations with a higher number of moments (13, 21, 29, 37, ...) are possible and take anisotropy and approximated correction terms (for cutting the infinite chain of hierarchy-equations) into account.

Energy equation

The equations of motion do not close, because at any order a new moment of the next higher order appears (closure problem), leading to a chain of equations. In the momentum equation the pressure tensor, **P**$_s$, is required, which can be obtained from taking the seond-order moment of Vlasov's equation. The results become complicated. Often only the trace of **P**$_s$, the isotropic pressure, p$_s$, is considered, and the traceless part, **P'**$_s$,the stress tensor is separated, which describes for example the shear stresses.

Full energy (temperature, heat transfer) equation:

$$\frac{3}{2} n_s k_B \left(\frac{\partial T_s}{\partial t} + \mathbf{v}_s \cdot \nabla T_s\right) + p_s \nabla \cdot \mathbf{v}_s \;=\; -\nabla \cdot \mathbf{q}_s - \left(\mathrm{P}'_s \cdot \nabla\right) \cdot \mathbf{v}_s$$

The sources or sinks on the right hand side are related to heat conduction, **q**$_s$, or mechanical stress, **P'**$_s$.

Equation of state

A truncation of the equation hierarchy can be achieved by assuming an equation of state, depending on the form of the pressure tensor.

If it is isotropic, **P**$_s$ = p_s**1**, with the unit dyade, **1**, and ideal gas equation, $p_s = n_s k_B T_s$, then we have a diagonal matrix:

$$\mathbf{P}_s = \begin{pmatrix} p_s & 0 & 0 \\ 0 & p_s & 0 \\ 0 & 0 & p_s \end{pmatrix}$$

- Isothermal plasma: T_s = const
- Adiabatic plasma: $T_s = T_{s0}\,(n_s/n_{s0})^{[redacted]1}$, with the adiabatic index [redacted] $= c_P/c_V = 5/3$ for a mono-atomic gas.
- Incompressible plasma Div **v** =0

Equation of state

Due to strong magnetization, the plasma pressure is often anisotropic, yet still gyrotropic, which implies the form:

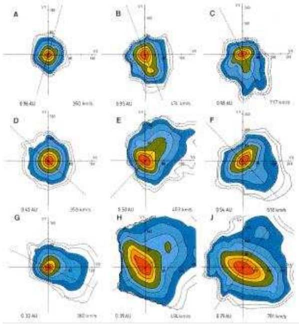
$$\mathbf{P}_s = p_{s\perp}\mathbf{I} + \left(p_{s\parallel} - p_{s\perp}\right)\frac{\mathbf{BB}}{B^2}$$

$$\mathbf{P}_s = \begin{pmatrix} p_{s\perp} & 0 & 0 \\ 0 & p_{s\perp} & 0 \\ 0 & 0 & p_{s\parallel} \end{pmatrix}$$

with a different pressure (temperature) parallel and perpendicular to the magnetic field. Then one has two energy equations, which yield (without sinks and sources) the double-adiabatic equations of state:

- T[redacted]B -> perpendicular heating in increasing field
- $T_{||}$ [redacted]$n/B)^2$ -> parallel cooling in declining density

Measured solar wind proton velocity distributions



$$\mathbf{P}_s = \begin{pmatrix} p_{s\perp} & 0 & 0 \\ 0 & p_{s\perp} & 0 \\ 0 & 0 & p_{s\parallel} \end{pmatrix}$$

Non-isotropic particle distribution functions lead to pressure tensor (instead of scalar pressure) in fluid approach.

Helios

Marsch et al., JGR, **87**, 52, 1982

One-fluid theory

Consider simplest possible plasma of fully ionized hydrogen with electrons with mass m_e and charge $q_e = -e$, and ions with mass m_i and charge $q_i = e$. We define charge and current density by:

$$\rho = e(n_i - n_e) \qquad \mathbf{j} = e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)$$

Usually ***quasineutrality*** applies, $n_e = n_i$, and space charges vanish, [redacted]= 0, but the plasma carries a finite current, i.e. we still need an equation for **j**. We introduce the *mean mass*, *density* and *velocity* in the single-fluid description as

$$m = m_e + m_i = m_i\left(1 + \frac{m_e}{m_i}\right)$$
$$n = \frac{m_e n_e + m_i n_i}{m_e + m_i}$$

$$\mathbf{v} = \frac{m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e}{m_e n_e + m_i n_i}$$

One-fluid momentum equation

Constructing the equation of motion is more difficult because of the nonlinear advection terms, $n_s\mathbf{v}_s\mathbf{v}_s$.

$$\frac{\partial(n_e\mathbf{v}_e)}{\partial t} + \nabla\cdot(n_e\mathbf{v}_e\mathbf{v}_e) = -\frac{1}{m_e}\nabla\cdot\mathrm{P}_e - \frac{n_e e}{m_e}(\mathrm{E} + \mathbf{v}_e\times\mathrm{B}) + \frac{\mathrm{R}}{m_e}$$
$$\frac{\partial(n_i\mathbf{v}_i)}{\partial t} + \nabla\cdot(n_i\mathbf{v}_i\mathbf{v}_i) = -\frac{1}{m_i}\nabla\cdot\mathrm{P}_i + \frac{n_i e}{m_i}(\mathrm{E} + \mathbf{v}_i\times\mathrm{B}) - \frac{\mathrm{R}}{m_i}$$

The equation of motion is obtained by adding these two equations and exploiting the definitions of [redacted] *m, n,* **v** and **j**. When multiplying the first by m_e and the second by m_i and adding up we obtain:

$$\begin{aligned} &- \nabla\cdot(\mathrm{P}_e + \mathrm{P}_i) + e(n_i - n_e)\mathrm{E} + e(n_i\mathbf{v}_i - n_e\mathbf{v}_e)\times\mathrm{B} \\ &= -\nabla\cdot\mathrm{P} + \rho\mathrm{E} + \mathbf{j}\times\mathrm{B} \end{aligned}$$

Here we introduced the total pressure tensor, **P** = **P**$_e$ + **P**$_i$. In the nonlinear parts of the advection term we can neglect the light electrons entirely.

Magnetohydrodynamics (MHD)

With these approximations, which are good for many quasineutral space plasmas, we have the MHD momentum equation, in which the space charge (electric field) term is also mostly disregarded.

$$\frac{\partial(nm\mathbf{v})}{\partial t} + \nabla\cdot(nm\mathbf{vv}) = -\nabla\cdot\mathbf{P} + \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}$$

Note that to close the full set, an equation for the current density is needed. For negligable displacement currents, we simply use Ampere's law in magnetohydrodynamics and **B** as a dynamic variable, and replace then the Lorentz force density by:

$$\mathbf{j}\times\mathbf{B} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})$$

Generalized Ohm's law

The evolution equation for the current density, **j**, is derived by use of the electron equation of motion and called generalized Ohm's law. It results from a subtraction of the ion and electron equation of motion. The nonlinear advection terms cancel in lowest order. The result is:

$$\boxed{\begin{aligned}\frac{m_e}{e}\frac{\partial \mathbf{j}}{\partial t} &= \nabla\cdot\left(\mathbf{P}_e - \frac{m_e}{m_i}\mathbf{P}_i\right) - \left(1+\frac{m_e}{m_i}\right)\mathbf{R} \\ &+ n_e e\left(1+\frac{m_e n_i}{m_i n_e}\right)\left[\mathbf{E} + \left(\mathbf{v}_e + \frac{m_e n_i}{m_i n_e}\mathbf{v}_i\right)\times\mathbf{B}\right]\end{aligned}}$$

The right hand sides still contain the individual densities, masses and speeds, which can be eliminated by using that $m_e/m_i << 1$, n_e[redacted]$_i$

Generalized Ohm's law

We obtain a simplified equation:

$$\boxed{\frac{m_e}{e}\frac{\partial \mathbf{j}}{\partial t} = \nabla\cdot\mathbf{P}_e + ne\left(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}\right) - \mathbf{R}}$$

Key features in single-fluid theory: Thermal effects on **j** enter only via, $\mathbf{P}_e$, i.e. the electron pressure gradient modulates the current. The Lorentz force term contains the electric field as seen in the electron frame of reference.

Generalized Ohm's law

Omitting terms of the order of the small mass ratio, the fluid bulk velocity is, $\mathbf{v}_i = \mathbf{v}$. Using this and the quasineutrality condition yields the electron velocity as: $\mathbf{v}_e = \mathbf{v} - \mathbf{j}/ne$. Finally, the collision term with frequency [redacted] can be assumed to be proportional to the velocity difference, and use of the resistivity, [redacted] m_e[redacted]$/ne^2$, permits us to write:

$$\boxed{\mathbf{R} = m_e n \nu_c (\mathbf{v}_i - \mathbf{v}_e)}$$

$$\boxed{\mathbf{R} = \eta n e \mathbf{j}}$$

Generalized Ohm's law

The resulting Ohm's law can then be written as:

$$\boxed{\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{ne}\mathbf{j}\times\mathbf{B} - \frac{1}{ne}\nabla\cdot\mathbf{P}_e + \frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t}}$$

The right-hand side contains, in a plasma in addition to the resistive term, three new terms: **Hall term,** ***electron pressure, contribution of electron inertia to current flow.***

Popular further simplifications are:

- Resistive MHD: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$
- Ideal MHD: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$

Magnetic tension

The Lorentz force or Hall term introduces a new effect in a plasma which is specific for magnetohydrodynamics: ***magnetic tension***, giving the conducting fluid stiffness. For slow variations Ampere's law can be used to derive:

$$\boxed{\mathbf{j}\times\mathbf{B} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})}$$

Applying some vector algebra to the right hand side gives:

$$\boxed{\mathbf{j}\times\mathbf{B} = -\nabla\left(\frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\nabla\cdot(\mathbf{BB})}$$

magnetic pressure $\boxed{p_B = \frac{B^2}{2\mu_0}}$

Plasma beta

Starting from the MHD equation of motion for a plasma at rest in a steady quasineutral state, we obtain the simple force balance:

$$\boxed{\nabla\cdot\mathbf{P} = -\frac{1}{\mu_0}\mathbf{B}\times(\nabla\times\mathbf{B})}$$

which expresses ***magnetohydrostatic equilibrium***, in which thermal pressure balances magnetic tension. If the particle pressure is nearly isotropic and the field uniform, this leads to the total pressure being constant:

$$\boxed{\nabla\left(p + \frac{B^2}{2\mu_0}\right) = 0}$$

The ratio of these two terms is called the ***plasma beta***: $\boxed{\beta = \frac{2\mu_0 p}{B^2}}$

Requirements for the validity of MHD

Variations must be large and slow, [redacted] [redacted] and $k < 1/r_{gi}$, which means fluid scales must be much larger than gyro-kinetic scales. Consider Ohm's law:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(\mathbf{v}\times\mathbf{B} - \frac{1}{ne}\mathbf{j}\times\mathbf{B} + \frac{1}{ne}\nabla\cdot\mathbf{P}_e - \frac{m_e}{ne^2}\frac{\partial \mathbf{j}}{\partial t} - \eta\mathbf{j}\right)}$$

Convection, Hall effect, thermoelectricity, polarization, resistivity

[redacted] [redacted]

Only in a strongly collisional plasma can the Hall term be dropped.

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}_e\times\mathbf{B})}$$

In collisionless MHD the electrons are frozen to the field.

MHD-equations (fluid part, including gravity)

a) Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad \text{(mass conservation)}$$

b) Momentum conservation equation (equation of motion)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\psi$$

c) Energy equation (various different forms possible)

$$\rho^\gamma\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v}\cdot\nabla\left(\frac{p}{\rho^\gamma}\right) = -(\gamma-1)\mathcal{L}$$

where

$$\mathcal{L} = \underbrace{\nabla\cdot\mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\frac{\mathbf{j}^2}{\sigma}}_{\text{Ohmic heating}} - \overbrace{H}^{\text{everything else}}.$$

source:
Neukirch 1998

MHD-equations (Maxwell-part + Ohm's law)

d) Ampère's law

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j}$$

(displacement current neglected).

e) Faraday's law

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

f) no magnetic monopoles

$$\nabla\cdot\mathbf{B} = 0$$

g) Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{R}$$

Summary: Fluid equations, MHD

- Multi-fluid theory
- Equation of state
- Single-fluid theory
- Generalised Ohm's law
- Magnetic tension and plasma beta
- Validity of magnetohydrodynamics

How to proceed?

- We will look at example solution of the MHD, which are relevant for space plasmas.
- First we compute static equilibria (no plasma-flow, no time dependence) => Magneto-statics
- Stationary solutions (no time dependence, but including stationary plasma flows)
- Sequences of equilibria (slow time dependence)
- Time dependent solutions, waves, instabilities and magnetic reconnection.

Exercises for Space Plasma Physics:
V. MHD

1. Remember the main steps we needed to derive the MHD-equations from a statistic particle model.

2. Under which conditions is the MHD-model valid?

3. Explain the different terms in the MHD-equations qualitatively.

4. Explain ideal and resistive MHD.

5. Are space plasmas typically ideal or resistive plasmas?

6. How can Ohm's law as used in MHD be derived from a two-fluid model?

7. In MHD one often compares different energy forms like the magnetic pressure $p_B = \frac{B^2}{2\mu_0}$ and the kinematic energy density $E_{\text{kin}} = \rho\frac{v^2}{2}$ with the plasma pressure p. How do p_B and E_{kin} evolve in time? Hint: derive the corresponding evolutionary equations from the MHD momentum-transport equation.

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Thomas Wiegelmann, 2012

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Magneto-Hydro-Statics (MHS)

- No time dependence and no plasma flows
- More precise: The dynamic terms in MHD are small compared with static forces (Lorentz-force, plasma pressure gradient, gravity)
- Sequences of equilibria (slow temporal changes) and equilibria with stationary plasma flow are studied in the next lecture.

Parts of this lecture are based on the course INTRODUCTION TO THE THEORY OF MHD EQUILIBRIA given by Thomas Neukirch in St. Andrews 1998.

Magneto-hydro-statics (MHS)

source: Neukirch 1998

a) Continuity equation

$$\xcancel{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0} \qquad \text{(mass conservation)}$$

b) Momentum conservation equation (equation of motion)

$$\xcancel{\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right)} = \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi$$

c) Energy equation (various different forms possible)

$$\xcancel{\rho^\gamma \frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \mathbf{v} \cdot \nabla\left(\frac{p}{\rho^\gamma}\right) = -(\gamma - 1)\mathcal{L}}$$

where

$$\mathcal{L} = \underbrace{\nabla \cdot \mathbf{q}}_{\text{heat flux}} + \overbrace{L_r}^{\text{radiative losses}} - \underbrace{\frac{\mathbf{j}^2}{\sigma}}_{\text{Ohmic heating}} - \overbrace{H.}^{\text{everything else}}$$

MHS-equations (Maxwell-part + Ohm‘s law)

d) Ampère's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

(displacement current neglected).

e) Faraday's law

$$\nabla \times \mathbf{E} = -\xcancel{\frac{\partial \mathbf{B}}{\partial t}}$$

f) no magnetic monopoles

$$\nabla \cdot \mathbf{B} = 0$$

g) Ohm's law

$$\mathbf{E} + \mathbf{v} \xcancel{\times \mathbf{B} = \mathbf{R}}$$

MHS

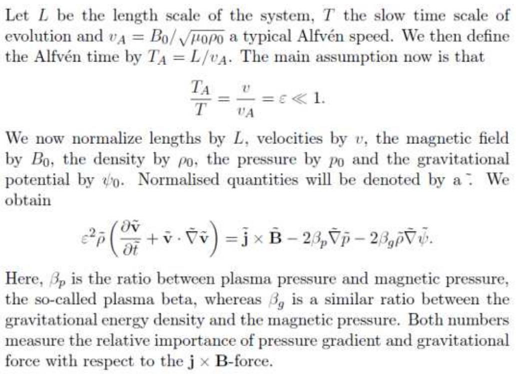
We finally end up with a set of three equations:

$$\begin{aligned} \mathbf{j} \times \mathbf{B} - \nabla p - \rho \nabla \psi &= 0 \\ \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} \\ \boxed{\nabla \cdot \mathbf{B} = 0} \end{aligned}$$

With the vector potential $\mathbf{B} = \nabla \times \mathbf{A}$ one equation (solenoidal condition) is already solved automatically.

Why is it useful to study MHS ?

- From a fundamental point of view we can regard the MHD equations as a set of equations describing extremely complicated dynamical systems. In the study of dynamical systems it is always useful to start with a study of the simplest solutions. These are usually the stationary states and their bifurcation properties, in the MHD case the static equilibria.
- From the point of view of modelling, many physical processes in plasma systems occur slowly, i.e. on time-scales which are much longer than the typical time-scale of the system.



Let L be the length scale of the system, T the slow time scale of evolution and $v_A = B_0/\sqrt{\mu_0 \rho_0}$ a typical Alfvén speed. We then define the Alfvén time by $T_A = L/v_A$. The main assumption now is that

$$\frac{T_A}{T} = \frac{v}{v_A} = \varepsilon \ll 1.$$

We now normalize lengths by L, velocities by v, the magnetic field by B_0, the density by ρ_0, the pressure by p_0 and the gravitational potential by ψ_0. Normalised quantities will be denoted by a $\tilde{}$. We obtain

$$\varepsilon^2 \tilde{\rho} \left(\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + \tilde{\mathbf{v}} \cdot \tilde{\nabla} \tilde{\mathbf{v}} \right) = \tilde{\mathbf{j}} \times \tilde{\mathbf{B}} - 2\beta_p \tilde{\nabla} \tilde{p} - 2\beta_g \tilde{\rho} \tilde{\nabla} \tilde{\psi}.$$

Here, β_p is the ratio between plasma pressure and magnetic pressure, the so-called plasma beta, whereas β_g is a similar ratio between the gravitational energy density and the magnetic pressure. Both numbers measure the relative importance of pressure gradient and gravitational force with respect to the $\mathbf{j} \times \mathbf{B}$-force.

Dimensionless Force-balance

Since ε is assume to be small, we obtain to lowest order

$$0 = \tilde{\mathbf{j}} \times \tilde{\mathbf{B}} - 2\beta_p \tilde{\nabla} \tilde{p} - 2\beta_g \tilde{\rho} \tilde{\nabla} \tilde{\psi}.$$

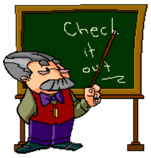
- The magnetostatic equations in dimensionless form can be used to evaluate the relative importance of the different terms.
- For a small β_g the gravity force can be neglected. This is usually fulfilled in magnetospheric plasmas.
- If β_p is small, too we can neglect also the plasma-pressure gradient in the force-balance
  => Force-free fields, a valid assumption in most parts of the solar corona.

- To lowest order we have the MHS force balance equation as fundamental equation and the time appears merely as a parameter.
- The fundamental importance of this quasi-static approximation lies in the fact that sequences of MHS equilibria can be used to model the slow evolution of plasma systems. (Growth phase of magnetospheric substorms, evolution of non-flaring solar active regions)
- Sequences have to satisfy the constraints imposed by the other equations, especially Ohm's law and the continuity equations.
- These constraints usually lead to very complicated integro-differential problems which are difficult to solve.

Systems with symmetries

- In the generic 3D case the MHS-equations are still difficult to solve due to their non-linearities.
- We consider now system with symmetries, e.g. in cartesian geometry (x,y,z) equilibria which are invariant in y. => 2D
- One can use also spherical, cylinder or helical symmetry, but the mathematics become a bit more complicated.
- We represent the magnetic field as:

$$\begin{aligned} \mathbf{B} &= \nabla A \times \mathbf{e}_y + \cancel{B_y \mathbf{e}_y} \\ &= \nabla \times (A\mathbf{e}_y) + \cancel{B_y \mathbf{e}_y} \end{aligned}$$



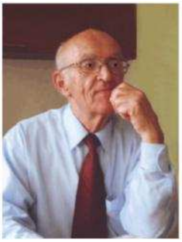
Grad-Shafranov-Equation

Harold Grad
1923-1986
Source: AIP.org

In 2D the MHS-equations (without gravity)

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) \times \vec{B} - \nabla P = 0$$
$$\nabla \cdot \vec{B} = 0$$

reduce to a single partial differential equation.



Vitalii Dmitrievich
Shafranov, born 1929
Source: Physics Uspekhi
53 (1) 101 (2010)

$$\boxed{-\Delta A = \mu_0 \frac{dp}{dA} + B_y \frac{dB_y}{dA}}$$

Grad-Shafranov-Equation

- We reduced the MHS to a single partial differential equations.
- The equation still contains the unknown pressure profile A -> p(A) and magnetic shear B_y
- Obviously p(A) has to be positive.
- For p(A)=A^2 the GSE becomes linear (popular choice)
- Within MHD p(A) is arbitrary, but it is possible to derive the pressure profile from kinetic theory.

Linear Grad-Shafranov-Equation

$\mu_0 j_y = k^2 A$

This form of j_y includes linear force-free fields. The Grad-Shafranov equation has the form

$$-\Delta A = k^2 A = \mu_0 \frac{d}{dA}\left(p(A) + \frac{B_y^2(A)}{2\mu_0}\right).$$

We get linear force-free fields if $p =$ constant, i.e.

$$-\Delta A = B_y \frac{dB_y}{dA} = k^2 A.$$

It follows that

$$B_y^2 = k^2 A^2 + B_{y0}^2.$$

Solutions can be obtained for example by separation of variables. In Cartesian coordinates we get

$$A = g(x)h(z)$$

leading to

$$-h\frac{d^2 g}{dx^2} - g\frac{d^2 h}{dz^2} = k^2\, g\, h$$

$$-\frac{1}{g}\frac{d^2 g}{dx^2} = k^2 + \frac{1}{h}\frac{d^2 h}{dz^2} = c^2$$

with c^2 constant. If we choose c^2 to be positive the solutions are

$$\begin{aligned} g &= g_1 \sin(cx) + g_2 \cos(cx) \\ h &= h_1 \exp(\sqrt{c^2 - k^2} z) + h_2 \exp(-\sqrt{c^2 - k^2} z) \end{aligned}$$

Here g_1, g_2, h_1 and h_2 are constants. In the case of linear force-free fields, we can replace k^2 by α^2. For all linear equations we can superpose different solutions to match boundary conditions for example. That is one of the reasons why the linear Grad-Shafranov equations are so popular.

Non-linear Grad-Shafranov-Equation

$\mu_0 j_y = \lambda \exp(2A)$

This is the first non-linear current density we investigate and is a very popular choice for two reasons:

(a) the complete set of solutions is known explicitly (Liouville, 1853) and the equation has particularly nice properties as conformal invariance;

(b) there is a physical justification for this current profile (!), because it results from a kinetic approach with Maxwellian distribution functions where the particles drift in the y-direction and the plasma is quasi-neutral. The plasma is then in local thermodynamic equilibrium. This argument does only apply, however, to a j_y caused by a pressure gradient and not to magnetic shear !

So the Grad-Shafranov equation is

$$-\Delta A = \lambda \exp(2A).$$

Kinetic theory for p(A)

- With invariance in y we have 2 constants of motion for the particles: Hamiltionian and momentum in y

$$\begin{aligned} H_s &= \frac{1}{2} m_s v^2 + q_s \phi \\ p_{ys} &= m_s v_y + q_s A \end{aligned}$$

- For a local thermic equilibrium the particle distribution function for each species is given by a drifting Maxwelian:

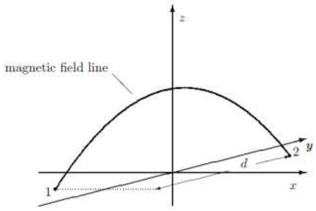
$$f_s(H_s, p_{ys}) \propto \exp\left(-\frac{H_s - u_s p_{ys}}{T_s}\right)$$

- Macroscopic (fluid) quantities like the particle density, current density and pressure for electrons and ions, we get by integration over the velocity space. The plasma pressure is isotropic per construction by a Maxwellian.
- We get the total pressure by adding the partial pressure of electron and ions.
- Assuming quasi-neutrality (n_i=n_e) allows to eliminate the electro static potential and we get:

$$p(A) = \lambda \exp\left(\frac{A}{\hat{A}}\right)$$

- We derived the GSE earlier from MHD, but it is valid also in kinetic theory and can be used to compute Vlasov-equilibria (e.g., intial states for kinetic simulations)

How to get B_y(A) ?



- The magnetic shear of a configuration like a coronal magnetic loop can in principle be computed from the displacement of footpoints.

Non-linear GSE, Liouville solution

$$-\Delta A = \lambda \exp(2A).$$

This equation is sometimes called Liouville's equation. The solutions to this equation are given by

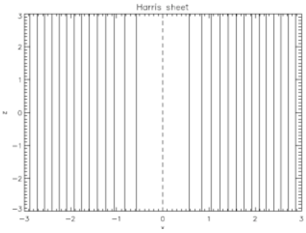
$$A = -\ln\left(\frac{1+\frac{1}{4}\lambda|\psi(u)|^2}{\left|\frac{d\psi}{du}\right|}\right)$$

where $u = x + iz$ and ψ is an analytic function, or written in a slightly different way

$$\lambda\exp(2A) = \frac{\lambda\left|\frac{d\psi}{du}\right|}{\left(1+\frac{1}{4}\lambda|\psi(u)|^2\right)^2}.$$

Non-linear GSE, 1D Harris (1962) Sheet

- We choose : $\psi = \frac{2}{\sqrt{\lambda}}\exp(\sqrt{\lambda}u)$
- And derive the solution: $A \;=\; -\ln\cosh\left(\sqrt{\lambda}x\right)$



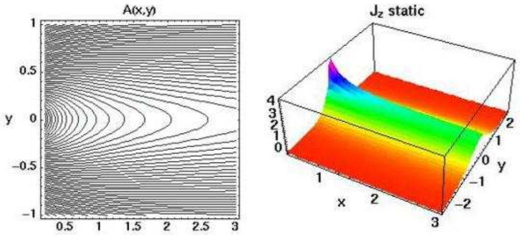
Used to describe current sheets in space plasmas. Valid also in kinetic approach.

Non-linear GSE, Magnetosphere solution

(Schindler & Birn 2004)

$$\Psi(u) = 2\hat{l}\exp\left(i\left(u/\hat{l} + \sqrt{\frac{u/\hat{l}}{\epsilon}}\right)\right) \quad A(x,y)/\hat{A} = \ln\left(\frac{\cosh\left(\frac{y}{\sqrt{2\epsilon}\sqrt{r+x}}+y\right)}{\sqrt{\frac{1}{r}\left(\frac{1}{4\epsilon}+\sqrt{\frac{r+x}{2\epsilon}}\right)+1}}\right)$$

$$r = \sqrt{x^2+y^2},$$



Source of pictures: Nickeler et al. 2010

Including gravity

- External gravity force: $\mathbf{f} = -\rho\nabla\Psi$
- Force-balance: $\mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\Psi = 0$
- Plasma pressure is not constant along field lines
  $$\mathbf{B}\cdot\nabla p = -\rho\mathbf{B}\cdot\nabla\Psi$$
- Pressure gradient needs to be compensated by gravity force, as the Lorentz-force vanishes parallel to the magnetic field

Including gravity

- We use again: $\mathbf{B} = \nabla A \times \mathbf{e}_y + B_y\mathbf{e}_y$
- Similar as for the Grad-Shafranov (GS) equation we get
  $$\mathbf{j}\times\mathbf{B} = \frac{1}{\mu_0}\left(-\Delta A - B_y\frac{dB_y}{dA}\right)\nabla A$$
- And the force balance:
  $$-\frac{1}{\mu_0}\left(\Delta A + B_y\frac{dB_y}{dA}\right)\nabla A - \nabla p - \rho\nabla\Psi$$

We now use the following argument. Each of the three vector fields ∇A, ∇p and $\nabla\Psi$ has only two components, namely in the x-z-plane. It follows that only two of these vector fields can be linearly independent.

Including gravity

We now assume that ∇A and $\nabla\Psi$ are linearly independent

$$\nabla p = p_A\,\nabla A + p_\Psi\,\nabla\Psi$$

with functions p_A and p_ψ as coefficients. Since

$$\nabla\times\nabla p = \mathbf{0}$$

we get

$$\nabla p_A \times \nabla A + \nabla p_\Psi \times \nabla\Psi = \mathbf{0}$$

This equation is fulfilled for:

$$p = F(A,\Psi),\; p_A = \partial F/\partial A$$
$$p_\Psi = \partial F/\partial\Psi$$

Including gravity

- Inserting this into the last equation:

$$\left(\frac{\partial^2 F}{\partial A^2}\nabla A + \frac{\partial^2 F}{\partial\psi\partial A}\nabla\Psi\right)\times\nabla A + \left(\frac{\partial^2 F}{\partial A\partial\Psi}\nabla A + \frac{\partial^2 F}{\partial A^2}\nabla\Psi\right)\times\nabla\Psi \;=\; \frac{\partial^2 F}{\partial A\partial\Psi}\left(\nabla\Psi\times\nabla A + \nabla A\times\nabla\Psi\right) = \mathbf{0}.$$

- And the force balance equation becomes:

$$\left[-\frac{1}{\mu_0}\left(\Delta A + B_y\frac{dB_y}{dA}\right) - \frac{\partial p}{\partial A}\right]\nabla A - \left(\frac{\partial p}{\partial\Psi} + \rho\right)\nabla\Psi = 0$$

- The coefficients must vanish because ∇A and $\nabla\Psi$ are linear independent.

Including gravity

- Finally we have to solve:

$$\begin{aligned} -\Delta A &= \mu_0\frac{\partial p}{\partial A} + B_y\frac{dB_y}{dA} \\ \frac{\partial p}{\partial\Psi} &= -\rho. \end{aligned}$$

- To proceed we make now assumptions about the equation of state, e.g. isothermal plasma (simplest approach and the only one we consider here explicitely)
- A polytropic equation of state is also possible.
- Most realistic (and most complicated) would be to solve the energy equation.

Including gravity, isothermal plasma

Ideal gas with $p = R\rho T$ with $T = T(A, \Psi)$ given. It follows that

$$\frac{\partial p}{\partial\psi} = -\frac{p}{RT}$$

and integrating once we get

$$p = p_0(A)\exp\left(-\int_{\Psi_0}^{\Psi}\frac{d\Psi'}{RT(A,\Psi')}\right).$$

In the special case of an isothermal ideal gas this can be written as

$$p = p_0(A)\exp\left(-\frac{\Psi-\Psi_0}{RT}\right).$$

This is the usual barometric formula with different base pressure $p_0(A)$ for each field line !

3D configurations

- Analytic we can find 3D-equilibria without symmetry only in special cases.
- We can use Euler potentials:

$$\mathbf{B} = \nabla\alpha\times\nabla\beta = \nabla\times(\alpha\nabla\beta)$$

- And transform the force balance to:

$$\begin{aligned} \mathbf{j}\times\mathbf{B} - \nabla p - \rho\nabla\Psi &= \mathbf{j}\times(\nabla\alpha\times\nabla\beta) - \nabla p - \rho\nabla\Psi \\ &= (\mathbf{j}\cdot\nabla\beta)\nabla\alpha - (\mathbf{j}\cdot\nabla\alpha)\nabla\beta - \nabla p - \rho\nabla\Psi \\ &= \mathbf{0}. \end{aligned}$$

3D configurations

As in the symmetric case, $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$ represent three linearly independent vector fields and we can split the force balance equation into three components along $\nabla\alpha$, $\nabla\beta$ and $\nabla\Psi$:

$$\begin{aligned} \nabla\beta\cdot\nabla\times(\nabla\alpha\times\nabla\beta) - \left(\frac{\partial p}{\partial\alpha}\right)_{\beta,\Psi} &= 0 \\ -\nabla\alpha\cdot\nabla\times(\nabla\alpha\times\nabla\beta) - \left(\frac{\partial p}{\partial\beta}\right)_{\alpha,\Psi} &= 0 \\ -\left(\frac{\partial p}{\partial\Psi}\right)_{\alpha,\beta} &= \rho \end{aligned}$$

Instead of a single partial differential equation (Grad-Shafranov) we get a system of coupled nonlinear equations => Extremely difficult to solve

Special cases in 3D: Force-free magnetic fields

A special equilibrium of ideal MHD (often used in case of the solar corona) occurs if the beta is low, such that the pressure gradient can be neglected. The stationary plasma becomes ***force free***, if the Lorentz force vanishes:

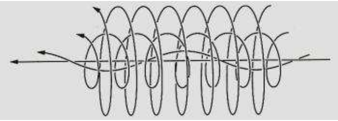
$$\boxed{\mathbf{j}\times\mathbf{B} = 0}$$

This condition is guaranteed if the current flows along the field:

$$\boxed{\mu_0\mathbf{j} = \alpha_L\mathbf{B}} \qquad \boxed{\nabla\times\mathbf{B} = \alpha_L\mathbf{B}}$$

By taking the divergence, one finds that [redacted]x) is constant along any field line:

$$\boxed{\mathbf{B}\cdot\nabla\alpha_L = 0}$$



Magneto-Hydro-Statics (MHS)

- Equilibrium structures (no time dependence, no plasma flow) are important and often approximately a reasonable assumption for space plasmas during quiet times.
- For symmetric configurations, MHS reduces to the Grad-Shafranov equation (GSE), a single (nonlinear) partial differential equation.
- GSE remains valid in kinetic theory.
- 3D MHS-equilibria have usually be computed numerically.

How to proceed?

- Study stationary states with plasma flow.
- The slow evolution of sequences of equilibria is often constraint, e.g., by the assumption of ideal MHD, which does not allow topology changes.
- Plasma waves are ubiquitos in space plasma.
- Discontinuities and current sheets.
- Plasma instabilities cause rapid changes and the equilibrium is lost (ideal and resistive instabilities).

Exercises for Space Plasma Physics:
VI. Magneto-statics

1. Under which conditions can a plasma described with magneto-hydrostatics (MHS)?
2. Which terms in MHD are neglected to derive the MHS-equations?
3. Give examples for solar system plasmas suitable for MHS models.
4. What is a force-free model? When can it be applied?
5. Can you give an example for a force-free plasmas?
6. Remember the main steps from a 2D-MHS-models to the Grad-Shafranov equation.
7. Now we extend the Grad-Shafranov theory to 2.5D, which means that all quantities are still invariant in the y-direction, but the magnetic field vector has three components. Use the ansatz

   $$\vec{B} = \nabla A \times e_y + B_y e_y$$

   and derive the corresponding Grad-Shafranov equation.
8. How does the Grad-Shafranov-equation (2.5D) look for a force-free plasma?

Space Plasma Physics

Thomas Wiegelmann, 2012

1. Basic Plasma Physics concepts
2. Overview about solar system plasmas

Plasma Models

3. Single particle motion, Test particle model
4. Statistic description of plasma, BBGKY-Hierarchy and kinetic equations
5. Fluid models, Magneto-Hydro-Dynamics
6. Magneto-Hydro-Statics
7. Stationary MHD and Sequences of Equilibria

Stationary MHD and Sequences of Equilibria

- Slowly varying sequences of equilibria.
- Topology conserving sequences of equilibria and formation of thin current sheets.
- Comparison: Stationary incompressible Hydro-dynamics and Magneto-hydro-statics
- Transformation from static MHD-equilibria to equilibria with plasma flow.

Sequences of Equilibria

- So far we studied static and stationary states as independent equilibria, which do not depend on time.
- Equilibria do, however, often depend on boundary conditions (like magnetic field in solar photosphere for coronal modelling or solar wind pressure for magnetotail models) which vary slowly in time.
- => We get a time-sequence of equilibria.
- We do not, however, understand how the transition between different equilibria takes place physically.
- For some cases (e.g. magnetospheric convection) we know that the plasma is ideal (no topology changes) in quiet times.

Sequences of Equilibria

- In the equilibrium theory developed so far, the different equilibria are not constraint.
- => For different boundary conditions we get different magneto-static equilibria, which might have a different magnetic topology.
- Such a sequence of equilibria CANNOT be considered as a physical meaningful slow evolution within ideal MHD.
- Can we constrain a sequence of static equilibria in a way that the ideal MHD-equation are obeyed?

Ideal MHD

Source: Schindler & Birn, JGR 1982

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

Continuity Eq.

$$\cancel{\rho \frac{\partial \underline{v}}{\partial t}} + \cancel{\rho \underline{v} \cdot \nabla \underline{v}} = -\nabla p + \underline{j} \times \underline{B}$$

Force-balance, Momentum equation

$$\underline{E} + \underline{v} \times \underline{B} = 0$$

Ideal Ohms Law

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^{\gamma}}\right) + \underline{v} \cdot \nabla\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Equation of state, Adiabatic convection

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{j}$$

$$\nabla \cdot \underline{B} = 0$$

We often can neglect the terms on the left side of the momentum equation (when they small compared to pressure gradient and Lorentz-force) We must still solve, however continuity equation, Ohms law and Eq. of state.

Sequences of Equilibria

- For special configurations (liked magnetospheric convection) it is possible to reformulate the ideal MHD equations in order to compute sequence of static equilibria under constrains of field line conservation.
- A principle way is:
  -compute an initial static equilibria
  -solve the time dependent ideal MHD-equations numerically and change the boundary conditions slowly in time.
  -If a nearby equilibrium exists, the MHD-code will very likely find it.
- The code will also find out if the configuration becomes unstable. One has to take care about algorithm to avoid artificial numerical diffussion.

Stationary incompressible MHD

$$\nabla \cdot (\rho \vec{\mathrm{v}}) = 0$$
$$\rho\,(\vec{\mathrm{v}} \cdot \nabla)\,\vec{\mathrm{v}} = \vec{j} \times \vec{B} - \nabla P$$
$$\nabla \times \left(\vec{\mathrm{v}} \times \vec{B}\right) = \vec{0}$$
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{\mathrm{v}} = 0$$

Subsets are
1.) No plasma flow: Magneto-Hydro-Statics (MHS)
2.) No magnetic field: Stationary Incom-pressibel Hydro-Dynamics (SIHD)

SIHD

$$\rho\,(\vec{\mathrm{v}} \cdot \nabla)\,\vec{\mathrm{v}} = -\nabla P$$
$$\nabla \cdot \vec{\mathrm{v}} = 0$$

vector identity:

$$(\vec{\mathrm{v}} \cdot \nabla)\,\vec{\mathrm{v}} = \nabla\left(\frac{1}{2}\mathrm{v}^2\right) - \vec{\mathrm{v}} \times (\nabla \times \vec{\mathrm{v}})$$

$$-\rho\vec{\mathrm{v}} \times (\nabla \times \vec{\mathrm{v}}) + \nabla\left(\frac{\rho}{2}\mathrm{v}^2\right) = -\nabla P$$
$$\nabla \cdot \vec{\mathrm{v}} = 0$$

SIHD

$$-\rho\,\underbrace{(\nabla \times \vec{\mathrm{v}})}_{\text{vorticity}} \times \vec{\mathrm{v}} - \nabla\left(\frac{\rho}{2}\mathrm{v}^2 + \mathrm{P}\right) = 0$$
$$\nabla \cdot \vec{\mathrm{v}} = 0$$

Magneto-hydro-statics

$$\frac{1}{\mu_0}\underbrace{(\nabla \times \vec{B})}_{\text{current density}} \times \vec{B} - \nabla P = 0$$
$$\nabla \cdot \vec{B} = 0$$

SIHD and MHS have same mathematical structure (Gebhardt&Kiessling 92):

$$\boxed{\begin{aligned} (\nabla \times \vec{c}) \times \vec{c} &= \nabla\Pi \\ \nabla \cdot \vec{c} &= 0 \end{aligned}}$$

MHS	SIHD
$\vec{c} = \vec{B}$	$\vec{c} = \sqrt{\rho}\,\vec{\mathrm{v}}$
$\Pi = \mu_0\, p$	$\Pi = -\frac{\rho}{2}\mathrm{v}^2 - p$

Stationary incompressible MHD

$$\nabla \cdot (\rho \vec{\mathrm{v}}) = 0$$
$$\rho\,(\vec{\mathrm{v}} \cdot \nabla)\,\vec{\mathrm{v}} = \vec{j} \times \vec{B} - \nabla P$$
$$\nabla \times \left(\cancel{\vec{\mathrm{v}} \times \vec{B}}\right) = \vec{0}$$
$$\nabla \times \vec{B} = \mu_0 \vec{j}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \cdot \vec{\mathrm{v}} = 0$$

If plasma flow parallel to magnetic field

Stationary incompressible MHD

- Mathematical structure of hydro-dynamics and magneto-static is similar.
- We assume, that we found solution of a magneto-static equilibrium (Grad-Shafranov Eq. in 2D = > Flux-function or Euler-potentials in 3D).
- We use the similar mathematical structure to find transformation equations (different flux-function) to solve stationary MHD.
- We introduce the Alfven velocity $v_A = B/\sqrt{\mu_0\,\rho}$ and the Alfven Machnumber $M_A = v/v_A$
- We limit to sub-Alfvenic flows here. Pure super-Alfvenic flows can be studied similar. Somewhat tricky are trans-Alfvenic flows.

Stationary incompressible MHD

- With this approach we can eliminate the plasma velocity in the SMHD equations and get:

$\vec{B} \cdot \vec{\nabla} M_A = 0$ => Alfven Machnumber is constant on field lines

$$\vec{\nabla}\Pi = \frac{(1-M_A^2)\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}}{\mu_0} - \frac{|\vec{B}|^2}{2\mu_0}\vec{\nabla}\left(1-M_A^2\right)$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

- Obviously the equations reduce to magnetostatics for M_A=0. The second equation changes it's sign from sub- to super-Alfvenic flows.

Stationary incompressible MHD in 2D

- Similar as in the magneto-static case we represent the magnetic field with a flux-function: $\alpha(x,z)$

$$\vec{B} = \vec{\nabla}\alpha \times \vec{e}_y$$

- Flux functions are not unique and we can transform to another flux function $A(x,z)$
- The Alfven mach number is constant on field lines:

$$M_A = M_A(\alpha) = M_A(A)$$

- We can now eliminate terms containing the Alfven Mach number by choosing:

$$(1-M_A^2)\left(\frac{\partial\alpha}{\partial A}\right)^2 \equiv 1$$

Grad-Shafranov-Equation with flow

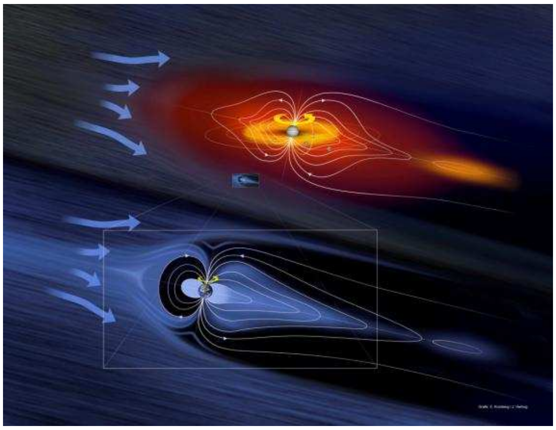
- The stationary incompressible MHD-equations reduce to a Grad-Shafranov equation

$$-\frac{1}{\mu_0}\Delta A = \frac{\partial\Pi}{\partial A}$$

- Any solution we found for the static case can be used to find a solution for equilibria with flow by:

$$\alpha = \pm\int\frac{dA}{\sqrt{1-M_A(A)^2}}$$

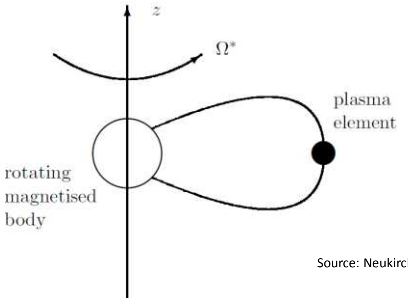
- Transformation can become complicated, however.



Jovian Magnetosphere

- Jupiter: fast rotation 10 h, mass-loading 1000 kg/s
- Dynamics driven largely by internal sources.
- Planetary rotation coupled with internal plasma loading from the moon Io may lead to additional currents, departure from equilibrium, magnetospheric instabilities and substorm-like processes.
- Stationary states of a fast rotating magnetosphere cannot be modeled with a magneto-static model.
- We have to incorporate the rotation => Equilibria with centrifugal force.

Planetary magnetospheres



Source: Neukirch 1998

We use a cylinder geometry and derive the corresponding Grad-Shafranov Equation.

Grad-Shafranov Eq. in cylinder-geometry

This section is based on a lecture by Neukirch 1998

- Rotational invariance without additional forces

$$\begin{aligned}\mathbf{B} &= \nabla A \times \nabla\phi + B_\phi \mathbf{e}_\phi \\ &= \frac{1}{\varpi}\nabla A \times \mathbf{e}_\phi + B_\phi \mathbf{e}_\phi \\ &= -\frac{1}{\varpi}\frac{\partial A}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial A}{\partial \varpi}\mathbf{e}_z + B_\phi \mathbf{e}_\phi\end{aligned}$$

Here A is *not* the ϕ-component of the vector potential, but A/ϖ is !

$$\mathbf{B}\cdot\nabla p = 0$$

$$p(\varpi, z) = F(A(\varpi, z)).$$

- The current density becomes:

$$\nabla\times\mathbf{B} = -\frac{\partial B_\phi}{\partial z}\mathbf{e}_\varpi - \left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}\right)\right]\mathbf{e}_\phi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi B_\phi\right)\mathbf{e}_z$$

- An for the Lorentz-force we get:

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= \mathbf{j}\times\left(\frac{1}{\varpi}\nabla A\times\mathbf{e}_\phi + B_\phi\mathbf{e}_\phi\right) \\ &= \frac{1}{\varpi}j_\phi\nabla A - \frac{1}{\varpi}\left(\mathbf{j}\cdot\nabla A\right)\mathbf{e}_\phi + B_\phi\mathbf{j}\times\mathbf{e}_\phi \\ &= -\frac{1}{\mu_0\varpi}\left\{\left[\frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2} + \frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}\right)\right]\nabla A - \right. \\ &\qquad \left[\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi B_\phi\right)\frac{\partial A}{\partial z} - \frac{\partial B_\phi}{\partial z}\frac{\partial A}{\partial\varpi}\right]\mathbf{e}_\phi + \\ &\qquad \left. B_\phi\nabla\left(\varpi B_\phi\right)\right\} \\ &= \frac{dp}{dA}\nabla A.\end{aligned}$$

Looking at the ϕ-component of this equation we see that

$$\begin{aligned}\frac{1}{\varpi}\left[\frac{\partial}{\partial\varpi}\left(\varpi B_\phi\right)\frac{\partial A}{\partial z} - \frac{\partial}{\partial z}\left(\varpi B_\phi\right)\frac{\partial A}{\partial\varpi}\right] &= \frac{1}{\varpi}\mathbf{B}\cdot\nabla\left(\varpi B_\phi\right) \\ \text{As the rotational gradient vanishes} \qquad &= 0.\end{aligned}$$

It follows that

$$b_\phi(\varpi, z) = \varpi B_\phi(\varpi, z) = G(A(\varpi, z)).$$

This allows us to write the rest of the equations as

$$-\frac{1}{\mu_0\varpi}\left[\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial A}{\partial\varpi}\right) + \frac{1}{\varpi}\frac{\partial^2 A}{\partial z^2}\right]\nabla A - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA}\nabla A = \frac{dp}{dA}\nabla A$$

- For non-vanishing gradient of A we get the Grad-Shafranov-Eq. in cylinder geometry:

$$\boxed{-\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) = \mu_0\frac{dp}{dA} + \frac{1}{\varpi^2}b_\phi\frac{db_\phi}{dA}}$$

Including the centrifugal force

- We now include a strictly rotational plasma flow

$$\mathbf{v} = \varpi\Omega\mathbf{e}_\phi$$

a) Continuity equation

$$\nabla\cdot(\rho\mathbf{v}) = \frac{1}{\varpi}\frac{\partial}{\partial\phi}(\varpi\rho\Omega) = 0$$

because of axisymmetry.

b) Ohm's law and Faraday's law

From Faraday's law

$$\nabla\times\mathbf{E} = \mathbf{0}$$

we conclude that

$$\mathbf{E} = -\nabla\Phi.$$

With

$$\mathbf{B} = \frac{1}{\varpi}\nabla A\times\mathbf{e}_\phi + B_\phi\mathbf{e}_\phi$$

(from $\nabla\cdot\mathbf{B} = 0$) we find that

$$\begin{aligned}\mathbf{v}\times\mathbf{B} &= \varpi\Omega\mathbf{e}_\phi\times\left(\frac{1}{\varpi}\nabla A\times\mathbf{e}_\phi + B_\phi\mathbf{e}_\phi\right) \\ &= \Omega\nabla A\end{aligned}$$

so that the ideal Ohm's law

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mathbf{0}$$

acquires the form

$$-\nabla\Phi + \Omega\nabla A = \mathbf{0}.$$

Taking the curl of this equation results in

$$\nabla\Omega\times\nabla A = \mathbf{0}$$

leading to the conclusion that

$$\Omega = H(A)$$

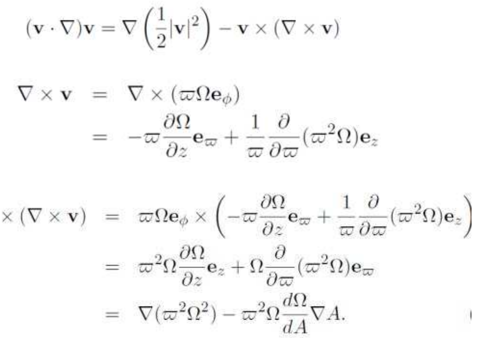
Ferraro's law of isorotation: The angular velocity is constant on magnetic field lines

Since Ω is a function of A, we also get

$$-\nabla\Phi + \Omega(A)\nabla A = \mathbf{0}$$

and find that the electric potential is a function of A as well.

We write the velocity more convenient



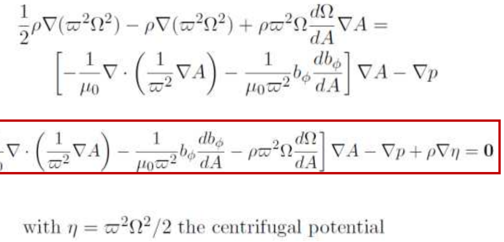
$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) - \mathbf{v}\times(\nabla\times\mathbf{v})$$

$$\begin{aligned}
\nabla\times\mathbf{v} &= \nabla\times(\varpi\Omega\mathbf{e}_\phi)\\
&= -\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}\times(\nabla\times\mathbf{v}) &= \varpi\Omega\mathbf{e}_\phi\times\left(-\varpi\frac{\partial\Omega}{\partial z}\mathbf{e}_\varpi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_z\right)\\
&= \varpi^2\Omega\frac{\partial\Omega}{\partial z}\mathbf{e}_z + \Omega\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\mathbf{e}_\varpi\\
&= \nabla(\varpi^2\Omega^2) - \varpi^2\Omega\frac{d\Omega}{dA}\nabla A.
\end{aligned}$$

Momentum Balance Equation

$$\boxed{\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{j}\times\mathbf{B} - \nabla p}$$

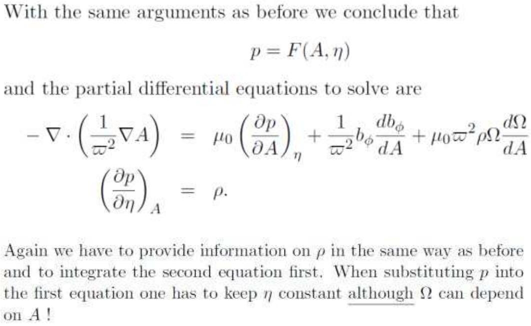


$$\begin{aligned}
&\frac{1}{2}\rho\nabla(\varpi^2\Omega^2) - \rho\nabla(\varpi^2\Omega^2) + \rho\varpi^2\Omega\frac{d\Omega}{dA}\nabla A =\\
&\qquad\left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA}\right]\nabla A - \nabla p
\end{aligned}$$

$$\boxed{\left[-\frac{1}{\mu_0}\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) - \frac{1}{\mu_0\varpi^2}b_\phi\frac{db_\phi}{dA} - \rho\varpi^2\Omega\frac{d\Omega}{dA}\right]\nabla A - \nabla p + \rho\nabla\eta = \mathbf{0}}$$

with $\eta = \varpi^2\Omega^2/2$ the centrifugal potential

Grad-Shafranov-eq. for rotating systems



With the same arguments as before we conclude that

$$p = F(A,\eta)$$

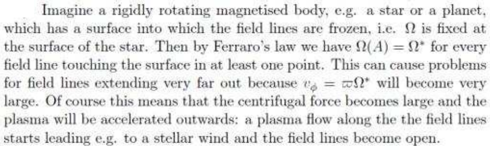
and the partial differential equations to solve are

$$\begin{aligned}
-\nabla\cdot\left(\frac{1}{\varpi^2}\nabla A\right) &= \mu_0\left(\frac{\partial p}{\partial A}\right)_\eta + \frac{1}{\varpi^2}b_\phi\frac{db_\phi}{dA} + \mu_0\varpi^2\rho\Omega\frac{d\Omega}{dA}\\
\left(\frac{\partial p}{\partial\eta}\right)_A &= \rho.
\end{aligned}$$

Again we have to provide information on ρ in the same way as before and to integrate the second equation first. When substituting p into the first equation one has to keep η constant although Ω can depend on A !

Grad-Shafranov-eq. for rotating systems

- Ferraro's law of isorotation restricts the angular velocity.



Imagine a rigidly rotating magnetised body, e.g. a star or a planet, which has a surface into which the field lines are frozen, i.e. Ω is fixed at the surface of the star. Then by Ferraro's law we have $\Omega(A) = \Omega^*$ for every field line touching the surface in at least one point. This can cause problems for field lines extending very far out because $v_\phi = \varpi\Omega^*$ will become very large. Of course this means that the centrifugal force becomes large and the plasma will be accelerated outwards: a plasma flow along the the field lines starts leading e.g. to a stellar wind and the field lines become open.

Sequences of equilibria

- One should not naively consider every sequences of static equilibria as a physical reasonable temporal evolution.
- Magnetostatic means, that velocity and time-dependence are small (and can be neglected) in the momentum transport.
- We still have to solve continuity equation, ideal Ohm’s law and an equation of state to obtain a physical meaningful time-sequence of equilibria.
- This can become involved.

Stationary MHD

- Magnetostatics and stationary Hydrodynamics are mathematical similar, also the terms have different physical meaning.
- We can use this property to transform solutions of MHS to stationary MHD for incompressible field line parallel plasma flows.
- Rotating systems are restricted by Ferraros law of isoration and we have to solve two coupled differential equations.

How to proceed?

- Study time dependent system:
- Plasma Waves
- Instabilities
- Discontinuities
- Waves and instabilities occur in MHD as well as in a kinetic model.
- In discontinuities the Fluid approach often breaks down and one has to apply kinetic models.

Exercises for Space Plasma Physics: VII. Stationary MHD

1. Discuss mathematical similarities and differences of MHS and stationary, incompressible hydro-dynamics.

2. Is this similarity useful to study stationary MHD? Under which conditions?

3. What are quasi-static equilibria? Is this a useful concept at all?

4. Euler potentials are defined as $\vec{B} = \nabla\alpha \times \nabla\beta$. Discuss briefly advantages and disadvantages of using Euler potentials.

5. For simplicity, one often investigates configuration in 2D, which are invariant in one spatial coordinate (say in y) and uses a flux function $A(x, z)$ with $\vec{B} = \nabla A \times \vec{e}_y$. Can you provide the vector potential and the Euler potentials for a given flux-function?

6. Now let's consider the full 3D case. Assume you have α and β given. Can you derive a coresponding vector potential $\vec{A}$?

Space Plasma Physics

Thomas Wiegelmann, 2012

Physical Processes

8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas

Electromagnetic wave in a plasma

Electromagnetic wave $\exp(i\omega t - \gamma x)$

With $\gamma = i\frac{\omega}{c}\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}$ $\qquad w_{pe} = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}} \propto \sqrt{n_e}$

γ becomes real for waves with a frequency $\omega < \omega_{pe}$

Electromagnetic waves with a frequency below the plasma frequency cannot travel into the plasma. (they become reflected) => Cutoff-frequeny

Exercise: How can this property be used to measure the (electron) density of a plasma with EM-waves?

Ideal MHD equations

Plasma equilibria can easily be perturbed and small-amplitude waves and fluctuations can be excited.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0$$

$$\frac{\partial (nm\mathbf{v})}{\partial t} + \nabla \cdot (nm\mathbf{v}\mathbf{v}) = -\nabla \cdot \left(\mathrm{P} + \frac{B^2}{2\mu_0}\mathrm{I}\right) + \frac{1}{\mu_0}\nabla \cdot (\mathbf{B}\mathbf{B})$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

$$\nabla \cdot \mathbf{B} = 0$$

Energy equation omitted, because not needed here.

MHD equilibrium and fluctuations

We assume **stationary** ideal **homogeneous** conditions as the intial state of the single-fluid plasma, with vanishing average electric and velocity fields, overal **pressure equilibrium** and no magnetic stresses. These assumptions yield:

$$\begin{aligned} \mathbf{v}_0 &= 0 \\ \mathbf{E}_0 &= 0 \\ \nabla\left(p_0 + B_0^2/2\mu_0\right) &= 0 \\ (\mathbf{B}_0 \cdot \nabla)\,\mathbf{B}_0 &= 0 \end{aligned}$$

These fields are decomposed as sums of their background initial values and space- and time-dependent **fluctuations** as follows:

$$\begin{aligned} n &= n_0 + \delta n \\ \mathbf{v} &= \delta\mathbf{v} \\ \mathbf{E} &= \delta\mathbf{E} \\ \mathbf{B} &= \mathbf{B}_0 + \delta\mathbf{B} \end{aligned}$$

Linear perturbation theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.
-> Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.
-> Then all variables can be expressed by one, say the magnetic field.

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys: $|\delta\mathbf{B}| \ll B_0$

In the uniform plasma with straight field lines, the field provides the only **symmetry axis** which may be chosen as z-axis of the coordinate system such that: **$\mathbf{B}_0$=$\mathbf{B}_0$ê**[redacted]

Linearized MHD equations

Linarization of the MHD equations leads to three equations for the three fluctuations, [redacted], [redacted], and [redacted]:

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta\mathbf{v} = 0$$

$$m_i n_0 \frac{\partial \delta\mathbf{v}}{\partial t} = -\nabla\left(\delta p + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \delta\mathbf{B}\right) + \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \nabla)\,\delta\mathbf{B}$$

$$\frac{\partial \delta\mathbf{B}}{\partial t} = (\mathbf{B}_0 \cdot \nabla)\,\delta\mathbf{B} - \mathbf{B}_0\,(\nabla \cdot \delta\mathbf{v})$$

Using the adiabatic pressure law, and the derived sound speed, $c_s^2 = p_0/(m_i n_0)$, leads to an equation for [redacted] and gives:

$$\frac{\partial \delta p}{\partial t} = m_i c_s^2 \frac{\partial \delta n}{\partial t} = -m_i n_0 c_s^2 \nabla \cdot \delta\mathbf{v}$$

Linearized MHD equations II

Inserting the continuity and pressure equations, and using the Alfvén velocity, $\mathbf{v}_A$=$\mathbf{B}_0$/([redacted]m$_i$)$^{1/2}$, two coupled vector equations result:

$$\frac{\partial \delta \mathbf{v}}{\partial t} = v_A^2 \nabla_{\|} \left(\frac{\delta \mathbf{B}_{\perp}}{B_0} \right) - \nabla \left(\frac{\delta p}{m_i n_0} \right)$$

$$\frac{\partial}{\partial t} \left(\frac{\delta \mathbf{B}}{B_0} \right) = \nabla_{\|} \delta \mathbf{v}_{\perp} - \hat{\mathbf{e}}_{\|} \left(\nabla_{\perp} \cdot \delta \mathbf{v}_{\perp} \right)$$

Time differentiation of the first and insertion of the second equation yields a second-order wave equation which can be solved by **Fourier transformation**.

$$\begin{aligned} \frac{\partial^2 \delta \mathbf{v}}{\partial t^2} &= c_{ms}^2 \nabla \left(\nabla \cdot \delta \mathbf{v} \right) \\ &+ v_A^2 \left(\nabla_{\|}^2 \delta \mathbf{v} - \nabla \nabla_{\|} \delta v_{\|} - \hat{\mathbf{e}}_{\|} \nabla_{\|} \nabla \cdot \delta \mathbf{v} \right) \end{aligned}$$

Dispersion relation

The ansatz of **travelling plane waves**,

$$\delta \mathbf{v} = \delta \mathbf{v}_0 \exp(i \mathbf{k} \cdot \mathbf{x} - i \omega t)$$

with arbitrary constant amplitude, [redacted]$_0$, leads to the

$$\left[\left(\omega^2 - k_{\|}^2 v_A^2 \right) \mathbf{I} - c_{ms}^2 \mathbf{k}\mathbf{k} + \left(\mathbf{k} \hat{\mathbf{e}}_{\|} + \hat{\mathbf{e}}_{\|} \mathbf{k} \right) k_{\|} v_A^2 \right] \cdot \delta \mathbf{v}_0 = 0$$

To obtain a nontrivial solution the determinant must vanish, which means

$$\begin{bmatrix} \omega^2 - v_A^2 k_{\|}^2 - c_{ms}^2 k_{\perp}^2 & 0 & -c_s^2 k_{\|} k_{\perp} \\ 0 & \omega^2 - v_A^2 k_{\|}^2 & 0 \\ -c_s^2 k_{\|} k_{\perp} & 0 & \omega^2 - c_s^2 k_{\|}^2 \end{bmatrix} \begin{bmatrix} \delta v_{0x} \\ \delta v_{0y} \\ \delta v_{0\|} \end{bmatrix} = 0$$

Here the **magnetosonic speed** is given by $c_{ms}^2 = c_s^2 + v_A^2$. The wave vector component perpendicular to the field is oriented along the x-axis, k= k[redacted]$_z$+ k[redacted]$_x$.

Alfvén waves

Inspection of the determinant shows that the fluctuation in the y-direction decouples from the other two components and has the linear dispersion

$$\omega_A = \pm k_{\|} v_A$$

This **transverse wave** travels parallel to the field. It is called **shear Alfvén wave**. It has no density fluctuation and a constant group velocity, $\mathbf{v}_{gr,A}$ = $\mathbf{v}_A$, which is always oriented along the background field, along which the wave energy is transported.

The transverse velocity and **magnetic field components are (anti)-correlated** according to: [redacted]$_y$/v$_A$ = ± [redacted]$_y$/B$_0$, for **parallel (anti-parallel) wave propagation**. The wave electric field points in the x-direction: [redacted]$_x$ = [redacted]$_y$/v$_A$

Magnetosonic waves

The remaing four matrix elements couple the fluctuation components, [redacted] and [redacted] The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_{\|}^2 = 0$$

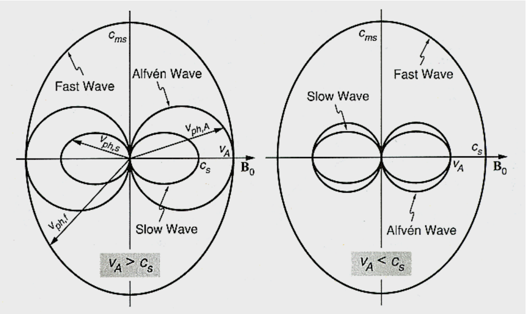
This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2} \left\{ c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2 \right)^2 + 4 v_A^2 c_s^2 \frac{k_{\perp}^2}{k^2} \right]^{1/2} \right\}$$

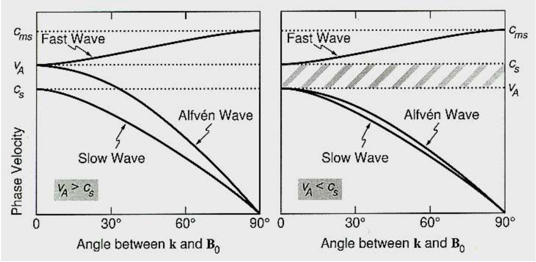
which are the phase velocities of the compressive **fast and slow magnetosonic waves**. They depend on the propagation angle [redacted] with k[redacted]/k^2 = sin^2[redacted] For [redacted] 90^0 we have: [redacted] kc$_{ms}$, and [redacted] 0^0:

$$\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \left(c_s^2 - v_A^2 \right) \right]$$

Phase-velocity polar diagram of MHD waves



Dependence of phase velocity on propagation angle



Magnetosonic wave dynamics

In order to understand what happens physically with the dynamic variables, [redacted]$_x$, [redacted]$_x$, [redacted], [redacted], [redacted], and [redacted], inspect again the equation of motion written in components:

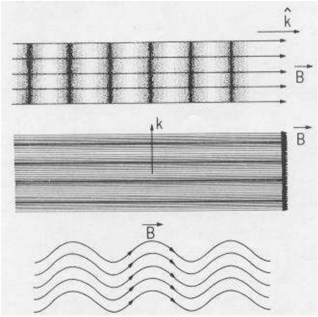
$$\omega\delta\mathbf{v} = \frac{\mathbf{k}}{m_i n_0}\left(\delta p + \frac{1}{\mu_0}\mathbf{B}_0\cdot\delta\mathbf{B}\right) - \frac{\mathbf{k}\cdot\mathbf{B}_0}{\mu_0 m_i n_0}\delta\mathbf{B}$$

Parallel direction: $\omega v_\| = \frac{k_\|\delta p}{m_i n_0}$ Parallel pressure variations cause parallel flow.

Oblique direction: $\omega\left(k_\| v_\| + k_\perp v_x\right) = \frac{k^2\delta p_{tot}}{m_i n_0}$

Total pressure variations (p_{tot}=p+B^2/2[redacted]) accelerate (or decelerate) flow, for in-phase (or out-of-phase) variations of [redacted] and [redacted], leading to the **fast** and **slow mode waves.**

Magnetohydrodynamic waves



- Magnetosonic waves

compressible

- parallel slow and fast
- perpendicular fast

$c_{ms} = (c_s^2+v_A^2)^{1/2}$

- Alfvén wave

incompressible

parallel and oblique

v_A = B/(4[redacted]$^{/2}$

Discontinuities and shocks

Changes occur perpendicular to the discontinuity, parallel the plasma is uniform. The normal vector, **n**, to the surface *S*(**x**) is defined as:

$$\mathbf{n} = -\frac{\nabla S}{|\nabla S|}$$

Any closed line integral (along a rectangular box tangential to the surface and crossing *S* from medium 1 to 2 and back) of a quantity *X* reduces to

$$\oint_S \frac{dX}{dn}dn = 2\int_1^2 \frac{dX}{dn}dn = 2\left(X_2 - X_1\right) = 2[X]$$

Since an integral over a conservation law vanishes, the gradient operation can be replaced by

$$\begin{aligned}\nabla X &\rightarrow \mathbf{n}[X]\\ \nabla\cdot\mathbf{X} &\rightarrow \mathbf{n}\cdot[\mathrm{X}]\\ \nabla\times\mathbf{X} &\rightarrow \mathbf{n}\times[\mathrm{X}]\end{aligned}$$

Transform to a frame moving with the discontinuity at local speed, **U**. Because of ***Galilean invariance***, the time derivative becomes:

$$\partial/\partial t = -\mathbf{U}\cdot\nabla = -U\cdot\mathbf{n}(\partial/\partial n)$$

Discontinuities and shocks

Continuity of the mass flux and magnetic flux:

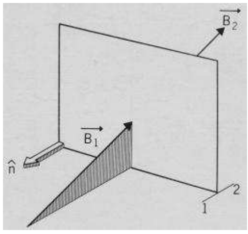
$B_n = B_{1n} = B_{2n}$

G_n = [redacted](V_{1n} - U) = [redacted](V_{2n} - U)

U is the speed of surface in the normal direction; **B** magnetic field vector; **V** the flow velocity. ***Mach number***, M = V/C. Here C is the wave phase speed.

Shock: G [redacted]

Discontinuity: G = 0



Contact discontinuity (CD)

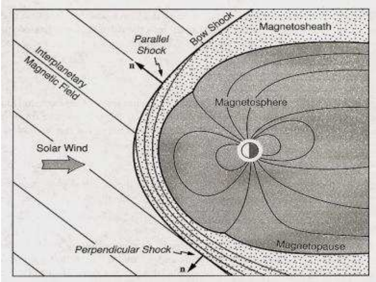
B does not change across the surface of the CD, but [redacted] and T_1 [redacted]$_2$.

Rankine-Hugoniot conditions

In the ***comoving frame*** (**v**' = **v** - **U**) the discontinuity (D) is stationary so that the time derivative can be dropped. We ***skip the prime*** and consider the situation in a frame where D is at rest. We assume an isotropic pressure, **P**=p**1**. Conservation laws transform into the ***jump conditions*** across D, reading:

$$\begin{aligned}\mathbf{n}\cdot[n\mathbf{v}] &= 0\\ \mathbf{n}\cdot[nm\mathbf{v}\mathbf{v}] + \mathbf{n}\left[p + \frac{B^2}{2\mu_0}\right] - \frac{1}{\mu_0}\mathbf{n}\cdot[\mathbf{B}\mathbf{B}] &= 0\\ [\mathbf{n}\times\mathbf{v}\times\mathbf{B}] &= 0\\ \mathbf{n}\cdot[\mathbf{B}] &= 0\end{aligned}$$

Bow shock



The most famous and mostly researched shock is the ***bow shock*** standing in front of the Earth as result of the interaction of the *magnetosphere* with the *supersonic solar wind,* with a high Machnumber, M_F [redacted]. Solar wind density and field jump by about a factor of 4 into the *magnetosheath*.

Plasma Instabilities

Because of a multitude of free-energy sources in space plasmas, a very large number of instabilities can develop.

If spatial the involved scale is:

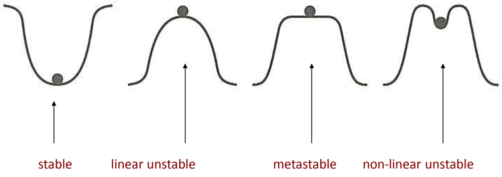
- comparable to macroscopic size (bulk scale of plasma,.....)
  -> *macroinstability* (affects plasma globally)
- comparable to microscopic scale (gyroradius, inertial length,...)
  -> *microinstability* (affects plasma locally)

Theoretical treatment:

- *macroinstability*, fluid plasma theory
- *microinstability*, kinetic plasma theory

Concept of instability

Generation of instability is the general way of redistributing energy which was accumulated in a non-equilibrium state.



Linear instability

The concept of linear instability arises from the consideration of a linear wave function. Assume any variable (density, magnetic field, etc.) here denoted by *A*, the fluctuation of which is [redacted], that can be Fourier decomposed as

$$\delta A = \sum_{\mathbf{k}} A_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

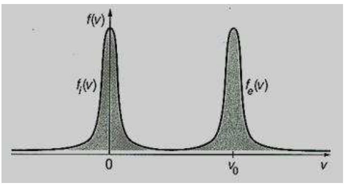
In general the dispersion relation (DR) has complex solutions: [redacted] [redacted] + [redacted] For real frequency the disturbances are oscillating waves. For complex solutions the sign of [redacted] decides whether the amplitude *A* growths ([redacted] *>0)* or decays ([redacted] *<0).*

$$A_{\mathbf{k}}(t) = A_{\mathbf{k}} \exp\left[\gamma(\omega_r, \mathbf{k})t\right]$$

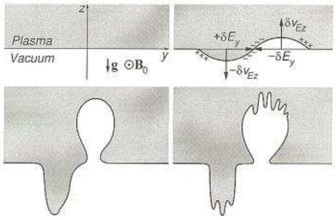
Buneman instability

The electron-ion two-stream instability, ***Buneman instability***, arises from a DR that can be written as (with ions at rest and electrons at speed v_0): => Current disruption

$$\epsilon(\omega, \mathbf{k}) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - kv_0)^2} = 0$$

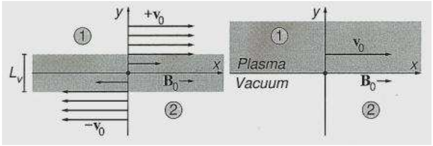


Rayleigh-Taylor instability



Consider a distortion of the boundary so the plasma density makes a sinusoidal excursion. The gravitational field causes an ion drift and current in the negative y direction, $v_{iy} = -m_i g/(eB_0)$, in which electrons do not participate; -> charge separation electric field [redacted]E_y evolves. Opposing drifts amplify the original distortions. The bubbles develop similar distortions on even smaller scales.

Kelvin-Helmholtz instability

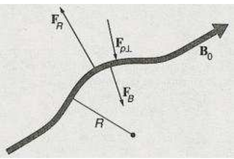


- Shear flow at magnetised plasma boundary may cause ripples on the surface that can grow
- The rigidity of the field provides the dominant restoring force

Consider shear flows (e.g., due to the solar wind) at a boundary, such as between Earth's magnetosheath and magnetopause. Linear perturbation analysis in both regions shows that incompressible waves confined to the interface can be excited.

Firehose instability

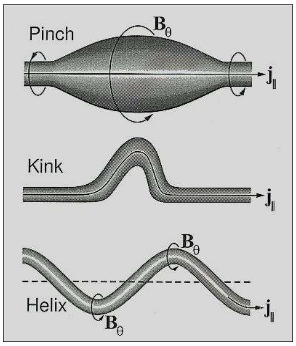
Mechanism of the firehose instability: Whenever the flux tube is slightly bent, the plasma exerts an outward centrifugal force (curvature radius, R), that tends to enhance the initial bending. The gradient force due to magnetic stresses and thermal pressure resists the centrifugal force.



The resulting instability condition for breaking equilibrium is:

$$p_{\|} > p_{\perp} + B_0^2/\mu_0$$

Flux tube instabilities

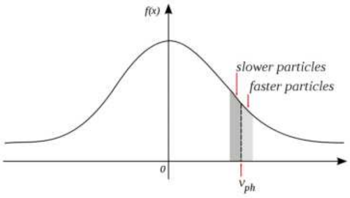


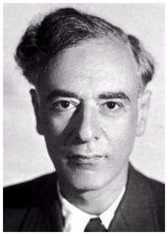
Current disruption

Bending of magnetic field line

Spiral formation of thin flux tube

Landau Damping (Lev Landau 1908-1968, Nobel prize 1962)

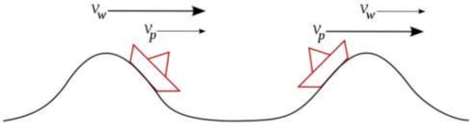




Source: Wikipedia

- Particles slightly slower than the wave gain energy from the wave (and wave looses energy)
- Particles slightly faster than the wave loose energy.
- For a Maxwell distribution there are more slower particles => Wave becomes damped (Landau damping)

Landau-damping



- Landau damping can be compared with a surfer.
- Left: Slow surfer gets catched by the wave and gains energy.
- Right: Fast surfer catches the wave and gives energy to the wave.

Source: Wikipedia

Waves, shocks and instabilities

- Wave occur naturally in fluid and kinetic models, e.g. Alfven waves and magnetosonic waves.
- Shocks are a special case of discontinuieties in fluid model with mass flux across the shock.
- Discontinuities happen in the fluid model, not in nature and for studying the physics of the discontinuity accurately, one has to apply kinetic models.
- Waves with growing amplitudes lead to instabilities, both in kinetic and fluid picture.
- A special instability ‘magnetic reconnection’ will be treated in the next lecture.

Exercises for Space Plasma Physics:
VIII. Waves, Shocks and instabilities

1. How does an electro magnetic wave travel in vacuum and in plasma?
2. What is the cut-off-frequency? How can it be used the derive the (electron) density of a plasma?
3. If one observes the Sun in visible light and long radio waves, will the sun (visible radius) look larger, smaller or the same in radio wavelength?
4. Discuss briefly the difference between a sound wave in a neutral gas and slow/fast magnet-sonic waves and Alfven waves in plasmas.
5. What is Landau damping?
6. What are macro and micro-instabilities?
7. Why are thin current sheets important for the dynamics of space plasmas?

Space Plasma Physics
Thomas Wiegelmann, 2012

Physical Processes

8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

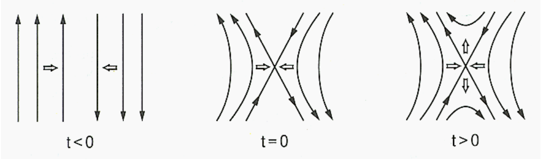
10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas

Magnetic reconnection

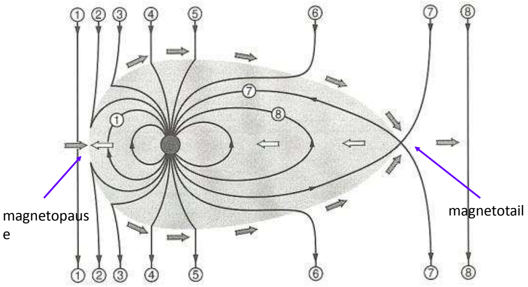
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma_0} \nabla^2 \mathbf{B}$$

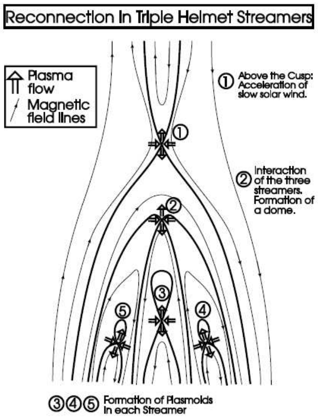
Assuming the plasma streams at bulk speed *V*, then the induction equation can be written in simple dimensional form as:

$$\frac{B}{\tau} = \frac{VB}{L_B} + \frac{B}{\tau_d}$$



Field line merging and reconnection in the Earth's magnetosphere





How to model magnetic reconnection?

- In ideal MHD, plasma is frozen into the magnetic field => field line topology conservation and magnetic reconnection is not possible.
- Resistive MHD in 2D
- Resistive MHD in 3D
- Collisionless reconnection, kinetic treatment
- Hybrid models, e.g. treat the ions with a kinetic model and electrons as fluid.

Magnetic reconnection, 2D MHD
Source: Priest et al. JGR 108, A7, 2003

- Kinematic approach (not to be confused with kinetic theory !!): One studies the induction equation and ignores the equation of motion for simplicity:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times (\eta \nabla \times \mathbf{B})$$

- We derived this equation in exercises earlier from Ohms law:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

- In most space plasmas the Reynolds number is very large and resistive effects can be neglected.

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{0}.$$

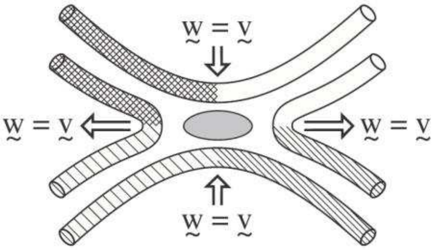
Magnetic reconnection, 2D MHD

- An exception are strong current concentrations with high gradients in the magnetic field. Here resistive effects become important and magnetic reconnection can occur.
- In generally we can describe the motion of magnetic flux, if a flux-conserving velocity **w** exists:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B})$$

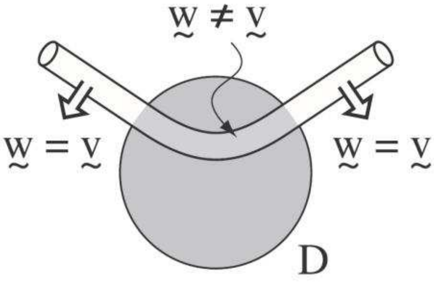
- How is flux-conserving velocity **w** related to the plasma flow velocity **v**?

Magnetic reconnection, 2D MHD



Breaking and rejoining of two flux tubes in 2-D to form two new flux tubes. Outside the diffusion region the flux-conserving velocity is identical with the plasma velocity.
Source: Priest et al. 2003

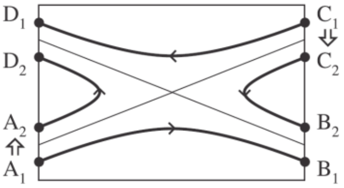
Magnetic reconnection, 2D MHD



Inside the diffusion region these two velocities are NOT identical, because the plasma is not frozen in the magnetic field here. Source: Priest et al. 2003

Properties of 2D MHD-reconnection

- A differentiable flux tube velocity **w** exists everywhere except at null points. The magnetic flux moves at the velocity **w** and slips through the plasma, which moves at **v**.
- Mapping of field lines are discontinuous



Moving over the separatrix from A_1 to A_2, there is a jump from B_1 to D_2
=> field lines break

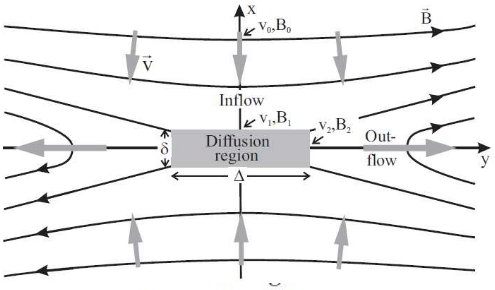
Source: Priest et al. 2003

Properties of 2D MHD-reconnection

- Field lines preserve their connections in the diffusion region. The exception is the X-point, where the field lines break and their connections change.
- Reconnecting flux tubes join perfectly and outside the diffusion region **v**=**w**.
- For a flux tube partly in the diffusion region, the part of the flux tube in the zone slips through the plasma. (Outside: frozen in)
- Different models have been suggested for 2D reconnection, which based on details of reconnection zone (e.g. Sweet-Parker model, Petschek reconnection)

Source: Priest et al. 2003

Sweet Parker 2D-model



$$L \geq \Delta \geq \breve{\delta}.$$

Source: Schindler & Hornig, 2001
Magnetic Reconnection, in Encyclopedia of Astronomy and Astrophysics

2D MHD-reconnection, Sweet Parker (1956)

- Reconnection is described as diffusion on scales smaller than typical macroscopic scales.
- Properties in the inflow region (flow velocity, magnetic field strength, convection electric field) are related to the outflow
  region and allow to calculate the reconnection rate (amount of flux reconnected per time, ratio of in and outflow velocity) $V_{out} \sim V_A \equiv \frac{B_{in}}{\sqrt{\mu_0 \rho}}$
- Reconnection rate is much slower as observed in space plasmas, but faster than global diffusion.

Source: Wikipedia

2D MHD-reconnection, Petschek model (1964)

- The reconnection rate in the Sweet-Parker model is too slow.
- Aspect-ratio of current sheet has to be large for high Lundquist numbers because of the relation
- Petschek refined the Sweet-Parker model in 1964. In and out-flow regions are separated by slow mode shocks.
- Reconnection does hardly depend on aspect ratio and fast reconnection is possible:

$$S \equiv \frac{\mu_0 L V_A}{\eta}.$$

$$\frac{V_{in}}{V_A} \sim \frac{1}{S^{1/2}}$$

$$\frac{V_{in}}{V_A} \approx \frac{\pi}{8 \ln S}.$$

Source: Wikipedia

Magnetic reconnection: energy conversion

- From the resistive MHD-simulations we derive equations for the balance of mechanical and electromagnetic energy:

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + u\right) + \nabla \cdot \left(\left(\frac{\rho v^2}{2} + u + p\right) v\right) = \boldsymbol{j} \cdot \boldsymbol{E}$$

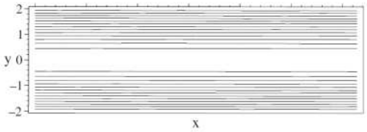
$$\frac{\partial}{\partial t}\left(\frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B}\right) = -\boldsymbol{j} \cdot \boldsymbol{E}.$$

- For magnetic reconnection j and E have the same sign (why? Think about it in exercise)
  => Electromagnetic energy decreases
  Thermic and kinetic energy increases

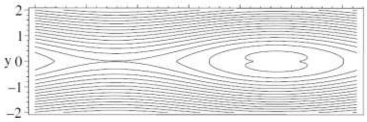
Source: Schindler & Hornig, 2001
Magnetic Reconnection, in Encyclopedia of Astronomy and Astrophysics

Time dependent reconnection

- Sweet-Parker and Petschek models investigated stationary reconnection.
- In Space plasmas reconnection happens often spontaneously as an instability process.



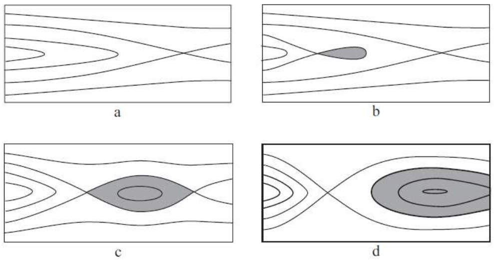
Unperturbed 1D-Harris sheet



Linear tearing instability leads to formation of a plasmoid.

Source: Schindler & Hornig, 2001

Time dependent reconnection



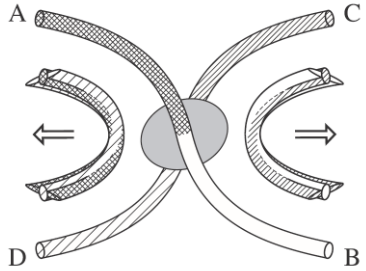
Streched configurations (modified Harris-sheet) are often used as models for the magnetotail and coronal helmet streamers. Time dependent reconnection occur in models for substorms and coronal mass ejections.

Source: Schindler & Hornig, 2001

Magnetic reconnection, 3D MHD

- Some concepts on 2D-reconnection cannot be generalized to the 3D-case.
- Reconnection at an X-point (magnetic null) becomes structual unstable in 3D:
  - 2D X-point becomes an x-line in 3D.
  -We can add a magnetic field parallel to this line (perpendicular to 2D reconnection-plane) and the x-line is NOT a magnetic null anymore.
- One cannot generally find a flux-conserving velocity **w** in 3D. (Proof by Priest et al. 2003, omitted here)

Magnetic reconnection, 3D MHD



In 3D the flux-tubes not necessarily match perfectly. In 3D the flux-tubes break and rejoin only partly forming four new flux tubes.

Source: Priest et al. 2003

GEM magnetic reconnection challenge

Source: Birn et al. JGR 106, A3, 1999

- The Geospace Environmental Modeling (GEM) Reconnection Challenge.
- Magnetic reconnection was studied in a simple Harris sheet configuration with a specified set of initial conditions, including a finite amplitude, magnetic island perturbation to trigger the dynamics.
- The evolution of the system is explored with a broad variety of codes, ranging from fully electromagnetic particle in cell (PIC) codes to resistive MHD.
- Aim: Identify the essential physics which is required to model collisionless magnetic reconnection.

GEM: Initial Harris sheet

- We computed the 1D Harris sheet profile in the magneto-static section, but it is a kinetic equilibrium and solves the stationary Vlasov equation as well.

$$B_x(z) = B_0 \tanh(z/\lambda)$$
$$n(z) = n_0 \mathrm{sech}^2(z/\lambda) + n_\infty$$

The electron and ion temperatures, T_e and T_i, are taken to be uniform in the initial state. The pressure balance condition gives $n_0(T_e + T_i) = B_0^2/8\pi$. The computation is carried out in a rectangular domain $-L_x/2 \leq x \leq L_x/2$ and $-L_z/2 \leq z \leq L_z/2$.

Source: Birn et al. 1999

- Used simulation parameters

$\lambda = 0.5,\ n_\infty/n_0 = 0.2,\ T_e/T_i = 0.2$

$m_i/m_e = 25,\ L_x = 25.6,$ and $L_z = 12.8$

- Physical quantities have been normalized with
  ion inertial length c/ω_{pi}
  ion cyclotron frequency $\Omega_i = eB_0/m_i c$
- So we have a thin current sheet (macroscopic length scale is of order of kinetic scales) and ions are hotter than electrons.
- Question: Why one chooses a mass ration of 25 instead of true mass ratio 1836 for protons and electrons?
- The initial Harris-sheet was perturbed:

$$\psi(x, z) = \psi_0 \cos(2\pi x/L_x) \cos(\pi z/L_z)$$

Source: Birn et al. 1999

BIRN ET AL.: GEM RECONNECTION CHALLENGE

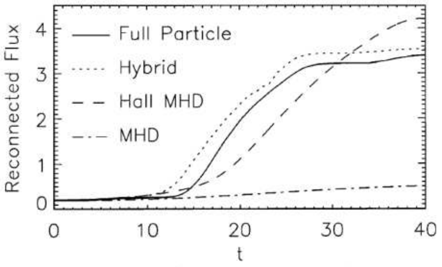


Figure 1. The reconnected magnetic flux versus time from a variety of simulation models: full particle, hybrid, Hall MHD, and MHD (for resistivity $\eta = 0.005$).

GEM: Results

- All models that include the Hall effect in the generalized Ohm's law produce essentially indistinguishable rates of reconnection, corresponding to nearly Alfvenic inflow velocities.
- Thus the rate of reconnection is insensitive to the specific mechanism which breaks the frozen-in condition, whether resistivity, electron inertia, or electron thermal motion.
- The reconnection rate in the conventional resistive MHD model, in contrast, is dramatically smaller unless a large localized or current dependent resistivity is used.

GEM: Results

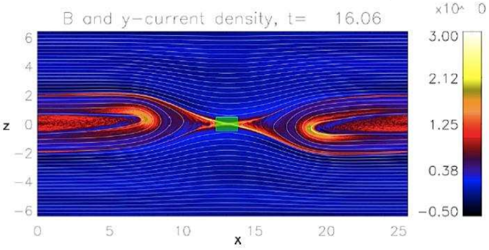
- Remember the generalized Ohm's law from our MHD-lecture:

$$\frac{4\pi}{\omega_{pe}^2}\frac{d\mathbf{j}}{dt} = \mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B} \underbrace{- \frac{1}{nec}\mathbf{j}\times\mathbf{B}}_{\text{Hall term}} + \frac{1}{ne}\nabla\cdot\vec{p_e} - \eta\mathbf{j},$$

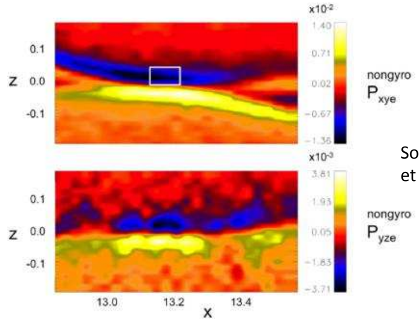
- The Hall term takes care about different motion of electrons and ions.
- Please note that in kinetic (PIC or Vlasov code) we do not have an Ohm's law, but this law was derived in the fluid picture by substraction the momentum equations for ions and electrons.
- => Effect is naturally there in kinetic theory.

The diffusion region

Source: Hesse et al. Space Sci Rev 2011

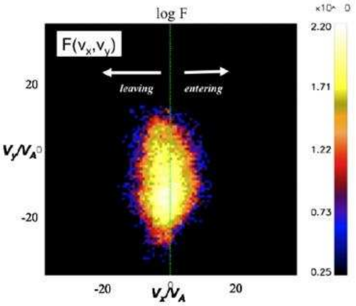


- As pointed out by Biskamp (Book:Nonlinear MHD 1993) the diffusion region is not treated correctly in Petschek-like-MHD-models)
- => Kinetic treatment necessary

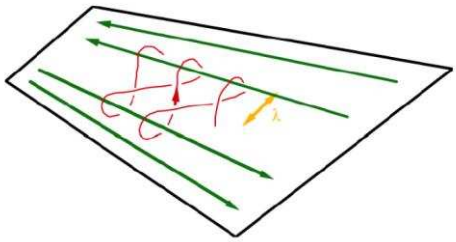


Source: Hesse et al. 2011

- Plasma pressure (electron pressure tensor) is not isotrop and not even gyrotropic in diffusion region.
- Simulations with mass ratio m_i/m_e=256, Hesse et al

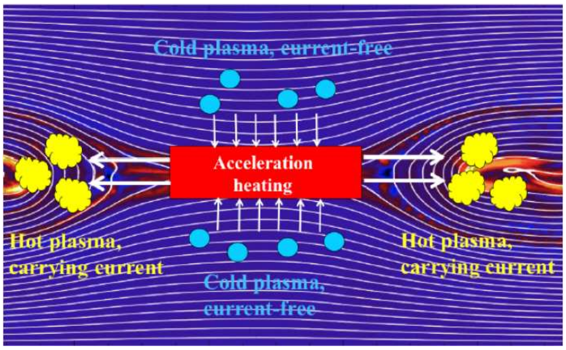


Reduced distribution F(vx, vy) in the outflow region. The distribution demonstrates the preferential loss of accelerated particles from the dissipation region. Source: Hesse et al. 2011



- Electrons become unmagnetized in diffusion region.
- Schematic picture of meandering electron orbits in a magnetic field reversal.

Source: Hesse et al. 2011



Source: Hesse et al. 2011

Magnetic reconnection

- Magnetic reconnection changes the magnetic field topology and magnetic energy is converted into thermal and kinetic energy.
- Reconnection was studied first in 2D resistive MHD with stationary and time-dependent models.
- In ideal MHD magnetic reconnection cannot happen.
- In the resistive (or diffusive) zone, basic assumptions of MHD break down and kinetic effects become important.
- Reconnection in 3D can happen at 3D-magnetic-Nulls and also for non-vanishing magnetic fields
  (active area of research and details are outside of scope of this indroductory lecture)

Exercises for Space Plasma Physics: IX. Magnetic Reconnection

1. What is magnetic reconnection?
2. Where in the solar system is magnetic reconnection important?
3. Can reconnection happen in laboratory plasmas?
4. Why does the current density **j** and the electric field **E** have the same sign in magnetic reconnection? Hint: study the 2D-case and apply Ohm's law at the X-point.
5. How is magnetic energy converted to other energy forms (which one) during magnetic reconnection?
6. Any idea why scientist use a reduced mass ration m_i/m_e of 25, 256 etc. for Vlasov-code or full particle simulations instead of the correct value 1836 for a proton electron plasma ?
7. Which MHD-assumptions break down in the diffusion zone?

Space Plasma Physics

Thomas Wiegelmann, 2012

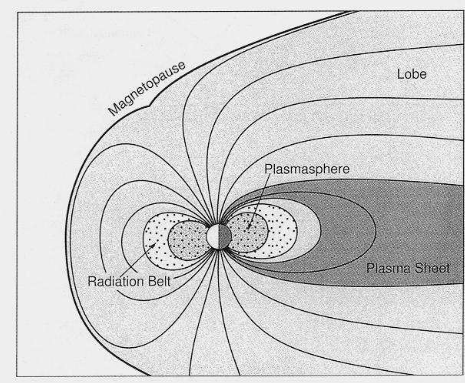
Physical Processes

8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas

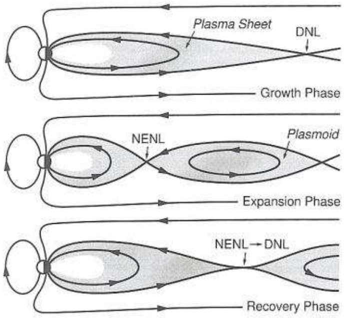
Earth magnetosphere



Magnetospheric substorm

Substorm phases:

- Growth
- Onset and expansion
- Recovery



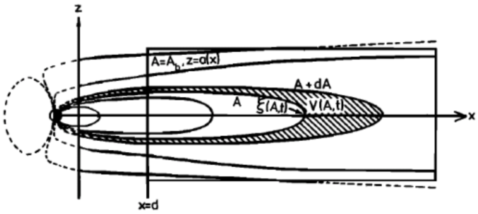
Show movie:
http://www.youtube.com/watch?v=BDZj1CmsJ64

Growth phase of magnetotail

- The growth phase (quiet evolution of magnetotail before a substorm) can be modelled as a sequence of magneto-static equilibria.
- Magnetic pressure in the lobe changes slowly (solar wind compresses field lines) and remains in equilibrium with plasma pressure in the plasma sheet.
- The magnetic field topology does not change in this phase (ideal MHD, no resistivity).
- Sequence of equilibria are not independent and the ideal MHD-evolution has to be implemented in the model.

Magnetospheric convection in quiet states

(Source: Schindler & Birn, JGR 1982)



Noon-midnight meridian plane of the magnetosphere.
V(A) is proportional to the differential volume
between field lines A and A + dA.

Ideal MHD

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$$ Continuity Eq.

$$\rho\frac{\partial\underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v} = -\nabla p + \underline{j}\times\underline{B}$$ Force-balance

$$\underline{E} + \underline{v}\times\underline{B} = 0$$ Ideal Ohms Law

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + \underline{v}\cdot\nabla\left(\frac{p}{\rho^\gamma}\right) = 0$$ Equation of state, Adiabatic convection

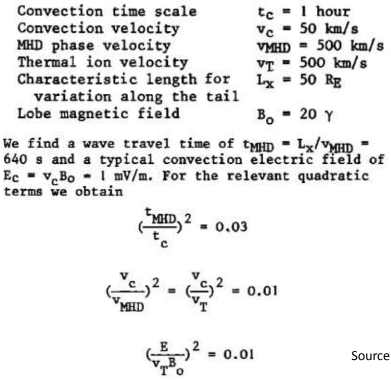
$$\left.\begin{aligned}\nabla\times\underline{E} &= -\frac{\partial\underline{B}}{\partial t}\\ \nabla\times\underline{B} &= \mu_0\underline{j}\\ \nabla\cdot\underline{B} &= 0\end{aligned}\right\}$$ Maxwell

Source: Schindler & Birn, JGR 1982

Basic assumptions for magnetotail

- 2D configuration, invariant in y, no B_y
- Tail-like structure, length scale L_x >> L_z. $\varepsilon = \frac{L_z}{L_x} \ll 1$ Terms of order ε^2 are neglected.
- Slow time dependence, $\partial/\partial t$, v, and E are small (order δ) and terms of order δ^2 are ignored.
- Physically this means that the convection time is much larger as the travel time of waves.
  => Quasistatic evolution
- Constant temperature and particle conservation.
- Inner (dipolar part) of magnetosphere is not included in model (Has been done later, Becker et al. 2001)

How well are these assumptions fulfilled?



Convection time scale	t_c = 1 hour
Convection velocity	v_c = 50 km/s
MHD phase velocity	v_{MHD} = 500 km/s
Thermal ion velocity	v_T = 500 km/s
Characteristic length for variation along the tail	L_x = 50 R_E
Lobe magnetic field	B_o = 20 γ

We find a wave travel time of $t_{MHD} = L_x/v_{MHD}$ = 640 s and a typical convection electric field of $E_c = v_c B_o$ = 1 mV/m. For the relevant quadratic terms we obtain

$$\left(\frac{t_{MHD}}{t_c}\right)^2 = 0.03$$

$$\left(\frac{v_c}{v_{MHD}}\right)^2 = \left(\frac{v_c}{v_T}\right)^2 = 0.01$$

$$\left(\frac{E}{v_T B_o}\right)^2 = 0.01$$

Source: Schindler & Birn, JGR 1982

$$-\nabla p + \underline{j} \times \underline{B} = 0$$ Force-balance

$$\underline{A} = (0, A(x,z,t), 0) \qquad \phi = \phi(x,z,t)$$

$$\underline{B} = \nabla A \times \underline{e}_y \qquad p(x,z,t) = p(A(x,z,t),t)$$

$$\underline{j} = \partial p(A,t)/\partial A \ \underline{e}_y \qquad \rho = \frac{m_i N(A)}{V(A,t)}$$

$$\boxed{-\Delta A(\underline{r},t) = \mu_o \frac{\partial p(A,t)}{\partial A}}$$

Grad-Shafranov equation, but now time-dependent.

$$\frac{\partial A}{\partial t} + \underline{v} \cdot \nabla A = \frac{dA}{dt} = 0$$ Ideal Ohms Law

$$\boxed{\frac{d}{dt}[p(A,t)^{1/\gamma} V(A,t)] = 0}$$

Equation of state, Adiabatic convection

$$V(A,t) = \int_A \frac{ds}{|\nabla A|}$$

Differential flux volume can be computed by differentiation along magnetic fieldlines.

For the stretched tail geometry the Grad-Shafranov-Eq. becomes 1D (in z), but depends parametrically on x and t.

$$\boxed{\frac{1}{2\mu_o}\left(\frac{\partial A}{\partial z}\right)^2 + p(A,t) = p_o(x,t)}$$

$$p_0(x,t) = \frac{B_0^2(x,t)}{2}$$ Boundary condition

The magnetic field in the lobe we get from observations:

$$B_0(x,t) = \sqrt{2}(1+t)\left(1+\Lambda(1+t)^2 x\right)^{-\frac{n}{2}}$$

The equation of state can now be written as an integral:

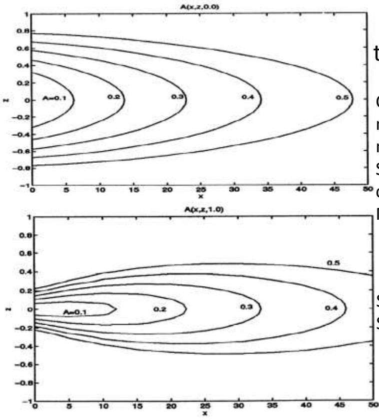
$$\frac{d}{dt}[p(A,t)^{1/\gamma} V(A,t)] = 0$$

$$\boxed{p(A,t)^{\frac{1}{\gamma}} \int_{p(A,t)}^{p_0(0,t)} \frac{dp_0}{-\frac{\partial p_0(x,t)}{\partial x}\sqrt{p_0 - p(A,t)}} = M(A)}$$

As intial pressure profile we use $p(A,0) = \exp(-2A)$ and sove the equation of state selfconsistently with the Grad-Shafranov equation (usually numerically).

$$\boxed{\frac{1}{2\mu_o}\left(\frac{\partial A}{\partial z}\right)^2 + p(A,t) = p_o(x,t)}$$

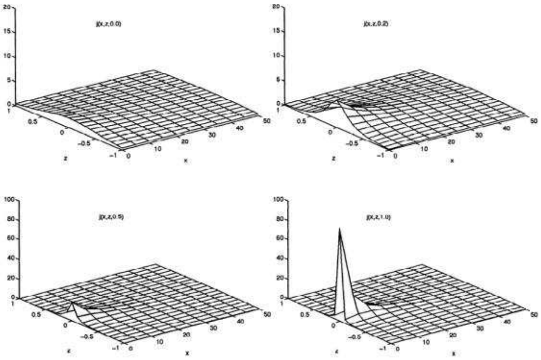
Solution of these equations



Changing of the magnetic field in the magnetotail. Sequence of equilibria constraint by ideal MHD.

Source: Wiegelmann & Schindler 1995

Formation of thin current sheets



Formation of thin current sheets

- Within our model assumptions strong current concentrations (thin current sheets) form.
- Such configurations are prone to current driven micro-instabilities (kinetic instabilities).
- Micro-instabilities cause resistivity on macroscopic MHD-scales.
- The assumption of ideal MHD is violated and we need (at least) use resistive MHD for further investigations.
- => Topology changes by magnetic reconnection and a fast dynamic evolution is possible (not quasistatic anymore) => Eruption phase of substorms start.

Resistive magnetotail MHD-simulations

(Otto et al. 1990, JGR)

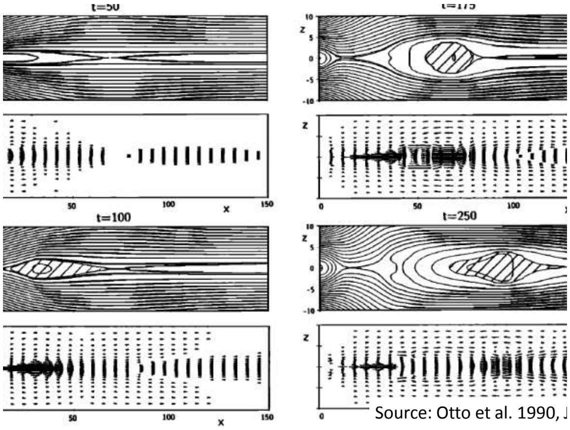
$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \left(\rho \mathbf{v}\mathbf{v} + \frac{1}{2}\left(p + \mathbf{b}^2\right)\underline{\mathbf{1}} - \mathbf{b}\mathbf{b} \right)$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b} - \eta \mathbf{j})$$

$$\frac{\partial h}{\partial t} = -\nabla \cdot (h\mathbf{v}) + \frac{\gamma - 1}{\gamma} h^{1-\gamma} \eta \mathbf{j}^2$$

$$\mathbf{j} = \nabla \times \mathbf{b}$$



Source: Otto et al. 1990, J

Thin current sheets

- We can understand the formation of strong current concentrations within MHD.
- Spatial scales in thin current sheets become very small and comparable with ion gyro-radius (about 500 km in magnetotail).
- => A basic assumption of MHD is violated and we need to apply kinetic theory (Vlasov equation).
- Stationary Vlasov equilibria.
- Time dependent evolution (either solve Vlasov-eq. directly or indirectly by particle in cell simulations.)

Vlasov-Maxwell Equations

kinetic description of a collisionless plasma

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \sum_\alpha \overline{n}_\alpha q_\alpha \int f_\alpha \, d\mathbf{v} + \frac{\rho_{q\,\mathrm{ext}}}{\epsilon_0}$$

$$\nabla \times \mathbf{B} = \mu_0 \sum_\alpha \overline{n}_\alpha q_\alpha \int \mathbf{v} \, f_\alpha \, d\mathbf{v} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}_{\mathrm{ext}}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

α stands for ions and electrons.

Properties of Vlasov Equation

- Vlasov Equation conserves particles
- Distribution functions remains positive
- Vlasov equation has many equilibrium solutions

$$\mathbf{v}\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{x}}+\frac{q_\alpha}{m_\alpha}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_{\alpha 0}}{\partial \mathbf{v}}=0$$

Index 0 stands for equilibrium. Any distribution function $f_{\alpha 0}$ which depends only on constants of motions (say $a(x,v)$, $b(x,v)$, ...) of the particle trajectories solves the stationary Vlasov equation. => Proof see Lecture 4, Kinetic Theory

$$f_{\alpha 0}(x,v)=f_{\alpha 0}\left(a(x,v),b(x,v),\dots\right)$$

Thin current sheets, Kinetic theory

Source: Schindler&Birn, JGR 2002

- Constants of motion (2D, x,z):
  - momentum in y $P_y = mv_y + qA(x,z)$
  - Hamiltonian $H=\frac{1}{2m}(P_x^2+P_z^2)+\frac{1}{2m}[P_y-qA(x,z)]^2+q\phi(x,z)$
- Stationary solution of Vlasov equation: $f_1 = F(H,P_y)$
- Macroscopic quantities (charge and current density) we get by integration over velocity space:

$$\sigma(A,\phi)=\sum_s q_s n_s=\sum_s q_s\int F_s(H_s,P_{ys})d^3v$$

$$j_y(A,\phi)=\sum_s q_s\int v_y F_s(H_s,P_{ys})d^3v$$

Thin current sheets, Kinetic theory

- Inserting into Maxwell equations: $-\Delta A=\mu_0 j_y(A,\phi)$
- Using the quasi-neutrality condition $\sigma(A,\phi)=0$ we can eliminate the electro-static potential and derive the Grad-Shafranov equation (like in MHS):

$$-\Delta A=\mu_0 j_y(A)$$

- Plasma pressure is isotropic (in x,z) for these form of distribution functions:

$$p(A,\phi)=\sum_s\int\frac{m_s}{2}(v_x^2+v_z^2)F_s(H_s,P_{ys})d^3v \qquad j_y=\frac{\partial p}{\partial A}$$

Thin current sheets, Kinetic theory

- Remember that any distribution function depending only on constants of motions solves the stationary Vlasov equation. Examles used in (Schindler & Birn 2002):
- LTE => Local Maxwellian of Hamiltonian:

$$F_s(H,P_y)=C_s\exp\left(-\frac{H_s}{k_BT_s}\right)g_s(P_y)$$

- The momentum in y causes electric currents (ions and electrons move in opposite direction, but the currents in the same. P_y profile can be derived from the result of sequences of magneto-static equlibria.
- Please note that this still leads to some freedom in the distribution functions, e.g. different temperatures for ions and electrons, thickness of sheet can be used.

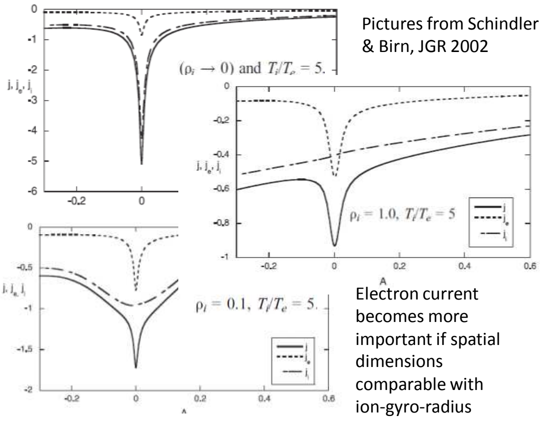
Thin current sheets, Kinetic theory

- Dimensionless parameters:

$$\tau_e=\frac{T_e}{T},\qquad \tau_i=\frac{T_i}{T},\qquad \boxed{\rho_i=\frac{\sqrt{m_ik_BT_i}}{eB_0L},}\qquad \rho_e=\frac{\sqrt{m_ek_BT_e}}{eB_0L},$$

Ratio of ion-gyro-radius to spatial dimension L of the current sheet.

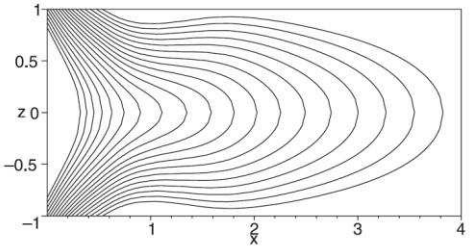
- Remember: MHD is valid only if spatial dimension L is much larger as the gyroradius. As larger this ratio, as more important are kinetic effects.



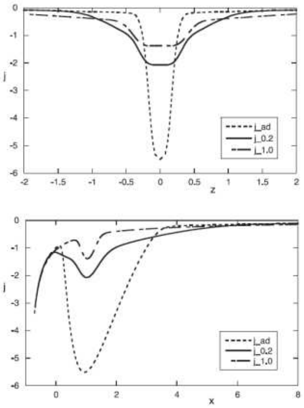
Pictures from Schindler & Birn, JGR 2002

Electron current becomes more important if spatial dimensions comparable with ion-gyro-radius

Thin current sheets, Kinetic theory



As in MHD, A(x,z) defines the magnetic field lines and the relation A-> J_y can be used to compute J_y(x,z). (Picture from Schindler & Birn, JGR 2002)



Thin current sheets, Kinetic theory

- j_ad corresponds to MHD case (gyro-radius infinitesimal small)
- Taking finite gyroradius effects into account localises the sheet in x, but smears it out in z.
- Ouside the current sheet, MHD is valid and the configurations are the same.

Pictures from Schindler & Birn, JGR 2002

Magnetosphere

- We can understand the growth phase of a substorm as a sequence of magneto-static equilibria.
- Interaction with the solar wind leads to stretching of the magnetotail and formation of thin current sheets in the center of the plasma-sheet.
- Thin current sheets cause resistivity on MHD-scales and lead to magnetic reconnection and initiation of the dynamic phase of a substorm.
- This eruptive phase can be simulated with time-dependent, resistive MHD-simulations.
- In the thin current sheets itself the MHD-approach is not valid and we need to apply a kinetic model.

Exercises for Space Plasma Physics:
X. Magnetosphere

1. How can the full MHD-equations be simplified to describe the quiet loading phase before a magnetic storm occurs?

2. Explain the physical processes which occur, when the slow quasistatic evolution of the magnetotail ends and a very dynamic eruptive phase starts.

3. Resistivity is a kind of friction in a plasma. How can it be that such kind of friction plays a key-role for the initiation of dynamic processes? For comparison: in classical mechanics friction slows down the motion of say a pendulum, but does not initiate the motion. How can it be that an ideal plasma remains calm, but erupts when resistivity occurs?

4. In the lecture we derived the Grad-Shafranov equation from magneto-hydro-statics. How can one find corresponding distribution functions which fulfill the stationary Vlasov-equation?

5. Scientists apply both MHD-models and kinetic models to study magnetospheric physics. Is this necessary? If MHD is not sufficient to understand some physical processes, why not study the whole magnetosphere with a kinetic model?

6. How could one in principle model planetary magnetospheres, like,e.g., the Jovian magnetosphere where fast planetary rotation and mass loading from the moon Io is assumed to play an important role?

Space Plasma Physics

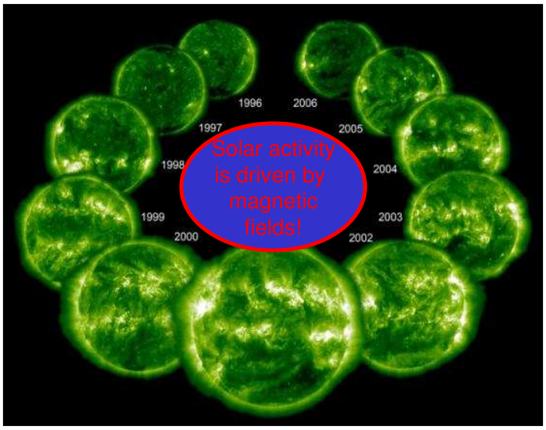
Thomas Wiegelmann, 2012

Physical Processes

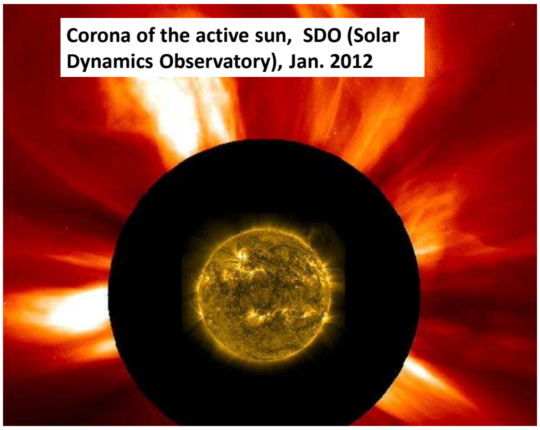
8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

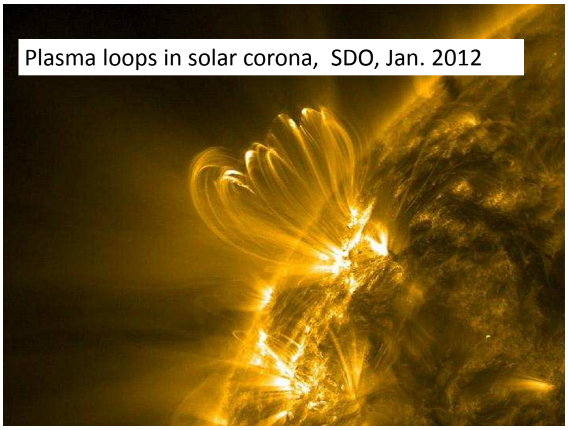
10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas



Solar activity is driven by magnetic fields!



Corona of the active sun, SDO (Solar Dynamics Observatory), Jan. 2012



Plasma loops in solar corona, SDO, Jan. 2012

Solar Eruptions

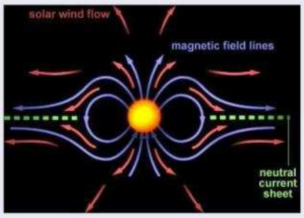
- The solar coronal plasma is frozen into the coronal magnetic field and plasma outlines the magnetic field lines.
- Coronal configurations are most of the time quasistatic and change only slowly.
- Occasionally the configurations become unstable and develop dynamically fast in time, e.g., in coronal mass ejections and flares.

Global magnetic field

Shape

- basic shape: dipole field $\sim 10^{-4}T$
- extension: IMF, heliosphere
- superimposed: complex series of local fields $\sim 0.1T$

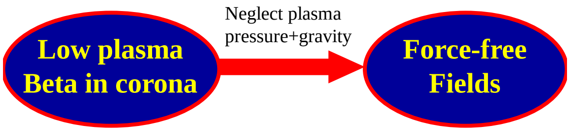
Interplanetary magnetic field



Encyclopedia of Science.

How to model the stationary Corona?

$$\underbrace{(\nabla \times \mathbf{B}) \times \mathbf{B}}_{\text{Lorentz force}} - \underbrace{\cancel{\mu_0 \nabla p}}_{\text{pressure gradient}} - \underbrace{\cancel{\mu_0 \rho \nabla \Psi}}_{\text{gravity}} = \mathbf{0}$$


Low plasma Beta in corona

Neglect plasma pressure+gravity

Force-free Fields

Force-Free Fields

$$\begin{aligned}(\nabla \times \mathbf{B}) \times \mathbf{B} &= \mathbf{0} \\ \nabla \cdot \mathbf{B} &= 0\end{aligned} \quad \xLeftrightarrow{\text{Equivalent}} \quad \begin{aligned}\nabla \times \mathbf{B} &= \alpha \mathbf{B} \\ \mathbf{B} \cdot \nabla \alpha &= 0\end{aligned}$$

Relation between currents and magnetic field.
Force-free functions is constant along field lines, but varies between field lines.
=> nonlinear force-free fields

Further simplifications

- Potential Fields (no currents)
- Linear force-free fields
  (currents globally proportional to B-field)

Potential and Linear Force-Free Fields

We have to solve the equations:

$$(\nabla \times \mathbf{B}) = \alpha \mathbf{B}$$
$$\nabla \cdot \mathbf{B} = 0$$

Here: global constant linear force-free parameter

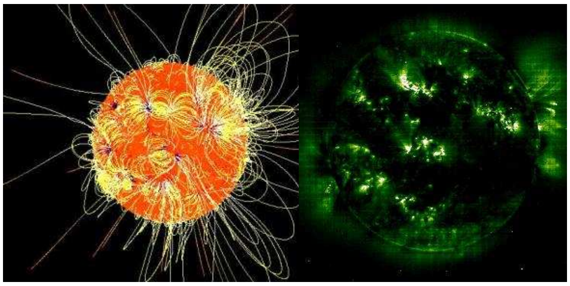
We take the curl of the first equation and apply a vector identity:

$$(\Delta + \alpha^2)\,\mathbf{B} = 0 \quad \text{(Helmholtz equation)}$$

We solve $(\Delta + \alpha^2) B_z = 0$ with observed (e.g. MDI) $B_z(x, y, 0)$ on the photosphere and get B_x and B_y from the other components of the Helmholtz equation and $\nabla \cdot \vec{B} = 0$.

A subclass of force-free fields are current-free potential fields.

$$\begin{aligned}\nabla \times \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{B} &= 0\end{aligned} \quad \Longrightarrow \quad \begin{aligned}\mathbf{B} &= \nabla \Phi \\ \Delta \Phi &= 0\end{aligned}$$



Global potential Field

Coronal Plasma seen in EUV.

Nonlinear Force-Free Fields

$$\begin{aligned}(\nabla \times \mathbf{B}) \times \mathbf{B} &= \mathbf{0} \\ \nabla \cdot \mathbf{B} &= 0\end{aligned} \quad \xLeftrightarrow{\text{Equivalent}} \quad \begin{aligned}\nabla \times \mathbf{B} &= \alpha \mathbf{B} \\ \mathbf{B} \cdot \nabla \alpha &= 0\end{aligned}$$

- For equilibria with symmetry we can reduce the force-free equations to a Grad-Shafranov equation (Low&Lou 1990).
- In 3D, the NLFFF-equations are solved numerically.
- Suitable boundary conditions are derived from measurements of the photospheric field vector.
  - B_n and J_n for positive or negative polarity on boundary (**Grad-Rubin**)
  - Magnetic field vector B_x B_y B_z on boundary (**Magnetofrictional, Optimization**)

Nonlinear force-free fields in 2.5D

(Source: Low&Lou, ApJ 1990)

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$$
$$\boldsymbol{B} \cdot \nabla \alpha = 0$$

- In spherical geometry (invariant in phi, but B-field has 3 components) the magnetic field can be represented as:

$$\boldsymbol{B} = \frac{1}{r \sin \theta}\left(\frac{1}{r}\frac{\partial A}{\partial \theta}\hat{\boldsymbol{r}} - \frac{\partial A}{\partial r}\hat{\boldsymbol{\theta}} + Q\hat{\boldsymbol{\phi}}\right) \qquad \alpha = \frac{dQ}{dA}$$

- The force-free equations reduce to a Grad-Shafranov Eq.:

$$\frac{\partial^2 A}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2 A}{\partial \mu^2} + Q\frac{dQ}{dA} = 0 \qquad \mu = \cos\theta.$$

- Low&Lou 1990 found separable solutions:

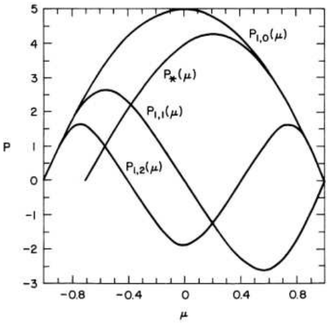
$$A = \frac{P(\mu)}{r^n}, \qquad (1-\mu^2)\frac{d^2P}{d\mu^2} + n(n+1)P + a^2\frac{1+n}{n}P^{1+2/n} = 0$$

$$Q(A) = aA^{1+1/n}$$

a and n are free parameters.

Nonlinear force-free fields in 2.5D

$$(1-\mu^2)\frac{d^2P}{d\mu^2}+n(n+1)P+a^2\frac{1+n}{n}P^{1+2/n}=0$$



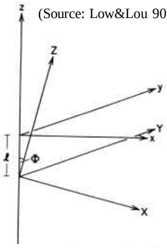
The angular functions P are computed by solving an ordinary (but nonlinear) differential equation. This can be easily done, e.g. with a Runge-Kutta method.

[redacted]onlinear force-free field in spherical geometry with symmetry in phi.

(Source: Low&Lou 90)

Nonlinear force-free fields in 2.5D
=> 3D looking fields in cartesian coordinates

A nice property of these equilibria is, that one get 3D-looking configurations by shifting and rotating geometry.
=> Model for a solar Active Region.



(Source: Low&Lou 90)

$$X = r\sin\theta\cos\phi,\ Y = r\sin\theta\sin\phi,\ Z = r\cos\theta$$

$$X = x\cos\Phi-(z+l)\sin\Phi$$

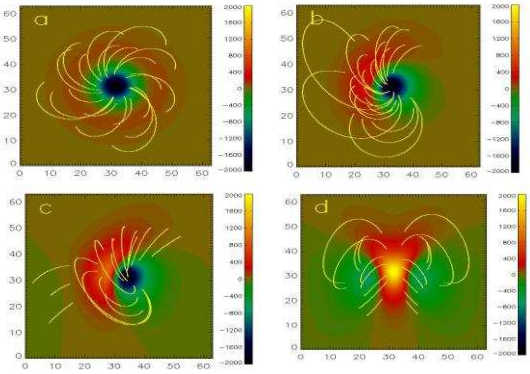
$$Y = y\,,$$

$$Z = x\sin\Phi+(z+l)\cos\Phi$$

$$\begin{aligned}
B_X &= B_r\sin\theta\cos\phi+B_\theta\cos\theta\cos\phi-B_\phi\sin\phi\,,\\
B_Y &= B_r\sin\theta\sin\phi+B_\theta\cos\theta\sin\phi+B_\phi\cos\phi\,,\\
B_Z &= B_r\cos\theta-B_\theta\sin\theta\,,
\end{aligned}$$

$$\begin{aligned}
B_x &= B_X\cos\Phi+B_Z\sin\Phi\,,\\
B_y &= B_Y\,,\\
B_z &= -B_X\sin\Phi+B_Z\cos\Phi\,.
\end{aligned}$$

Low&Lou-model active regions, PHI was varied



Some properties for force-free fields in 3D
(Molodensky 1969, Aly 1989)

$$\int_V \nabla\cdot\mathbf{B}\,d^3x=0\Rightarrow\oint_S\mathbf{B}\,d\mathbf{S}\ =0$$ Boundary flux balanced

$$\boxed{\begin{aligned}
&\int_V(\nabla\times\mathbf{B})\times\mathbf{B}\,d^3x=0\\
&\int_V\nabla\cdot T\,d^3x=0\Rightarrow\oint_S T d\mathbf{S}\ =0\\
&T_{ij}=B_iB_j-\frac{1}{2}\mathbf{B}^2\delta_{ij}
\end{aligned}}$$

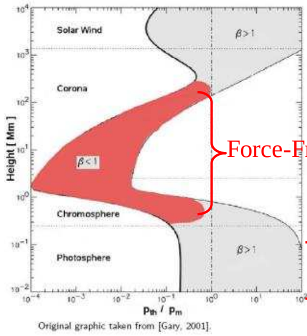
Maxwell Stress
torque

No net force on boundaries

$$\boxed{\begin{aligned}
&\int_V \mathrm{r}\times[(\nabla\times\mathbf{B})\times\mathbf{B}]\ d^3x=0\\
&\int_V\nabla\cdot\tilde{T}\,d^3x=0\Rightarrow\oint_S\tilde{T}d\mathbf{S}\ =0\\
&\tilde{T}_{ij}=\epsilon_{jkl}\,r_k\,T_{lj}
\end{aligned}}$$

No net torque on boundaries

Extrapolation domain



Original graphic taken from [Gary, 2001].

Force-Free

B-Field Measurements, non-force-free

Plasma beta

$$\beta\ =\ \frac{p_{th}}{p_m}\qquad(3)$$
$$\simeq\ \frac{n\,K_B\,T}{B^2}$$

MHD-relaxation
Chodura & Schlueter 1981

$$\begin{aligned}
\nu\mathbf{v} &= (\nabla\times\mathbf{B})\times\mathbf{B}\\
\mathbf{E}+\mathbf{v}\times\mathbf{B} &= \mathbf{0}\\
\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}\\
\nabla\cdot\mathbf{B} &= 0\\
\nu &= \frac{1}{\mu}|\mathbf{B}|^2
\end{aligned}$$

The equations are combined to

$$\boxed{\frac{\partial\mathbf{B}}{\partial t}\ =\ \mu\,\mathbf{F}_{\rm MHD}}$$

$$\mathbf{F}_{\rm MHD}\ =\nabla\times\left(\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{B^2}\right)$$

Optimization
Wheatland et al. 2000

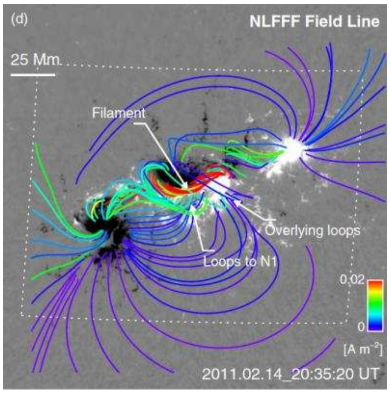
$$L\ =\ \int_V\left[B^{-2}|(\nabla\times\mathbf{B})\times\mathbf{B}|^2+|\nabla\cdot\mathbf{B}|^2\right]\ d^3V$$

Take functional derivative:

$$\Rightarrow\frac{1}{2}\frac{dL}{dt}\ =\ -\int_V\frac{\partial\mathbf{B}}{\partial t}\cdot\tilde{\mathbf{F}}\,d^3x-\int_S\frac{\partial\mathbf{B}}{\partial t}\cdot\tilde{\mathbf{G}}\,d^2x$$

$$\boxed{\frac{\partial\mathbf{B}}{\partial t}\ =\ \mu\,\mathbf{F}}$$

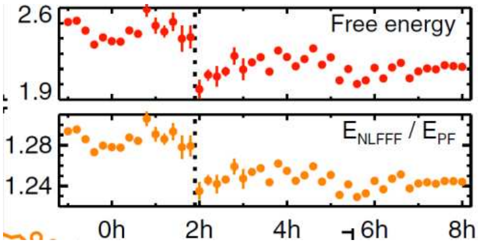
$$\begin{aligned}
\mathbf{F}\ &=\nabla\times\left(\frac{[(\nabla\times\mathbf{B})\times\mathbf{B}]\times\mathbf{B}}{B^2}\right)\\
&\quad+\left\{-\nabla\times\left(\frac{((\nabla\cdot\mathbf{B})\,\mathbf{B})\times\mathbf{B}}{B^2}\right)\right.\\
&\quad-\mathbf{\Omega}\times(\nabla\times\mathbf{B})-\nabla(\mathbf{\Omega}\cdot\mathbf{B})\\
&\quad\left.+\mathbf{\Omega}(\nabla\cdot\mathbf{B})+\mathbf{\Omega}^2\,\mathbf{B}\right\}\\
\Omega\ &=\ B^{-2}\ [(\nabla\times\mathbf{B})\times\mathbf{B}-(\nabla\cdot\mathbf{B})\,\mathbf{B}]
\end{aligned}$$



Nonlinear force-free coronal magnetic fields extrapolated with optimization method from SDO measurements. Selected field lines are colour coded by the amount of electric current they contain. 5h after this snapshot a X-class (large!) Flare occured.

Source:Sun et al., ApJ 2012

Nonlinear force-free fields in 3D



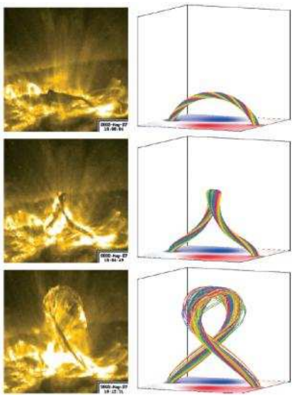
Monitoring of free magnetic energy before and after the X-class Flare (dotted vertical line) by a sequence of force-free equilibria. During the Flare magnetic energy is converted to thermal and kinetic energy => Dynamic phase, cannot be modelled by sequences of equilibria. Source:Sun et al., 2012

Time dependent evolution of force-free fields

In the limit of a vanishing plasma beta the ideal, time dependent MHD-equations are:

$$\begin{aligned}
\partial_t \rho &= -\nabla\cdot(\rho\,\mathbf{u}),\\
\rho\,\partial_t \mathbf{u} &= -\rho\,(\mathbf{u}\cdot\nabla)\,\mathbf{u} + \mathbf{j}\times\mathbf{B},\\
\partial_t \mathbf{B} &= \nabla\times(\mathbf{u}\times\mathbf{B}),\\
\mathbf{j} &= \mu_0^{-1}\nabla\times\mathbf{B}.
\end{aligned}$$

Nonlinear force-free equilibria contain free energy and can become unstable, e.g. an ideal kink-instability.
Source: Kliem and Török, ESASP 2004



Time dependent evolution of force-free fields

Left: Confined filament eruption observed with Trace in May 2002.
Right: Magnetic field lines of a kink-unstable flux robe.

Source: Kliem&Török, ApJL 2005

Movies: http://www.lesia.obspm.fr/perso/tibor-torok/res.html

Coronal plasma and magnetic field

$$\mathbf{B}\bullet\,\underbrace{(\nabla\times\mathbf{B})\times\mathbf{B}}_{\text{Lorentz force}} - \underbrace{\mu_0\nabla p}_{\text{pressure gradient}} - \underbrace{\mu_0\,\rho\,\nabla\Psi}_{\text{gravity}} = \mathbf{0}$$

for vertical scales < 0.1 R_S
(gravitational scale height) =>
Pressure constant on field lines

$$\boxed{\mathbf{B}\cdot[\nabla p} + \cancel{\rho\nabla\Psi}] = 0$$

Force-free equilibria and coronal plasma

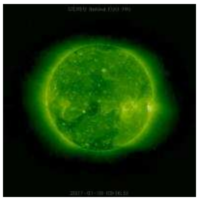
$$\underbrace{\mathbf{j}\times\mathbf{B}}_{\text{small}} = \underbrace{\nabla p + \rho\nabla\Psi}_{\text{small}}$$

$$\beta \ll 1, \quad \text{but} \quad \beta \neq 0$$

$$|\mathbf{j}\times\mathbf{B}| \ll |\mathbf{j}|\,|\mathbf{B}|$$

$$j_\perp \ll j_\parallel$$

$j_\perp$ structures the plasma.



Coronal plasma loop modelling

- The magnetic field structures the solar corona and due to the low plasma-beta we can neglect plasma effects when modelling the coronal magnetic field.
- Modeling the coronal plasma can be done with 1D-hydrodynamics models along the magnetic field lines. (Question/Exercise: Why not MHD along loops?)
- Time-dependent processes (like reconnection) are thought to be important for heating (and flares, eruptions etc.)
- Here we concentrate on stationary solution.

1D plasma loop modelling

Source: Aschwanden&Schrijver, ApJ 2002

$$\frac{1}{A}\frac{d}{ds}(nvA) = 0$$ Continuity Equation

$$mnv\frac{dv}{ds} = -\frac{dp}{ds} + \frac{dp_{\text{grav}}}{dr}\left(\frac{dr}{ds}\right)$$ Momentum Eq.

$$\frac{1}{A}\frac{d}{ds}(nvA[\epsilon_{\text{enth}} + \epsilon_{\text{kin}} + \epsilon_{\text{grav}}] + \underbrace{AF_C}_{\text{conduction}}) = \underbrace{E_H}_{\text{heating}} + \underbrace{E_R}_{\text{radiative cooling}}$$

Energy Eq.

A is the loop cross section and s a coordinate along the loop.

$$\epsilon_{\text{enth}}(s) = \tfrac{5}{2}k_B T(s) \qquad \epsilon_{\text{kin}}(s) = \tfrac{1}{2}mv^2(s)$$

$$\epsilon_{\text{grav}}(r) = -\frac{GM_\odot m}{r} = -mg_\odot\left(\frac{R_\odot^2}{r}\right)$$

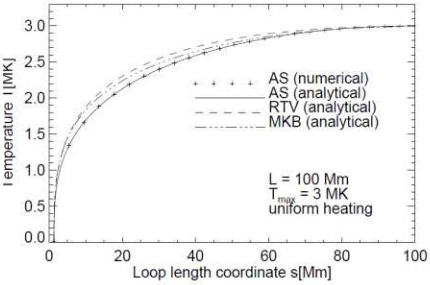
$$F_C(s) = \left[-\kappa T^{5/2}(s)\frac{dT(s)}{ds}\right]$$ conductive flux

$$E_H(s) = E_0 \exp\left(-\frac{s}{s_H}\right)$$ heating rate, not very well known, E_0 is heating at base

$$E_R(s) = -n_e^2(s)\Lambda[T(s)]$$ radiative cooling, proportional to square of electron density and a factor (Temperature dependent, not well known)

Source: Aschwanden&Schrijver 2002

Analytic approx. of the numerical solution



$$T(s) = T_{\max}\left[1 - \left(\frac{L-s}{L-s_0}\right)^a\right]^b$$

Computing the numerical solution is expensive and analytic approximations useful. (L is loop half-length, a and b are fitting parameters.)

Source: Aschwanden&Schrijver 2002

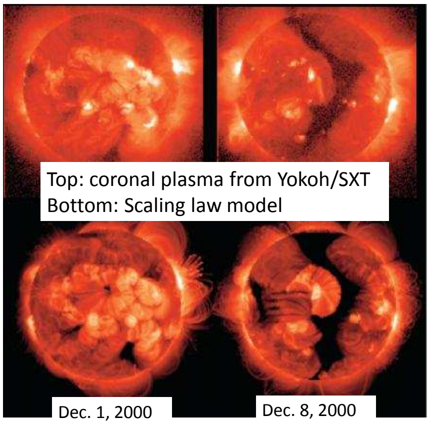
Scaling laws

- Analytic expression help to compute the plasma along many field lines, which is numerically very expensive. Plasma pressure can be derived by integration.
- Analytic models are simplifications, based on circular loops and static equilibria without flow.
- In a subsequent work Schrijver et al. 2005 derived scaling laws and related the heating flux density to magnetic field strength at loop base B_base and loop half length L:

$$F_H = \alpha B_{\text{base}}^{\beta} L_{\text{half}}^{\lambda} f(B_{\text{base}})$$

- Best agreement was found (in CGS-units) for:

$$F_H \approx 4\times10^{14} B^{1.0\pm0.3} / L^{1.0\pm0.5}$$



Top: coronal plasma from Yokoh/SXT
Bottom: Scaling law model

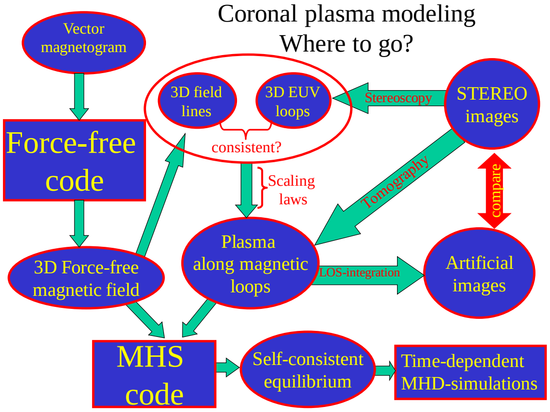
Dec. 1, 2000

Dec. 8, 2000

The coronal plasma model in the bottom images is based on a potential field model for the magnetic field and scaling laws for the plasma.

(Source: Schrijver et al. 2005)

Coronal plasma modeling
Where to go?



Vector magnetogram

Force-free code

3D Force-free magnetic field

3D field lines

3D EUV loops

consistent?

Stereoscopy

STEREO images

Scaling laws

Tomography

compare

Plasma along magnetic loops

LOS-integration

Artificial images

MHS code

Self-consistent equilibrium

Time-dependent MHD-simulations

Summary: coronal plasma

- Due to the low plasma-beta in the solar corona the coronal magnetic field is force-free.
- With nonlinear force-free models we can monitor the total and free energy of active regions.
- Configurations with free energy can become unstable and erupt => Flares, mass ejections.
- Eruptive phase can be modelled with time-dependent MHD-simulations.
- Coronal plasma is frozen into the magnetic field and can be modelled by 1D hydrodynamics along the magnetic field lines.

Exercises for Space Plasma Physics:
XI. Solar Corona

1. Why is the solar magnetic field important for the structure and dynamics of the solar corona?

2. How can the MHD-equations be simplified to study the evolution of the solar corona in quiet and active times?

3. The solar corona is much hotter as the solar surface. How can this happen. Our normal life experience tells us that it becomes colder as farer one moves away from a heat source.

4. What is the difference between an ideal and resistive MHD-instability?

5. If plasma and magnetic field (plasmoids) are ejected in a coronal mass ejection, magnetic reconnection must occur at some point. Why? It is not entirely clear, however, if reconnection is the driver of the eruption or just occurs in the aftermath. Describe briefly both scenarios.

6. Is the heat conduction parallel and perpendicular to magnetic loops the same or different?

7. What are scaling laws and are they useful?

Space Plasma Physics

Thomas Wiegelmann, 2012

Physical Processes

8. Plasma Waves, instabilities and shocks
9. Magnetic Reconnection

Applications

10. Planetary Magnetospheres
11. Solar activity
12. Transport Processes in Plasmas

Transport in neutral gases

- Transport processes in gases have first been studied by Maxwell and Boltzmann, the founders of kinetic theory.
- Aim of transport theory is to understand non-equilibrium processes from first principles.
- For neutral gases this approach was successful in, e.g., pridicting of transport coefficients like:
  - diffusion
  - heat conduction
  - viscosity
  -thermo-diffusion (cross-coefficients for different species)
- Motor for irreversibility are (mainly binary) collisions.

Transport in plasmas

- Theory was driven mainly to understand fully ionized plasmas in fusion experiments (since mid 20th century)
- Naturally scientists applied the well known methods from gas-kinetic-theory to plasmas.
- But there are two major differences

1.) Plasma contains charged particles and (long-ranging) Coulomb interaction dominate interaction of particles.

2.) External electric and magnetic fields act on the charged particle.

Classical transport theory in plasmas

- A formalism (Chapman-Enskog theory for multicomponent gases) treats non-equilibrium gases as (small) perturbations from Maxwell-Boltzmann distribution.
- Incorporating electromagnetic fields, Coulomb-collisions etc. in this formalism lead to the **classical transport theory for plasmas** (Braginskii 1965, Balescu 1988)
- So far so good. But then the theory was compared with experiments and this was really disappointing:
- Classical transport theory grossly underestimated the transport-coefficients, often **by several orders of magnitude!!**

Transport in space-plasmas

- Limitations of classical transport theory apply also to space plasmas.
- Classical theory was succesfull to calculate transport coefficients like conductivity (or it's inverse the resistivity) for weakly ionized plasmas like the ionosphere. (Spitzer resistivity)
- For fully ionized space plasmas we face the same problem as in fusion theory: the transport coefficients (say resistivity) are orders of magnitudes to low.
- From classical theory most space plasmas would be ideal and resistive processes (like reconnection) would not play any role.

Plasma resistivity in partly ionized plasma

In the presence of collisions we have to add a collision term in the equation of motion. Assume collision partners moving at velocity ***u***.

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - m\nu_c(\mathbf{v} - \mathbf{u})$$

In a steady state collisional friction balances electric acceleration. Assume there is no magnetic field, **B = 0**. Then we get for the electrons with ions at rest:

$$\mathbf{E} = -\frac{m_e\nu_c}{e}\mathbf{v}_e$$

Since electrons move with respect to the ions, they carry the current density, **j** = $-en_e\mathbf{v}_e$. Combining this with the above equation yields, **E** = η **j**, with the resistivity:

$$\eta = \frac{m_e\nu_c}{n_e e^2}$$

Conductivity in a magnetized plasma

In a steady state collisional friction balances the Lorentz force. Assume the ions are at rest, $\mathbf{v}_i = \mathbf{0}$. Then we get for the electron bulk velocity:

$$\mathbf{E} + \mathbf{v}_e \times \mathbf{B} = -\frac{m_e \nu_c}{e}\mathbf{v}_e$$

Assume for simplicity that, $\mathbf{B}=B\mathbf{e}_z$. Then we can solve for the electron bulk velocity and obtain the current density, which can in components be written as:

$$\begin{aligned} j_x &= \sigma_0 E_x + \frac{\omega_{ge}}{\nu_c} j_y \\ j_y &= \sigma_0 E_y - \frac{\omega_{ge}}{\nu_c} j_x \\ j_z &= \sigma_0 E_z \end{aligned}$$

$$\sigma_0 = \frac{n_e e^2}{m_e \nu_c}$$

The current can be expressed in the form of Ohm's law in vector notation as: $\mathbf{j} = \boldsymbol{\sigma}\,\mathbf{E}$, with the dyadic conductivity tensor $\boldsymbol{\sigma}$.

Conductivity in a magnetized plasma

For a magnetic field in *z* direction the conductivity tensor $\boldsymbol{\sigma}$ reads:

$$\sigma = \begin{pmatrix} \sigma_P & -\sigma_H & 0 \\ \sigma_H & \sigma_P & 0 \\ 0 & 0 & \sigma_{\|} \end{pmatrix}$$

$$\mathbf{j} = \sigma_{\|}\mathbf{E}_{\|} + \sigma_P \mathbf{E}_{\perp} - \sigma_H (\mathbf{E}_{\perp} \times \mathbf{B})/B$$

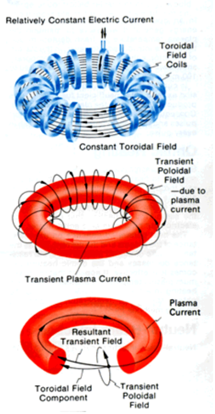
The tensor elements are the *Pedersen*,σ_D, the *Hall*,σ_H, and the *parallel* conductivity. In a weak magnetic field the Hall conductivity is small and the tensor diagonal, i.e. the current is then directed along the electric field.

$$\begin{aligned} \sigma_P &= \frac{\nu_c^2}{\nu_c^2 + \omega_{ge}^2}\sigma_0 \\ \sigma_H &= -\frac{\omega_{ge}\nu_c}{\nu_c^2 + \omega_{ge}^2}\sigma_0 \\ \sigma_{\|} &= \sigma_0 = \frac{n_e e^2}{m_e \nu_c} \end{aligned}$$

Successful method for weakly ionized plasmas like inosphere

Which assumptions of classical transport theory are invalid in fully ionized plasma?

- 1.) Collisions cannot be considered as binary interactions due to the long ranging Coulomb force.
- => For weakly non-ideal (fusion) plasma this problem was treated by the Lenard-Balescu equation, which takes the many-particle character of Coulomb collisions (Debye-Shielding) into account.
- 2.) Influence of external magnetic fields on the free motion of particles in fusion devices (toroidal magnetic field and space plasmas.
- => Gyration, Drifts etc. which we studies in test-particle approach. But these effects have large influence on computing transport coefficients.



Tokamak

A Tokamak contains strong magnetic fields in a toroidal geometry in order to trap plasma for nuclear fusion.

But as we just learned: transport coefficients have been underestimated by orders of magnetide
⇒ Plasma transported away and not trapped long enough for fusion

Neo-classical transport theory

- Strong and inhomogenous magnetic fields occur both in tokamaks (fusion machines) and space plasmas.
- Effect of these strong magnetic fields has been first investigated for toroidal magnetic fields. This research was clearly dedicated for fusion plasmas.
- Combining the effects of strong magnetic fields with methods of classical plasma transport lead to significantly higher transport coefficients.
- The new approach was dubbed **neo-classical transport theory**. (Galeev&Sgdeec 1968, Hinton & Hazeltine 1976, Balescu 1988)

Neo-classical transport theory

- Neo-classic theory was an important step forward in fusion theory (never become that popular for space plasmas) and showed reasonable agreement with some experiments.
- But: Still not satisfactory to explain the often observed leaks in tokamaks.
- Problem: A real plasma is never in a quiescent state (like the static and stationary equilibria we calculated) but collective nature of plasma dominates it's behaviour.
- At high temperatures (both in fusion and space plasmas) particle collisions are less important than self-organized effects like waves, vortices, modes etc.

Collective effects

- Such self-organized coherent structures are more efficients to tranport plasma and energy than interaction of individual particles. (For space-plasma physicists it's interesting to study these dynamics, but fusion-people loose the plasma in their machines due to these effects => no fusion.)
- Remember the many waves we studied in space plasmas (and we studied only a small fraction of all known plasma waves and in simplified geometries)
- Some waves become unstable (growing amplitudes) and we studied instabilities mainly with linear theory.
- For large amplitudes nonlinear effects become important.

Complexity and turbulence

- For low aplitude waves linear superposition can be used to study the combined effects.
- As amplitudes grow, nonlinear interactions become very important => One gets a large number of new modes, saturation of amplitudes.
- => The plasma becomes more and more complex and it's behavious unpredictable.
- => Turbulent or chaotic motion.
- In chaos small differences in location and velocity can influence significantly the evolution (like the butterfly-effect in weather prediction).

Non-equilibrium steady state

- Under quiet condition the random collisions in a gas lead to a steady state thermal equilibrium.
- For the Boltzmann-equation (Vlasov-equation with Boltzmann collision term) it has been proofed that the final stationary state solution is the Maxwell-Boltzmann distribution for the distribution functions.
- Turbulent, collective interaction (e.g. of particles and waves) will lead to a non-equilibrium steady state.
- Turbulent dissipation becomes very important and does of course influence (or even dominate!) the transport coefficients (additional to classical and neo-classical effects)
- => **Anomalous Transport Theory**

Anomalous vs. strange transport

- In anomalous transport theory the transport coefficients are computed from turbulent/chaotic collective particle motions.
- In many cases the relations between macroscopic plasma variables remain simple, e.g. a linear dependence of fluxes and forces.
- One can still use fluid theory, but with transport coefficients computed by turbulent theory.
  => Macroscopic laws are the same as in classical theory
- In some cases, however, turbulence can lead to strange results, e.g. the transport coefficients are either zero or infinity => **strange transport**.
  (maybe called strange because we not understand it)

Anomalous transport: Quasi linear theory

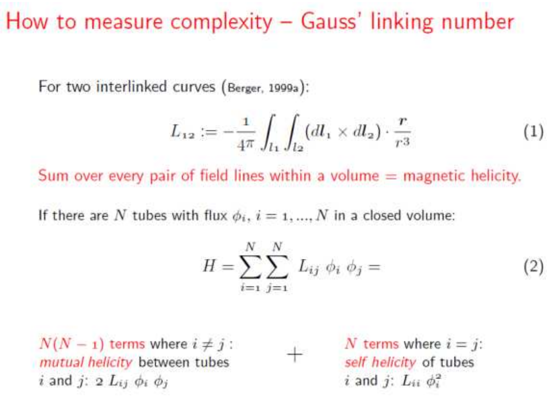
- Quasi-linear theory, valid for weak turbulence.
- Particle distribution function and fields can be split into a slowly evolving part (f_{s0}, $\mathbf{E}_0$, $\mathbf{B}_0$) and small fluctuations (δf_s, $\delta\mathbf{E}$ and $\delta\mathbf{B}$)
- We get the Vlasov equation for the slowly evolving part:

$$\frac{\partial f_{s0}}{\partial t} + \mathrm{v}\cdot\nabla f_{s0} + \frac{q_s}{m_s}(\mathrm{v}\times\mathrm{B})\cdot\frac{\partial f_{s0}}{\partial \mathrm{v}} = -\frac{q_s}{m_s}\left\langle(\delta\mathrm{E}+\mathrm{v}\times\delta\mathrm{B})\cdot\frac{\partial \delta f_s}{\partial \mathrm{v}}\right\rangle$$

- As we calculated in the excercises, the entropy remains constant in Vlasov systems. If one defines, however, an entropy only for the slowly evolving part, this (new defined) entropy can increase and be used to measure the disorder of a system

Numerical simulations

- In many cases we cannot apply quasi-linear theory or other analytic methods and use numerical simulations.
- Are **simulations** theory or experiments?
- For experimental physicists (or observers in space physics) often everything not related to experiments or observations is theory.
- But: Often not true and we speak of so called: **numerical experiments**. Simulations produces often such huge amount of data, that methods similar as for observational data are used to analyze them.
- If we able to correctly simulate a physical system, this does not necessarily mean that we understand all physics. (But it helps, as we can switch on/off physical terms.)



How to measure complexity – Gauss' linking number

For two interlinked curves (Berger, 1999a):

$$L_{12} := -\frac{1}{4\pi}\int_{l_1}\int_{l_2}(d\boldsymbol{l}_1 \times d\boldsymbol{l}_2)\cdot\frac{\boldsymbol{r}}{r^3} \qquad (1)$$

Sum over every pair of field lines within a volume = magnetic helicity.

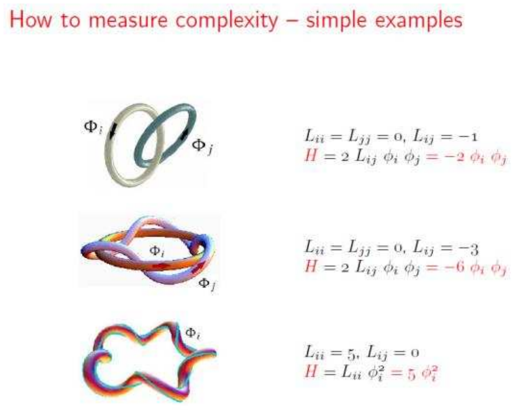
If there are N tubes with flux ϕ_i, $i = 1, ..., N$ in a closed volume:

$$H = \sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}\ \phi_i\ \phi_j = \qquad (2)$$

$N(N-1)$ terms where $i \neq j$:
mutual helicity between tubes
i and j: $2\ L_{ij}\ \phi_i\ \phi_j$

$+$

N terms where $i = j$:
self helicity of tubes
i and j: $L_{ii}\ \phi_i^2$

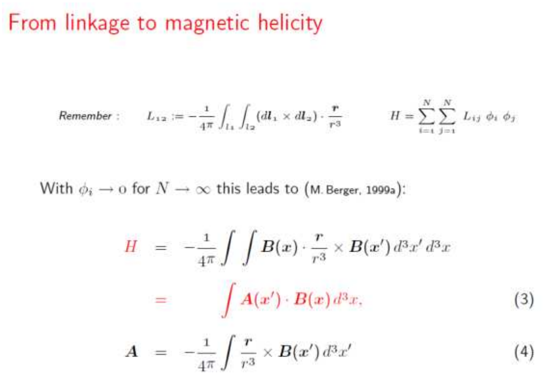


How to measure complexity – simple examples

$L_{ii} = L_{jj} = 0$, $L_{ij} = -1$
$H = 2\ L_{ij}\ \phi_i\ \phi_j = -2\ \phi_i\ \phi_j$

$L_{ii} = L_{jj} = 0$, $L_{ij} = -3$
$H = 2\ L_{ij}\ \phi_i\ \phi_j = -6\ \phi_i\ \phi_j$

$L_{ii} = 5$, $L_{ij} = 0$
$H = L_{ii}\ \phi_i^2 = 5\ \phi_i^2$



From linkage to magnetic helicity

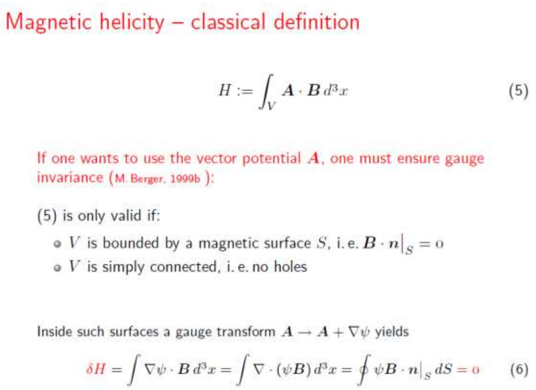
Remember : $L_{12} := -\frac{1}{4\pi}\int_{l_1}\int_{l_2}(d\boldsymbol{l}_1 \times d\boldsymbol{l}_2)\cdot\frac{\boldsymbol{r}}{r^3}$ $\qquad H = \sum_{i=1}^{N}\sum_{j=1}^{N} L_{ij}\ \phi_i\ \phi_j$

With $\phi_i \to 0$ for $N \to \infty$ this leads to (M. Berger, 1999a):

$$H = -\frac{1}{4\pi}\int\int \boldsymbol{B}(\boldsymbol{x})\cdot\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{B}(\boldsymbol{x}')\,d^3x'\,d^3x$$

$$= \int \boldsymbol{A}(\boldsymbol{x}')\cdot\boldsymbol{B}(\boldsymbol{x})\,d^3x, \qquad (3)$$

$$\boldsymbol{A} = -\frac{1}{4\pi}\int\frac{\boldsymbol{r}}{r^3}\times\boldsymbol{B}(\boldsymbol{x}')\,d^3x' \qquad (4)$$



Magnetic helicity – classical definition

$$H := \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,d^3x \qquad (5)$$

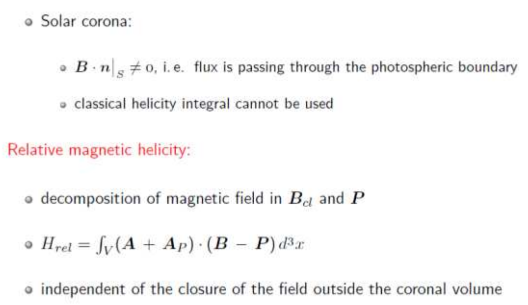
If one wants to use the vector potential $\boldsymbol{A}$, one must ensure gauge invariance (M. Berger, 1999b):

(5) is only valid if:

- V is bounded by a magnetic surface S, i. e. $\boldsymbol{B}\cdot\boldsymbol{n}|_S = 0$
- V is simply connected, i. e. no holes

Inside such surfaces a gauge transform $\boldsymbol{A} \to \boldsymbol{A} + \nabla\psi$ yields

$$\delta H = \int \nabla\psi\cdot\boldsymbol{B}\,d^3x = \int \nabla\cdot(\psi\boldsymbol{B})\,d^3x = \oint \psi\boldsymbol{B}\cdot\boldsymbol{n}|_S\,dS = 0 \qquad (6)$$



- Solar corona:
  - $\boldsymbol{B}\cdot\boldsymbol{n}|_S \neq 0$, i. e. flux is passing through the photospheric boundary
  - classical helicity integral cannot be used

Relative magnetic helicity:

- decomposition of magnetic field in $\boldsymbol{B}_{cl}$ and $\boldsymbol{P}$
- $H_{rel} = \int_V(\boldsymbol{A} + \boldsymbol{A}_P)\cdot(\boldsymbol{B} - \boldsymbol{P})\,d^3x$
- independent of the closure of the field outside the coronal volume

Magnetic Helicity

- Magnetic helicity is a measure on the complexity of magnetic fields.
- Helicity is conserved for ideal MHD-processes and also for some resistive processes like 2D reconnection.
- For 3D reconnection, the helicity can change, but is dissipated much slower as the energy.
- Approximate helicity conservation constrains (additional to energy conservation) which physical processes are possible.
- E.g., A nonlinear force-free magnetic field cannot relax to a potential-field (which would have zero helicity), but only to a linear force-free field with same helicity. (Or the helicity must be transported away, e.g. by a coronal mass ejection).

Fluid vs. Vlasov approach

- In Vlasov theory we study distribution functions f(x,v,t) in 6D-phase space.
- In Fluid equations we deal with macroscopic plasma variables like density n(x,t), flow velocity, V(x,t), plasma pressure p(x,t) in the 3D configuration space.
- What is gained and lost by these approaches?
- The fluid equations are simpler and easier to solve.
- The Vlasov approach is more complete and includes the microphysics.

Fluid vs. Vlasov approach
What is gained by fluid approach?

- It is easy to extend the fluid equations to include collisions, e.g. momentum transfer between ions and electrons or also the inclusion of neural gas atoms. (Computing a collision term in kinetic theory is much more complicated, in particular for anomalous transport)
- One can simplify the fluid equations further by reasonable (for studied system) assumption, e.g. using a scalar pressure instead of tensor or neglection the pressure for low-Beta force-free fields.

Fluid vs. Vlasov approach
What is lost in the fluid approach?

- Fluid equations are not closed (momentum hierarchy) => We have more variables than equations.
- Often it is not easy, however, to make assumptions for pressure, equation of state or energy equation.
- If we can solve f(x,v,t) from Vlasov theory however, computing the pressure tensor and higher moments is just a simple integration => One can deduce an equation of state from Vlasov theory.
- Microphysics effects are lost in fluid theory, in particular all effects in velocity space like Landau damping and kinetic instabilities.

What is missing in Vlasov theory?

- Vlasov equation is basically a fluid flow in 6D-phase-space
- Not included in Vlasov theory are all effects, where the discreteness of the plasma (that plasma is a collection of individual particles) is important:
- What is the level of electromagnetic fluctuations in a stable plasma?
- What is the emission rate and spectrum of radiation from a plasma?
- What is the cross section for scattering of radiation from a plasma?
- => Need higher order kinetic equations.

Space Plasma Physics
Th. Wiegelmann
Summer Semester 2012

Links to movies

- Solar wind's effects on Earth
- Charge in uniform magnetic field
- Magnetic Bottle-1
- Magnetic Bottle-2
- Radiation Belt as Magnetic Bottle
- Radiation belt song
- Debye Sphere
- Alfven wave in corona observed by Hinode
- Alfven waves in corona
- Rayleigh-Taylor instability
- Kelvin Helmholtz
- Landau damping
- Substorm
- Magnetic Reconnection (Hesse)
- Vortex Reconnection
- Solar eruptions, Kink-instability
- Tokamak